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CONSEQUENCES OF TERZAGHI'S EFFECTIVE STRESS DECOMPOSITION IN THE CONTEXT OF FINITE STRAIN PORO-MECHANICS

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Abstract. Poro-mechanics is a branch of mechanics considering the hydro-mechanical behaviour of a porous solid medium whose pores are saturated by a fluid. The presence of both these constituents significantly influences the overall macro-response of the material.

Regardless of the application, Terzaghi's effective stress decomposition is a well-established hypothesis regarding how the total stress tensor is decomposed into a part directly related to strain (effective stress) and a part borne by the fluid phase (interstitial pressure). The assumption of an incompressible solid medium physically justifies Terzaghi's effective decomposition. However, under the further assumption of finite strain mechanics, a more restrictive constraint on the volume change arises, which the vast majority of formulations neglect.

This work details how disregarding this constraint on the volume change can lead to the violation of solid mass balance, which can be counted among the fundamental principles of continuum mechanics. Furthermore, to address this issue practically, an ad hoc stress-strain relationship is designed to respect Terzaghi stress decomposition and solid mass balance in the context of finite-strain hyper-elasticity.

Key words: *Poro-mechanics; Finite strain-mechanics; Mass conservation; Hyper-elasticity*

1 Introduction

A plethora of materials with a wide range of applications see their mechanical behaviour as severely influenced by the presence of an interstitial fluid. The subject that aims to study this type of material is poromechanics, which Biot first described in [1], and it has been later expanded to finite strain.

If focusing on the hydro-mechanical behaviour of porous media, three mechanisms govern the volume change: the compressibility of the two constituents (i.e., the solid and the fluid phases) and the allowed drainage. In particular, this last phenomenon deals with the conditions applied on the boundary of the sample, which, in turn, are a function of the considered problem. On the other hand, modelling the constituents as compressible (or not) depends on the specific material. In geomechanics, it is often assumed that both these constituents, i.e., the soil grains and the water, are incompressible since the drainage conditions drive the main volume change. In particular, this work investigates how the assumption of incompressible solid phase and the definition of the Eulerian porosity n constraint the values of the Jacobian J[1]. When violated, the solid mass is not conserved, and the considered equations lose physical meaning. The following developments heavily draw on Pretti *et al.* [2].

[1] A rigorous definition of the Eulerian porosity n and the Jacobian J are given below in Section 2.

2 Main assumptions and framework

When considering a material composed of two phases, there exist multiple assumptions regarding each constituent and the way they interact. This work adopts the following assumptions:

A.1 thermal effects are not considered;

A.2 one fluid fully saturates the solid porous skeleton. These are juxtaposed continua;

A.3 the phases do not exchange mass;

A.4 the solid phase is incompressible;

A.5 the fluid is non-viscous;

A.6 the material undergoes finite strain deformations and rotations;

A.7 the considered continuum is isotropic;

Owing to hypothesis **A.6**, the considered material's configuration varies with time. Hence, $\boldsymbol{x}$ denotes the position shared by the two constituents in the current configuration (assumption **A.2**). $\boldsymbol{X}$ indicates the original position of the solid phase, while the original position of the fluid is unnecessary for this work. A mixed particle[2] occupies an initial volume $d\Omega = d\Omega^{sk} \cup d\Omega^{f}$ and a current one $d\omega = d\omega^{sk} \cup d\omega^{f}$. Since the solid phase is described in a Lagrangian way, the same particle is considered for the volumes $d\Omega^{sk}$, $d\omega^{sk}$, while fluid phase volumes $d\Omega^{f}$ and $d\omega^{f}$ are not occupied by the same fluid. The motion of the mixed particle is therefore tracked by following its solid constituent, i.e., $\boldsymbol{x} = \boldsymbol{\varphi}\left(\boldsymbol{X}^{sk}, t\right)$, with $\boldsymbol{\varphi}$ being the mapping between the original and current configurations. This mapping allows to define the deformation gradient $\boldsymbol{F} := \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}^{sk},t)}{\partial \boldsymbol{X}^{sk}}$, whose determinant, named Jacobian, describes the volume change of the mixed particle, i.e., $d\Omega = J\, d\omega$. For mathematical (invertibility of the mapping) and physical reasons (volumes cannot be negative), the Jacobian must satisfy the condition $J > 0$. To describe which portion of the current volume is occupied by the fluid phase, the Eulerian porosity is defined as $n := \frac{d\omega}{d\omega^{f}}$, with n_0 being its initial value. The definition of the Eulerian porosity entails that its values are always bounded between zero and one, i.e.,

$$0 < n < 1. \tag{1}$$

As demonstrated by Borja and Alarcón [3] for assumption **A.6**, the simultaneous adoption of the above assumptions **A.1**-**A.5** leads to the Terzaghi effective decompositions, i.e., the total stress tensor can be additively decomposed into an effective part (related to a strain measure) and a fluid interstitial pressure.

3 Constraint on the Jacobian

Owing to hypothesis **A.3**, the equations of mass conservation can be written separately for the two constituents. If the solid mass conservation is considered, this is expressed by

$$\frac{d}{dt}\Big|_{X^{sk}} \left(\int_{\omega} \rho^{sk} (1-n)\, dv \right) = 0, \tag{2}$$

where $\frac{d}{dt}\Big|_{X^{sk}} (\bullet) = (\dot{\bullet})$ indicates the material derivative tracking the solid phase, and ρ^{sk} the current solid density. After a few manipulations and owing to assumption **A.4**, the above equation can be rewritten as

$$n = 1 - \frac{1}{J}(1 - n_0). \tag{3}$$

[2] A mixed particle is a particle in which, according to hypothesis **A.2**, both consistents coexist.

If the inequalities presented in (1) are substituted in the above equation, it follows that

$$\begin{cases} J > (1-n_0); & \text{(4a)} \\ \dfrac{1}{J}(1-n_0) > 0. & \text{(4b)} \end{cases}$$

While (4b) is trivially satisfy by $J > 0$ and $0 < n_0 < 1$, it can be seen that (4a) places a more restrictive constraint on the Jacobian. Inequality (4a) suggests that the progressive expulsion of the fluid makes the material gradually evolve only into its solid constituent, incompressible as assumed by **A.4**.

4 Hyper-elastic constitutive relationship

There are different ways to include the constraint provided by Eq. (4a) into the equations. Well-established practices to (weakly) include a constraint are the Lagrange multiplier or the penalty method (see to Pretti *et al.* [2] for a list of advantages and disavantages of these).

Regardless, this work adopts another technique to enforce the constraint (4a), which consists of modifying the stress-strain relationship. This adaptation is remarkably straightforward for those materials exhibiting an decomposition into a volumetric and the deviatoric part. If an isotropic (assumption **A.7**) Hencky material is considered, its effective free energy function can be adapted to include (4a) as follows

$$\hat{\Psi}^{sk}_{inc}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \frac{K}{2n}(\varepsilon^e_v)^2 + \frac{3}{2}G\left(\varepsilon^e_q\right)^2, \quad \text{with } \varepsilon_v := \ln J, \quad \text{and } \varepsilon_q := \sqrt{\frac{2}{3}\left(\boldsymbol{\varepsilon} - \frac{\varepsilon_v}{3}\boldsymbol{I}^{(2)}\right) : \left(\boldsymbol{\varepsilon} - \frac{\varepsilon_v}{3}\boldsymbol{I}^{(2)}\right)} \quad (5)$$

where the invariants of $\boldsymbol{\varepsilon} := \frac{1}{2}\ln(\boldsymbol{F}\boldsymbol{F}^T)$ have been adopted, $\boldsymbol{I}^{(2)}$ denotes the second-order identity tensor, and **:** the double contraction. $K > 0$ and $G > 0$ indicate the bulk parameter and the shear modulus.

5 Numerical example & Conclusions

Further assumptions: the above constitutive relationship has been implemented into a Material Point Method (MPM) $\boldsymbol{u} - p^f$ formulation. Further assumption necessary to run the example are as follows:

- **A.8** the fluid barotropic and obeys to this constitutive law $\dot{\rho^f} = \frac{\rho^f}{K^f}\dot{p^f}$, with ρ^f being the current fluid density, K^f the fluid bulk modulus and p^f Cauchy fluid pressure;
- **A.9** the flow exhibits low *Reynolds numbers*, which permits to consider the *Darcy* equation for the fluid flow, i.e., $\boldsymbol{q}^f = -\frac{\kappa}{g}\left(\frac{\partial p^f}{\partial \boldsymbol{x}} - \rho^f \boldsymbol{f}\right)$, with $\boldsymbol{q}^f$ being the relative fluid flux, κ the hydraulic conductivity, $\frac{\partial \bullet}{\partial \boldsymbol{x}}$ the gradient with respect to the current position, and $\boldsymbol{f}$ the self-weight acceleration;
- **A.10** hydraulic conductivity obeys to the *Kozeny-Carman* formula, $\kappa = c\frac{n^3}{(1-n)^2}$, with c constant.

Example scope: the aim of this example is to demonstrate the difference between a standard Hencky material and one described by Eq. (5). **Setup:** the column illustrated in Figure 1 is subjected to a gravitational acceleration $b = 1{,}000$ m s^{-2}, which ramps linearly from the start of the simulation until the $1{,}000$ steps and kept constant for the remaining $2{,}000$ steps of the simulation[3]. The bulk modulus for the original Hencky material $\bar{K}$ is set up to match the initial tangent elastic modulus given by Eq. (5),

[3]The reason for such a substantial acceleration lies in the opportunity to cut the costs of the simulation. Since the difference between the two materials lies in the effective stresses, the column must consolidate, and this load accelerate this process.

Table 1: Parameters considered in the analyses in Section 5.

Solid phase		Fluid phase	
G	$3 \cdot 10^5$ Pa	K^f	$2.2 \cdot 10^9$ Pa
ρ^s	2650 kg m^{-3}	ρ_0^f	1000 kg m^{-3}
Porous material			
κ_0		$1 \cdot 10^{-5}$ m s^{-1}	
n_0		0.3	
Simulation Parameters			
H, n_z^{els}		1 m, 20	
mmp		4	

† mmp is the number of material points per direction per element.

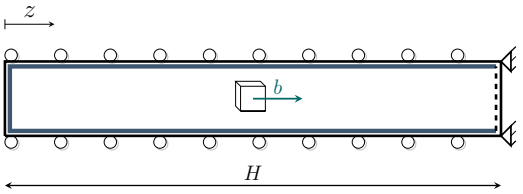
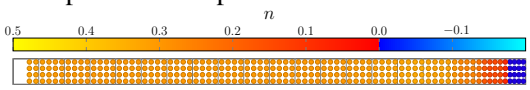
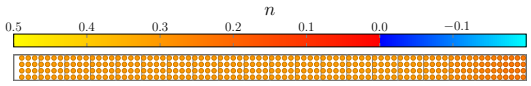


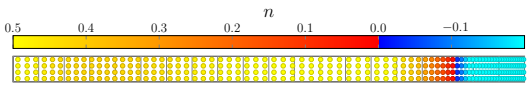
Figure 1: Illustration of the elastic column under self-weight. Rollers are applied on the top and bottom sides, while atmospheric pressure is applied at the right-hand side of the problem. Figure reproduced from Pretti *et al.* [2].



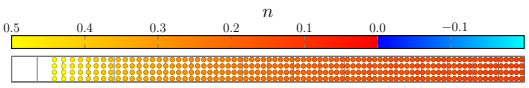
(a) Eulerian porosity for the original Hencky material, $\tilde{t} = 157^{th}$ time-step.



(b) Eulerian porosity for the improved Hencky material, $\tilde{t} = 157^{th}$ time-step.



(c) Eulerian porosity for the original Hencky material, final time-step.



(d) Eulerian porosity for the improved Hencky material, final time-step.

Figure 2: Columns at $\tilde{t} = 157^{th}$ step and the end of each simulation. Figure from Pretti *et al.* [2].

i.e., $K = \bar{K} n_0 = 5 \cdot 10^5$ Pa. **Results discussion:** as it can appreciated from Figure 2a, the original Hencky material results in negative porosities quite early in the analysis. From this point, this simulation loses physical meaning, and the column surprisingly swells before failing (Figure 2c). On the other hand, the new material does not fail for the quite sustantial time of the simulation (Figures 2d). **Conclusions:** Overall, when considering an incompressible solid phase of a porous material, the Jacobian is subjected to a more severe constraint, inherited from the Eulerian porosity. Disregarding this restriction leads to the solid mass violation. A material respectful of this law was introduced and its effectiveness tested.

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