

DCTP/01/04, IPPP/01/02, MADPH-00-1210, hep-ph/yymmnnn

Two-loop QCD corrections to massless quark-gluon scattering*

C. Anastasiou^a, E. W. N. Glover^a, C. Oleari^b and M. E. Tejeda-Yeomans^a

^a*Department of Physics, University of Durham, Durham DH1 3LE, England*

^b*Department of Physics, University of Wisconsin, 1150 University Avenue
Madison WI 53706, U.S.A.*

E-mail: Ch.Anastasiou@durham.ac.uk, E.W.N.Glover@durham.ac.uk,
Oleari@pheno.physics.wisc.edu, M.E.Tejeda-Yeomans@durham.ac.uk

ABSTRACT: We present the $\mathcal{O}(\alpha_s^4)$ virtual QCD corrections to the scattering process of massless quark $q\bar{q} \rightarrow gg$ due to the interference of tree and two-loop amplitudes and to the self-interference of one-loop amplitudes. We work in conventional dimensional regularisation and our results are renormalised in the $\overline{\text{MS}}$ scheme. The structure of the infrared divergences agrees with that predicted by Catani while expressions for the finite remainder are given for the $q\bar{q} \rightarrow gg$ and the $qg \rightarrow qg$ ($g\bar{q} \rightarrow g\bar{q}$) scattering processes in terms of logarithms and polylogarithms that are real in the physical region. These results, together with those previously obtained for quark-quark scattering, are important ingredients in the next-to-next-to-leading order contribution to inclusive jet production at hadron colliders.

KEYWORDS: QCD, Jets, LEP HERA and SLC Physics, NLO and NNLO Computations.

*Work supported in part by the UK Particle Physics and Astronomy Research Council and by the EU Fourth Framework Programme ‘Training and Mobility of Researchers’, Network ‘Quantum Chromodynamics and the Deep Structure of Elementary Particles’, contract FMRX-CT98-0194 (DG 12 - MIHT). C.A. acknowledges the financial support of the Greek government and M.E.T. acknowledges financial support from CONACyT and the CVCP. We thank the British Council and German Academic Exchange Service for support under ARC project 1050.

1. Introduction

In hadronic collisions, jet cross sections are computed as a convolution of hard partonic cross sections with parton-distribution functions, followed by the fragmentation of the final-state partons into hadrons. The theoretical prediction can be improved by including higher order corrections which have the effect of reducing the unphysical renormalisation- and factorisation-scale dependences and by improving the matching of the parton-level theoretical jet algorithm with the hadron-level experimental jet algorithm. At present jet production is described at next-to-leading order (NLO) and several numerical programs are available [1, 2] which have been extensively used to compare with data from the TEVATRON and CERN $Spp\bar{S}$.

Improving the theoretical prediction to next-to-next-to-leading order (NNLO) requires several ingredients. First, the parton-density functions are needed to NNLO accuracy which in turn requires knowledge of the three-loop splitting functions. At present, the even moments of the splitting functions are known for the flavour singlet and non-singlet structure functions F_2 and F_L up to $N = 12$ while the odd moments up to $N = 13$ are known for F_3 [3, 4]. The numerically small N_F^2 non-singlet contribution is also known [5]. These moments are sufficient to parameterise the splitting functions in x -space [6, 7] and NNLO global analyses [8] are starting to appear. Second, the hard scattering matrix elements can be computed at NNLO. In the high energy limit, the quarks can be assumed to be massless and at this order, there are contributions from $2 \rightarrow 4$ tree-level diagrams [9, 10, 11, 12], from $2 \rightarrow 3$ one-loop-level diagrams [13, 14, 15] and two-loop $2 \rightarrow 2$ diagrams.

The evaluation of the two-loop diagrams has been a challenge for the past few years due to the presence of two-loop planar and crossed boxes. In the massless parton limit and in dimensional regularisation, analytic expressions for these basic scalar integrals have now been provided by Smirnov [16] and Tausk [17] as series in $\epsilon = (4 - D)/2$, where D is the space-time dimension, together with constructive procedures for reducing tensor integral to a basis set of known scalar (master) integrals [18, 19]. This makes the calculation of the two-loop amplitudes for $2 \rightarrow 2$ QCD scattering processes possible. Bern, Dixon and Kosower [20] were the first to address such scattering processes and provided analytic expressions for the maximal-helicity-violating two-loop amplitude for $gg \rightarrow gg$. Subsequently, Bern, Dixon and Ghinculov [21] completed the two-loop calculation of physical $2 \rightarrow 2$ scattering amplitudes for the QED processes $e^+e^- \rightarrow \mu^+\mu^-$ and $e^+e^- \rightarrow e^-e^+$. More recently, we have provided expressions relevant for unlike- and like-quark scattering in the massless limit in Refs. [22, 23] respectively. The corresponding matrix elements for the self-interference of one-loop $2 \rightarrow 2$ quark scattering processes are given in [24].

In this paper, we address the $\mathcal{O}(\alpha_s^4)$ one- and two-loop corrections to the QCD process

$$q + \bar{q} \rightarrow g + g, \tag{1.1}$$

together with the time-reversed and crossed processes

$$q + g \rightarrow q + g, \quad (1.2)$$

$$g + \bar{q} \rightarrow g + \bar{q}, \quad (1.3)$$

$$g + g \rightarrow q + \bar{q}. \quad (1.4)$$

As is in Refs. [22, 23, 24], we use the $\overline{\text{MS}}$ renormalisation scheme to remove the ultraviolet singularities and conventional dimensional regularisation, where all external particles are treated in D dimensions. We provide expressions for both the interference of tree-level and two-loop graphs as well as the self-interference of one-loop amplitudes. In each case, we find that the infrared pole structure agrees with that obtained using Catani's general factorisation formulae [25]. The finite remainders are the main new results presented in this paper and we give explicit analytic expressions valid for each of the processes of Eqs. (1.1)–(1.4) in terms of logarithms and polylogarithms that are real in the physical domain. For simplicity, we decompose our results according to the powers of the number of colours N and the number of light-quark flavours N_F .

Our paper is organised as follows. We first establish our notation in Sec. 2. Analytic expressions for the interference of the two-loop and tree-level amplitudes are given in Sec. 3, while formulae describing the self-interference of one-loop graphs are given in Sec. 4. In Sec. 3.1 we adopt the notation used in Ref. [25], to isolate the infrared singularity structure of the two-loop amplitudes in the $\overline{\text{MS}}$ scheme and we demonstrate that the anticipated singularity structure agrees with our explicit calculation. The finite $\mathcal{O}(\epsilon^0)$ remainder of the two-loop graphs is one of the main results of our paper and expressions appropriate for the $q\bar{q} \rightarrow gg$, $qg \rightarrow qg$ ($g\bar{q} \rightarrow g\bar{q}$) and $gg \rightarrow q\bar{q}$ scattering processes are given in Sec. 3.2 in terms of logarithms and polylogarithms that have no imaginary parts. Expressions for the self-interference of one-loop graphs are given in Sec. 4 in terms of the one-loop bubble integral in $D = 4 - 2\epsilon$ and the one-loop box integral in $D = 6 - 2\epsilon$. Analytic formulae connecting these integrals in the various kinematic regions are given in Appendix A. As for the two-loop contributions, the singularity structure agrees with that obtained using Catani's formalism. Finally we conclude with a brief summary of the results in Sec. 5.

2. Notation

For calculational purposes, the process we consider is

$$q(p_1) + \bar{q}(p_2) + g(p_3) + g(p_4) \rightarrow 0, \quad (2.1)$$

where the particles are all incoming and carry light-like momenta, satisfying

$$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0, \quad p_i^2 = 0.$$

The associated Mandelstam variables are given by

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_1 + p_3)^2, \quad s + t + u = 0. \quad (2.2)$$

We work in conventional dimensional regularisation treating all external quark and gluon states in D dimensions and renormalise the ultraviolet divergences in the $\overline{\text{MS}}$ scheme. The bare coupling α_0 is related to the running coupling $\alpha_s \equiv \alpha_s(\mu^2)$ at renormalisation scale μ , by

$$\alpha_0 S_\epsilon = \alpha_s \left[1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right], \quad (2.3)$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma}, \quad \gamma = 0.5772\dots = \text{Euler constant} \quad (2.4)$$

is the typical phase-space volume factor in $D = 4 - 2\epsilon$ dimensions. The first two coefficients of the QCD beta function, β_0 and β_1 , for N_F (massless) quark flavours, are

$$\beta_0 = \frac{11C_A - 4T_R N_F}{6}, \quad \beta_1 = \frac{17C_A^2 - 10C_A T_R N_F - 6C_F T_R N_F}{6}, \quad (2.5)$$

where N is the number of colours and

$$C_F = \frac{N^2 - 1}{2N}, \quad C_A = N, \quad T_R = \frac{1}{2}. \quad (2.6)$$

The renormalised four point amplitude in the $\overline{\text{MS}}$ scheme is thus

$$|\mathcal{M}\rangle = 4\pi\alpha_s \left[|\mathcal{M}^{(0)}\rangle + \left(\frac{\alpha_s}{2\pi} \right) |\mathcal{M}^{(1)}\rangle + \left(\frac{\alpha_s}{2\pi} \right)^2 |\mathcal{M}^{(2)}\rangle + \mathcal{O}(\alpha_s^3) \right], \quad (2.7)$$

where the $|\mathcal{M}^{(i)}\rangle$ represents the colour-space vector describing the i -loop amplitude. The dependence on both renormalisation scale μ and renormalisation scheme is implicit.

We denote the squared amplitude summed over spins and colours by

$$\langle \mathcal{M} | \mathcal{M} \rangle = \sum |\mathcal{M}(q + \bar{q} \rightarrow g + g)|^2 = \mathcal{C}(s, t, u). \quad (2.8)$$

which is symmetric under the exchange of t and u .

The squared matrix elements for the crossed processes are obtained by exchanging the Mandelstam variables and introducing a minus sign for each quark change between initial and final states

$$\sum |\mathcal{M}(g + g \rightarrow q + \bar{q})|^2 = \mathcal{C}(s, t, u), \quad (2.9)$$

$$\sum |\mathcal{M}(q + g \rightarrow q + g)|^2 = -\mathcal{C}(u, t, s), \quad (2.10)$$

$$\sum |\mathcal{M}(g + \bar{q} \rightarrow g + \bar{q})|^2 = -\mathcal{C}(u, t, s). \quad (2.11)$$

The function \mathcal{C} can be expanded perturbatively to yield

$$\mathcal{C}(s, t, u) = 16\pi^2 \alpha_s^2 \left[\mathcal{C}^4(s, t, u) + \left(\frac{\alpha_s}{2\pi} \right) \mathcal{C}^6(s, t, u) + \left(\frac{\alpha_s}{2\pi} \right)^2 \mathcal{C}^8(s, t, u) + \mathcal{O}(\alpha_s^3) \right], \quad (2.12)$$

where

$$\begin{aligned} \mathcal{C}^4(s, t, u) &= \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle = \\ &= 2 \frac{N^2 - 1}{N} (1 - \epsilon) \left(\frac{N^2 - 1}{ut} - 2 \frac{N^2}{s^2} \right) (t^2 + u^2 - \epsilon s^2), \end{aligned} \quad (2.13)$$

$$\mathcal{C}^6(s, t, u) = \left(\langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(1)} | \mathcal{M}^{(0)} \rangle \right), \quad (2.14)$$

$$\mathcal{C}^8(s, t, u) = \left(\langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle \right). \quad (2.15)$$

Expressions for \mathcal{C}^6 are given in Ref. [26] using dimensional regularisation to isolate the infrared and ultraviolet singularities.

In the following sections, we present expressions for the infrared singular and finite contributions to \mathcal{C}^8 and the crossed processes. For convenience, we divide $\mathcal{C}^8(s, t, u)$ into two pieces

- the pure two-loop contributions

$$\mathcal{C}^{8(2 \times 0)}(s, t, u) = \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle, \quad (2.16)$$

described in Sec. 3 and

- the self-interference of the one-loop amplitude

$$\mathcal{C}^{8(1 \times 1)}(s, t, u) = \langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle, \quad (2.17)$$

described in Sec. 4.

As in Refs. [22, 23], we use QGRAF [27] to produce the two-loop Feynman diagrams to construct $|\mathcal{M}^{(2)}\rangle$. We then project by $\langle \mathcal{M}^{(0)} |$ and perform the summation over colours and spins. It should be noted that when summing over the gluon polarisations, we ensure that the polarisations states are transversal (i.e. physical) by using an axial gauge

$$\sum_{\text{spins}} \epsilon_i^\mu \epsilon_i^{\nu*} = -g^{\mu\nu} + \frac{n_i^\mu p_i^\nu + n_i^\nu p_i^\mu}{n_i \cdot p_i} \quad (2.18)$$

where p_i is the momentum of gluon i and n_i is an arbitrary light-like 4-vector. For simplicity, we choose $n_3^\mu = p_4^\mu$ and $n_4^\mu = p_3^\mu$. Finally, the trace over the Dirac matrices is carried out in D dimensions using conventional dimensional regularisation. It is then straightforward to identify the scalar and tensor integrals present and replace them with combinations of the basis set of master integrals using the tensor reduction

of two-loop integrals described in [18, 19, 28], based on integration-by-parts [29] and Lorentz invariance [30] identities. The final result is a combination of master integrals in $D = 4 - 2\epsilon$ for which the expansions around $\epsilon = 0$ are given in [16, 17, 18, 19, 28, 31, 32, 33, 34].

3. Two-loop contribution

We further decompose the two-loop contributions as a sum of two terms

$$\mathcal{C}^{8(2\times 0)}(s, t, u) = \mathcal{Poles}(s, t, u) + \mathcal{Finite}(s, t, u). \quad (3.1)$$

\mathcal{Poles} contains infrared singularities that will be analytically canceled by the infrared singularities occurring in radiative processes of the same order (ultraviolet divergences are removed by renormalisation). \mathcal{Finite} is the remainder which is finite as $\epsilon \rightarrow 0$.

3.1 Infrared Pole Structure

Following the procedure outlined in Ref. [25], we can write the infrared pole structure, renormalised in the $\overline{\text{MS}}$ scheme as

$$\begin{aligned} \mathcal{Poles}(s, t, u) = \text{Re} \left\{ \right. \\ & \left[\frac{V^2}{4N} \left(-\mathbf{A}(\epsilon, s, t, u)^2 - \mathbf{B}(\epsilon, s, t, u)\mathbf{D}(\epsilon, s, t, u) + 2\mathcal{R}\mathbf{A}(2\epsilon, s, t, u) - 4\frac{\beta_0}{\epsilon}\mathbf{A}(\epsilon, s, t, u) \right) \right. \\ & \quad + \frac{V}{4} \left(-\mathbf{A}(\epsilon, s, t, u)\mathbf{B}(\epsilon, s, t, u) - \mathbf{B}(\epsilon, s, t, u)\mathbf{C}(\epsilon, s, t, u) + 2\mathcal{R}\mathbf{B}(2\epsilon, s, t, u) \right. \\ & \quad \left. \left. - 4\frac{\beta_0}{\epsilon}\mathbf{B}(\epsilon, s, t, u) \right) + \frac{V}{4N}\mathbf{B}(\epsilon, s, t, u)\mathbf{D}(\epsilon, s, u, t) \right] \mathcal{T}_1(s, t, u) \\ & + \left[\frac{V}{4N} \left(\mathbf{A}(\epsilon, s, t, u)^2 + \mathbf{B}(\epsilon, s, t, u)\mathbf{D}(\epsilon, s, t, u) + 4\frac{\beta_0}{\epsilon}\mathbf{A}(\epsilon, s, t, u) - 2\mathcal{R}\mathbf{A}(2\epsilon, s, t, u) \right) \right. \\ & \quad + \frac{V}{4} \left(-\mathbf{A}(\epsilon, s, t, u)\mathbf{B}(\epsilon, s, t, u) - \mathbf{B}(\epsilon, s, t, u)\mathbf{C}(\epsilon, s, t, u) - 4\frac{\beta_0}{\epsilon}\mathbf{B}(\epsilon, s, t, u) \right. \\ & \quad \left. \left. + 2\mathcal{R}\mathbf{B}(2\epsilon, s, t, u) \right) - \frac{V^2}{4N}\mathbf{B}(\epsilon, s, t, u)\mathbf{D}(\epsilon, s, u, t) \right] \mathcal{T}_2(s, t, u) \\ & + \left[\frac{V^2}{4N}\mathbf{A}(\epsilon, s, t, u) + \frac{V}{4}\mathbf{B}(\epsilon, s, t, u) \right] \mathcal{L}_1(s, t, u) \\ & + \left[-\frac{V}{4N}\mathbf{A}(\epsilon, s, t, u) + \frac{V}{4}\mathbf{B}(\epsilon, s, t, u) \right] \mathcal{L}_2(s, t, u) \\ & + \left[\frac{V^2}{4N}\mathbf{D}(\epsilon, s, t, u) + \frac{V}{4}\mathbf{C}(\epsilon, s, t, u) - \frac{V}{4N}\mathbf{D}(\epsilon, s, u, t) \right] \mathcal{L}_3(s, t, u) + \mathcal{H}_2 \\ & \left. + (u \leftrightarrow t) \right\}, \quad (3.2) \end{aligned}$$

where

$$\mathbf{A}(\epsilon, s, t, u) = - \left\{ \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \left[N \left(-\frac{\mu^2}{u} \right)^\epsilon + C_F \left(-\frac{\mu^2}{s} \right)^\epsilon \right] + N \left(\frac{\beta_0}{2N\epsilon} - \frac{3}{4\epsilon} \right) \left[\left(-\frac{\mu^2}{s} \right)^\epsilon + \left(-\frac{\mu^2}{u} \right)^\epsilon \right] \right\} \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \quad (3.3)$$

$$\mathbf{B}(\epsilon, s, t, u) = \left(\frac{1}{\epsilon^2} + \frac{3}{4\epsilon} + \frac{\beta_0}{2N\epsilon} \right) \left[\left(-\frac{\mu^2}{t} \right)^\epsilon - \left(-\frac{\mu^2}{s} \right)^\epsilon \right] \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \quad (3.4)$$

$$\mathbf{C}(\epsilon, s, t, u) = - \left[\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) C_F + \left(\frac{1}{\epsilon^2} + \frac{\beta_0}{N\epsilon} \right) N \right] \left(-\frac{\mu^2}{s} \right)^\epsilon \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \quad (3.5)$$

$$\mathbf{D}(\epsilon, s, t, u) = \left(\frac{1}{\epsilon^2} + \frac{3}{4\epsilon} + \frac{\beta_0}{2N\epsilon} \right) \left[\left(-\frac{\mu^2}{t} \right)^\epsilon - \left(-\frac{\mu^2}{u} \right)^\epsilon \right] \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \quad (3.6)$$

and

$$V = N^2 - 1, \quad (3.7)$$

$$\mathcal{R} = e^{-\epsilon\gamma} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\beta_0}{\epsilon} + K \right), \quad (3.8)$$

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R N_F. \quad (3.9)$$

The tree-type structures, \mathcal{T}_1 and \mathcal{T}_2 , are given by

$$\mathcal{T}_1(s, t, u) = \frac{t}{u} \mathcal{T}_2(s, t, u), \quad (3.10)$$

$$\mathcal{T}_2(s, t, u) = 8(1-\epsilon) \frac{(t^2 + u^2 - \epsilon s^2)}{s^2}, \quad (3.11)$$

while the interference of tree with one-loop structures are represented by \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3

$$\begin{aligned} \mathcal{L}_1(s, t, u) &= \frac{N^2 + 1}{N} f_2(s, t, u) - \frac{t}{Nu} f_1(s, u, t) + f_3(s, t, u) \\ &\quad - 6\beta_0 \frac{(1-\epsilon)}{3-2\epsilon} \mathcal{T}_1(s, t, u) \text{Bub}(s) \\ &\quad - 2\epsilon(1-2\epsilon) \left[\frac{N}{\epsilon^2} \text{Bub}(u) + \frac{N}{2} \left(\frac{1}{\epsilon^2} - \frac{2\beta_0}{N\epsilon} + \frac{3}{2\epsilon} \right) \text{Bub}(s) \right. \\ &\quad \left. - \frac{1}{2N} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \text{Bub}(s) \right] \mathcal{T}_1(s, t, u) \end{aligned} \quad (3.12)$$

$$\mathcal{L}_2(s, t, u) = \frac{t}{u} \mathcal{L}_1(s, u, t) \quad (3.13)$$

$$\begin{aligned} \mathcal{L}_3(s, t, u) &= 16(1-2\epsilon) \frac{1}{u} \left[t^2 + u^2 + (ut - 2t^2 - 2u^2) \epsilon + (t^2 + u^2 + 3ut) \epsilon^2 \right] \text{Box}^6(t, u) \\ &\quad + 2 \left(\frac{1}{\epsilon} - 2 \right) [\text{Bub}(t) - \text{Bub}(s)] \mathcal{T}_1(s, t, u) + f_1(s, t, u) \\ &\quad + 2 \left(\frac{1}{\epsilon} - 2 \right) [\text{Bub}(u) - \text{Bub}(s)] \mathcal{T}_2(s, t, u) + \frac{t}{u} f_1(s, u, t), \end{aligned} \quad (3.14)$$

where the infrared-finite functions f_1, f_2 and f_3 are

$$f_1(s, t, u) = 8(1 - 2\epsilon) \frac{1}{s^2} \left[u \left(2u^2 + 5t^2 + 3tu \right) + \epsilon \left(3t^3 - 4u^3 - 3tu^2 \right) - \epsilon^2 s \left(4t^2 + 2u^2 + 5tu \right) + s^2 t \epsilon^3 \right] \text{Box}^6(s, t) \quad (3.15)$$

$$f_2(s, t, u) = 8(1 - 2\epsilon) \frac{t}{us^2} \left[(t - u) \left(t^2 + 2u^2 + tu \right) + \epsilon \left(-2t^3 + ut^2 + 4tu^2 + 5u^3 \right) - s^3 \epsilon^2 \right] \text{Box}^6(s, u) \quad (3.16)$$

$$f_3(s, t, u) = -4(1 - \epsilon) \frac{t}{su} \left\{ \frac{V}{N} \left[2s - t - \epsilon(2s - 3u) - 3s\epsilon^2 \right] + 4N(u - u\epsilon + s\epsilon^2) \right\} [\text{Bub}(u) - \text{Bub}(s)] + \frac{4t}{s^2 u (1 - \epsilon)(3 - 2\epsilon)} \left\{ -N \left[18u^2 + 15t^2 - 3t(s - t) - \epsilon \left(78u^2 - 36t(s - t) + st \right) + \epsilon^2 \left(80u^2 + 10s(s - t) - 69st \right) \right] - \frac{1}{N} \left[-24u^2 + 3tu - 21t^2 + \epsilon \left(85u^2 - 43t(s - t) + 3st \right) - \epsilon^2 \left(112u^2 + 6s(s - t) - 109st \right) \right] + \beta_0 \left[20s^2 - 40tu - 2\epsilon \left(38s^2 - 3us - 62tu \right) + 4\epsilon^2 \left(27s^2 - 26tu \right) \right] \right\} \epsilon \text{Bub}(s) + \mathcal{O}(\epsilon^3). \quad (3.17)$$

These expressions are valid in all kinematic regions. However, to evaluate the pole structure in a particular region, the one-loop bubble graph Bub and the one-loop box integral in $D = 6 - 2\epsilon$ dimensions, Box^6 , must be expanded as a series in ϵ . This analytic expansion is given in Appendix A.

The function \mathcal{H}_2 , that appears in Eq. (3.2), exhibits only a single pole in ϵ and is given by

$$\mathcal{H}_2 \equiv \langle \mathcal{M}^{(0)} | \mathbf{H}^{(2)}(\epsilon) | \mathcal{M}^{(0)} \rangle = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1 - \epsilon)} H^{(2)} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle \quad (3.18)$$

where the constant $H^{(2)}$ is

$$H^{(2)} = \left(\zeta_3 + \frac{5}{6} + \frac{11}{72} \pi^2 \right) C_A^2 + \left(13\zeta_3 + \frac{245}{108} - \frac{23}{24} \pi^2 \right) C_A C_F + \frac{10}{27} N_F^2 + \left(-\frac{\pi^2}{36} - \frac{58}{27} \right) C_A N_F + \left(\frac{\pi^2}{12} + \frac{29}{54} \right) N_F C_F + \left(-\frac{3}{4} + \pi^2 - 12\zeta_3 \right) C_F^2, \quad (3.19)$$

and ζ_n is the Riemann Zeta function with $\zeta_2 = \pi^2/6$ and $\zeta_3 = 1.202056\dots$. We note that $H^{(2)}$ is renormalisation-scheme dependent and Eq. (3.19) is valid in the $\overline{\text{MS}}$ scheme. We also note that Eq. (3.19) differs from the corresponding expressions found in the singularity structure of two-loop quark-quark scattering in all but the

C_F^2 coefficient. This is due to the presence of infrared emissions from gluons which modify the terms involving either C_A or N_F .

It can be easily noted that the leading infrared singularity in Eq. (3.2) is $\mathcal{O}(1/\epsilon^4)$. It is a very stringent check on the reliability of our calculation that the pole structure obtained by computing the Feynman diagrams directly and introducing series expansions in ϵ for the scalar master integrals agrees with Eq. (3.2) through to $\mathcal{O}(1/\epsilon)$. We therefore construct the finite remainder by subtracting Eq. (3.2) from the full result.

3.2 Finite contributions

The finite two-loop contribution to $\mathcal{C}^8(s, t, u)$ is defined as

$$\mathcal{F}inite(s, t, u) = 2 \operatorname{Re} \left[\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)\text{fin}} \rangle \right]. \quad (3.20)$$

In hadronic collisions, all parton scattering processes (Eqs. (1.1)–(1.4)) contribute simultaneously. We therefore need to evaluate $\mathcal{F}inite(s, t, u)$ for the $q\bar{q} \rightarrow gg$ and $gg \rightarrow q\bar{q}$ process (which we denote as the s -channel since, although the tree-level process contains graphs in all three channels, the squared tree matrix elements are proportional to $1/s^2$) and $\mathcal{F}inite(u, t, s)$ for the QCD Compton processes $qg \rightarrow qg$ and $g\bar{q} \rightarrow g\bar{q}$ (which we label as the u -channel).

Of course, the analytic expressions for the various processes are related by crossing symmetry. However, the master crossed boxes have cuts in all three channels yielding complex parts in all physical regions. The analytic continuation is therefore rather involved and prone to error. We therefore choose to give expressions describing $\mathcal{C}^8(s, t, u)$ and $\mathcal{C}^8(u, t, s)$ which are directly valid in the physical region $s > 0$ and $u, t < 0$, and are given in terms of logarithms and polylogarithms that have no imaginary parts.

As usual, the polylogarithms $\operatorname{Li}_n(w)$ are defined by

$$\begin{aligned} \operatorname{Li}_n(w) &= \int_0^w \frac{dt}{t} \operatorname{Li}_{n-1}(t) \quad \text{for } n = 2, 3, 4 \\ \operatorname{Li}_2(w) &= - \int_0^w \frac{dt}{t} \log(1-t). \end{aligned} \quad (3.21)$$

Using the standard polylogarithm identities [35], we retain the polylogarithms with arguments x , $1-x$ and $(x-1)/x$, where

$$x = -\frac{t}{s}, \quad y = -\frac{u}{s} = 1-x, \quad z = -\frac{u}{t} = \frac{x-1}{x}. \quad (3.22)$$

For convenience, we also introduce the following logarithms

$$X = \log\left(\frac{-t}{s}\right), \quad Y = \log\left(\frac{-u}{s}\right), \quad S = \log\left(\frac{s}{\mu^2}\right), \quad U = \log\left(\frac{-u}{\mu^2}\right), \quad (3.23)$$

where μ is the renormalisation scale.

For each channel, we choose to present our results by grouping terms according to the power of the number of colours N and the number of light quarks N_F , so that in the generic c -channel we write

$$\begin{aligned} \mathcal{F}inite_c(s, t, u) &= (N^2 - 1) \\ &\times \left(N^3 A_c + N B_c + \frac{1}{N} C_c + \frac{1}{N^3} D_c + N_F N^2 E_c + N_F F_c + \frac{N_F}{N^2} G_c + N_F^2 N H_c + \frac{N_F^2}{N} I_c \right). \end{aligned} \quad (3.24)$$

3.2.1 $\mathcal{F}inite(s, t, u)$: the s -channel process

We first give expressions valid for the annihilation processes, $q\bar{q} \rightarrow gg$ and $gg \rightarrow \bar{q}q$. We find that

$$\begin{aligned} A_s &= \left\{ \left(-\frac{36077}{864} - \frac{109}{144} \pi^4 + \frac{241}{12} \zeta_3 - \frac{5}{2} \pi^2 + \frac{2327}{216} S + 20 \text{Li}_4(x) + \frac{11}{72} \pi^2 S - 9 \zeta_3 S \right. \right. \\ &\quad + \frac{121}{9} S^2 - 21 \text{Li}_4(y) - 20 \text{Li}_4(z) - \frac{35}{6} \text{Li}_3(y) + \frac{1}{3} X Y^3 + \frac{35}{12} Y^2 X + \frac{10}{3} Y X \pi^2 \\ &\quad + 4 \pi^2 \text{Li}_2(x) + 16 Y \zeta_3 + \frac{1}{2} Y^2 \text{Li}_2(y) + \frac{35}{6} Y \text{Li}_2(y) + \frac{121}{9} Y S - 5 X^2 Y^2 \\ &\quad + \frac{505}{72} Y^2 - 5 X^2 - \frac{77}{36} Y^3 + \frac{1}{8} Y^4 - \frac{5}{6} X^4 + \frac{10}{3} Y X^3 - \frac{2}{3} \pi^2 Y^2 - \frac{5}{3} \pi^2 X^2 \\ &\quad \left. - \frac{143}{36} Y \pi^2 - \frac{11}{3} Y^2 S - 20 Y \text{Li}_3(x) + \frac{2273}{216} Y \right) \frac{t}{u} \\ &\quad + \left(\frac{11}{90} \pi^4 + \frac{9}{2} \zeta_3 + \frac{169}{36} \pi^2 + \frac{457}{144} - \frac{11}{6} S + \frac{33}{4} X \text{Li}_2(x) - 8 Y \zeta_3 + \frac{33}{8} X^2 Y \right. \\ &\quad \left. - \frac{33}{4} \text{Li}_3(x) - \frac{2417}{144} X + \frac{2}{3} Y \pi^2 + 8 Y \text{Li}_3(x) - \frac{17}{24} X^3 + \frac{141}{16} Y^2 + \frac{55}{6} Y S + X^2 Y^2 \right) \\ &\quad + \left(\frac{29}{72} \pi^4 - \frac{205}{6} \zeta_3 - 2 \pi^2 - \frac{1535}{108} S + \frac{30377}{432} - 22 \text{Li}_4(x) - \frac{11}{36} \pi^2 S + 18 \zeta_3 S \right. \\ &\quad - \frac{242}{9} S^2 + 18 \text{Li}_4(y) + 16 \text{Li}_4(z) + \frac{5}{12} \text{Li}_3(y) + \frac{57}{4} \text{Li}_3(x) - \frac{2}{3} X Y^3 - \frac{5}{24} Y^2 X \\ &\quad - \frac{8}{3} Y X \pi^2 + \frac{977}{144} X - 4 \pi^2 \text{Li}_2(x) - Y^2 \text{Li}_2(y) - \frac{5}{12} Y \text{Li}_2(y) - \frac{55}{6} X S \\ &\quad + \frac{22}{3} S X^2 - 8 \text{Li}_3(y) X - \frac{715}{18} Y S + 2 X^2 Y^2 - \frac{57}{4} X \text{Li}_2(x) - \frac{57}{8} X^2 Y \\ &\quad + 4 X \text{Li}_3(x) + 4 X \zeta_3 - X^2 \text{Li}_2(x) + \frac{40}{9} X \pi^2 - \frac{3847}{144} Y^2 - \frac{793}{144} X^2 + \frac{401}{72} Y^3 \\ &\quad - \frac{1}{4} Y^4 + \frac{1}{12} X^4 + \frac{431}{72} X^3 - \frac{8}{3} Y X^3 + \frac{4}{3} \pi^2 Y^2 + 4 \pi^2 X^2 + \frac{47}{18} Y \pi^2 + \frac{22}{3} Y^2 S \\ &\quad \left. + 8 Y \text{Li}_3(x) - \frac{3059}{432} Y \right) \frac{t^2}{s^2} \Big\} + \left\{ u \leftrightarrow t \right\} \end{aligned} \quad (3.25)$$

$$\begin{aligned}
B_s = & \left\{ \left(\frac{317}{180} \pi^4 - \frac{733}{18} \zeta_3 - \frac{1055}{72} \pi^2 - \frac{7543}{216} S + 2Y \operatorname{Li}_3(y) + 8 \operatorname{Li}_4(x) - 20YX \operatorname{Li}_2(y) \right. \right. \\
& - \frac{44}{3} YXS + \frac{65}{9} \pi^2 S + 10 \zeta_3 S - \frac{121}{9} S^2 - 79 \operatorname{Li}_4(y) - 74 \operatorname{Li}_4(z) - \frac{71}{6} \operatorname{Li}_3(y) \\
& + \frac{25}{3} \operatorname{Li}_3(x) + \frac{164771}{2592} + \frac{5}{3} XY^3 + \frac{1}{4} Y^2 X + \frac{19}{3} YX \pi^2 - \frac{22}{27} X - \frac{19}{3} \pi^2 \operatorname{Li}_2(x) \\
& + 74Y \zeta_3 + \frac{1}{2} Y^2 \operatorname{Li}_2(y) + \frac{71}{6} Y \operatorname{Li}_2(y) + \frac{16}{9} XY - \frac{220}{9} XS + 24 \operatorname{Li}_3(y) X \\
& - \frac{22}{9} YS - \frac{49}{2} X^2 Y^2 - \frac{25}{3} X \operatorname{Li}_2(x) + \frac{55}{6} X^2 Y + 44X \operatorname{Li}_3(x) - 36X \zeta_3 \\
& - 11X^2 \operatorname{Li}_2(x) - \frac{551}{36} X \pi^2 - \frac{125}{72} Y^2 + \frac{583}{36} X^2 + \frac{127}{36} Y^3 - \frac{3}{8} Y^4 - \frac{25}{12} X^4 - \frac{17}{2} X^3 \\
& + 9YX^3 - \frac{8}{3} \pi^2 Y^2 - \frac{16}{3} \pi^2 X^2 - \frac{83}{36} Y \pi^2 + \frac{22}{3} Y^2 S - 70Y \operatorname{Li}_3(x) - \frac{5765}{216} Y \left. \right) \frac{t}{u} \\
& + \left(- \frac{229}{240} \pi^4 + \frac{118}{3} \zeta_3 - \frac{799}{72} \pi^2 - \frac{11}{6} S + \frac{139}{36} - 18Y \operatorname{Li}_3(y) + \frac{65}{2} \operatorname{Li}_4(y) \right. \\
& - \frac{299}{6} \operatorname{Li}_3(x) + \frac{2}{3} XY^3 - \frac{5}{6} YX \pi^2 + \frac{14}{9} X + 19Y \zeta_3 + \frac{7}{4} Y^2 \operatorname{Li}_2(y) + \frac{73}{4} XY \\
& - \frac{55}{6} YS - \frac{13}{8} X^2 Y^2 + \frac{299}{6} X \operatorname{Li}_2(x) + \frac{62}{3} X^2 Y - \frac{311}{36} Y^2 - \frac{17}{48} Y^4 + \frac{73}{18} X^3 \\
& \left. - \frac{47}{12} \pi^2 Y^2 - \frac{55}{6} Y \pi^2 - \frac{11}{6} Y^2 S - 9Y \operatorname{Li}_3(x) \right) \\
& + \left(- \frac{17}{60} \pi^4 + \frac{443}{18} \zeta_3 + \frac{16}{9} \pi^2 + \frac{1502}{27} S + 20Y \operatorname{Li}_3(y) + \frac{7}{2} \operatorname{Li}_4(x) + 8YX \operatorname{Li}_2(y) \right. \\
& - \frac{1}{12} \pi^2 S - 2 \zeta_3 S - \frac{79}{2} \operatorname{Li}_4(y) - 22 \operatorname{Li}_4(z) + \frac{71}{6} \operatorname{Li}_3(y) - \frac{71}{6} \operatorname{Li}_3(x) - \frac{16}{3} XY^3 \\
& + \frac{32}{3} Y^2 X - \frac{8}{3} YX \pi^2 + \frac{359}{36} X + 7 \pi^2 \operatorname{Li}_2(x) - Y \zeta_3 - \frac{45}{4} Y^2 \operatorname{Li}_2(y) - \frac{71}{6} Y \operatorname{Li}_2(y) \\
& + \frac{391}{18} XY + \frac{11}{6} XS + \frac{11}{6} SX^2 + 7 \operatorname{Li}_3(y) X - \frac{11}{6} YS + \frac{13}{4} X^2 Y^2 + \frac{71}{6} X \operatorname{Li}_2(x) \\
& - 24X^2 Y + 16X \operatorname{Li}_3(x) + X \zeta_3 - \frac{27}{4} X^2 \operatorname{Li}_2(x) + \frac{103}{18} X \pi^2 - \frac{203}{36} Y^2 - \frac{277}{12} X^2 \\
& - \frac{101}{18} Y^3 + \frac{31}{48} Y^4 - \frac{33}{16} X^4 + \frac{341}{18} X^3 + 3YX^3 + \frac{11}{4} \pi^2 Y^2 + \frac{11}{4} \pi^2 X^2 - \frac{103}{18} Y \pi^2 \\
& \left. - \frac{11}{6} Y^2 S - 7Y \operatorname{Li}_3(x) - \frac{19139}{324} + \frac{601}{36} Y \right) \frac{t^2}{s^2} \\
& \left. - X^2 \frac{t^4}{u^2 s^2} \right\} + \left\{ u \leftrightarrow t \right\} \tag{3.26}
\end{aligned}$$

$$C_s = \left\{ \left(- \frac{41393}{2592} + \frac{301}{720} \pi^4 + \frac{173}{36} \zeta_3 - \frac{133}{36} \pi^2 + \frac{5297}{216} S + 4Y \operatorname{Li}_3(y) - 20 \operatorname{Li}_4(x) \right. \right.$$

$$\begin{aligned}
& -4Y X \operatorname{Li}_2(y) - \frac{13}{24} \pi^2 S + 5 \zeta_3 S + \operatorname{Li}_4(y) + 8 \operatorname{Li}_4(z) - \frac{15}{2} \operatorname{Li}_3(y) - 3 \operatorname{Li}_3(x) \\
& - \frac{2}{3} X Y^3 - \frac{11}{12} Y^2 X - \frac{14}{3} Y X \pi^2 + \frac{59}{3} X - \frac{8}{3} \pi^2 \operatorname{Li}_2(x) - 12 Y \zeta_3 - \frac{1}{2} Y^2 \operatorname{Li}_2(y) \\
& + \frac{15}{2} Y \operatorname{Li}_2(y) + 2 X Y - \frac{22}{3} S X^2 + 10 \operatorname{Li}_3(y) X - 11 Y S + \frac{7}{2} X^2 Y^2 + 3 X \operatorname{Li}_2(x) \\
& - \frac{10}{3} X^2 Y + 10 X \operatorname{Li}_3(x) - 6 X \zeta_3 - 4 X^2 \operatorname{Li}_2(x) + \frac{49}{18} X \pi^2 - \frac{425}{72} Y^2 + \frac{179}{18} X^2 \\
& - \frac{23}{36} Y^3 + \frac{3}{8} Y^4 + \frac{1}{3} X^4 - \frac{65}{18} X^3 - \frac{8}{3} Y X^3 + \frac{5}{6} \pi^2 Y^2 + \frac{3}{2} \pi^2 X^2 + \frac{5}{18} Y \pi^2 \\
& - \frac{11}{3} Y^2 S + 14 Y \operatorname{Li}_3(x) + \frac{271}{24} Y \Big) \frac{t}{u} \\
& + \left(\frac{271}{240} \pi^4 + \frac{3}{2} \zeta_3 - \frac{127}{72} \pi^2 + \frac{13}{4} X \operatorname{Li}_2(x) + 26 Y \operatorname{Li}_3(y) - \frac{21}{4} Y^2 \operatorname{Li}_2(y) - 25 Y \zeta_3 \right. \\
& - \frac{119}{24} X^2 Y - \frac{13}{4} \operatorname{Li}_3(x) + \frac{3133}{144} X - \frac{4}{3} X Y^3 + \frac{7}{4} \pi^2 Y^2 + \frac{55}{18} Y \pi^2 - \frac{55}{12} X Y \\
& - \frac{83}{2} \operatorname{Li}_4(y) - \frac{7}{6} Y X \pi^2 - \frac{22}{3} Y^2 S + 15 Y \operatorname{Li}_3(x) + \frac{23}{16} - \frac{209}{72} X^3 + \frac{919}{144} Y^2 \\
& \left. + \frac{9}{16} Y^4 - \frac{22}{3} Y S + \frac{23}{8} X^2 Y^2 \right) \\
& + \left(\frac{13}{360} \pi^4 + 15 \zeta_3 + \frac{1}{3} \pi^2 - \frac{3}{4} S + \frac{11}{2} \operatorname{Li}_4(x) + \pi^2 S - 12 \zeta_3 S - \frac{11}{2} \operatorname{Li}_4(y) - 6 \operatorname{Li}_4(z) \right. \\
& + \frac{11}{4} \operatorname{Li}_3(y) - \frac{11}{4} \operatorname{Li}_3(x) - \frac{255}{16} - \frac{25}{8} Y^2 X + \frac{2}{3} Y X \pi^2 - \frac{21}{16} X + \frac{1}{3} \pi^2 \operatorname{Li}_2(x) \\
& + 3 Y \zeta_3 - \frac{1}{4} Y^2 \operatorname{Li}_2(y) - \frac{11}{4} Y \operatorname{Li}_2(y) - \frac{11}{2} X Y + 3 \operatorname{Li}_3(y) X - \frac{3}{4} X^2 Y^2 \\
& + \frac{11}{4} X \operatorname{Li}_2(x) + \frac{25}{8} X^2 Y - 3 X \zeta_3 + \frac{1}{4} X^2 \operatorname{Li}_2(x) - \frac{7}{6} X \pi^2 + \frac{83}{16} Y^2 - \frac{11}{16} X^2 \\
& + \frac{13}{24} Y^3 - \frac{5}{48} Y^4 - \frac{7}{48} X^4 - \frac{13}{24} X^3 + Y X^3 + \frac{1}{12} \pi^2 Y^2 - \frac{7}{12} \pi^2 X^2 + \frac{7}{6} Y \pi^2 \\
& \left. - 3 Y \operatorname{Li}_3(x) + \frac{21}{16} Y \right) \frac{t^2}{s^2} + X^2 \frac{t^4}{u^2 s^2} \Big\} + \left\{ u \leftrightarrow t \right\} \tag{3.27}
\end{aligned}$$

$$\begin{aligned}
D_s = & \left\{ \left(-\frac{1}{90} \pi^4 + \frac{5}{2} \zeta_3 + \frac{5}{8} \pi^2 - \frac{3}{8} S + 2 Y \operatorname{Li}_3(y) + \frac{1}{2} \pi^2 S - 6 \zeta_3 S + 11 \operatorname{Li}_4(y) \right. \right. \\
& + 14 \operatorname{Li}_4(z) - \frac{3}{2} \operatorname{Li}_3(y) + 8 \operatorname{Li}_3(x) - \frac{187}{32} + \frac{3}{4} Y^2 X - \frac{7}{3} Y X \pi^2 + X + \pi^2 \operatorname{Li}_2(x) \\
& - 18 Y \zeta_3 - \frac{1}{2} Y^2 \operatorname{Li}_2(y) + \frac{3}{2} Y \operatorname{Li}_2(y) + 2 \operatorname{Li}_3(y) X + 4 X^2 Y^2 - 8 X \operatorname{Li}_2(x) \\
& - \frac{11}{2} X^2 Y - 6 X \operatorname{Li}_3(x) + 4 X \zeta_3 - X^2 \operatorname{Li}_2(x) + \frac{10}{3} X \pi^2 + \frac{5}{8} Y^2 - \frac{9}{4} X^2 - \frac{3}{4} Y^3 \\
& \left. - \frac{1}{8} Y^4 - \frac{1}{12} X^4 - \frac{7}{3} Y X^3 + \frac{1}{2} \pi^2 Y^2 + \frac{1}{2} \pi^2 X^2 + 2 Y \pi^2 + 16 Y \operatorname{Li}_3(x) + \frac{39}{8} Y \right) \frac{t}{u}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{4} - \frac{4}{45} \pi^4 - 2 \zeta_3 + \frac{43}{12} \pi^2 - 19 X \operatorname{Li}_2(x) - 4 Y \operatorname{Li}_3(y) - Y^2 \operatorname{Li}_2(y) - 12 Y \zeta_3 \right. \\
& - \frac{15}{2} X^2 Y + 19 \operatorname{Li}_3(x) + \frac{29}{4} X - \frac{1}{3} \pi^2 Y^2 + \frac{4}{3} Y \pi^2 - \frac{7}{2} X Y + 10 \operatorname{Li}_4(y) \\
& \left. - \frac{4}{3} Y X \pi^2 + 16 Y \operatorname{Li}_3(x) - \frac{11}{6} X^3 + 4 Y^2 - \frac{7}{12} Y^4 + 2 X^2 Y^2 \right) \\
& + \left(-3 X + \frac{7}{2} X^2 + 3 Y + \frac{3}{2} \pi^2 + \frac{1}{2} Y^2 - 3 X Y \right) \frac{t^2}{s^2} \\
& - X^2 \frac{t^4}{u^2 s^2} \left. \right\} + \left\{ u \leftrightarrow t \right\} \tag{3.28}
\end{aligned}$$

$$\begin{aligned}
E_s = & \left\{ \left(\frac{14}{45} \pi^4 - \frac{17}{6} \zeta_3 + \frac{77}{54} \pi^2 + \frac{185}{54} S - 8 \operatorname{Li}_4(x) - \frac{1}{36} \pi^2 S - \frac{44}{9} S^2 + 8 \operatorname{Li}_4(y) \right. \right. \\
& + 8 \operatorname{Li}_4(z) + \frac{4}{3} \operatorname{Li}_3(y) - \frac{2}{3} Y^2 X - \frac{4}{3} Y X \pi^2 - \frac{4}{3} \pi^2 \operatorname{Li}_2(x) - 8 Y \zeta_3 - \frac{4}{3} Y \operatorname{Li}_2(y) \\
& - \frac{44}{9} Y S + 2 X^2 Y^2 - \frac{43}{18} Y^2 + 2 X^2 + \frac{7}{18} Y^3 + \frac{1}{3} X^4 - \frac{4}{3} Y X^3 + \frac{2}{3} \pi^2 X^2 + \frac{13}{18} Y \pi^2 \\
& \left. + \frac{2}{3} Y^2 S + 8 Y \operatorname{Li}_3(x) + \frac{10}{9} Y + \frac{1307}{216} \right) \frac{t}{u} \\
& + \left(-\frac{37}{36} - \frac{11}{360} \pi^4 - 4 \zeta_3 - \frac{49}{36} \pi^2 + \frac{1}{3} S - 4 X \operatorname{Li}_2(x) + 2 Y \zeta_3 - 2 X^2 Y + 4 \operatorname{Li}_3(x) \right. \\
& \left. + \frac{91}{18} X + \frac{2}{3} Y \pi^2 - 2 Y \operatorname{Li}_3(x) - 2 Y^2 - \frac{5}{3} Y S - \frac{1}{4} X^2 Y^2 \right) \\
& + \left(-\frac{17}{180} \pi^4 + \frac{17}{3} \zeta_3 - \frac{32}{27} \pi^2 - \frac{221}{27} S + 4 \operatorname{Li}_4(x) + \frac{1}{18} \pi^2 S + \frac{88}{9} S^2 - 4 \operatorname{Li}_4(y) \right. \\
& - 4 \operatorname{Li}_4(z) + \frac{4}{3} \operatorname{Li}_3(y) - 4 \operatorname{Li}_3(x) - \frac{2}{3} Y^2 X + \frac{2}{3} Y X \pi^2 - \frac{19}{18} X + \frac{2}{3} \pi^2 \operatorname{Li}_2(x) \\
& + 2 Y \zeta_3 - \frac{4}{3} Y \operatorname{Li}_2(y) - \frac{863}{108} + \frac{5}{3} X S - \frac{4}{3} S X^2 + 2 \operatorname{Li}_3(y) X + \frac{109}{9} Y S \\
& - \frac{1}{2} X^2 Y^2 + 4 X \operatorname{Li}_2(x) + 2 X^2 Y - 2 X \zeta_3 - \frac{16}{9} X \pi^2 + \frac{70}{9} Y^2 + \frac{1}{9} X^2 - \frac{7}{9} Y^3 \\
& \left. - \frac{1}{6} X^4 - \frac{7}{9} X^3 + \frac{2}{3} Y X^3 - \frac{1}{3} \pi^2 X^2 - \frac{7}{9} Y \pi^2 - \frac{4}{3} Y^2 S - 2 Y \operatorname{Li}_3(x) - \frac{31}{6} Y \right) \frac{t^2}{s^2} \left. \right\} \\
& + \left\{ u \leftrightarrow t \right\} \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
F_s = & \left\{ \left(-\frac{28}{45} \pi^4 - \frac{10}{9} \zeta_3 + \frac{347}{108} \pi^2 + \frac{7}{9} S + 16 \operatorname{Li}_4(x) + \frac{8}{3} Y X S - \frac{11}{9} \pi^2 S + \frac{44}{9} S^2 \right. \right. \\
& \left. - 16 \operatorname{Li}_4(y) - 16 \operatorname{Li}_4(z) + \frac{4}{3} \operatorname{Li}_3(y) + 4 \operatorname{Li}_3(x) + \frac{1}{2} Y^2 X + \frac{8}{3} Y X \pi^2 - \frac{161}{18} X \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{8}{3} \pi^2 \text{Li}_2(x) + 16 Y \zeta_3 - \frac{4}{3} Y \text{Li}_2(y) - \frac{1}{2} X Y - \frac{3661}{324} + \frac{62}{9} X S + \frac{26}{9} Y S \\
& - 4 X^2 Y^2 - 4 X \text{Li}_2(x) - \frac{2}{3} X^2 Y - \frac{37}{18} X \pi^2 + \frac{35}{18} Y^2 - \frac{79}{18} X^2 - \frac{7}{9} Y^3 - \frac{2}{3} X^4 \\
& + \frac{2}{3} X^3 + \frac{8}{3} Y X^3 - \frac{4}{3} \pi^2 X^2 + \frac{1}{18} Y \pi^2 - \frac{4}{3} Y^2 S - 16 Y \text{Li}_3(x) + \frac{23}{9} Y \Big) \frac{t}{u} \\
& + \left(-\frac{11}{90} \pi^4 + \frac{52}{3} \zeta_3 + \frac{35}{12} \pi^2 + \frac{1}{3} S + \frac{52}{3} X \text{Li}_2(x) + 8 Y \zeta_3 + \frac{47}{6} X^2 Y - \frac{52}{3} \text{Li}_3(x) \right. \\
& - \frac{50}{9} X + \frac{1}{18} Y \pi^2 - \frac{1}{3} X Y + \frac{1}{3} Y^2 S - 8 Y \text{Li}_3(x) - \frac{11}{36} X^3 + \frac{49}{18} Y^2 - \frac{19}{36} \\
& \left. + \frac{5}{3} Y S - X^2 Y^2 \right) \\
& + \left(-\frac{263}{27} S - \frac{1}{3} X S + \frac{3}{2} X \pi^2 - \frac{1}{3} S X^2 + \frac{5}{2} X^2 Y + \frac{1}{3} Y S - \frac{1}{6} \pi^2 S \right. \\
& + \frac{1}{3} Y^2 S + \frac{7}{9} X + \frac{4}{3} X \text{Li}_2(x) + \frac{211}{18} X^2 - \frac{4}{3} \text{Li}_3(x) + \frac{4085}{324} + \frac{4}{3} \text{Li}_3(y) - \frac{1}{9} \zeta_3 \\
& - \frac{31}{9} Y - \frac{4}{3} Y \text{Li}_2(y) - \frac{3}{2} Y \pi^2 - \frac{73}{36} X^3 - \frac{7}{6} Y^2 X + \frac{25}{36} Y^3 + \frac{25}{18} \pi^2 - \frac{41}{18} Y^2 \\
& \left. - \frac{58}{9} X Y \right) \frac{t^2}{s^2} - 3 X^2 \frac{t^4}{u^2 s^2} \Big\} + \left\{ u \leftrightarrow t \right\} \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
G_s = & \left\{ \left(\frac{28}{15} \pi^4 + \frac{143}{18} \zeta_3 + \frac{19}{36} \pi^2 - \frac{227}{54} S - 48 \text{Li}_4(x) - \frac{1}{12} \pi^2 S + 48 \text{Li}_4(y) + 48 \text{Li}_4(z) \right. \right. \\
& - 8 \text{Li}_3(x) + \frac{3401}{648} + \frac{1}{6} Y^2 X - 8 Y X \pi^2 + \frac{101}{6} X - 8 \pi^2 \text{Li}_2(x) - 48 Y \zeta_3 \\
& + \frac{1}{2} X Y + \frac{4}{3} S X^2 + 2 Y S + 12 X^2 Y^2 + 8 X \text{Li}_2(x) + \frac{1}{3} X^2 Y + \frac{64}{9} X \pi^2 \\
& + \frac{4}{9} Y^2 + \frac{89}{9} X^2 + \frac{7}{18} Y^3 + 2 X^4 + \frac{7}{9} X^3 - 8 Y X^3 + 4 \pi^2 X^2 + \frac{2}{9} Y \pi^2 + \frac{2}{3} Y^2 S \\
& \left. + 48 Y \text{Li}_3(x) - \frac{11}{3} Y \right) \frac{t}{u} \\
& + \left(\frac{11}{30} \pi^4 - 52 \zeta_3 - \frac{107}{18} \pi^2 - 52 X \text{Li}_2(x) - 24 Y \zeta_3 - \frac{71}{3} X^2 Y + 52 \text{Li}_3(x) + \frac{71}{9} X \right. \\
& \left. + \frac{46}{9} Y \pi^2 - \frac{2}{3} X Y + \frac{4}{3} Y^2 S + 24 Y \text{Li}_3(x) + \frac{7}{9} X^3 - \frac{37}{9} Y^2 + \frac{4}{3} Y S + 3 X^2 Y^2 \right) \\
& + \left(-\frac{10}{3} X \pi^2 - 8 X - 4 X \text{Li}_2(x) - 18 X^2 + 4 \text{Li}_3(x) - 4 \text{Li}_3(y) + 8 Y + 4 Y \text{Li}_2(y) \right. \\
& \left. + \frac{10}{3} Y \pi^2 + 9 Y^2 \right) \frac{t^2}{s^2} + 9 X^2 \frac{t^4}{u^2 s^2} \Big\} + \left\{ u \leftrightarrow t \right\} \tag{3.31}
\end{aligned}$$

$$H_s = \left\{ \left(-\frac{4}{27} \pi^2 + \frac{4}{9} S^2 - \frac{20}{27} S - \frac{10}{27} Y + \frac{4}{9} Y S + \frac{1}{6} Y^2 \right) \frac{t}{u} \right.$$

$$+ \left(\frac{8}{27} \pi^2 + \frac{40}{27} S - \frac{8}{9} S^2 + \frac{20}{27} Y - \frac{8}{9} Y S - \frac{1}{3} Y^2 \right) \frac{t^2}{s^2} \Big\} + \left\{ u \leftrightarrow t \right\} \quad (3.32)$$

$$I_s = \left\{ \left(-\frac{1}{9} X Y + \frac{10}{27} Y - \frac{1}{6} Y^2 + \frac{20}{27} S - \frac{4}{9} X S - \frac{4}{9} S^2 - \frac{1}{18} X^2 - \frac{7}{54} \pi^2 \right. \right. \\ \left. \left. - \frac{4}{9} Y S + \frac{10}{27} X \right) \frac{t}{u} - \frac{2}{9} X \left(X - Y \right) \frac{t^2}{s^2} \right\} + \left\{ u \leftrightarrow t \right\} \quad (3.33)$$

3.2.2 *Finite*(u, t, s): the u -channel process

Similarly, for the Compton scattering processes, $qg \rightarrow qg$ and $g\bar{q} \rightarrow g\bar{q}$, we find that the coefficients for the finite part in Eq. (3.24) are given by

$$A_u = \left(-\frac{2417}{144} X + \frac{11}{45} \pi^4 - \frac{19}{144} \pi^2 + \frac{3}{4} \zeta_3 - \frac{11}{3} U + \frac{33}{4} \text{Li}_3(x) + \frac{457}{72} + 8 \text{Li}_3(x) Y \right. \\ - \frac{55}{3} U Y - \frac{10}{3} \pi^2 Y + 8 \zeta_3 Y - \frac{2}{3} \pi^2 Y^2 - 2 X^2 Y - \frac{33}{4} X \text{Li}_2(x) - 8 X \zeta_3 \\ - \frac{35}{24} X \pi^2 + \frac{4}{3} X Y^3 + 8 \text{Li}_3(y) X + \frac{55}{6} U X + \frac{141}{16} X^2 - \frac{17}{24} X^3 - \frac{141}{8} Y X \\ \left. + 2 Y^2 X + 2 X^2 Y^2 - \frac{8}{3} \pi^2 X Y - \frac{2}{3} Y^4 - \frac{4}{3} Y^3 + \frac{141}{8} Y^2 + \frac{2417}{72} Y \right) \\ + \left(\frac{121}{9} U^2 - \frac{36077}{864} - 9 U \zeta_3 - \frac{121}{72} U \pi^2 + \frac{2273}{432} X - \frac{181}{720} \pi^4 - \frac{215}{144} \pi^2 + \frac{103}{6} \zeta_3 \right. \\ + \frac{2327}{216} U + \frac{1}{2} \text{Li}_4(z) + \frac{1}{2} \text{Li}_4(y) + \frac{35}{12} \text{Li}_3(x) - 10 \text{Li}_3(x) Y - \frac{121}{9} U Y - \frac{29}{18} \pi^2 Y \\ - 6 \zeta_3 Y + \frac{7}{12} \pi^2 Y^2 + \frac{7}{4} X^2 Y - \frac{35}{12} X \text{Li}_2(x) + 8 X \zeta_3 - \frac{1}{4} X^2 \text{Li}_2(x) - \frac{55}{36} X \pi^2 \\ - \frac{11}{6} U X^2 - \frac{13}{6} X Y^3 + \frac{1}{12} \pi^2 X^2 - \frac{1}{4} \pi^2 \text{Li}_2(x) - \frac{11}{3} U Y^2 - 10 \text{Li}_3(y) X \\ + \frac{121}{18} U X + \frac{1}{12} X^4 + \frac{145}{144} X^2 - \frac{77}{72} X^3 - \frac{1}{2} X^3 Y - \frac{145}{72} Y X - \frac{7}{4} Y^2 X - \frac{17}{8} X^2 Y^2 \\ + \frac{37}{12} \pi^2 X Y + \frac{11}{3} U X Y - \frac{1}{2} \text{Li}_2(y) X Y + \frac{13}{12} Y^4 + \frac{7}{6} Y^3 + \frac{145}{72} Y^2 \\ \left. - \frac{2273}{216} Y \right) \left[\frac{t^2 + s^2}{st} \right] \\ + \left(\frac{11}{6} U \pi^2 - \frac{2273}{432} X - \frac{35}{36} \pi^4 - \frac{865}{144} \pi^2 + \frac{35}{12} \zeta_3 - \frac{41}{2} \text{Li}_4(z) - \frac{35}{6} \text{Li}_3(y) + 20 \text{Li}_4(x) \right. \\ + \frac{41}{2} \text{Li}_4(y) - \frac{35}{12} \text{Li}_3(x) - 10 \text{Li}_3(x) Y - \frac{7}{4} \pi^2 Y - 20 \text{Li}_3(y) Y + 10 \zeta_3 Y - \frac{13}{4} \pi^2 Y^2 \\ - \frac{7}{4} X^2 Y + \frac{35}{12} X \text{Li}_2(x) - 8 X \zeta_3 + \frac{1}{4} X^2 \text{Li}_2(x) + \frac{55}{36} X \pi^2 + \frac{11}{6} U X^2 - \frac{3}{2} X Y^3 \\ \left. - \frac{7}{4} \pi^2 X^2 + \frac{35}{6} \text{Li}_2(y) Y - \frac{15}{4} \pi^2 \text{Li}_2(x) + 10 \text{Li}_3(y) X - \frac{121}{18} U X - \frac{11}{12} X^4 - \frac{865}{144} X^2 \right)$$

$$\begin{aligned}
& + \frac{77}{72} X^3 - \frac{1}{2} \text{Li}_2(y) Y^2 + \frac{23}{6} X^3 Y + \frac{865}{72} Y X + \frac{14}{3} Y^2 X - \frac{23}{8} X^2 Y^2 + \frac{1}{4} \pi^2 X Y \\
& - \frac{11}{3} U X Y + \frac{1}{2} \text{Li}_2(y) X Y \Big) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(-\frac{242}{9} U^2 + 18 U \zeta_3 + \frac{253}{36} U \pi^2 - \frac{4}{27} X - \frac{7}{40} \pi^4 + \frac{35}{9} \pi^2 - \frac{161}{6} \zeta_3 - \frac{1535}{108} U \right. \\
& + 2 \text{Li}_4(z) + 2 \text{Li}_4(y) - \frac{22}{3} \text{Li}_3(x) + \frac{30377}{432} + 2 \text{Li}_3(x) Y + \frac{440}{9} U Y + \frac{155}{18} \pi^2 Y \\
& - 6 \zeta_3 Y + \frac{5}{3} \pi^2 Y^2 - 2 X \text{Li}_3(x) - \frac{41}{3} X^2 Y + \frac{22}{3} X \text{Li}_2(x) + 4 X \zeta_3 + X^2 \text{Li}_2(x) \\
& + \frac{223}{36} X \pi^2 + \frac{22}{3} U X^2 + 2 X Y^3 - \frac{1}{3} \pi^2 X^2 + \pi^2 \text{Li}_2(x) + \frac{44}{3} U Y^2 - 2 \text{Li}_3(y) X \\
& - \frac{220}{9} U X - \frac{1}{3} X^4 - \frac{145}{9} X^2 + \frac{52}{9} X^3 + \frac{5}{3} X^3 Y + \frac{290}{9} Y X + \frac{41}{3} Y^2 X - \frac{3}{2} X^2 Y^2 \\
& + \pi^2 X Y - \frac{44}{3} U X Y + 2 \text{Li}_2(y) X Y - Y^4 - \frac{82}{9} Y^3 - \frac{290}{9} Y^2 + \frac{8}{27} Y \Big) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(\frac{2995}{432} X + \frac{38}{45} \pi^4 + \frac{509}{48} \pi^2 + \frac{83}{12} \zeta_3 + 20 \text{Li}_4(z) - \frac{83}{6} \text{Li}_3(y) - 16 \text{Li}_4(x) - 20 \text{Li}_4(y) \right. \\
& - \frac{83}{12} \text{Li}_3(x) + 10 \text{Li}_3(x) Y + \frac{17}{6} \pi^2 Y + 20 \text{Li}_3(y) Y - 10 \zeta_3 Y + 2 \pi^2 Y^2 - 2 X \text{Li}_3(x) \\
& + \frac{17}{6} X^2 Y + \frac{83}{12} X \text{Li}_2(x) + 4 X \zeta_3 + \frac{37}{24} X \pi^2 + 2 X Y^3 + \frac{4}{3} \pi^2 X^2 + \frac{83}{6} \text{Li}_2(y) Y \\
& + 4 \pi^2 \text{Li}_2(x) - 10 \text{Li}_3(y) X + \frac{275}{18} U X + \frac{2}{3} X^4 + \frac{509}{48} X^2 + \frac{5}{24} X^3 - 3 X^3 Y \\
& \left. - \frac{509}{24} Y X + \frac{49}{12} Y^2 X + 2 X^2 Y^2 \right) \left[\frac{t^2 - s^2}{u^2} \right] \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
B_u = & \left(-\frac{11}{6} U \pi^2 + \frac{14}{9} X + \frac{119}{180} \pi^4 + 24 \pi^2 + \frac{173}{6} \zeta_3 - \frac{11}{3} U - \frac{65}{2} \text{Li}_4(z) - \frac{65}{2} \text{Li}_4(y) \right. \\
& + \frac{299}{6} \text{Li}_3(x) - 27 \text{Li}_3(x) Y + \frac{55}{3} U Y - \frac{29}{6} \pi^2 Y - 11 \zeta_3 Y + \frac{1}{4} \pi^2 Y^2 \\
& + 18 X \text{Li}_3(x) - \frac{197}{6} X^2 Y - \frac{299}{6} X \text{Li}_2(x) + X \zeta_3 - \frac{7}{4} X^2 \text{Li}_2(x) - \frac{11}{6} X \pi^2 \\
& - \frac{11}{6} U X^2 - \frac{23}{6} X Y^3 - \frac{5}{4} \pi^2 X^2 - \frac{7}{4} \pi^2 \text{Li}_2(x) - \frac{11}{3} U Y^2 + 9 \text{Li}_3(y) X - \frac{55}{6} U X \\
& - \frac{41}{24} X^4 - \frac{311}{36} X^2 + \frac{73}{18} X^3 + \frac{37}{6} X^3 Y - \frac{173}{9} Y X + \frac{73}{3} Y^2 X - \frac{55}{8} X^2 Y^2 \\
& + \frac{27}{4} \pi^2 X Y + \frac{11}{3} U X Y - \frac{7}{2} \text{Li}_2(y) X Y + \frac{139}{18} + \frac{23}{12} Y^4 - \frac{146}{9} Y^3 + \frac{173}{9} Y^2 \\
& \left. - \frac{28}{9} Y \right) \\
& + \left(\frac{164771}{2592} - \frac{121}{9} U^2 + 10 U \zeta_3 - \frac{34}{9} U \pi^2 - \frac{5941}{432} X - \frac{4}{3} \pi^4 + \frac{1123}{144} \pi^2 - \frac{1529}{36} \zeta_3 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{7543}{216}U + \frac{71}{2}\text{Li}_4(z) + \frac{71}{2}\text{Li}_4(y) + \frac{7}{4}\text{Li}_3(x) + \frac{242}{9}UY - \frac{347}{36}\pi^2 Y - 38\zeta_3 Y \\
& -\frac{3}{4}\pi^2 Y^2 - 23X\text{Li}_3(x) + \frac{11}{4}X^2 Y - \frac{7}{4}X\text{Li}_2(x) + 42X\zeta_3 + \frac{21}{4}X^2\text{Li}_2(x) \\
& -\frac{91}{72}X\pi^2 + \frac{11}{3}UX^2 - \frac{23}{6}XY^3 + \frac{5}{12}\pi^2 X^2 + \frac{21}{4}\pi^2\text{Li}_2(x) - \frac{22}{3}UY^2 - 46\text{Li}_3(y)X \\
& -\frac{121}{9}UX + \frac{43}{24}X^4 + \frac{347}{48}X^2 - \frac{179}{72}X^3 - \frac{19}{3}X^3 Y - \frac{1169}{72}YX + \frac{59}{12}Y^2 X \\
& +\frac{37}{8}X^2 Y^2 + \frac{31}{4}\pi^2 XY + \frac{22}{3}UXY + \frac{21}{2}\text{Li}_2(y)XY + \frac{23}{12}Y^4 - \frac{59}{18}Y^3 \\
& +\frac{1169}{72}Y^2 + \frac{5941}{216}Y \Big) \left[\frac{t^2 + s^2}{st} \right] \\
& + \left(-\frac{11}{3}U\pi^2 + \frac{207}{16}X - \frac{319}{90}\pi^4 + \frac{1291}{144}\pi^2 + \frac{121}{12}\zeta_3 - \frac{87}{2}\text{Li}_4(z) - \frac{121}{6}\text{Li}_3(y) \right. \\
& +74\text{Li}_4(x) + \frac{87}{2}\text{Li}_4(y) - \frac{121}{12}\text{Li}_3(x) - 26\text{Li}_3(x)Y + \frac{163}{12}\pi^2 Y - 52\text{Li}_3(y)Y \\
& +26\zeta_3 Y - \frac{19}{4}\pi^2 Y^2 - 21X\text{Li}_3(x) + \frac{163}{12}X^2 Y + \frac{121}{12}X\text{Li}_2(x) - 34X\zeta_3 \\
& +\frac{23}{4}X^2\text{Li}_2(x) + \frac{43}{24}X\pi^2 - \frac{11}{3}UX^2 + \frac{1}{6}XY^3 - \frac{17}{4}\pi^2 X^2 + \frac{121}{6}\text{Li}_2(y)Y \\
& -\frac{95}{12}\pi^2\text{Li}_2(x) + 26\text{Li}_3(y)X - 11UX - \frac{9}{8}X^4 + \frac{1291}{144}X^2 - \frac{433}{72}X^3 + \frac{17}{2}\text{Li}_2(y)Y^2 \\
& +7X^3 Y - \frac{1291}{72}YX - \frac{7}{2}Y^2 X - \frac{89}{8}X^2 Y^2 + \frac{25}{4}\pi^2 XY + \frac{22}{3}UXY \\
& \left. -\frac{17}{2}\text{Li}_2(y)XY \right) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(-2U\zeta_3 - \frac{1}{12}U\pi^2 + \frac{40}{3}X - \frac{883}{360}\pi^4 + \frac{329}{36}\pi^2 + \frac{443}{18}\zeta_3 + \frac{1502}{27}U - \frac{19139}{324} \right. \\
& +18\text{Li}_4(z) + 18\text{Li}_4(y) + 18\text{Li}_3(x)Y + \frac{40}{3}\pi^2 Y - 18\zeta_3 Y + 4\pi^2 Y^2 - 18X\text{Li}_3(x) \\
& -\frac{40}{3}X^2 Y + 18X\zeta_3 + 9X^2\text{Li}_2(x) + \frac{20}{3}X\pi^2 + \frac{11}{3}\pi^2 X^2 + 9\pi^2\text{Li}_2(x) - 18\text{Li}_3(y)X \\
& +\frac{1}{2}X^4 - \frac{517}{36}X^2 + \frac{20}{3}X^3 + X^3 Y + 7YX + \frac{19}{2}X^2 Y^2 + \pi^2 XY \\
& \left. +18\text{Li}_2(y)XY - 7Y^2 - \frac{80}{3}Y \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(\frac{11}{6}U\pi^2 - \frac{121}{36}X - \frac{119}{180}\pi^4 - \frac{157}{18}\pi^2 - \frac{71}{6}\zeta_3 - \frac{43}{2}\text{Li}_4(z) + \frac{71}{3}\text{Li}_3(y) \right. \\
& +22\text{Li}_4(x) + \frac{43}{2}\text{Li}_4(y) + \frac{71}{6}\text{Li}_3(x) - 9\text{Li}_3(x)Y - \frac{39}{2}\pi^2 Y - 18\text{Li}_3(y)Y + 9\zeta_3 Y \\
& -\frac{17}{4}\pi^2 Y^2 + 2X\text{Li}_3(x) - \frac{39}{2}X^2 Y - \frac{71}{6}X\text{Li}_2(x) - X\zeta_3 - \frac{9}{4}X^2\text{Li}_2(x) \\
& \left. -\frac{137}{18}X\pi^2 + \frac{11}{6}UX^2 - \frac{5}{2}XY^3 - \frac{11}{4}\pi^2 X^2 - \frac{71}{3}\text{Li}_2(y)Y - \frac{5}{4}\pi^2\text{Li}_2(x) \right)
\end{aligned}$$

$$\begin{aligned}
& +9 \operatorname{Li}_3(y) X + \frac{11}{6} U X - \frac{43}{24} X^4 - \frac{157}{18} X^2 + \frac{221}{18} X^3 - \frac{7}{2} \operatorname{Li}_2(y) Y^2 + \frac{29}{6} X^3 Y \\
& + \frac{157}{9} Y X + \frac{23}{3} Y^2 X - \frac{13}{8} X^2 Y^2 - \frac{37}{12} \pi^2 X Y - \frac{11}{3} U X Y \\
& + \frac{7}{2} \operatorname{Li}_2(y) X Y \left[\frac{t^2 - s^2}{u^2} \right] - Y^2 \frac{s^4}{t^2 u^2} + \left(-Y^2 + 2 Y X - \pi^2 - X^2 \right) \frac{t^4}{s^2 u^2} \quad (3.35)
\end{aligned}$$

$$\begin{aligned}
C_u = & \left(-\frac{22}{3} U \pi^2 + \frac{3133}{144} X - \frac{203}{180} \pi^4 + \frac{401}{48} \pi^2 - \frac{1}{4} \zeta_3 + \frac{83}{2} \operatorname{Li}_4(z) + \frac{23}{8} + \frac{83}{2} \operatorname{Li}_4(y) \right. \\
& + \frac{13}{4} \operatorname{Li}_3(x) + 41 \operatorname{Li}_3(x) Y + \frac{44}{3} U Y + \frac{41}{9} \pi^2 Y + 9 \zeta_3 Y - \frac{13}{12} \pi^2 Y^2 - 26 X \operatorname{Li}_3(x) \\
& + \frac{41}{3} X^2 Y - \frac{13}{4} X \operatorname{Li}_2(x) + X \zeta_3 + \frac{21}{4} X^2 \operatorname{Li}_2(x) - \frac{299}{72} X \pi^2 - \frac{22}{3} U X^2 + \frac{3}{2} X Y^3 \\
& + \frac{25}{12} \pi^2 X^2 + \frac{21}{4} \pi^2 \operatorname{Li}_2(x) - \frac{44}{3} U Y^2 - 11 \operatorname{Li}_3(y) X - \frac{22}{3} U X + \frac{55}{24} X^4 + \frac{919}{144} X^2 \\
& - \frac{209}{72} X^3 - \frac{47}{6} X^3 Y - \frac{259}{72} Y X - \frac{161}{6} Y^2 X + \frac{125}{8} X^2 Y^2 - \frac{33}{4} \pi^2 X Y \\
& \left. + \frac{44}{3} U X Y + \frac{21}{2} \operatorname{Li}_2(y) X Y - \frac{3}{4} Y^4 + \frac{161}{9} Y^3 + \frac{259}{72} Y^2 - \frac{3133}{72} Y \right) \\
& + \left(5 U \zeta_3 - \frac{145}{24} U \pi^2 + \frac{743}{48} X - \frac{113}{240} \pi^4 + \frac{1631}{144} \pi^2 - \frac{4}{9} \zeta_3 + \frac{5297}{216} U + \frac{19}{2} \operatorname{Li}_4(z) \right. \\
& + \frac{19}{2} \operatorname{Li}_4(y) - \frac{41393}{2592} + \frac{21}{4} \operatorname{Li}_3(x) + 19 \operatorname{Li}_3(x) Y + 11 U Y + \frac{5}{2} \pi^2 Y - \zeta_3 Y \\
& + \frac{5}{12} \pi^2 Y^2 - 7 X \operatorname{Li}_3(x) + \frac{17}{2} X^2 Y - \frac{21}{4} X \operatorname{Li}_2(x) - 2 X \zeta_3 + \frac{9}{4} X^2 \operatorname{Li}_2(x) \\
& - \frac{27}{8} X \pi^2 - \frac{11}{2} U X^2 + \frac{5}{6} X Y^3 + \frac{3}{4} \pi^2 X^2 + \frac{9}{4} \pi^2 \operatorname{Li}_2(x) - 11 U Y^2 + 5 \operatorname{Li}_3(y) X \\
& - \frac{11}{2} U X + \frac{7}{12} X^4 + \frac{97}{48} X^2 - \frac{17}{8} X^3 - \frac{4}{3} X^3 Y - \frac{145}{24} Y X - 18 Y^2 X + \frac{57}{8} X^2 Y^2 \\
& - \frac{61}{12} \pi^2 X Y + 11 U X Y + \frac{9}{2} \operatorname{Li}_2(y) X Y - \frac{5}{12} Y^4 + 12 Y^3 + \frac{145}{24} Y^2 \\
& \left. - \frac{743}{24} Y \right) \left[\frac{t^2 + s^2}{st} \right] \\
& + \left(-\frac{11}{6} U \pi^2 + \frac{67}{16} X + \frac{14}{45} \pi^4 + \frac{1141}{144} \pi^2 + \frac{9}{4} \zeta_3 + \frac{21}{2} \operatorname{Li}_4(z) - \frac{9}{2} \operatorname{Li}_3(y) - 8 \operatorname{Li}_4(x) \right. \\
& - \frac{21}{2} \operatorname{Li}_4(y) - \frac{9}{4} \operatorname{Li}_3(x) + 5 \operatorname{Li}_3(x) Y + \frac{17}{3} \pi^2 Y + 10 \operatorname{Li}_3(y) Y - 5 \zeta_3 Y + \frac{3}{4} \pi^2 Y^2 \\
& - 3 X \operatorname{Li}_3(x) + \frac{17}{3} X^2 Y + \frac{9}{4} X \operatorname{Li}_2(x) + 6 X \zeta_3 + \frac{7}{4} X^2 \operatorname{Li}_2(x) + \frac{19}{72} X \pi^2 - \frac{11}{6} U X^2 \\
& + \frac{3}{2} X Y^3 + \frac{3}{4} \pi^2 X^2 + \frac{9}{2} \operatorname{Li}_2(y) Y + \frac{5}{12} \pi^2 \operatorname{Li}_2(x) - 5 \operatorname{Li}_3(y) X + \frac{11}{2} U X + \frac{1}{4} X^4 \\
& \left. + \frac{1141}{144} X^2 - \frac{107}{72} X^3 + \frac{1}{2} \operatorname{Li}_2(y) Y^2 - \frac{2}{3} X^3 Y - \frac{1141}{72} Y X - \frac{41}{12} Y^2 X + \frac{3}{8} X^2 Y^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\pi^2 XY + \frac{11}{3}UXY - \frac{1}{2}\text{Li}_2(y)XY \Big) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(-\frac{3}{4}U - \frac{255}{16} + 15\zeta_3 + \frac{11}{90}\pi^4 - \frac{35}{12}\pi^2 + \frac{9}{4}X^2 + YX - Y^2 + U\pi^2 \right. \\
& \left. - 12U\zeta_3 \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(-\frac{21}{16}X - \frac{47}{180}\pi^4 - \frac{47}{16}\pi^2 - \frac{11}{4}\zeta_3 - \frac{11}{2}\text{Li}_4(z) + \frac{11}{2}\text{Li}_3(y) + 6\text{Li}_4(x) + \frac{11}{2}\text{Li}_4(y) \right. \\
& + \frac{11}{4}\text{Li}_3(x) - 3\text{Li}_3(x)Y - \frac{3}{2}\pi^2Y - 6\text{Li}_3(y)Y + 3\zeta_3Y - \frac{1}{4}\pi^2Y^2 - \frac{3}{2}X^2Y \\
& - \frac{11}{4}X\text{Li}_2(x) - 3X\zeta_3 - \frac{1}{4}X^2\text{Li}_2(x) + \frac{17}{24}X\pi^2 - \frac{1}{2}XY^3 - \frac{5}{12}\pi^2X^2 - \frac{11}{2}\text{Li}_2(y)Y \\
& - \frac{7}{12}\pi^2\text{Li}_2(x) + 3\text{Li}_3(y)X - \frac{1}{8}X^4 - \frac{47}{16}X^2 - \frac{13}{24}X^3 + \frac{1}{2}\text{Li}_2(y)Y^2 + \frac{1}{2}X^3Y \\
& \left. + \frac{47}{8}YX - \frac{5}{4}Y^2X - \frac{5}{8}X^2Y^2 + \frac{1}{4}\pi^2XY - \frac{1}{2}\text{Li}_2(y)XY \right) \left[\frac{t^2 - s^2}{u^2} \right] \\
& + Y^2 \frac{s^4}{t^2u^2} + \left(Y^2 - 2YX + \pi^2 + X^2 \right) \frac{t^4}{s^2u^2} \tag{3.36}
\end{aligned}$$

$$\begin{aligned}
D_u = & \left(\frac{1}{2} + \frac{29}{4}X + \frac{3}{5}\pi^4 + \frac{25}{6}\pi^2 + 15\zeta_3 - 10\text{Li}_4(z) - 10\text{Li}_4(y) - 19\text{Li}_3(x) \right. \\
& + 12\text{Li}_3(x)Y + \frac{7}{3}\pi^2Y + 12\zeta_3Y + \frac{1}{3}\pi^2Y^2 + 4X\text{Li}_3(x) + 13X^2Y + 19X\text{Li}_2(x) \\
& - 16X\zeta_3 + X^2\text{Li}_2(x) - \frac{1}{6}X\pi^2 + 2XY^3 - \frac{2}{3}\pi^2X^2 + \pi^2\text{Li}_2(x) + 20\text{Li}_3(y)X \\
& - X^4 + 4X^2 - \frac{11}{6}X^3 + 4X^3Y - YX - 9Y^2X + \frac{5}{2}X^2Y^2 - \frac{7}{3}\pi^2XY \\
& \left. + 2\text{Li}_2(y)XY - Y^4 + 6Y^3 + Y^2 - \frac{29}{2}Y \right) \\
& + \left(-6U\zeta_3 + \frac{1}{2}U\pi^2 + \frac{47}{16}X + \frac{7}{18}\pi^4 - \frac{3}{16}\pi^2 + \frac{23}{4}\zeta_3 - \frac{3}{8}U - \frac{11}{2}\text{Li}_4(z) \right. \\
& - \frac{11}{2}\text{Li}_4(y) - \frac{13}{4}\text{Li}_3(x) - \frac{187}{32} + 7\text{Li}_3(x)Y + \frac{7}{6}\pi^2Y + 7\zeta_3Y + \frac{5}{12}\pi^2Y^2 \\
& + 2X\text{Li}_3(x) + \frac{7}{2}X^2Y + \frac{13}{4}X\text{Li}_2(x) - 9X\zeta_3 + \frac{3}{4}X^2\text{Li}_2(x) + \frac{1}{24}X\pi^2 + \frac{7}{6}XY^3 \\
& - \frac{5}{12}\pi^2X^2 + \frac{3}{4}\pi^2\text{Li}_2(x) + 11\text{Li}_3(y)X - \frac{5}{8}X^4 - \frac{13}{16}X^2 - \frac{3}{8}X^3 + \frac{5}{2}X^3Y + \frac{13}{8}YX \\
& - 5Y^2X + \frac{11}{8}X^2Y^2 - \frac{13}{12}\pi^2XY + \frac{3}{2}\text{Li}_2(y)XY - \frac{7}{12}Y^4 + \frac{10}{3}Y^3 - \frac{13}{8}Y^2 \\
& \left. - \frac{47}{8}Y \right) \left[\frac{t^2 + s^2}{st} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{31}{16} X + \frac{103}{180} \pi^4 - \frac{23}{16} \pi^2 + \frac{19}{4} \zeta_3 + \frac{11}{2} \text{Li}_4(z) - \frac{19}{2} \text{Li}_3(y) - 14 \text{Li}_4(x) \right. \\
& - \frac{11}{2} \text{Li}_4(y) - \frac{19}{4} \text{Li}_3(x) + 3 \text{Li}_3(x) Y + 2 \pi^2 Y + 6 \text{Li}_3(y) Y - 3 \zeta_3 Y + \frac{1}{4} \pi^2 Y^2 \\
& + 4 X \text{Li}_3(x) + 2 X^2 Y + \frac{19}{4} X \text{Li}_2(x) + 7 X \zeta_3 + \frac{1}{4} X^2 \text{Li}_2(x) + \frac{7}{24} X \pi^2 + \frac{1}{2} X Y^3 \\
& + \frac{3}{4} \pi^2 X^2 + \frac{19}{2} \text{Li}_2(y) Y - \frac{3}{4} \pi^2 \text{Li}_2(x) - 3 \text{Li}_3(y) X - \frac{1}{24} X^4 - \frac{23}{16} X^2 + \frac{3}{8} X^3 \\
& - \frac{1}{2} \text{Li}_2(y) Y^2 + \frac{1}{6} X^3 Y + \frac{23}{8} Y X + \frac{11}{4} Y^2 X + \frac{5}{8} X^2 Y^2 - \frac{19}{12} \pi^2 X Y \\
& \left. + \frac{1}{2} \text{Li}_2(y) X Y \right) \left[\frac{t^2 - s^2}{st} \right] + \left(2 X^2 - Y X + \frac{1}{2} \pi^2 + Y^2 \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(\frac{3}{2} X^2 - 3 Y X + \frac{3}{2} \pi^2 - 3 X \right) \left[\frac{t^2 - s^2}{u^2} \right] - Y^2 \frac{s^4}{t^2 u^2} \\
& + \left(-Y^2 + 2 Y X - \pi^2 - X^2 \right) \frac{t^4}{s^2 u^2} \tag{3.37}
\end{aligned}$$

$$\begin{aligned}
E_u = & \left(\frac{91}{18} X - \frac{11}{180} \pi^4 - \frac{25}{18} \pi^2 - 4 \zeta_3 + \frac{2}{3} U - 4 \text{Li}_3(x) - 2 \text{Li}_3(x) Y + \frac{10}{3} U Y \right. \\
& + \frac{2}{3} \pi^2 Y - 2 \zeta_3 Y + \frac{1}{6} \pi^2 Y^2 - \frac{37}{18} + 2 X^2 Y + 4 X \text{Li}_2(x) + 2 X \zeta_3 + \frac{2}{3} X \pi^2 \\
& - \frac{1}{3} X Y^3 - 2 \text{Li}_3(y) X - \frac{5}{3} U X - 2 X^2 + 4 Y X - 2 Y^2 X - \frac{1}{2} X^2 Y^2 + \frac{2}{3} \pi^2 X Y \\
& \left. + \frac{1}{6} Y^4 + \frac{4}{3} Y^3 - 4 Y^2 - \frac{91}{9} Y \right) \\
& + \left(-\frac{44}{9} U^2 + \frac{11}{36} U \pi^2 + \frac{5}{9} X + \frac{11}{90} \pi^4 + \frac{133}{108} \pi^2 - \frac{13}{6} \zeta_3 + \frac{185}{54} U - \frac{2}{3} \text{Li}_3(x) \right. \\
& + 4 \text{Li}_3(x) Y + \frac{44}{9} U Y + \frac{13}{36} \pi^2 Y + 4 \zeta_3 Y - \frac{1}{3} \pi^2 Y^2 - \frac{1}{4} X^2 Y + \frac{2}{3} X \text{Li}_2(x) \\
& - 4 X \zeta_3 + \frac{5}{18} X \pi^2 + \frac{1}{3} U X^2 + \frac{2}{3} X Y^3 + \frac{2}{3} U Y^2 + 4 \text{Li}_3(y) X - \frac{22}{9} U X \\
& - \frac{7}{36} X^2 + \frac{7}{36} X^3 + \frac{7}{18} Y X + \frac{1}{4} Y^2 X + X^2 Y^2 - \frac{4}{3} \pi^2 X Y - \frac{2}{3} U X Y \\
& \left. - \frac{1}{3} Y^4 - \frac{1}{6} Y^3 - \frac{7}{18} Y^2 + \frac{1307}{216} - \frac{10}{9} Y \right) \left[\frac{t^2 + s^2}{st} \right] \\
& + \left(-\frac{1}{3} U \pi^2 - \frac{5}{9} X + \frac{11}{30} \pi^4 + \frac{79}{36} \pi^2 - \frac{2}{3} \zeta_3 + 8 \text{Li}_4(z) + \frac{4}{3} \text{Li}_3(y) - 8 \text{Li}_4(x) \right. \\
& - 8 \text{Li}_4(y) + \frac{2}{3} \text{Li}_3(x) + 4 \text{Li}_3(x) Y + \frac{1}{4} \pi^2 Y + 8 \text{Li}_3(y) Y - 4 \zeta_3 Y + \pi^2 Y^2 \\
& + \frac{1}{4} X^2 Y - \frac{2}{3} X \text{Li}_2(x) + 4 X \zeta_3 - \frac{5}{18} X \pi^2 - \frac{1}{3} U X^2 + \frac{2}{3} X Y^3 + \frac{2}{3} \pi^2 X^2 \\
& \left. - \frac{4}{3} \text{Li}_2(y) Y + \frac{4}{3} \pi^2 \text{Li}_2(x) - 4 \text{Li}_3(y) X + \frac{22}{9} U X + \frac{1}{3} X^4 + \frac{79}{36} X^2 - \frac{7}{36} X^3 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{4}{3}X^3Y - \frac{79}{18}YX - \frac{11}{12}Y^2X + X^2Y^2 + \frac{2}{3}UXY \Big) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(\frac{88}{9}U^2 - \frac{28}{9}X - \frac{221}{27}U + \frac{4}{3}\text{Li}_3(x) + \frac{13}{3}\zeta_3 - \frac{67}{54}\pi^2 - \frac{124}{9}UY - \frac{17}{18}X\pi^2 \right. \\
& + \frac{5}{3}X^2Y - \frac{4}{3}X\text{Li}_2(x) - \frac{4}{3}UX^2 - \frac{8}{3}UY^2 + \frac{62}{9}UX + \frac{71}{18}X^2 - \frac{7}{9}X^3 - \frac{71}{9}YX \\
& - \frac{5}{3}Y^2X + \frac{8}{3}UXY + \frac{10}{9}Y^3 + \frac{71}{9}Y^2 - \frac{10}{9}\pi^2Y - \frac{863}{108} - \frac{23}{18}U\pi^2 \\
& \left. + \frac{56}{9}Y \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(\frac{37}{18}X - \frac{11}{60}\pi^4 - \frac{23}{6}\pi^2 - \frac{8}{3}\zeta_3 - 4\text{Li}_4(z) + \frac{16}{3}\text{Li}_3(y) + 4\text{Li}_4(x) + 4\text{Li}_4(y) \right. \\
& + \frac{8}{3}\text{Li}_3(x) - 2\text{Li}_3(x)Y - \frac{4}{3}\pi^2Y - 4\text{Li}_3(y)Y + 2\zeta_3Y - \frac{1}{2}\pi^2Y^2 - \frac{4}{3}X^2Y \\
& - \frac{8}{3}X\text{Li}_2(x) - 2X\zeta_3 - \frac{1}{2}X\pi^2 - \frac{1}{3}XY^3 - \frac{1}{3}\pi^2X^2 - \frac{16}{3}\text{Li}_2(y)Y - \frac{2}{3}\pi^2\text{Li}_2(x) \\
& + 2\text{Li}_3(y)X - \frac{47}{9}UX - \frac{1}{6}X^4 - \frac{23}{6}X^2 + \frac{2}{3}X^3Y + \frac{23}{3}YX - \frac{4}{3}Y^2X \\
& \left. - \frac{1}{2}X^2Y^2 \right) \left[\frac{t^2 - s^2}{u^2} \right] \tag{3.38}
\end{aligned}$$

$$\begin{aligned}
F_u = & \left(\frac{1}{3}U\pi^2 - \frac{50}{9}X - \frac{11}{45}\pi^4 + \frac{41}{9}\pi^2 + \frac{52}{3}\zeta_3 + \frac{2}{3}U - \frac{19}{18} + \frac{52}{3}\text{Li}_3(x) - 8\text{Li}_3(x)Y \right. \\
& - \frac{10}{3}UY - \frac{85}{36}\pi^2Y - 8\zeta_3Y + \frac{2}{3}\pi^2Y^2 - \frac{83}{12}X^2Y - \frac{52}{3}X\text{Li}_2(x) + 8X\zeta_3 \\
& - \frac{115}{36}X\pi^2 + \frac{1}{3}UX^2 - \frac{4}{3}XY^3 + \frac{2}{3}UY^2 - 8\text{Li}_3(y)X + \frac{5}{3}UX + \frac{49}{18}X^2 \\
& - \frac{11}{36}X^3 - \frac{43}{9}YX + \frac{21}{4}Y^2X - 2X^2Y^2 + \frac{8}{3}\pi^2XY - \frac{2}{3}UXY + \frac{2}{3}Y^4 - \frac{7}{2}Y^3 \\
& \left. + \frac{43}{9}Y^2 + \frac{100}{9}Y \right) \\
& + \left(\frac{44}{9}U^2 - \frac{3661}{324} + \frac{7}{9}U\pi^2 - \frac{115}{36}X - \frac{11}{45}\pi^4 - \frac{367}{108}\pi^2 + \frac{14}{9}\zeta_3 + \frac{7}{9}U - \frac{8}{3}\text{Li}_3(x) \right. \\
& - 8\text{Li}_3(x)Y - \frac{88}{9}UY - \frac{1}{12}\pi^2Y - 8\zeta_3Y + \frac{2}{3}\pi^2Y^2 + \frac{1}{4}X^2Y + \frac{8}{3}X\text{Li}_2(x) \\
& + 8X\zeta_3 - \frac{2}{3}UX^2 - \frac{4}{3}XY^3 + \frac{4}{3}UY^2 - 8\text{Li}_3(y)X + \frac{44}{9}UX - \frac{11}{9}X^2 - \frac{1}{18}X^3 \\
& + \frac{53}{18}YX + \frac{9}{4}Y^2X - 2X^2Y^2 + \frac{8}{3}\pi^2XY - \frac{4}{3}UXY + \frac{2}{3}Y^4 - \frac{3}{2}Y^3 - \frac{53}{18}Y^2 \\
& \left. + \frac{115}{18}Y \right) \left[\frac{t^2 + s^2}{st} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2}{3} U \pi^2 - \frac{23}{4} X - \frac{11}{15} \pi^4 - \frac{19}{6} \pi^2 + \frac{4}{3} \zeta_3 - 16 \text{Li}_4(z) - \frac{8}{3} \text{Li}_3(y) + 16 \text{Li}_4(x) \right. \\
& + 16 \text{Li}_4(y) - \frac{4}{3} \text{Li}_3(x) - 8 \text{Li}_3(x) Y - \frac{19}{12} \pi^2 Y - 16 \text{Li}_3(y) Y + 8 \zeta_3 Y - 2 \pi^2 Y^2 \\
& - \frac{19}{12} X^2 Y + \frac{4}{3} X \text{Li}_2(x) - 8 X \zeta_3 - \frac{1}{18} X \pi^2 + \frac{2}{3} U X^2 - \frac{4}{3} X Y^3 - \frac{4}{3} \pi^2 X^2 \\
& + \frac{8}{3} \text{Li}_2(y) Y - \frac{8}{3} \pi^2 \text{Li}_2(x) + 8 \text{Li}_3(y) X + 2 U X - \frac{2}{3} X^4 - \frac{19}{6} X^2 + \frac{13}{18} X^3 + \frac{8}{3} X^3 Y \\
& \left. + \frac{19}{3} Y X + \frac{35}{12} Y^2 X - 2 X^2 Y^2 - \frac{4}{3} U X Y \right) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(-\frac{4}{3} X - \frac{263}{27} U - \frac{1}{9} \zeta_3 - \frac{1}{3} \pi^2 - \frac{2}{3} X \pi^2 + \frac{4}{3} X^2 Y + \frac{85}{18} X^2 - \frac{2}{3} X^3 - 3 Y X \right. \\
& \left. + 3 Y^2 - \frac{4}{3} \pi^2 Y + \frac{4085}{324} - \frac{1}{6} U \pi^2 + \frac{8}{3} Y \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(\frac{19}{9} X + \frac{8}{3} \text{Li}_3(y) + \frac{4}{3} \text{Li}_3(x) - \frac{4}{3} \zeta_3 + 7 \pi^2 + \frac{5}{12} X \pi^2 + \frac{9}{4} X^2 Y - \frac{4}{3} X \text{Li}_2(x) \right. \\
& - \frac{1}{3} U X^2 - \frac{8}{3} \text{Li}_2(y) Y - \frac{1}{3} U X + 7 X^2 - \frac{49}{36} X^3 - 14 Y X - \frac{43}{12} Y^2 X \\
& \left. + \frac{2}{3} U X Y + \frac{9}{4} \pi^2 Y - \frac{1}{3} U \pi^2 \right) \left[\frac{t^2 - s^2}{u^2} \right] - 3 Y^2 \frac{s^4}{t^2 u^2} \\
& + \left(-3 \pi^2 - 3 X^2 - 3 Y^2 + 6 Y X \right) \frac{t^4}{s^2 u^2} \tag{3.39}
\end{aligned}$$

$$\begin{aligned}
G_u = & \left(\frac{4}{3} U \pi^2 + \frac{71}{9} X + \frac{11}{15} \pi^4 - 20 \pi^2 - 52 \zeta_3 - 52 \text{Li}_3(x) + 24 \text{Li}_3(x) Y - \frac{8}{3} U Y \right. \\
& + \frac{64}{9} \pi^2 Y + 24 \zeta_3 Y - 2 \pi^2 Y^2 + \frac{64}{3} X^2 Y + 52 X \text{Li}_2(x) - 24 X \zeta_3 + \frac{85}{9} X \pi^2 \\
& + \frac{4}{3} U X^2 + 4 X Y^3 + \frac{8}{3} U Y^2 + 24 \text{Li}_3(y) X + \frac{4}{3} U X - \frac{37}{9} X^2 + \frac{7}{9} X^3 + \frac{86}{9} Y X \\
& \left. - \frac{50}{3} Y^2 X + 6 X^2 Y^2 - 8 \pi^2 X Y - \frac{8}{3} U X Y - 2 Y^4 + \frac{100}{9} Y^3 - \frac{86}{9} Y^2 - \frac{142}{9} Y \right) \\
& + \left(\frac{11}{12} U \pi^2 + \frac{79}{12} X + \frac{11}{15} \pi^4 + \frac{151}{36} \pi^2 + \frac{71}{18} \zeta_3 + \frac{3401}{648} - \frac{227}{54} U + 4 \text{Li}_3(x) \right. \\
& + 24 \text{Li}_3(x) Y - 2 U Y + \frac{5}{3} \pi^2 Y + 24 \zeta_3 Y - 2 \pi^2 Y^2 - 2 X^2 Y - 4 X \text{Li}_2(x) \\
& - 24 X \zeta_3 - \frac{1}{12} X \pi^2 + U X^2 + 4 X Y^3 + 2 U Y^2 + 24 \text{Li}_3(y) X + U X + \frac{31}{6} X^2 \\
& + \frac{7}{12} X^3 - \frac{65}{6} Y X - \frac{3}{2} Y^2 X + 6 X^2 Y^2 - 8 \pi^2 X Y - 2 U X Y - 2 Y^4 + Y^3 \\
& \left. + \frac{65}{6} Y^2 - \frac{79}{6} Y \right) \left[\frac{t^2 + s^2}{st} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{3} U \pi^2 + \frac{41}{4} X + \frac{11}{5} \pi^4 + \frac{85}{18} \pi^2 - 4 \zeta_3 + 48 \text{Li}_4(z) + 8 \text{Li}_3(y) - 48 \text{Li}_4(x) \right. \\
& - 48 \text{Li}_4(y) + 4 \text{Li}_3(x) + 24 \text{Li}_3(x) Y - \frac{2}{3} \pi^2 Y + 48 \text{Li}_3(y) Y - 24 \zeta_3 Y + 6 \pi^2 Y^2 \\
& - \frac{2}{3} X^2 Y - 4 X \text{Li}_2(x) + 24 X \zeta_3 - \frac{17}{36} X \pi^2 + \frac{1}{3} U X^2 + 4 X Y^3 + 4 \pi^2 X^2 \\
& - 8 \text{Li}_2(y) Y + 8 \pi^2 \text{Li}_2(x) - 24 \text{Li}_3(y) X - U X + 2 X^4 + \frac{85}{18} X^2 + \frac{7}{36} X^3 - 8 X^3 Y \\
& \left. - \frac{85}{9} Y X - \frac{10}{3} Y^2 X + 6 X^2 Y^2 - \frac{2}{3} U X Y \right) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(-\frac{9}{2} X^2 + 9 Y X - \frac{9}{2} \pi^2 - 9 Y^2 \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(4 X \text{Li}_2(x) - 4 \text{Li}_3(x) - 8 \text{Li}_3(y) + 4 \zeta_3 - \frac{27}{2} X^2 + 27 Y X + 4 Y^2 X + \frac{2}{3} X \pi^2 \right. \\
& \left. + 8 \text{Li}_2(y) Y - \frac{27}{2} \pi^2 - 8 X \right) \left[\frac{t^2 - s^2}{u^2} \right] + 9 Y^2 \frac{s^4}{t^2 u^2} \\
& + \left(9 X^2 + 9 \pi^2 - 18 Y X + 9 Y^2 \right) \frac{t^4}{s^2 u^2} \tag{3.40}
\end{aligned}$$

$$\begin{aligned}
H_u = & \left(\frac{1}{12} X^2 - \frac{5}{27} X + \frac{1}{6} Y^2 - \frac{1}{6} Y X - \frac{4}{9} U Y - \frac{20}{27} U - \frac{7}{108} \pi^2 + \frac{2}{9} U X + \frac{4}{9} U^2 \right. \\
& \left. + \frac{10}{27} Y \right) \left[\frac{t^2 + s^2}{st} \right] \\
& + \left(-\frac{1}{12} \pi^2 - \frac{1}{12} X^2 + \frac{5}{27} X + \frac{1}{6} Y X - \frac{2}{9} U X \right) \left[\frac{t^2 - s^2}{st} \right] \\
& + \left(-\frac{8}{9} U^2 + \frac{40}{27} U - \frac{1}{6} X^2 - \frac{20}{27} Y + \frac{1}{3} Y X + \frac{7}{54} \pi^2 - \frac{1}{3} Y^2 + \frac{10}{27} X \right. \\
& \left. + \frac{8}{9} U Y - \frac{4}{9} U X \right) \left[\frac{t^2 + s^2}{u^2} \right] \\
& + \left(\frac{1}{6} X^2 - \frac{1}{3} Y X + \frac{1}{6} \pi^2 - \frac{10}{27} X + \frac{4}{9} U X \right) \left[\frac{t^2 - s^2}{u^2} \right] \tag{3.41}
\end{aligned}$$

$$\begin{aligned}
I_u = & \left(\frac{8}{9} U Y + \frac{1}{3} Y X - \frac{4}{9} U X + \frac{10}{27} X + \frac{5}{54} \pi^2 - \frac{1}{9} X^2 + \frac{20}{27} U - \frac{20}{27} Y \right. \\
& \left. - \frac{1}{3} Y^2 - \frac{4}{9} U^2 \right) \left[\frac{t^2 + s^2}{st} \right] + \left(\frac{1}{18} \pi^2 + \frac{1}{18} X^2 - \frac{1}{9} Y X \right) \left[\frac{t^2 - s^2}{st} \right] \\
& - \frac{1}{9} \left(X^2 - \pi^2 \right) \left[\frac{t^2 + s^2}{u^2} \right] + \left(\frac{2}{9} Y X - \frac{1}{9} \pi^2 - \frac{1}{9} X^2 \right) \left[\frac{t^2 - s^2}{u^2} \right] \tag{3.42}
\end{aligned}$$

4. One-loop self-interference

In this section, we present the self-interference of the one-loop amplitude of Eq. (2.17) in terms of the one-loop box in $D = 6 - 2\epsilon$ dimensions and the one-loop bubble in $D = 4 - 2\epsilon$ dimensions. The ϵ expansion of these integrals is given in the Appendix and can be inserted directly into the following expressions. We can write

$$\mathcal{C}^{8(1\times 1)}(s, t, u) = -\mathcal{P}_S(s, t, u) \mathcal{I}_2(s, t, u) s^2 t u + 2 \operatorname{Re} \{ \mathcal{P}_L(s, t, u) \} + \mathcal{F}_T(s, t, u), \quad (4.1)$$

where the self-interference of the singular terms of the amplitude is

$$\begin{aligned} \mathcal{P}_S(s, t, u) &= \frac{VN}{4} |\mathcal{P}_L(s, t, u)|^2 \\ &+ \frac{V}{4} 2 \operatorname{Re} \left\{ \mathcal{P}_L(s, t, u)^\dagger [\mathcal{P}_A(s, t, u) + \mathcal{R}_A(s, t, u) + (t \leftrightarrow u)] \right\} \\ &+ \frac{V^2}{4N} \left\{ |\mathcal{P}_A(s, t, u) + \mathcal{R}_A(s, t, u)|^2 + (t \leftrightarrow u) \right\} \\ &- \frac{V}{4N} 2 \operatorname{Re} \left\{ [\mathcal{P}_A(s, t, u)^\dagger + \mathcal{R}_A(s, t, u)^\dagger] [\mathcal{P}_A(s, u, t) + \mathcal{R}_A(s, u, t)] \right\}, \end{aligned} \quad (4.2)$$

and the interference of the singular terms with the one-loop amplitude is

$$\begin{aligned} \mathcal{P}_L(s, t, u) &= -\frac{s}{2} \left\{ \frac{VN}{4} \mathcal{P}_L(s, t, u)^\dagger u \mathcal{L}_3(s, t, u) \right. \\ &+ \frac{V}{4} \left[(\mathcal{P}_A(s, t, u)^\dagger + \mathcal{R}_A(s, t, u)^\dagger) u \mathcal{L}_3(s, t, u) \right. \\ &+ \left. \mathcal{P}_L(s, t, u)^\dagger u \mathcal{L}_2(s, t, u) + (t \leftrightarrow u) \right] \\ &+ \frac{V^2}{4N} \left[(\mathcal{P}_A(s, t, u)^\dagger + \mathcal{R}_A(s, t, u)^\dagger) u \mathcal{L}_2(s, t, u) + (t \leftrightarrow u) \right] \\ &\left. - \frac{V}{4N} \left[(\mathcal{P}_A(s, u, t)^\dagger + \mathcal{R}_A(s, u, t)^\dagger) u \mathcal{L}_2(s, t, u) + (t \leftrightarrow u) \right] \right\}, \end{aligned} \quad (4.3)$$

where the functions \mathcal{L}_2 and \mathcal{L}_3 are defined in Eqs. (3.13) and (3.14) respectively. In Eqs. (4.2) and (4.3)

$$\begin{aligned} \mathcal{P}_A(s, t, u) &= \frac{1 - 2\epsilon}{2st} \left\{ \beta_0 [\operatorname{Bub}(s) + \operatorname{Bub}(t)] + \left[\frac{V}{N} \left(\frac{1}{\epsilon} + \frac{3}{2} \right) - \frac{3N}{2} \right] \operatorname{Bub}(s) \right. \\ &\left. + N \left(\frac{2}{\epsilon} + \frac{3}{2} \right) \operatorname{Bub}(t) \right\}, \end{aligned} \quad (4.4)$$

$$\mathcal{P}_L(s, t, u) = -\frac{1 - 2\epsilon}{\epsilon} \left[\frac{1}{tu} \operatorname{Bub}(s) + \frac{1}{su} \operatorname{Bub}(t) + \frac{1}{st} \operatorname{Bub}(u) \right] \quad (4.5)$$

are infrared divergent while

$$\mathcal{R}_A(s, t, u) = -\frac{1}{2st} \beta_0 \left\{ \frac{(2 - \epsilon)(3 - 8\epsilon + 8\epsilon^2)}{(1 - \epsilon)(3 - 2\epsilon)} \operatorname{Bub}(s) - \frac{2}{\epsilon} \right\}, \quad (4.6)$$

originates from ultraviolet terms which are rendered finite after renormalisation in the $\overline{\text{MS}}$ scheme.

Finally, the finite contribution is given by

$$\mathcal{F}_T(s, t, u) = \frac{VN}{4} \mathcal{F}_A(s, t, u) + \frac{V}{2} 2 \text{Re} \{ \mathcal{F}_B(s, t, u) \} + \frac{V^2}{4N} \mathcal{F}_D(s, t, u) - \frac{V}{4N} \mathcal{F}_C(s, t, u) + (t \leftrightarrow u), \quad (4.7)$$

where

$$\begin{aligned} \mathcal{F}_A(s, t, u) = & \left[\frac{2u(8t^4 + 4s^4 + 12st^3 + 13s^2t^2 + 4s^3t)}{s^2t} \right] |\text{Box}^6(s, t)|^2 \\ & + \left[\frac{4u(2s^2 + ts + 4t^2)}{t} \right] 2 \text{Re} \{ \text{Box}^6(s, t)^\dagger \text{Box}^6(t, u) \} \\ & + \left[\frac{4tu(5s^2 + 4ts + 4t^2)}{s^2} \right] \text{Box}^6(s, t)^\dagger \text{Box}^6(s, u) \\ & + \left[\frac{4s^2(t^2 + u^2)}{tu} \right] |\text{Box}^6(t, u)|^2, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \mathcal{F}_B(s, t, u) = & \beta_0 \mathcal{F}_E(t, s)^\dagger \left\{ \left[\frac{2(t^2 + u^2)}{t} \right] \text{Box}^6(t, u) + \left[\frac{u^2(2s^2 + ts + 4t^2)}{s^2t} \right] \text{Box}^6(s, t) \right. \\ & + \left. \left[\frac{t(4t^2 + 5s^2 + 7ts)}{s^2} \right] \text{Box}^6(s, u) \right\} \\ & + \left[\frac{(u-t)(2s^4 + 3s^3t - 7sut^2 + 4t^4)N}{s^2t} + \frac{2s^4 + 2t^4 + ts^3 - 4sut^2}{stN} \right] |\text{Box}^6(s, t)|^2 \\ & + \left[\frac{(u-t)t(4t^2 - 5su)N}{s^2} + \frac{t(2t^2 + 3ts + 5s^2)}{sN} \right] \text{Box}^6(s, u)^\dagger \text{Box}^6(s, t) \\ & + \left[\frac{2(u-t)(2t^2 - su)N}{t} + \frac{2s(t^2 + s^2)}{tN} \right] \text{Box}^6(t, u)^\dagger \text{Box}^6(s, t) \\ & + \left[(6t + s)N + \frac{s(t + 3s)}{tN} \right] \text{Box}^6(t, u)^\dagger \mathcal{F}_E(t, s) \\ & + \left[\frac{-(6u^2 + s^2)N}{t} + \frac{2(5s^2 - 8tu)}{Nt} \right] \epsilon \text{Box}^6(t, u)^\dagger \text{Bub}(s) \\ & + \left[\frac{(12t^3 + 17t^2s + 13ts^2 + 2s^3)N}{2s^2} + \frac{(t + 3s)(2t^2 + ts + 2s^2)}{2stN} \right] \text{Box}^6(s, t)^\dagger \mathcal{F}_E(t, s) \\ & + \left[\frac{(12t^2 + 11ts + 9s^2)tN}{2s^2} + \frac{(t + 3s)t}{sN} \right] \text{Box}^6(s, u)^\dagger \mathcal{F}_E(t, s) \\ & + \left[-\frac{(12t^2 + 33ts + 23s^2)tN}{2s^2} + \frac{2(8t^2 + 14ts + 11s^2)t}{s^2N} \right] \epsilon \text{Box}^6(s, u)^\dagger \text{Bub}(s) \\ & + \left[-\frac{(12t^4 + 39t^3s + 38t^2s^2 + 31ts^3 + 14s^4)N}{2s^2t} \right] \end{aligned}$$

$$+ \frac{16t^4 + 36t^3s + 34t^2s^2 + 21ts^3 + 10s^4}{s^2tN} \left] \epsilon \text{Box}^6(s, t)^\dagger \text{Bub}(s), \quad (4.9)$$

$$\begin{aligned} \mathcal{F}_C(s, t, u) = & 2\beta_0^2 \left\{ \left[\frac{t^2 + u^2}{s^2} \right] \mathcal{F}_E(t, s)^\dagger \mathcal{F}_E(u, s) - \frac{2ut}{s^2} \epsilon^2 |\text{Bub}(s)|^2 \right\} \\ & + \beta_0 \left\{ \left[\frac{2(t^2 + 8ut + u^2)N}{s^2} + \frac{2}{N} \right] \epsilon^2 |\text{Bub}(s)|^2 \right. \\ & + \left[-\frac{(6t^2 + s^2)N}{s^2} + \frac{2(5s^2 - 8tu)}{s^2N} \right] 2\epsilon \text{Re} [\text{Bub}(s)^\dagger \mathcal{F}_E(t, s)] \\ & + \left[\frac{2(u-t)(2t^2 - su)N}{s^2} + \frac{2(t^2 + s^2)}{sN} \right] 2\text{Re} [\text{Box}^6(s, t)^\dagger \mathcal{F}_E(u, s)] \\ & + \left[\frac{(5s^2 - 12tu)N}{s^2} + \frac{5}{N} \right] \mathcal{F}_E(t, s)^\dagger \mathcal{F}_E(u, s) \left. \right\} \\ & + \left[-\frac{4(s^2 - ut)(t-u)^2N^2}{s^2} + \frac{4s^2}{N^2} \right] \text{Box}^6(s, t)^\dagger \text{Box}^6(s, u) \\ & + \left[\frac{3(s^2 - 3tu)N^2}{s^2} + \frac{15}{2} \right] \mathcal{F}_E(t, s)^\dagger \mathcal{F}_E(u, s) \\ & + \left[\frac{(u-t)(6t^2 + 8ts + 5s^2)N^2}{s^2} + \frac{5t^2 + 2ts + 3s^2}{s} + \frac{2s-t}{N^2} \right] 2\text{Re} \{ \text{Box}^6(s, t)^\dagger \mathcal{F}_E(u, s) \} \\ & + \left[\frac{(t-u)(6t^2 - 3ts + s^2)N^2}{s^2} - \frac{32t^3 + 35t^2s + 22ts^2 + 11s^3}{s^2} \right. \\ & + \left. \frac{2(4t^2 + 5s^2)}{sN^2} \right] 2\epsilon \text{Re} \{ \text{Box}^6(s, t)^\dagger \text{Bub}(s) \} \\ & + \left[\frac{3(2s^2 + 5ts - 6t^2)N^2}{2s^2} + \frac{48t^2 + 5ts - 3s^2}{2s^2} + \frac{4(t+3s)}{sN^2} \right] 2\epsilon \text{Re} \{ \mathcal{F}_E(t, s)^\dagger \text{Bub}(s) \} \\ & + \left[\frac{(7s^2 - 36tu)N^2}{2s^2} - \frac{43s^2 - 48tu}{s^2} + \frac{47s^2 - 64tu}{s^2N^2} \right] \epsilon^2 |\text{Bub}(s)|^2 \quad (4.10) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_D(s, t, u) = & \beta_0^2 \left\{ \frac{4ut}{s^2} \epsilon^2 |\text{Bub}(s)|^2 + \frac{2u(t^2 + u^2)}{s^2t} |\mathcal{F}_E(t, s)|^2 \right\} \\ & + \beta_0 \left\{ \left[\frac{2u(6t+s)N}{s^2} + \frac{2u(t+3s)}{stN} \right] |\mathcal{F}_E(t, s)|^2 \right. \\ & + \left[-\frac{u(6t^2 + 12ts + 7s^2)N}{s^2t} + \frac{2u(5s^2 - 8tu)}{s^2tN} \right] 2\epsilon \text{Re} [\text{Bub}(s)^\dagger \mathcal{F}_E(t, s)] \\ & + \left[\frac{2u(u-t)(2t^2 - su)N}{s^2t} + \frac{2u(t^2 + s^2)}{stN} \right] 2\text{Re} [\text{Box}^6(s, t)^\dagger \mathcal{F}_E(t, s)] \\ & + \left. \left[\frac{4u(s-3t)N}{s^2} + \frac{4u}{sN} \right] \epsilon^2 |\text{Bub}(s)|^2 \right\} \end{aligned}$$

$$\begin{aligned}
& + \left[\frac{(t-u)(6t^4 + 21t^3s + 25t^2s^2 + 19ts^3 + 7s^4)N^2}{s^2tu} \right. \\
& \left. - \frac{32t^5 + 99t^4s + 130t^3s^2 + 100t^2s^3 + 58ts^4 + 17s^5}{s^2tu} \right. \\
& \left. + \frac{2(4t^4 - 8t^2su + 8ts^3 + 5s^4)}{stuN^2} \right] 2\epsilon \operatorname{Re} \left\{ \operatorname{Box}^6(s, t)^\dagger \operatorname{Bub}(s) \right\} \\
& + \left[\frac{2(t-u)^2(2t^4 - 4t^2su + 2ts^3 + s^4)N^2}{s^2tu} + \frac{4(u-t)(t^4 - 2t^2su - s^3u)}{stu} \right. \\
& \left. + \frac{2(s^4 + t^2u^2)}{tuN^2} \right] \left| \operatorname{Box}^6(s, t) \right|^2 \\
& + \left[\frac{(u-t)(6t^3 + 10t^2s + 7ts^2 + s^3)N^2}{s^2u} + \frac{t^4 - 3t^3s - 6t^2s^2 - 9ts^3 - 3s^4}{stu} \right. \\
& \left. + \frac{s^2(t+3s)}{tuN^2} \right] 2 \operatorname{Re} \left\{ \operatorname{Box}^6(s, t)^\dagger \mathcal{F}_\mathcal{E}(t, s) \right\} \\
& + \left[\frac{(36t^4 + 114t^3s + 191t^2s^2 + 160ts^3 + 51s^4)N^2}{2s^2tu} \right. \\
& \left. - \frac{2(24t^4 + 99t^3s + 156t^2s^2 + 117ts^3 + 34s^4)}{s^2tu} \right. \\
& \left. + \frac{64t^4 + 192t^3s + 241t^2s^2 + 162ts^3 + 51s^4}{s^2tuN^2} \right] \epsilon^2 \left| \operatorname{Bub}(s) \right|^2 \\
& + \left[-\frac{(18t^3 + 57t^2s + 42ts^2 + 7s^3)N^2}{2s^2u} + \frac{48t^4 + 101t^3s + 49t^2s^2 - 33ts^3 - 21s^4}{2s^2tu} \right. \\
& \left. + \frac{(t+3s)(4u^2 + s^2)}{stuN^2} \right] 2\epsilon \operatorname{Re} \left\{ \operatorname{Bub}(s)^\dagger \mathcal{F}_\mathcal{E}(t, s) \right\} \\
& + \left[\frac{t(9t^2 + 12ts + 5s^2)N^2}{s^2u} + \frac{(t+3s)(s+3t)}{su} + \frac{(t+3s)^2}{2tuN^2} \right] \left| \mathcal{F}_\mathcal{E}(t, s) \right|^2, \quad (4.11)
\end{aligned}$$

with

$$\mathcal{F}_\mathcal{E}(t, s) = \operatorname{Bub}(t) - (1 + \epsilon)\operatorname{Bub}(s). \quad (4.12)$$

Only the first term in the expansions of the box integrals are required in Eqs. (4.8)–(4.11), where we have systematically discarded contributions of $\mathcal{O}(\epsilon)$. The bubble integrals must be expanded through to $\mathcal{O}(\epsilon^0)$.

In this section, we have isolated the infrared divergences of the one-loop amplitude into the terms $\mathcal{P}_\mathcal{A}$ and $\mathcal{P}_\mathcal{C}$. Both functions depend on one-loop bubble integrals and diverge as $1/\epsilon^2$. The separation of the singular terms becomes evident when we complete our basis of one-loop master integrals with the finite box in $D = 6 - 2\epsilon$. In the context of one-loop integrals, this form for the divergences arises naturally. The singular behaviour of the one-loop amplitude can also be predicted applying the

formalism of [25] which yields

$$\mathcal{P}_{\mathcal{A},\mathcal{C}}(s, t, u) = -\frac{1}{st}\mathbf{A}(\epsilon, s, u, t) \quad (4.13)$$

and

$$\mathcal{P}_{\mathcal{C},\mathcal{C}}(s, t, u) = -\frac{1}{su}\mathbf{B}(\epsilon, s, t, u) - \frac{1}{st}\mathbf{B}(\epsilon, s, u, t), \quad (4.14)$$

where $\mathbf{A}(\epsilon, s, t, u)$ and $\mathbf{B}(\epsilon, s, t, u)$ are defined in Eqs. (3.3) and (3.4). The singular structure of $\mathcal{P}_{\mathcal{A},\mathcal{C}}$ and $\mathcal{P}_{\mathcal{C},\mathcal{C}}$ around $\epsilon = 0$ precisely matches that given in Eq. (4.4) and (4.5) and the two formalisms differ only in the finite remainder. In order to make the agreement explicit one could replace $\mathcal{P}_{\mathcal{A}}(\mathcal{P}_{\mathcal{C}})$ with $\mathcal{P}_{\mathcal{A},\mathcal{C}}(\mathcal{P}_{\mathcal{C},\mathcal{C}})$ in Eqs. (4.2) and (4.3) with appropriate modifications due to the finite differences $\mathcal{P}_{\mathcal{A}} - \mathcal{P}_{\mathcal{A},\mathcal{C}}$ and $\mathcal{P}_{\mathcal{C}} - \mathcal{P}_{\mathcal{C},\mathcal{C}}$ in Eqs. (4.3) and (4.7).

5. Summary

In this paper we presented the $\mathcal{O}(\alpha_s^4)$ QCD corrections to the $2 \rightarrow 2$ scattering processes $q\bar{q} \rightarrow gg$, $gg \rightarrow \bar{q}q$ and the associated crossed processes $qg \rightarrow qg$ and $\bar{q}g \rightarrow \bar{q}g$ in the high energy limit, where the quark masses can be ignored. We computed renormalised analytic expressions for the interference of the tree-level diagrams with the two-loop ones and for the self-interference of one-loop graphs in the $\overline{\text{MS}}$ scheme. Throughout we employed conventional dimensional regularisation.

The renormalised matrix elements are infrared divergent and contain poles down to $\mathcal{O}(1/\epsilon^4)$. The singularity structure of one- and two-loop diagrams has been thoroughly studied by Catani [25] who provided a procedure for predicting the infrared behaviour of renormalised amplitudes. The anticipated pole structure agrees exactly with that obtained by direct Feynman diagram evaluation. In fact Catani's method does not determine the $1/\epsilon$ poles exactly, but expects that the remaining unpredicted $1/\epsilon$ poles are non-logarithmic and proportional to constants (colour factors, π^2 and ζ_3). We find that this is indeed the case, and the constant $H^{(2)}$ is given in Eq. (3.19). This provides a very strong check on the reliability of our results. Similarly, the infrared divergent structure of the squared one-loop diagrams we found by direct evaluation agrees with the expected pole structure.

The pole structure of the two-loop contribution is given in Eq. (3.2) while expressions for the finite parts in the s -channel and u -channels according to the colour decomposition of Eq. (3.24) are given in Secs. 3.2.1 and 3.2.2 respectively. Similarly, the infrared divergent one-loop contributions along with the remaining finite parts are detailed in Sec. 4. The expressions for the two-loop pole structure and the singular and finite parts of the self-interference of one-loop graphs are analytic and are given in terms of the one-loop bubble graph and one-loop box graph in $D = 6 - 2\epsilon$ dimensions. To evaluate these formulae in the appropriate physical region requires

the insertion of the series expansion of the one-loop graphs around $\epsilon = 0$. These expansions are given in Appendix A.

In this paper, we have concentrated only on QCD processes. However, the QED processes $e^+e^- \rightarrow \gamma\gamma$ and Compton scattering $e^-\gamma \rightarrow e^-\gamma$ as well as $\gamma\gamma \rightarrow e^-e^+$ are also of interest. We note that the expressions given here are related to those for the corresponding QED processes in the limit where the electron mass can be ignored by the alteration of various colour factors. We expect to address this in a separate article.

The results presented here, together with those previously computed for quark-quark scattering [22, 23, 24] are necessary ingredients for the next-to-next-to-leading order predictions for jet cross sections in hadron-hadron collisions. On their own, they are insufficient to make physical predictions and much work remains to be done. First, at the matrix element level, similar expressions to those presented here for gluon-gluon scattering are needed. Given the recent progress in the field, we anticipate that this problem will soon be solved. Second, a systematic procedure for analytically canceling the infrared divergences between the tree-level $2 \rightarrow 4$, the one-loop $2 \rightarrow 3$ and the $2 \rightarrow 2$ processes needs to be established for semi-inclusive jet cross sections. Again, recent progress in determining the singular limits of tree-level matrix elements when two particles are unresolved [36, 37] and the soft and collinear limits of one-loop amplitudes [38, 39], together with the analytic cancellation of the infrared singularities in the somewhat simpler case of $e^+e^- \rightarrow \text{photon} + \text{jet}$ at next-to-leading order [40], suggest that the technical problems will soon be solved for generic $2 \rightarrow 2$ scattering processes. There are additional problems due to initial state radiation. However, the recent steps taken towards the determination of the three-loop splitting functions [3, 4, 5] are also promising.

Third, a numerical implementation of the various contributions must be developed. The next-to-leading order programs for three jet production that have already been written provide a first step in this direction [41, 42]. We are confident that the problem of the numerical cancellation of residual infrared divergences will soon be addressed thereby enabling the construction of numerical programs to provide next-to-next-to-leading order QCD estimates of jet production in hadron collisions.

Acknowledgements

C.A. acknowledges the financial support of the Greek Government and M.E.T. acknowledges financial support from CONACyT and the CVCP. We gratefully acknowledge the support of the British Council and German Academic Exchange Service under ARC project 1050. This work was supported in part by the EU Fourth Framework Programme ‘Training and Mobility of Researchers’, Network ‘Quantum Chromodynamics and the Deep Structure of Elementary Particles’, contract FMRX-CT98-0194 (DG-12-MIHT), in part by the U.S. Department of Energy under Grant

No. DE-FG02-95ER40896 and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation.

A. One-loop master integrals

In this appendix, we list the expansions for the one-loop box integrals in $D = 6 - 2\epsilon$. We remain in the physical region $s > 0$, $u, t < 0$, and write coefficients in terms of logarithms and polylogarithms that are real in this domain. More precisely, we use the notation of Eqs. (3.22) and (3.23) to define the arguments of the logarithms and polylogarithms. The polylogarithms are defined as in Eq. (3.21).

We find that the box integrals have the expansion,

$$\begin{aligned} \text{Box}^6(u, t) = & \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{2s\Gamma(1-2\epsilon)(1-2\epsilon)} \left(\frac{\mu^2}{s}\right)^\epsilon \left\{ \frac{1}{2} [(X-Y)^2 + \pi^2] \right. \\ & + 2\epsilon \left[\text{Li}_3(x) - X\text{Li}_2(x) - \frac{1}{3}X^3 - \frac{\pi^2}{2}X \right] \\ & - 2\epsilon^2 \left[\text{Li}_4(x) + Y\text{Li}_3(x) - \frac{1}{2}X^2\text{Li}_2(x) - \frac{1}{8}X^4 - \frac{1}{6}X^3Y + \frac{1}{4}X^2Y^2 \right. \\ & \left. \left. - \frac{\pi^2}{4}X^2 - \frac{\pi^2}{3}XY - \frac{\pi^4}{45} \right] + (u \leftrightarrow t) \right\} + \mathcal{O}(\epsilon^3), \end{aligned} \quad (\text{A.1})$$

and

$$\begin{aligned} \text{Box}^6(s, t) = & \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{2u\Gamma(1-2\epsilon)(1-2\epsilon)} \left(-\frac{\mu^2}{u}\right)^\epsilon \left\{ (X^2 + 2i\pi X) \right. \\ & + \epsilon \left[\left(-2\text{Li}_3(x) + 2X\text{Li}_2(x) - \frac{2}{3}X^3 + 2YX^2 - \pi^2X + 2\zeta_3 \right) \right. \\ & \left. \left. + i\pi \left(2\text{Li}_2(x) + 4YX - X^2 - \frac{\pi^2}{3} \right) \right] \right. \\ & + \epsilon^2 \left[\left(2\text{Li}_4(z) + 2\text{Li}_4(y) - 2Y\text{Li}_3(x) - 2X\text{Li}_3(y) + (2XY - X^2 - \pi^2)\text{Li}_2(x) \right. \right. \\ & \left. \left. + \frac{1}{3}X^4 - \frac{5}{3}X^3Y + \frac{3}{2}X^2Y^2 + \frac{2}{3}\pi^2X^2 - 2\pi^2XY + 2Y\zeta_3 + \frac{1}{6}\pi^4 \right) \right. \\ & \left. \left. + i\pi \left(-2\text{Li}_3(x) - 2\text{Li}_3(y) + 2Y\text{Li}_2(x) + \frac{1}{3}X^3 - 2X^2Y + 3XY^2 \right. \right. \right. \\ & \left. \left. \left. - \frac{\pi^2}{3}Y + 2\zeta_3 \right) \right] \right\} + \mathcal{O}(\epsilon^3). \end{aligned} \quad (\text{A.2})$$

$\text{Box}^6(s, u)$ is obtained from Eq. (A.2) by exchanging u and t .

Finally, the one-loop bubble integral in $D = 4 - 2\epsilon$ dimensions is given by

$$\text{Bub}(s) = \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)\epsilon} \left(-\frac{\mu^2}{s}\right)^\epsilon. \quad (\text{A.3})$$

References

- [1] S.D. Ellis, Z. Kunszt and D.E. Soper, Phys. Rev. Lett. **69** (1992) 3615 [hep-ph/9208249].
- [2] W.T. Giele, E.W.N. Glover and D.A. Kosower, Phys.Rev. **D52** (1995) 1486 [hep-ph/9412338].
- [3] S.A. Larin, T. van Ritbergen, and J.A.M. Vermaseren, Nucl. Phys. **B427** (1994) 41; S.A. Larin, P. Nogueira, T. van Ritbergen, and J.A.M. Vermaseren, Nucl. Phys. **B492** (1997) 338 [hep-ph/9605317].
- [4] A. Retey and J.A.M. Vermaseren, hep-ph/0007294.
- [5] J. A. Gracey, Phys. Lett. **B322** (1994) 141 [hep-ph/9401214].
- [6] W.L. van Neerven and A. Vogt, Nucl. Phys. **B568** (2000) 263 [hep-ph/9907472]; hep-ph/0006154.
- [7] W.L. van Neerven and A. Vogt, Phys. Lett. **B490** (2000) 111 [hep-ph/0007362].
- [8] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, Eur. Phys. J. **C18** (2000) 117 [hep-ph/0007099].
- [9] J.F. Gunion and J. Kalinowski, Phys. Rev. **D34** (1986) 2119; S.J. Parke and T.R. Taylor, Nucl. Phys. **B269** (1986) 410; F.A. Berends and W.T. Giele, Nucl. Phys. **B294** (1987) 700; M. Mangano, S.J. Parke and Z. Xu, Nucl. Phys. **B298** (1988) 653.
- [10] Z. Kunszt, Nucl. Phys. **B271** (1986) 333; S.J. Parke and T.J. Taylor, Phys. Rev. **D35** (1987) 313.
- [11] J.F. Gunion and Z. Kunszt, Phys. Lett. **159B** (1985) 167.
- [12] J.F. Gunion and Z. Kunszt, Phys. Lett. **176B** (1986) 163.
- [13] Z. Bern, L. Dixon, D.A. Kosower, Phys. Rev. Lett **70** (1993) 2677 [hep-ph/9302280].
- [14] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. **B437** (1995) 259 [hep-ph/9409393].
- [15] Z. Kunszt, A. Signer and Z. Trócsányi, Phys. Lett. **B336** (1994) 529 [hep-ph/9405386].
- [16] V.A. Smirnov, Phys. Lett. **B460** (1999) 397 [hep-ph/9905323].
- [17] J.B. Tausk, Phys. Lett. **B469** (1999) 225 [hep-ph/9909506].
- [18] V.A. Smirnov and O.L. Veretin, Nucl. Phys. **B566** (2000) 469 [hep-ph/9907385].
- [19] C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, Nucl. Phys. **B580** (2000) 577 [hep-ph/0003261].
- [20] Z. Bern, L. Dixon and D.A. Kosower, JHEP **0001** (2000) 027 [hep-ph/0001001].

- [21] Z. Bern, L. Dixon and A. Ghinculov, hep-ph/0010075.
- [22] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0010212.
- [23] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0011094.
- [24] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0012007.
- [25] S. Catani, Phys. Lett. **B427** (1998) 161 [hep-ph/9802439].
- [26] R.K. Ellis and J.C. Sexton, Nucl. Phys. **B269** (1986) 445.
- [27] P. Nogueira, J.Comput.Phys. **105** (1993) 279.
- [28] C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. **B575** (2000) 416, Erratum-
ibid. **B585** (2000) 763 [hep-ph/9912251].
- [29] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, Nucl. Phys. **B174** (1980) 345 ;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159.
- [30] T. Gehrmann and E. Remiddi, Nucl. Phys. **B580** (2000) 485 [hep-ph/9912329].
- [31] C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. **B572** (2000) 307 [hep-
ph/9907494].
- [32] R.J. Gonsalves, Phys. Rev. **D28** (1983) 1542;
G. Kramer and B. Lampe, J. Math. Phys. **28** (1987) 945.
- [33] T. Gehrmann and E. Remiddi, Nucl. Phys. Proc. Suppl. **89** (2000) 2512 [hep-
ph/0005232].
- [34] C. Anastasiou, J.B. Tausk and M.E. Tejeda-Yeomans, Nucl. Phys. Proc. Suppl. **89**
(2000) 262 [hep-ph/0005328].
- [35] K.S. Kölbig, J.A. Mignaco and E. Remiddi, B.I.T. **10** (1970) 38.
- [36] J.M. Campbell and E.W.N. Glover, Nucl. Phys. **B527** (1998) 264 [hep-ph/9710255];
S. Catani and M. Grazzini, Phys. Lett. **446B** (1999) 143 [hep-ph/9810389];
S. Catani and M. Grazzini, Nucl. Phys. **B570** (2000) 287 [hep-ph/9908523].
- [37] F.A. Berends and W.T. Giele, Nucl. Phys. **B313** (1989) 595;
S. Catani, in Proceedings of the workshop on *New Techniques for Calculating Higher
Order QCD Corrections*, report ETH-TH/93-01, Zurich (1992).
- [38] Z. Bern, V. Del Duca and C.R. Schmidt, Phys. Lett. **445B** (1998) 168 [hep-
ph/9810409];
Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, Phys. Rev. **D60** (1999) 116001
[hep-ph/9903516];
S. Catani and M. Grazzini, hep-ph/0007142.

- [39] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. **B425** (1994) 217 [hep-ph/9403226];
D.A. Kosower, Nucl. Phys. **B552** (1999) 319;
D.A. Kosower and P. Uwer, Nucl. Phys. **B563** (1999) 477 [hep-ph/9903515].
- [40] A. Gehrmann-De Ridder and E.W.N. Glover, Nucl. Phys. **B517** (1998) 269 [hep-ph/9707224].
- [41] Z. Trocsanyi, Phys. Rev. Lett. **77** (1996) 2182 [hep-ph/9610499].
- [42] W.B. Kilgore and W.T. Giele, Phys. Rev. **D55** (1997) 7183 [hep-ph/9610433]; hep-ph/0009193.