Graphs with minimum degree-entropy*

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Abstract

We continue studying extremal values of the degree-entropy, which is an information-theoretic measure defined as the Shannon entropy based on the information functional involving vertex degrees. For a graph with a given number of vertices and edges achieving the minimum entropy value, we show its unique structure. Also, a tight lower bound for the entropy in bipartite graphs with a given number of vertices and edges is proved. Our result directly derive the result of Cao et al. (2014) that for a tree with a given number of vertices, the minimum value of the entropy is attained if and only if the tree is the star.

Keywords: Complexity measure; Graph entropy; Extremal value

1 Introduction

All graphs considered in this paper are finite, simple and undirected. The logarithms here are base 2. We use the convention that $0 \log 0 = 0$.

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A system consisting of many interacting microscopic components seems to be complex always [33]. In order to describe the structural complexity or the information content of a system, various graph entropies were introduced (refer to [4, 16] for reviewing). Reshevsky is the pioneer to quantify the complexity of a system by the so-called topological information content, which is the earliest graph entropy measurement [32]. Since then, various graph invariants, for instance, degrees, distances and the number of subgraphs [8, 9, 11, 36], have been manipulated to construct entropy based measurements. Compared with other entropies, Cao et al. [9] proposed an easily computable graph entropy called degree-entropy, which is of significance to apply in mathematical chemistry, information theory and complexity networks [2, 3].

The study of graph entropy, especially degree-entropy, is of great significance at national level as it affects multiple domains and enhances understanding of complex systems. Investigating graph entropy is relevant for assessing vulnerabilities in military and defense networks. It helps identify potential weak points, understand the impact from targeted attacks, and improve overall security for national defense infrastructure. Structural entropy extends degree-based entropy by helping identify vulnerable nodes and assesses overall network resilience [24, 38]. Degree-related entropy offers a quantitative measure for determining the structural intricacy and connectivity within software systems. This information proves valuable when making informed decisions regarding software architecture, maintenance, security, and optimization efforts that ultimately contribute to developing more robust and efficient software solutions. For example, the degree distribution entropy is employed as a method for measuring software quality [34]. Graph entropy also helps to model the spread of infectious diseases across borders, facilitating a coordinated international response and mitigating the impact on global health. The potential severity of COVID-19 was found using the entropy measurement of the pandemic tree compared to the 1918 Spanish flu [30]. The efficient functioning of global economic, social, and political activities heavily relies on interconnected communication and information networks. Graph entropy techniques facilitate the analysis of network efficiency, resilience, and susceptibilities, ensuring reliable communication pathways even during emergencies or critical situations. Quantifying data transmission through transfer entropy allows us to characterize emergent computations within this system while highlighting their association with source vertex degrees, importantly, hubs emerge as crucial sources [27].

Let $G = (V, E)$ be a graph. The degree of a vertex v in G is the number of edges incident with this vertex, denoted by $d_G(v)$. The degree-entropy is one kind of degree-based entropy defined as follows [9].

Definition 1 ([9])**.** Let k be a real number, and let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *degree-based (graph) entropy* of G is defined as

$$I_d^k(G) = -\sum_{i=1}^{n} \frac{d_G^k(v_i)}{\sum_{j=1}^{n} d_G^k(v_j)} \log \frac{d_G^k(v_i)}{\sum_{j=1}^{n} d_G(v_j)}.$$

In [9], they defined a special degree-based entropy, called first-order degree-based entropy. In this paper, the first order degree-based entropy is called degree-entropy as a shorthand, defined as follows.

Definition 2 ([9])**.** Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *degree-entropy* of G is defined as

$$I_d(G) = -\sum_{i=1}^{n} \frac{d_G(v_i)}{\sum_{j=1}^{n} d_G(v_j)} \log \frac{d_G(v_i)}{\sum_{j=1}^{n} d_G(v_j)}. \quad (1)$$

It is not hard to find that $I_d(G) = I_d^1(G)$.

This paper focuses on studying the extremal problem of degree-entropy, which can be formulated as an optimization problem to determine the maximum and minimum values of degree entropy for a graph under certain constraints. This problem belongs to the field of extremal graph theory, which lies at the intersection of extremal value combinatorics and graph theory. Investigating the extreme value problem of degree entropy is valuable for uncovering underlying principles in graph structures and advancing the theory development of extreme value combination and related aspects in graph theory. The theoretical significance of studying extremal problems on degree-entropy is profound. We focus on general graphs and bipartite graphs, which has been arisen researchers' interesting. Here we provide references to papers addressing extremal problems on different parameters on general graphs [1, 5, 10, 14, 15, 17, 31] and bipartite graphs [1, 12, 39] for interested readers.

The following is a study of the existing extremal results of degree-entropy. Cao et al. [9] proved the minimum and maximum degree-based entropies of some graph families, such as trees, unicyclic, bicyclic and chemical graphs, and proposed conjectures to determine extremal values. Shortly afterwards Ilić [22] proved one part of the conjectures. Ghalavand et al. [19] applied majorization to prove the graphs which minimize or maximize the degree-entropy for some families of graphs. By characterizing corresponding degree sequences, Dong et al. [18] characterized the extremal bipartite graphs with n vertices and m edges attaining the maximum value of the degree-entropy.

Our study enriches the existing results of the extremal problems of degree-entropy, and extend the study to general graphs and bipartite graphs. Among trees with a given number of vertices, Cao et al. [9] proved that the degree-entropy of the path attains the maximum value and the degree-entropy of the star attains the minimum value. Since there is no odd cycles in a tree, a tree is also a bipartite graph. If a complete bipartite graph is a tree, then it must be a star. This implies that it easy to derive their extremal result from our result (the star attains the minimum value) of bipartite graphs (complete bipartite graphs attain the minimum value). In terms of research methods, they use the graph operation method of moving edges for proving the extremal value [9] or some partial order relations [19]. Our proofs are mainly a synthesis of different methods including graph operations, inequalities and convex optimization methods.

Mathematical research employs various methods and techniques depending on context-specific problems addressed in papers. Although some applications may not appear immediately clear or obvious, mathematics has consistently played a crucial role in advancing scientific disciplines across diverse fields. Furthermore, mathematics provides a theoretical foundation for data mining and data management. Statistical learning refers to a set of tools for modeling and understanding complex data sets. They provide a large number of mathematical methods and language to solve and translate statistical learning problems in [23]. For example, they use the likelihood function to estimate regression coefficients based on available training data. However, it should be noted that only data cannot fully solve all mathematical problems due to inherent limitations imposed by available datasets when dealing with the large number. Mathematical research and data processing should be a complementary relationship. For example, we can use partial data results to verify that our results are correct on a small number of vertices.

An (n, m)-*graph* (resp. (n, m)-*bipartite graph*) is a graph (resp. bipartite graph) with n vertices and m edges. Any two vertices of a graph are joint by an edge is said to be the *complete graph* and denoted by K_n if it has n vertices. A bipartite graph with bipartition (X, Y) satisfying $xy \in E$ for any pair in $\{(x, y) : x \in X, y \in Y\}$ is a *complete bipartite graph* and is denoted by $K_{s,t}$ if $|X| = s$ and $|Y| = t$. The *complement* $\bar{G}$ of $G = (V, E)$ is the graph with $E(\bar{G}) = E(K_{|V(G)|}) \setminus E(G)$ and $V(\bar{G}) = V(G)$. The *union* of graphs G and G', denoted by $G \cup G'$, is the graph with $E(G \cup G') = E(G) \cup E(G')$ and $V(G \cup G') = V(G) \cup V(G')$. Let m and k be two integers, and let $k^* = \max\{k : \binom{k}{2} \leq m\}$ and $t^* = m - \binom{k^*}{2}$. By $\omega(G)$ denote the clique number of G. It is trivial that $\omega(G) \leq k^*$ if G is a graph with m edges. The graph $K(k, t)$ is obtained by adding a vertex adjacent to t vertices of K_k. Let

$$\sigma(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Among (n, m)-graphs, the graph minimizes the degree-entropy is characterized.

Theorem 1. *Let $n \geq 2$, $1 \leq m \leq \binom{n}{2}$ be integers. If G is an (n, m)-graph, then*

$$I_d(G) \geq \log(2m) - \frac{t^* k^* \log k^* + (k^* - t^*)(k^* - 1)\log(k^* - 1) + t^* \log t^*}{2m},$$

with equality if and only if $G \cong K(k^*, t^*) \cup \bar{K}_{n-k^*-\sigma(t^*)}$.

The following theorem that characterizes all the extremal graphs achieving a lower bound among $(n.m)$-bipartite graphs.

Theorem 2. *Let $n \geq 2$ and $1 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ be integers. If G is an (n, m)-bipartite graph, then $I_d(G) \geq 1 + \log \sqrt{m}$, with equality if and only if $G \cong K_{q,b} \cup \bar{K}_{n-q-b}$, where q and b satisfy $qb = m$ and $q + b \leq n$.*

The next result can be immediately deduced by Theorem 2.

Corollary 1 ([9])**.** *If T is a tree on n vertices, then $I_d(T) \geq 1 + \log \sqrt{n-1}$, with equality if and only if $T \cong K_{1,n-1}$.*

The remaining part of this paper is organized as follows. In Section 2, we give preliminary results and notations that are going to be used. Main results will be proved in Section 3. Conclude with a problem and some remarks in Section 4.

2 Preliminaries

Let $G = (V, E)$ be a graph, and let S be a subset of $V(G)$. If the number of vertices with degree d_i is a_i for $i = 0, 1, \ldots, k$, then we denote by $D(S) = [d_k^{a_k}, d_{k-1}^{a_{k-1}}, \ldots, d_1^{a_1}]$ the degree sequence of S in which $0 = d_0 < d_1 < \cdots < d_k$ and $a_0 + a_1 + \cdots + a_k = |S|$. We use $D(G)$ to represent the degree sequence $D(V(G))$. A *graphical sequence* is the degree sequence of a simple graph. A graph with degree sequence D is called a *realization* of D. If all realizations of a degree sequence are isomorphic, then this degree sequence is *unigraphic*. Let $0 < d_1 < \cdots < d_s$ be all positive distinct degrees of G, and let $d_0 = 0$. Let $D_i = \{v \in V : d_G(v) = d_i\}$ for $i = 0, 1, \ldots, s$. The sequence $D_0, D_1, \ldots, D_s$ is called the *degree partition* of G. By $\Delta(S)$ and $\delta(S)$ denote the maximum degree and minimum degree of vertices in S, respectively. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and the minimum degree of G, respectively. Denote by $N(v)$ the set of neighbors of vertex v in G. Let $N[v] = N(v) \cup \{v\}$ for vertex v in G.

Let G be a graph with m edges and vertex set $V = \{v_1, v_2, \ldots, v_n\}$. By the hand shaking lemma,

$$\sum_{j=1}^{n} d_G(v_j) = 2m.$$

From Equality (1), we infer

$$\begin{aligned} I_d(G) &= \log \sum_{j=1}^{n} d_G(v_j) - \frac{1}{\sum_{j=1}^{n} d_G(v_j)} \sum_{i=1}^{n} d_G(v_i) \log d_G(v_i) \\ &= \log(2m) - \frac{1}{2m} \sum_{i=1}^{n} d_G(v_i) \log d_G(v_i). \end{aligned}$$

We define a function

$$h_d(G) = \sum_{i=1}^{n} d_G(v_i) \log d_G(v_i).$$

Equation (1) implies that $\min_{G\in\mathcal{G}} I_d(G) = \log(2m) - \frac{1}{2m} \max_{G\in\mathcal{G}} h_d(G)$ for a certain family of graphs $\mathcal{G}$.

For solving some inequality problems, the concept majorization is a technique to use [29]. Let $A = [a_1, a_2, \ldots, a_n]$ and $B = [b_1, b_2, \ldots, b_n]$ be non-increasing integer sequences of length n. Then A *majorizes* B if

$$\sum_{i=1}^{s} a_i \geq \sum_{i=1}^{s} b_i, \quad s = 1, 2, \ldots, n-1$$

and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

We use $A \succeq B$ to denote that A majorizes B. If there is a strict inequality, then it is the strict majorization, and is denoted by $A \succ B$.

Some results on the minimum and maximum values of the degree-entropy were proved by the next theorem [19].

Theorem 3 ([19]). *Let G and H be (n, m)-graphs. If $D(G) \succeq D(H)$, then $I_d(G) \leq I_d(H)$, with equality if and only if $D(G) = D(H)$.*

Chvátal and Hammer [13] proposed the family of threshold graphs. A graph G is called a *threshold graph* if each vertex v_i of G can be assigned a non-negative real number w_i, and G can be assigned a non-negative real number r such that $v_i \in V(G)$ and $v_j \in V(G)$ are adjacent if and only if $w_i + w_j > r$. By definition, Chvátal and Hammer [13] proved the following result.

Fact 1 ([13]). *If G is a threshold graph, then every induced subgraph of G is a threshold graph.*

A degree sequence is the *threshold sequence* if it is the degree sequence of a threshold graph.

Theorem 4 ([28]). *A graphical sequence is a threshold sequence if and only if it has a unique labeled realization.*

Let G be a graph with degree partition $D_0, D_1, \ldots, D_s$. Some basic characterizations of threshold graphs are listed.

Theorem 5 ([13, 28]). *There are three equivalent conditions:*

(a) *the graph G is a threshold graph;*

(b) *the graph G does not have an alternating 4-cycle (i.e., there are no four vertices $u, v, w, x \in V(G)$ such that $uw, vx \notin E(G)$ and $uv, wx \in E(G)$);*

(c) *for each $v \in D_k$,*

$$N(v) = \cup_{j=1}^{k} D_{s+1-j} \text{ for } k = 1, 2, \ldots, \lfloor \frac{s}{2} \rfloor;$$

$$N[v] = \cup_{j=1}^{k} D_{s+1-j} \text{ for } k = \lfloor \frac{s}{2} \rfloor + 1, \lfloor \frac{s}{2} \rfloor + 2, \ldots, s,$$

in other words, for $u \in D_i$ and $v \in D_j$, u is adjacent to v if and only if $i + j > s$; Figure 1 illustrates this with $s = 6$ and $s = 7$.

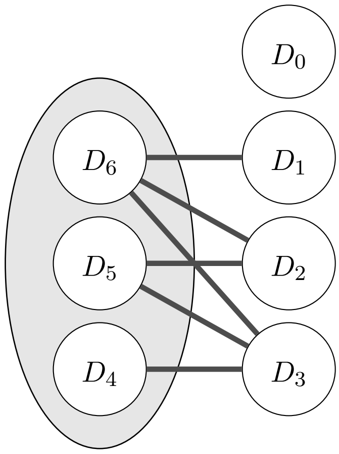
Figure 1 illustrates the degree partitions of two threshold graphs with $s = 6$ and $s = 7$, respectively. A line between D_i and D_j indicates that every vertex of D_i is adjacent to every vertex of D_j. An oval indicates that the included vertices form a clique.

Theorem 5 (c) indicates the following theorem which shows the relation between degrees and degree partitions [28].

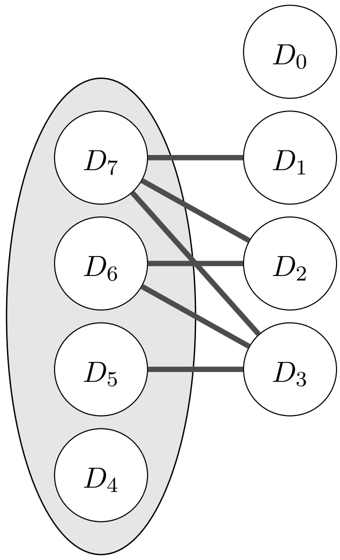
Theorem 6 ([28]). *For any threshold graph, we have*

$$d_{k+1} = d_k + |D_{s-k}| \text{ for } k = 0, 1, \ldots, s, \; k \neq \lfloor s/2 \rfloor;$$

$$d_{k+1} = d_k + |D_{s-k}| - 1 \text{ for } k = \lfloor s/2 \rfloor.$$



(a) $s = 6$



(b) $s = 7$

Figure 1: An illustration of the degree partitions of two threshold graphs with $s = 6$ and $s = 7$

By Theorem 4, we verify that a threshold graph is uniquely determined by its degree sequence. Using Theorem 6, a threshold graph is also uniquely determined by its degree partition.

Hammer et al. [21] proposed the family of difference graphs. A graph G is called a *difference graph* if each vertex v_i of G can be assigned $w_i \in \mathbb{R}$ and G can be assigned $r \in \mathbb{R}^+$ such that

(i) $|w_i| < r$ for any i;

(ii) $v_i \in V(G)$ and $v_j \in V(G)$ are adjacent if and only if $|w_i - w_j| \geq r$.

There are some basic characterizations of difference graphs [28].

Theorem 7 ([28])**.** *Let G be a bipartite graph with bipartition (X, Y). There are three equivalent conditions as follows:*

(a) *the graph G is a difference graph;*

(b) *there are no $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1y_1, x_2y_2 \in E(G)$ and $x_1y_2, x_2y_1 \notin E(G)$;*

(c) *without isolated vertices, every induced subgraph has a domination vertex on each side of the bipartition.*

Let $c_1, c_2, \ldots, c_n$ be integers satisfying $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. The *conjugate sequence* of a sequence $C = [c_1, c_2, \ldots, c_n]$ is $C^* = [c_1^*, c_2^*, \ldots, c_n^*]$ in which $c_i^* = |\{j : c_j \geq i\}|$. The following result is one characterization of difference graphs by the degree sequences.

Theorem 8 ([28])**.** *A pair of non-negative and non-increasing integer sequences* $D(X) = [d_G(x_1), d_G(x_2), \ldots, d_G(x_{|X|})]$ *and* $D(Y) = [d_G(y_1), d_G(y_2), \ldots, d_G(y_{|Y|})]$ *with* $d_G(y_1) \leq |X|$ *is a pair of degree sequences of a difference graph* G *with bipartition* (X, Y) *if and only if* $D(X) = D^*(Y)$ *in which* $D^*(Y)$ *is the conjugate of* $D(Y)$.

For proving Theorem 1, we give the following four lemmas.

Lemma 1. *If* G *is a threshold graph with the degree partition* $D_0, D_1, \ldots, D_s$*, then* $|D_{\lceil \frac{s}{2} \rceil}| \geq 2$.

Proof. We prove it by contradiction. Suppose that $|D_{\lceil \frac{s}{2} \rceil}| = 1$. It follows that $\lceil \frac{s+1}{2} \rceil = \lfloor \frac{s}{2} \rfloor + 1 > \lfloor \frac{s}{2} \rfloor$. Let d_i be the degree of the vertices in D_i for $i = 0, 1, \ldots, s$. By Theorem 5 (c), we have $d_{\lceil \frac{s+1}{2} \rceil} = \sum_{i=\lceil \frac{s}{2} \rceil}^{s} |D_i| - 1 = \sum_{i=\lceil \frac{s}{2} \rceil + 1}^{s} |D_i| = d_{\lfloor \frac{s}{2} \rfloor}$, which contradicts $d_i > d_j$ for $i > j$. $\square$

We use Lagrange mean value theorem to prove the following result which will be used to prove Theorem 1.

Lemma 2. *Let* a *be a positive integer, and let* $A = [a_1, a_2, \ldots, a_n]$ *be a positive integer sequence of length* n*. If* f *is strictly concave and* $\sum_{i=1}^{n} a_i = a$*, then* $\sum_{i=1}^{n} f(a_i)$ *attains the maximum value if and only if* $a_i = \lceil \frac{a}{n} \rceil$ *or* $a_i = \lfloor \frac{a}{n} \rfloor$ *for* $i = 1, 2, \ldots, n$.

Proof. We prove it by contradiction. Suppose that $\sum_{i=1}^{n} a_i$ attains the maximum value and $a_i - a_j \geq 2$. Let $A' = [a_1', a_2', \ldots, a_n']$ be a sequence such that $a_i' = a_i - 1$, $a_j' = a_i + 1$ and $a_k' = a_k$ for $k \neq i, j$. Because f is strictly concave (i.e., the first derivative f' is strictly decreasing), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(a_i') \\
=& (f(a_i) - f(a_i - 1)) - (f(a_j + 1) - f(a_j)) \\
=& f'(\xi_1) - f'(\xi_2) \\
<& 0,
\end{aligned}
$$

where $\xi_1 \in (a_i - 1, a_i)$ and $\xi_2 \in (a_j, a_j + 1)$. $\square$

By Jensen's inequality, we have the following result.

Lemma 3. *Let a be a positive integer, and let $A = [a_1, a_2, \ldots, a_n]$ be a positive integer sequence of length n. If f is strictly concave and $\sum_{i=1}^{n} a_i = a$, then $\sum_{i=1}^{n} f(a_i) \leq nf(\frac{a}{n})$ with equality if and only if $a_i = \frac{a}{n}$ for $i = 1, 2, \ldots, n$.*

We prove the following result using the alternating 4-cycles used to describe threshold graphs.

Lemma 4. *Let $n \geq 2$ and $1 \leq m \leq \binom{n}{2}$ be integers, and let G be an (n, m)-graph. If $I_d(G)$ achieves the minimum value among (n, m)-graphs, then G is a threshold graph.*

Proof. We assume that G is not a threshold graph. By Theorem 5 (b), there are four vertices $u, v, w, x \in V(G)$ such that $uv, wx \in E(G)$ and $ux, vw \notin E(G)$. We assume w.l.o.g. that $d_G(v) \geq d_G(x)$. Set $G' = G - wx + wv$. Hence $D(G') \succ D(G)$. By Theorem 3, we have $I_d(G') < I_d(G)$, a contradiction. $\square$

To prove Theorems 2, the following lemma can be used.

Lemma 5. *Let $n \geq 2$ and $1 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ be integers, and let G be an (n, m)-bipartite graph. If $I_d(G)$ achieves the minimum value among (n, m)-bipartite graphs, then G is a difference graph.*

Proof. Suppose that G is not a difference graph. Let X and Y be parts of G. By Theorem 7 (b), there are four vertices $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1y_1, x_2y_2 \in E(G)$ and $x_1y_2, x_2y_1 \notin E(G)$. We assume w.l.o.g. that $d_G(x_1) \geq d_G(x_2)$. Set $G' = G - x_2y_2 + x_1y_2$. Hence $D(G') \succ D(G)$. By Theorem 3, we have $I_d(G') < I_d(G)$, a contradiction. $\square$

3 Proofs

Proof of Theorem 1. We assume that G^* is an (n, m)-graph with $I_d(G^*) = \min\{I_d(G) : G$ is an (n, m)-graph$\}$. Let $D_0, D_1, \ldots, D_s$ be the degree partition of G^*. By Lemma 4, G^* is a threshold graph. Therefore, $K(k^*, t^*) \cup \bar{K}_{n-k^*-\sigma(t^*)}$, a threshold graph, is uniquely determined by its degree partition (or degree sequence):

(Ai) for $t^* = 0$ (i.e., $m = \binom{k^*}{2}$), $|D_0| = n - k^*$ and $|D_1| = k^*$ (or $D(K_{k^*} \cup \bar{K}_{n-k^*}) = [(k^* - 1)^{k^*}]$);

(Aii) for $1 \leq t^* < k^* - 1$ (i.e., $1 \leq m - \binom{k^*}{2} < k^* - 1$), $|D_0| = n - k^* - 1$, $|D_1| = 1$, $|D_2| = k^* - t^*$ and $|D_3| = t^*$ (or $D(K(k^*, t^*) \cup \bar{K}_{n-k^*-1}) = [(k^*)^{t^*}, (k^* - 1)^{k^*-t^*}, t^*]$);

(Aiii) for $t^* = k^* - 1$ (i.e., $m - \binom{k^*}{2} = k^* - 1$), $|D_0| = n - k^* - 1$, $|D_1| = 2$ and $|D_2| = k^* - 1$ (or $D(K(k^*, k^* - 1) \cup \bar{K}_{n-k^*-1}) = [(k^*)^{k^*-1}, (k^* - 1)^2]$).

Because $|D_0|$ can be obtained by $|D_i|$ (i.e., $|D_0| = n - \sum_{i=1}^{s} |D_i|$), it is sufficient to show $|D_i|$ for $i = 1, 2, \ldots, s$. So we omit $|D_0|$ in the following. Let $K = \cup_{i=\lceil \frac{s+1}{2} \rceil}^{s} D_i$, and let $k = \omega(G^*)$. By Theorem 5 (c), we have $k = \sum_{i=\frac{s+2}{2}}^{s} |D_i| + 1$ (resp., $k = \sum_{i=\frac{s+1}{2}}^{s} |D_i|$) for s even (resp., odd). So we have $|K| = k - 1$ (resp., $|K| = k$) for s even (resp. odd). We state a claim.

Claim 1. If $\Delta(K) - \delta(K) \leq 1$, then $s \leq 4$.

Proof. We prove it by contradiction. Suppose that $s \geq 5$. There are at least three distinct degrees of vertices in K of G^*. This implies $\Delta(K) - \delta(K) \geq 2$, a contradiction. $\square$

There exists a non-negative integer c such that $(c-1)(k-1) < m - \binom{k}{2} \leq c(k-1)$ (i.e., $c = \lceil \frac{m-\binom{k}{2}}{k-1} \rceil$). We state a claim.

Claim 2. We have $\Delta(K) - \delta(K) \leq 1$ if and only if

(i) $|D_1| = k$ for $m - \binom{k}{2} = c(k-1)$ and $c = 0$;

(ii) $|D_1| = c + 1$ and $|D_2| = k - 1$ for $m - \binom{k}{2} = c(k-1)$ and $c \geq 1$;

(iii) $|D_1| = 1$, $|D_2| = \binom{k+1}{2} - m$ and $|D_3| = m - \binom{k}{2}$ for $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c = 1$;

(iv) $|D_1| = 1$, $|D_2| = c$, $|D_3| = c(k-1) + \binom{k}{2} - m$ and $|D_4| = m - \binom{k}{2} - (c-1)(k-1)$ for $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c \geq 2$.

Proof. $\Longleftarrow$) By Theorem 6, we have

(i) $\Delta(K) = \delta(K) = d_1 = k - 1$ for $m - \binom{k}{2} = c(k-1)$ and $c = 0$;

(ii) $\Delta(K) = \delta(K) = d_2 = c + k - 1$ for $m - \binom{k}{2} = c(k-1)$ and $c \geq 1$;

(iii) $\Delta(K) = d_3 = k$ and $\delta(K) = d_2 = k - 1$ for $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c = 1$;

(iv) $\Delta(K) = d_4 = c+k-1$ and $\delta(K) = d_3 = c+k-2$ for $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c \geq 2$.

Thus $\Delta(K) - \delta(K) \leq 1$.

$\Longrightarrow$) By Claim 1, we distinguish four cases.

Case 1. $s = 1$.

We have $K = D_1$ and $|D_1| = k$. By Theorem 6, we have $\Delta(K) = \delta(K) = d_1 = k - 1$. By Theorem 5 (c), every vertex in D_1 is adjacent to other vertices in D_1, which implies $m = \binom{k}{2}$, that is, $m - \binom{k}{2} = c(k-1)$ and $c = 0$.

Case 2. $s = 2$.

We have $K = D_2$ and $|D_2| = k - 1$. By Theorem 6, we have $\Delta(K) = \delta(K) = d_2 = d_1 + |D_1| - 1$ and $d_1 = d_0 + |D_2| = k - 1$. Since $d_1|D_1| + d_2|D_2| = 2m$, we have $|D_1| = \frac{m-\binom{k-1}{2}}{k-1} = \frac{m-\binom{k}{2}}{k-1} + 1$. This implies $c = \frac{m-\binom{k}{2}}{k-1}$. We have $m - \binom{k}{2} = c(k-1)$. By Lemma 1, we have $|D_1| \geq 2$, which implies $c \geq 1$.

Case 3. $s = 3$.

We have $K = D_2 \cup D_3$ and $|K| = k$. This implies $\Delta(K) = d_3$, $\delta(K) = d_2$ and $|D_2| + |D_3| = k$. By Theorem 6, we have $d_1 = d_0 + |D_3| = |D_3|$, $d_2 = d_1 + |D_2| - 1$ and $d_3 = d_2 + |D_1|$. If $|D_1| \geq 2$, then $d_3 - d_2 = |D_1| \geq 2$, which contradicts $\Delta(K) - \delta(K) \leq 1$. So we have $|D_1| = 1$. Because $|D_2| + |D_3| = k$, $d_2 = d_1 + |D_2| - 1$ and $d_1 = |D_3|$, we have $d_2 = k - 1$. Since $d_3 = d_2 + 1$, $d_3 = k$. We deduce $2m = d_1|D_1| + d_2|D_2| + d_3|D_3| = kd_1 + (k-1)(k-d_1) + d_1$. By calculating, we have $d_1 = m - \binom{k}{2}$. It follows from $|D_3| = d_1 = m - \binom{k}{2}$ and $|D_2| + |D_3| = k$ that $|D_2| = k - (m - \binom{k}{2}) = \binom{k+1}{2} - m$. By Lemma 1, we have $|D_2| \geq 2$. Because $|D_2| + |D_3| = k$, we have $1 \leq d_1 \leq k - 2$. This implies $0 < m - \binom{k}{2} < k - 1$ (i.e., $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c = 1$).

Case 4. $s = 4$.

We have $K = D_3 \cup D_4$ and $|K| = k - 1$. This implies $\Delta(K) = d_4$, $\delta(K) = d_3$ and $|D_3| + |D_4| = k - 1$. By Theorem 6, we have $d_1 = d_0 + |D_4| = |D_4|$, $d_2 = d_1 + |D_3|$, $d_3 = d_2 + |D_2| - 1$ and $d_4 = d_3 + |D_1|$. Since $|D_3| + |D_4| = k - 1$, $d_1 = |D_4|$ and $d_2 = d_1 + |D_3|$, we have $d_2 = k - 1$. If $|D_1| \geq 2$, then $d_4 - d_3 = |D_1| \geq 2$, which contradicts $\Delta(K) - \delta(K) \leq 1$. So we have $|D_1| = 1$. Because $d_2 = d_1 + |D_3|$, $d_1 = |D_4|$ and $|D_3| + |D_4| = k - 1$, we have $d_2 = k - 1$. It follows from $d_3 = d_2 + |D_2| - 1$ and $d_2 = k - 1$ that $d_3 = |D_2| + k - 2$. Since $2m = d_1|D_1| + d_2|D_2| + d_3|D_3| + d_4|D_4| = 2d_1 + 2(k-1)|D_2| + (k-1)(k-2)$, $|D_2| = \frac{m-\binom{k-1}{2}-d_1}{k-1}$. Because $0 < d_1 < d_2 = k-1$, we have $|D_2| = \lfloor \frac{m-\binom{k-1}{2}}{k-1} \rfloor = \lfloor \frac{m-\binom{k}{2}}{k-1} + 1 \rfloor = \lceil \frac{m-\binom{k}{2}}{k-1} \rceil$. Thus $c = \lceil \frac{m-\binom{k}{2}}{k-1} \rceil = |D_2| = \frac{m-\binom{k-1}{2}-d_1}{k-1}$. So we have $d_1 = m - \binom{k-1}{2} - c(k-1) = m - (\binom{k-1}{2} + k - 1) + k - 1 - c(k-1) = m - \binom{k}{2} - (c-1)(k-1) = |D_4|$. This implies $|D_3| = k - 1 - |D_4| = c(k-1) + \binom{k}{2} - m$. By Lemma 1, we have $|D_2| \geq 2$, which implies $c \geq 2$. Thus $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c \geq 2$. $\square$

Let u be a vertex in $D_{\lceil \frac{s}{2} \rceil}$. By Theorem 6, we have $d_{G^*}(u) = k-1$. Set $G' = G^* - u$. Since G' is an induced subgraph of G^*, by Fact 1, we have G' is a threshold graph. Let $\omega(G') = k'$. If s is even, then there exists a maximum clique without vertex u since we may use other vertex in $D_{\frac{s}{2}}$ to replace u. So we have $k' = k$ if s is even. If s is odd, then $k' = k-1$ since u must be in the maximum clique. We state a claim.

Claim 3. The graph G^* satisfies

(i) $|D_1| = k$ for $m - \binom{k}{2} = c(k-1)$ and $c = 0$;

(ii) $|D_1| = c+1$, $|D_2| = k-1$ for $m - \binom{k}{2} = c(k-1)$ and $c \geq 1$;

(iii) $|D_1| = 1$, $|D_2| = \binom{k+1}{2} - m$ and $|D_3| = m - \binom{k}{2}$ for $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c = 1$;

(iv) $|D_1| = 1$, $|D_2| = c$, $|D_3| = c(k-1) + \binom{k}{2} - m$ and $|D_4| = m - \binom{k}{2} - (c-1)(k-1)$ for $(c-1)(k-1) < m - \binom{k}{2} < c(k-1)$ and $c \geq 2$.

Proof. By induction on m. It is trivial for $m = 1$. For $m = 2$, $G^* \cong K_{1,2} \cup \bar{K}_{n-3}$, that is, $|D_1| = 2$ and $|D_2| = 1$ (i.e., $m - \binom{k}{2} = k-1$ and $k = 2$).

It follows that G' is a threshold graph with $n' = n-1$ vertices, $m' = m+1-k$ edges and $\omega(G') = k'$. Let $D'_0, D'_1, \ldots, D'_{s'}$ be the degree partition of G'. Let $c' = \lceil \frac{m' - \binom{k'}{2}}{k'-1} \rceil$. We assume that G' satisfies

(i) $|D'_1| = k'$ for $m' - \binom{k'}{2} = c'(k'-1)$ and $c' = 0$;

(ii) $|D'_1| = c'+1$, $|D'_2| = k'-1$ for $m' - \binom{k'}{2} = c'(k'-1)$ and $c' \geq 1$;

(iii) $|D'_1| = 1$, $|D'_2| = \binom{k'+1}{2} - m'$ and $|D'_3| = m' - \binom{k'}{2}$ for $(c'-1)(k'-1) < m' - \binom{k'}{2} < c'(k'-1)$ and $c' = 1$;

(iv) $|D'_1| = 1$, $|D'_2| = c'$, $|D'_3| = c'(k'-1) + \binom{k'}{2} - m'$ and $|D'_4| = m' - \binom{k'}{2} - (c'-1)(k'-1)$ for $(c'-1)(k'-1) < m - \binom{k'}{2} < c'(k'-1)$ and $c' \geq 2$.

We have a recurrence relation

$$\begin{aligned} h_d(G^*) =& h_d(G') + (k-1)\log(k-1) \\ &+ \sum_{v \in K \setminus \{u\}} (d_{G^*}(v) \log d_{G^*}(v) - (d_{G^*}(v) - 1)\log(d_{G^*}(v) - 1)). \end{aligned}$$

Let $f(x) = x \log x - (x-1)\log(x-1)$ for $x \geq 2$. It follows that $f(x)$ is strictly concave. Since $\sum_{v \in K \setminus \{u\}} d_{G^*}(v) = m + \binom{k-1}{2}$ and $|K \setminus \{u\}| = k-1$, by Lemma 2,

$\sum_{v\in K\setminus\{u\}}(d_{G^*}(v)\log d_{G^*}(v)-(d_{G^*}(v)-1)\log(d_{G^*}(v)-1))$ attains the maximum value if and only if the maximum degree exceeds the minimum degree by at most 1, that is, $\Delta(K)-\delta(K)\leq 1$. By Claim 1 and the induction hypothesis, the claim does hold. $\square$

We now prove that $k=k^*$. By Claim 3, for $s=1$, we have $|D_1|=k^*$ in which $\binom{k^*}{2}=m$; for $s=3$, we have $|D_0|=n-k^*-1$, $|D_1|=1$, $|D_2|=k^*-t^*$ and $|D_3|=t^*$ in which $1\leq m-\binom{k^*}{2}<k^*-1$. By Claim 3, if $m-\binom{k}{2}=c(k-1)$, then the degree sequence of G^* is $[(c+k-1)^{k-1},(k-1)^{c+1}]$ (i.e., $\Delta(K)-\delta(K)=0$). Let $G_{n,m,k}$ be the (n,m)-graph satisfying $D(G_{n,m,k})=[(c+k-1)^{k-1},(k-1)^{c+1}]$ in which $c=\frac{m-\binom{k}{2}}{k-1}$. Notice that there are $n-c-k$ isolated vertices of $G_{n,m,k}$. By Theorem 4, this threshold sequence has a unique labeled realization. Therefore $G_{n,m,k}\cong G^*$ if $m-\binom{k}{2}=c(k-1)$. Let $g(m,k)=(k-1)(c+k-1)\log(c+k-1)+(c+1)(k-1)\log(k-1)$ in which $c=\frac{m-\binom{k}{2}}{k-1}$. It follows that $h_d(G_{n,m,k})=g(m,k)$. We state a claim.

Claim 4. We have $h_d(G^*)\leq g(m,k)$, with equality if and only if $G^*\cong G_{n,m,k}$.

Proof. Because $\sum_{v\in K\setminus\{u\}}d_{G^*}(v)=m+\binom{k-1}{2}$, $|K\setminus\{u\}|=k-1$ and $f(x+1)=(x+1)\log(x+1)-x\log x$ is strictly concave, by Lemma 3, we have

$$\begin{aligned}&\sum_{v\in K\setminus\{u\}}(d_{G^*}(v)\log d_{G^*}(v)-(d_{G^*}(v)-1)\log(d_{G^*}(v)-1))\\ \leq&(k-1)\left(\left(\frac{m+\binom{k}{2}}{k-1}\right)\log\left(\frac{m+\binom{k}{2}}{k-1}\right)+\left(\frac{m+\binom{k}{2}}{k-1}-1\right)\log\left(\frac{m+\binom{k}{2}}{k-1}-1\right)\right),\end{aligned}$$

with equality if and only if $d_{G^*}(v)=\frac{m+\binom{k-1}{2}}{k-1}$. For $m=2$, $G^*\cong K_{1,2}\cup\bar{K}_{n-3}$ and $h_d(G^*)=2=g(2,2)$. By the similar induction of Claim 3, we assume $h_d(G')\leq g(m',k')$. By the induction hypothesis, we have $h_d(G^*)\leq g(m,k)$, with equality if and only if $G^*\cong G_{n,m,k}$. $\square$

Next we prove $g(m,k)$ is strictly increasing in k.

Claim 5. We have $h_d(G_{n,m,k})\leq h_d(G_{n,m,k^*})$, with equality if and only if $k=k^*$.

Proof. By calculating, we have

$$\begin{aligned}h_d(G_{n,m,k})&=g(m,k)\\&=\left(m+\binom{k-1}{2}\right)\log\left(\frac{m+\binom{k-1}{2}}{k-1}\right)+\left(m-\binom{k-1}{2}\right)\log(k-1).\end{aligned}$$

For $k\geq 2$ and $m\geq\binom{k}{2}$, by calculating, we obtain $\frac{\partial g(m,k)}{\partial k}=\left(\frac{2k-3}{2}\right)\log\frac{(k-1)(k-2)+2m}{2(k-1)^2}+\frac{\log e}{2}>0$ in which e is the natural constant. So we have $h_d(G_{n,m,k})$ is strictly increasing in k. Since $k \leq k^*$, we have $h_d(G_{n,m,k}) = g(m,k) \leq h_d(G_{n,m,k^*})$, with equality if and only if $k = k^*$. $\square$

We now finish the proof by going through the cases of Claim 3. Since $\omega(G^*) \leq k^*$, $0 \leq t^* = m - \binom{k^*}{2} \leq k^* - 1$. In particular, this shows that Case (iv) cannot occur. By observation, Cases (i) and (iii) are the same as items (Ai) and (Aii), respectively. We are done for $t^* = 0$ (i.e., item (Ai)) and $0 < t^* < k^* - 1$ (i.e., item (Aii)). So we consider $t^* = m - \binom{k^*}{2} = k^* - 1$ (i.e., $\frac{m-\binom{k^*}{2}}{k^*-1} = 1$) in the following. By Claims 4 and 5, we have $h_d(G^*) \leq g(m,k^*)$, with equality if and only if $G^* \cong G_{n,m,k^*}$. We have $D(G_{n,m,k}) = [(c+k-1)^{k-1}, (k-1)^{c+1}]$ in which $c = \frac{m-\binom{k}{2}}{k-1}$. Clearly, $D(G_{n,m,k^*}) = [(c^*+k^*-1)^{k^*-1}, (k^*-1)^{c^*+1}]$ in which $c^* = \frac{m-\binom{k^*}{2}}{k^*-1} = 1$. By calculating, we have $D(G_{n,m,k^*}) = [(k^*)^{k^*-1}, (k^*-1)^2]$. Thus $|D_1| = 2$ and $|D_2| = k^* - 1$ for $t^* = k^* - 1$ (i.e., item (Aiii)).

$\square$

Proof of Theorem 2. Suppose that G^* is an (n,m)-bipartite graph with $I_d(G^*) = \min\{I_d(G) : G \text{ is an } (n,m)\text{-bipartite graph}\}$. Let X and Y be parts of G^*. We denote the vertices of X (resp. Y) by $x_1, x_2, \ldots, x_{|X|}$ (resp. $y_1, y_2, \ldots, y_{n-|X|}$). We first consider $n \geq m+1$. From Lemma 5, G^* is a difference graph. By Theorem 7 (c), without isolated vertices, G^* has a domination vertex on each side of the bipartition. Assume w.l.o.g. that x_1 is the domination vertex in X and $x_1 y_j \in E(G^*)$ for $j = 1, 2, \ldots, b$. So we have $d_{G^*}(x_1) = b$. If $m = b$, then $G^* \cong K_{1,b} \cup \bar{K}_{n-b-1}$.

We consider $m > b$. By induction on m. For $m = 1$, $G^* \cong K_{1,1} \cup \bar{K}_{n-2}$. For $m \geq 2$, set $G' = G^* - x_1y_1 - x_1y_2 - \cdots - x_1y_b$ and assume that $G' \cong K_{q-1,b} \cup \bar{K}_{n-q-b}$.

We have a recurrence relation

$$
\begin{aligned}
h_d(G^*) =& \sum_{i=1}^{|X|} d_{G^*}(x_i) \log d_{G^*}(x_i) + \sum_{j=1}^{n-|X|} d_{G^*}(y_j) \log d_{G^*}(y_j) \\
=& \sum_{i=1}^{|X|} d_{G^*}(x_i) \log d_{G^*}(x_i) + \sum_{j=1}^{b} d_{G^*}(y_j) \log d_{G^*}(y_j) \\
=& \sum_{i=2}^{|X|} d_{G^*}(x_i) \log d_{G^*}(x_i) + \sum_{j=1}^{b} (d_{G^*}(y_j) - 1) \log(d_{G^*}(y_j) - 1) + \sum_{j=1}^{b} d_{G^*}(y_j) \log d_{G^*}(y_j) \\
& - \sum_{j=1}^{b} (d_{G^*}(y_j) - 1) \log(d_{G^*}(y_j) - 1) + b \log b \\
=& h_d(G') + \sum_{j=1}^{b} d_{G^*}(y_j) \log d_{G^*}(y_j) - \sum_{j=1}^{b} (d_{G^*}(y_j) - 1) \log(d_{G^*}(y_j) - 1) + b \log b.
\end{aligned}
$$

For $\sum_{j=1}^{b} z_j = \sum_{j=1}^{b} d_{G^*}(y_j) = m > b$, we analyze conditional extremums of the following function

$$f(z_1, z_2, \ldots, z_b) = \sum_{j=1}^{b} z_j \log z_j - \sum_{j=1}^{b}(z_j - 1)\log(z_j - 1)$$

and corresponding Lagrangian function

$$L(z_1, z_2, \ldots, z_b, \lambda) = f(z_1, z_2, \ldots, z_b) + \lambda \left(\sum_{j=1}^{b} z_j - m \right)$$

with the additional boundary condition $z_j > 1$. For each of its arguments on the closed region, this function is differentiable and well-defined. This means that extremal values are either on the boundary or critical points. It follows that

$$\frac{\partial}{\partial \lambda} L = \sum_{j=1}^{b} z_j - m = 0$$

and

$$\frac{\partial}{\partial z_j} L = \log z_j - \log(z_j - 1) + \lambda = 0$$

for $j = 1, 2, \ldots, b$. By the set of equations, the unique critical point satisfies $z_1 = z_2 = \cdots = z_b = \frac{m}{b} > 1$. It is a local maximum since

$$\frac{\partial^2}{\partial z_j^2} L = -\frac{1}{\ln 2 z_j (z_j - 1)} < 0.$$

For variables $d_{G^*}(y_1), d_{G^*}(y_2), \ldots, d_{G^*}(y_b)$, the function $f(d_{G^*}(y_1), d_{G^*}(y_2), \ldots, d_{G^*}(y_b))$ attains the maximum value if and only if $d_{G^*}(y_j) = \frac{m}{b}$ for $j = 1, 2, \ldots, b$. So we have $D(Y) = [q^b]$ in which $q = \frac{m}{b}$. By Theorem 8, we obtain $D(X) = D^*(Y)$ in which $D^*(Y)$ is the conjugate of $D(Y)$. This implies that the degree sequence of X is $D(X) = [b^q]$. Obviously, this pair of degree sequences is unigraphic, that is, $K_{q,b} \cup \bar{K}_{n-q-b}$ is the only realization of this pair of degree sequences up to isomorphism. There exist two integers b and q such that $m = qb$ and $n \geq q + b$ if $n \geq m + 1$. By induction hypothesis, $G^* \cong K_{q,b} \cup \bar{K}_{n-q-b}$ for $n \geq m + 1$. By calculating, we have

$$\begin{aligned} I_d(K_{q,b} \cup \bar{K}_{n-q-b}) &= \log(2m) - \frac{1}{2m}(bq \log q + qb \log b) \\ &= \log(2m) - \frac{1}{2m}(m \log m) \\ &= 1 + \log \sqrt{m}. \end{aligned}$$

Therefore we have $I_d(G) \leq 1 + \log \sqrt{m}$, with equality if and only if $G \cong K_{q,b} \cup \bar{K}_{n-q-b}$, where q and b satisfy $qb = m$ and $q + b \leq n$.

Now we consider $n \leq m$ in the following. It is sufficient to prove $I_d(G^*) > 1 + \log \sqrt{m}$ if $G^* \cong K_{q,b} \cup \bar{K}_{n-q-b}$ does not hold for any integers b and q. Contrarily, we assume that $I_d(G^*) \leq 1 + \log \sqrt{m}$. It follows that $I_d(G^*) = I_d(G^* \cup \bar{K}_{m+1-n}) \geq I_d(K_{1,m}) = 1 + \log \sqrt{m}$. So we have $I_d(G^*) = 1 + \log \sqrt{m}$ and $G^* \cup \bar{K}_{m+1-n}$ is not isomorphic to $K_{q,b} \cup \bar{K}_{n-q-b}$ for any integers q and b, which contradicts that $G^* \cong I_d(K_{q,b} \cup \bar{K}_{n-q-b})$ for $n \geq m+1$. $\square$

4 Concluding remarks

We modified the proof of Theorem 1 in this new version due to Cambie and Mazzamurro providing a counter-example for the proof idea of Theorem 1 in the previous version. Last time, we used the Lagrange multiplier method treating degrees as real numbers, which produces a "graph" with non-integer vertex degrees whose entropy is lower than the minimum entropy, while such a graph does not exist. In order to avoid the conversion between discrete and continuous, we stop using the Lagrange multiplier method to prove Theorem 1 this time. Cambie and Mazzamurro [6] proved the same result by using Karamata's inequality, and they pointed out extremal graphs called colex graphs.

The Lagrange multiplier method can be used to prove Theorem 2 for presenting a lower bound. Among (n, m)-bipartite graphs, we prove $I_d(K_{q,b} \cup \bar{K}_{n-q-b})$ attaining the lower bound $1 + \log \sqrt{m}$, where $qb = m$ and $q + b \leq n$. We consider extremal results for bipartite graphs with no more than 6 vertices. Before we list the results, we define a special bipartite graph. Let n, m and b be three integers with $n \geq 2$ and $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor \geq m \geq b \geq 1$. Set $q = \lfloor \frac{m}{b} \rfloor$, $r = m - b\lfloor \frac{m}{b} \rfloor$. Let $B(n, m, b)$ be the (n, m)-bipartite graph with parts $X = \{x_1, x_2, \ldots, x_{|X|}\}$, $Y = \{y_1, y_2, \ldots, y_{|Y|}\}$, where $|X| \geq q + \sigma(r)$, $|Y| \geq b$, $x_i y_j \in E(B(n, m, b))$ for $i = 1, 2, \ldots, q$, $j = 1, 2, \ldots, b$ and $x_{q+1} y_s \in E(B(n, m, b))$ for $s = 1, 2, \ldots, r$. We list the extremal graphs with at most 6 vertices in the following.

By observation, we find $I_d(B(6, 7, 3)) < I_d(B(6, 7, 4))$. Does $I_d(B(n, m, b))$ decrease as the variable b increases for any given n and m? The answer is no since $I_d(B(7, 7, 4)) < I_d(B(7, 7, 3)) < I_d(B(7, 7, 5))$. Let $\mathcal{B}(n, m, b)$ be the set of (n, m)-bipartite graphs satisfying that the maximum degree of one part is b. Does the graph G with the minimum degree-entropy in $\mathcal{B}(n, m, b)$ satisfy $G \cong B(n, m, b)$? The answer is no since $I_d(B(7, 10, 4)) > I_d(B(7, 10, 3))$ and $\mathcal{B}(7, 10, 3) \subseteq \mathcal{B}(7, 10, 4)$ up to isomorphism. So we pose a general problem as follows.

Table 1: The bipartite graphs with the minimum value of degree-entropy.

m \ n	2	3	4	5	6
1	$K_{1,1}$	$K_{1,1} \cup \bar{K}_1$	$K_{1,1} \cup \bar{K}_2$	$K_{1,1} \cup \bar{K}_3$	$K_{1,1} \cup \bar{K}_4$
2		$K_{1,2}$	$K_{1,2} \cup \bar{K}_1$	$K_{1,2} \cup \bar{K}_2$	$K_{1,2} \cup \bar{K}_3$
3			$K_{1,3}$	$K_{1,3} \cup \bar{K}_1$	$K_{1,3} \cup \bar{K}_2$
4			$K_{2,2}$	$K_{2,2} \cup \bar{K}_1$ and $K_{1,4}$	$K_{2,2} \cup \bar{K}_2$ and $K_{1,4} \cup \bar{K}_1$
5				$B(5,5,3)$	$K_{1,5}$
6				$K_{2,3}$	$K_{2,3} \cup \bar{K}_1$
7					$B(6,7,4)$
8					$K_{2,4}$
9					$K_{3,3}$

Problem 1. If there does not exist a complete bipartite graph $K_{q,b}$ satisfying $qb = m$ and $q + b \leq n$, then how to find the ones attaining the minimum degree-entropy among (n, m)-bipartite graphs?

The latest results on this problem can be found in [7]. They proved that extremal graphs are complete bipartite graphs or nearly complete bipartite. Because this problem is related to the number theory, the authors concluded that the general characterization of the extremal graphs is a difficult problem.

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References

[1] R. Ahlswede and G.O.H. Katona, Graphs with maximal number of adjacent pairs of edges, *Acta Math. Acad. Sci. Hungar.*, **32** (1978) 97–120.

[2] B. Bollobás and V. Nikiforov, Degree powers in graphs with forbidden subgraphs, *Electron. J. Comb.*, **11** (2004) #R42.

[3] B. Bollobás and V. Nikiforov, Degree powers in graphs: the Erdős-Stone theorem, *Comb. Probab. Comput.*, **21** (2012) 89–105.

[4] D. Bonchev and D.H. Rouvray, *Complexity in Chemistry, Biology and Ecoclogy*, Springer, New York, 2005.

[5] D. de Caen, An upper bound on the sum of squares of degrees in a graph, *Discrete Math.,* **185** (1998) 245–248.

[6] S. Cambie and M. Mazzamurro, Minimum entropy of graphs with given size, *arXiv e-prints*, arXiv:2204.08251, Apr. 2022.

[7] S. Cambie, Y. Dong and M. Mazzamurro, Extremal values of degree-based entropies of bipartite graphs, *arXiv e-prints*, arXiv:2206.00922, Jun. 2022.

[8] S. Cao, M. Dehmer and Z. Kang, Network entropies based on independent sets and matchings, *Appl. Math. Comput.*, **307** (2017) 265–270.

[9] S. Cao, M. Dehmer and Y. Shi, Extremality of degree-based graph entropies, *Inf. Sci.*, **278** (2014) 22–33.

[10] G. Caporossi, I. Gutman, P. Hansen and L. Pavlović, Graphs with maximum connectivity index, *Comput. Biol. Chem.*, **27** (2003) 85–90.

[11] Z. Chen, M. Dehmer and Y. Shi, A note on distance-based graph entropies, *Entropy*, **16** (2014) 5416–5427.

[12] T.C.E. Cheng, Y. Guo, S. Zhang and Y. Du, Extreme values of the sum of squares of degree of bipartite graphs, *Discrete Math.,* **309** (2009) 1557–1564.

[13] V. Chvátal and P.L. Hammer, Aggregation of inequalities in integer programming, In P.L. Hammer, E.L. Johnson, B.H. Korte and G.L. Nemhauser, editors, *Studies in integer programming*, pp. 145–162. North-Holland, New York, 1977. *Annals of Discrete Mathematics 1.*

[14] K.C. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math.,* **285** (2004) 57–66.

[15] K. Das and S.-G. Lee, On the Estrada index conjecture, *Linear Algebra Appl.,* **431** (2009) 1351–1359.

[16] M. Dehmer and A. Mowshowitz, A history of graph entropy measures, *Inf. Sci.*, **181** (2011) 57–78.

[17] H. Dong and X. Guo, Character of graphs with extremal Balaban index, *MATCH Commun. Math. Comput. Chem.*, **63** (2010) 799–812.

[18] Y. Dong S. Qiao, B. Cheng, P. Wan and S. Zhang, Maximum values of degree-based entropies of bipartite graphs, *Appl. Math. Comput.*, **401** (2021) 126094.

[19] A. Ghalavand, M. Eliasi and A.R. Ashrafi, First degree-based entropy of graphs, *J. Appl. Math. Comput.*, **59** (2019) 37–46.

[20] I. Gutman, Degree-based topological indices, *Croat. Chem. Acta*, **86** (2013) 351–361.

[21] P.L. Hammer, U.N. Peled and X. Sun, Difference graphs, *Discrete Appl. Math.*, **28** (1990) 35–44.

[22] A. Ilić, On the extremal values of general degree-based graph entropies, *Inf. Sci.*, **370–371** (2016) 424–427.

[23] G. James, D. Witten, T. Hastie and R. Tibshirani, *An Introduction to Statistical Learning*, Springer, New York, 2013.

[24] A. Li and Y. Pan, Structural information and dynamical complexity of networks, *IEEE Trans. Inf. Theory*, **63** (2016) 3290–3339.

[25] X. Li, Y Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[26] N. Linial and E. Rozenman, An extremal problem on degree sequences of graphs, *Graph. Comb.,* **18** (2002) 573–582.

[27] J.T. Lizier, *The Local Information Dynamics of Distributed Computation in Complex Systems*, University of Sydney, Springer, Heidelberg, 2013.

[28] N.V.R. Mahadev and U.N. Peled, *Threshold Graphs and Related Topics*, Elsevier Science B.V., Amsterdam, 1995.

[29] A.W. Marshall, I. Olkin and B.C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer, New York, 2011.

[30] G.K. Nandini, R.S. Rajan, A.A. Shantrinal, T.M. Rajalaxmi, I. Rajasingh and K. Balasubramanian, Topological and thermodynamic entropy measures for COVID-19 pandemic through graph theory, *Symmetry*, **12** (2020) 1992.

[31] U.N. Peled, R. Petreschi and A. Sterbini, (n, e)-graphs with maximum sum of squares of degrees, *J. Graph Theory,* **31** (1999) 283–295.

[32] N. Rashevsky, Life, information theory, and topology, *Bull. Math. Biophys.*, **17** (1955) 229–235.

[33] H.A. Simon, The architecture of complexity, *Proceedings of the American Philosophical Society*, **106** (1962) 467–482.

[34] I. Turnu, M. Marchesi and R. Tonelli, Entropy of the degree distribution and object-oriented software quality, *WETSoM 2012*, (2012) 77–82.

[35] D. Vukičević and B. Furtula, Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges, *J. Math. Chem.,* **46** (2009) 1369–1376.

[36] P. Wan, X. Chen, J. Tu, M. Dehmer, S. Zhang and F. Emmert-Streib, On graph entropy measures based on the number of independent sets and matchings, *Inf. Sci.*, **516** (2020) 491–504.

[37] S. Wagner and I. Gutman, Maxima and minima of the Hosoya index and the Merrifield-Simmons index: A survey of results and techniques, *Acta Appl. Math.,* **112** (2010) 323–346.

[38] X. Zeng, H. Peng, A. Li, C. Liu, L. He and P.S. Yu, Hierarchical State Abstraction based on Structural Information Principles, *IJCAI-23*, **8** (2023) 4549–4557.

[39] S. Zhang and C. Zhou, Bipartite graphs with the maximum sum of squares of degrees, *Acta Math. Appl. Sin. Engl. Ser.,* **30** (2014) 801–806.

[40] L. Zhong, The harmonic index for graphs, *Appl. Math. Lett.,* **25** (2013) 562–566.