Renormalons and multiloop estimates in scalar correlators, Higgs decay and quark-mass sum rules

D. J. Broadhurst[a,1], A. L. Kataev[b,2] and C. J. Maxwell[c,3]

[a] Physics Department, Open University, Milton Keynes MK7 6AA, UK

[b] Institute for Nuclear Research of the Academy of Sciences of Russia,
117312 Moscow, Russia

[c] Centre for Particle Theory, University of Durham, Durham, DH1 3LE, UK

Abstract. The single renormalon-chain contribution to the correlator of scalar currents in QCD is calculated in the $\overline{\rm MS}$-scheme in the limit of a large number of fermions, N_f. At n-loop order we find that in the $\overline{\rm MS}$-scheme the factorial growth of the perturbative coefficients due to renormalons takes over almost immediately in the euclidean region. The essential differences between the large-order growth of perturbative coefficients in the present scalar case, and in the previously-studied vector case are analysed. In the timelike region a stabilization of the corresponding perturbative series for the imaginary part, with n-loop behaviour $S_n/[\log(s/\Lambda^2)]^{n-1}$, where S_n is essentially constant for $n \leq 6$, is observed. Only for $n \geq 7$ does one discern the factorial growth and alternations of sign. We use the new all-orders results to scrutinize the performance of multiloop estimates , using a large-$\beta_0 = (11N_c - 2N_f)/12$ approximation, the so-called "naive nonabelianization" procedure, and within the effective charges approach. The asymptotic behaviour of perturbative coefficients, in both large-N_f and large-N_c limits, is analysed both in the spacelike and timelike regions. A contour-improved resummation technique in the time-like region is developed. Some subtleties connected with scheme-dependence are analysed , and illustrated using results in the $\overline{\rm MS}$ and V-schemes. The all-orders series under investigation are summed up with the help of the Borel resummation method. The results obtained are relevant to the analysis of the theoretical uncertainties in the 4-loop extractions of the running and invariant s-quark masses from QCD sum rules, and in calculations of the Higgs boson decay width into a quark-antiquark pair.

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[1] D.Broadhurst@open.ac.uk; http://physics.open.ac.uk/˜dbroadhu

[2] Kataev@ms2.inr.ac.ru

[3] C.J.Maxwell@durham.ac.uk.

1 Introduction

Thanks to fine work by Chetyrkin [1], we have information about the imaginary part of the correlator of a pair of scalar currents at the 4-loop level of QCD, i.e. to the same order in perturbation theory as for the vector channel [2]. The role of the vector correlator, in the annihilation of an electron-positron pair into hadrons, has been much studied. Here we focus on the scalar channel, which is even more intriguing. There are at least 6 potent reasons for studying scalar correlators.

1. Quark masses: Consider the scalar divergence, $i(m_a - m_b)\overline{\psi}_a\psi_b$, of a flavour-changing vector current. In perturbative QCD, its correlator is known exactly [3] to two-loop order, where it involves trilogarithms of two variables: m_a^2/Q^2 and m_b^2/Q^2, at euclidean momentum Q^2. At 3 loops, it is was possible to obtain the first two [4, 5, 6, 7] terms of the expansion in $1/Q^2$; at 4 loops, only the imaginary part is known, and then only to leading order in $1/Q^2$. The coefficients of quark and gluon condensates are known, to lesser accuracy [8]. Combining this information with experimental data on K_0^* mesons, coupling to the scalar form factor of semi-leptonic K_{l3} decay, and with further theoretical input from chiral perturbation theory, the scale-dependent strange-quark mass of the modified minimal subtraction ($\overline{\text{MS}}$) scheme has been determined [9] at a scale of $\mu = 1$ GeV, from whence it may be evolved to higher scales, using the anomalous quark-mass dimension $\gamma_m(\alpha_s) = d\log m(\mu^2)/d\log(\mu^2)$.

2. Higgs decay: At the opposite extreme of very high energy, 4-loop perturbative analysis of the scalar channel yields radiative corrections [1] to the decay of the Higgs boson of the standard model into quark-antiquark pairs, with a coupling to each flavour proportional to the mass of the quark (the 3-loop corrections are known from the results of Refs.[5, 6, 7, 10])).

3. Renormalization group: As is clear from these two important phenomenological investigations, mass renormalization is of the essence in the scalar channel. In the vector channel, we may ignore quark-mass effects at high energy; in the scalar channel they abide, since the only form of scalar current that has meaning is $\overline{\psi}_a M_{a,b}\psi_b$, where M is a mass matrix. Hence the vertex renormalization of the scalar current $\overline{\psi}_a\psi_b$ is precisely the inverse of mass renormalization. It follows that the anomalous quark-mass dimension $\gamma_m(\alpha_s)$ is ever present in the renormalization-group equations for correlators of scalar currents, while in the vector case it is inactive at very high energy, provided the order $O(m^2/Q^2)$ corrections are neglected.

Thanks to the dedicated labour and great ingenuity of colleagues, we have been provided with the 4-loop anomalous quark-mass dimension [11, 12] and the 4-loop beta function [13] of QCD. In our scalar analysis, these are inextricably intertwined.

4. Renormalons: It is thus of great interest to try to extend our understanding of perturbative quantum field theory by studying the interplay of coupling-constant and mass renormalization. In this respect, we noticed an apparent omission, concerning behaviour at higher orders in the scalar channel. Let N_f be the number of quark flavours and $\alpha_s(\mu^2)$ be the strong coupling at scale μ^2. In the limit $N_f \to \infty$, with $b = N_f\alpha_s/6\pi$ held fixed,

the vector correlator is known [14] to all orders in b, at order $1/N_f$. At first sight, this limit appears remote from asymptotically free QCD, where the beta function is dominated by gluons. However, it has become common practice to transform large-N_f results to so-called large-β_0 results, where $\beta_0 \equiv (11N_c - 2N_f)/12$, with $N_c = 3$ colours and $N_f = 3, 4, 5$ active flavours, gives the one-loop term in the QCD beta function

$$\beta(\alpha_s) \equiv \frac{d \log \alpha_s}{d \log \mu^2} = -\sum_{n \geq 0} \beta_n \left(\frac{\alpha_s}{\pi}\right)^{n+1}. \tag{1}$$

By the simplistic device $N_f \to N_f - \frac{33}{2}$, called naive nonabelianization (NNA) in [15], one transforms the irrelevant large-N_f ultraviolet (UV) factorial perturbative growth of an abelian theory, like QED, into the highly pertinent large-β_0 asymptotic perturbative growth of QCD series, since $-b$ is then replaced by $\overline{b} = \beta_0 \alpha_s(\mu^2)/\pi \approx 1/\log(\mu^2/\Lambda^2)$. Then the so-called renormalon structure (i.e. the perturbative factorial growth) of the vector result [14] is related to the way that long-distance physics is absorbed into condensates in the operator-product expansion (OPE) of the vector correlator of QCD (for the studies in QCD see e.g. [16, 17, 18] and [19, 20] for the reviews).

The virtue of an all-orders large-N_f result is to provide a smooth map from the full two-loop result, which it exactly reproduces, to the large-order behaviour, which it reproduces at leading order in $O(1/N_f)$. It was therefore natural to inquire whether the vector analysis [14] had yet been extended to the scalar case. We did not find such a work. Undertaking the task ourselves, we came to understand why it is so much more difficult in the scalar channel: the ultraviolet (UV) infinities of mass renormalization must be included to all orders in the coupling. In Sec. 2, we achieve this, after taking guidance from the study of critical phenomena.

5. Multiloop estimates: Effort has been expended in estimating effects in electron-positron annihilation, τ-decay and in deep-inelastic scattering characteristics beyond the orders of perturbation theory that are exactly computed (for different approaches see Refs.[21, 22, 20]). A similar attempt was made in the case of the decay width of the Higgs boson [23] following the ideas of Ref.[21] and by the authors of Ref.[24] using the asymptotic Padé-approximant method of Ref.[25]. This can only be inspired guesswork, informed by past experience and hopeful intuition. To study in detail the ideas and approximations, lying beyond such guesswork, one must test them in detail in as many processes as possible. In Sec. 3 we submit a variety of suggestions to detailed scrutiny in the scalar channel, by taking account of existing 4-loop input from [1, 11, 12, 13] and by exploiting our new all-orders results at large-N_f. Special attention is paid to the results of application of the NNA procedure, closely connected to the large-N_f expansion. We also investigate the structure of the 3- and 4-loop coefficients we are interested in within the "dual NNA" procedure, which is exact in the large number of colours N_c limit.

6. Analytical continuation: Direct multiloop calculations are usually performed in the euclidean space-like region. However, for the scalar correlator the quantities of phenomenological interest, namely the spectral functions of the QCD sum rules (see e.g. Refs.[26, 27, 28]) and the decay width of the Higgs boson are proportional to its imaginary part, defined in the minkowskian time-like region. In higher orders the coefficients of perturbative expansions contain π^2-contributions, which generally speaking, are not small

and can affect the asymptotic structure of the perturbative series in powers of α_s/π. These effects have been much studied (see e.g. Refs.[29]-[40],[4]) in attempts to resum them in all-orders of perturbation theory. In Sec. 3, using the results of Sec. 2, we develop further this approach to the case of fractional powers of α_s, which appear in the relation between the $\overline{\text{MS}}$-scheme running quark mass $m(\mu^2)$ and the scheme-invariant mass $\hat{m}$, introduced in Refs.[41, 42] (recent analogous independent considerations were given in Ref.[43]). The NNA approximation and the Borel resummation technique will be essential theoretical cornerstones of our analysis.

In the Conclusions we discuss the theoretical uncertainties of the various approaches and results, considered in the previous sections, and summarize the phenomenological relevance of the results obtained.

2   Renormalon analysis at large N_f

First, we briefly review the much easier vector case, where the Ward identity $Z_1 = Z_2$ protects the renormalon chain from UV disturbance, by cancellations between the two-loop skeletons into which the chain is inserted. All of these vector methods are necessary here, though they are not sufficient. Then we turn to the scalar case, which requires more powerful techniques, since there is no Ward identity to protect the insertions. Our aim is to handle both UV-divergent two-loop skeletons exactly, and to renormalize the mass to all orders in $N_f\alpha_s$ at large N_f, so as to achieve an $\overline{\text{MS}}$-renormalized result that connects the full two-loop result to the large-N_f renormalons, checking en route the $O(1/N_f)$ terms in the 3-loop correlator and 4-loop imaginary part that were given in [1].

In the interests of transparency, we describe the situation at large-N_f, in the first instance, since this is a well-defined limit. Only later do we allow ourselves the prevailing luxury of believing that this has anything to do with QCD. By presenting things this way, we allow the reader to distinguish hard (and also new) analysis from more easy (and also old) conjecture.

2.1   Combinatorics of vector resummation

Consider the formal series

$$\Pi_V(b,\varepsilon) = -\sum_{n>1}\left(\frac{b}{b-\varepsilon}\right)^{n-1}\frac{L(\varepsilon,n\varepsilon)}{n} \tag{2}$$

where the multiloop generating function

$$L(\varepsilon,\delta) = \sum_{j,k\geq 0}L_{j,k}\varepsilon^j\delta^k \tag{3}$$

is regular near $\varepsilon = \delta = 0$. Here b stands for a renormalized coupling chosen such that the large-N_f beta function vanishes at $b = \varepsilon$ in $d \equiv 4-2\varepsilon$ dimensions. In (2), the denominator

<div align="center">3</div>

$b - \varepsilon$ comes from transforming the bare to the renormalized coupling. For example, in minimally subtracted large-N_f QED, the bare charge e_0 is related to $b = N_f \alpha(\mu^2)/3\pi$ by

$$\frac{e_0^2}{4\pi^2} = \frac{3b}{1 - b/\varepsilon} \frac{(\mu^2/4\pi)^\varepsilon \Gamma(1 - 2\varepsilon)}{[\Gamma(1 - \varepsilon)]^2 \Gamma(1 + \varepsilon)} \tag{4}$$

where μ^2 makes b dimensionless, and the Γ functions make one-loop massless two-point diagrams rational in ε at euclidean momentum $\mu^2 = Q^2$.

Now we resum the series by collecting powers of the renormalized coupling, obtaining

$$\Pi_V(b, \varepsilon) = \sum_{n>1} b^{n-1} \sum_{k<n} \frac{C_{n,k}}{\varepsilon^k} \tag{5}$$

where the Laurent series at $n > 1$ loops starts, by assumption, with $1/\varepsilon^{n-1}$. At any order in renormalized perturbation theory, there are only two significant terms, namely [14]

$$C_{n,1} = \frac{L_{n-2,0}}{n(n - 1)} \tag{6}$$

$$C_{n,0} = \frac{L_{n-1,0}}{n(n - 1)} + (-1)^n (n - 2)! L_{0,n-1} \tag{7}$$

which are obtained by formal combinatorics. The first result determines the n-loop contribution to the anomalous dimension; the second gives the finite part. Thus, in this simple vector case, protected by the Ward identity $Z_1 = Z_2$, it is not necessary to know everything about the master n-loop integral $L(\varepsilon, n\varepsilon)$, resulting from a chain of $n - 2$ fermion loops in a pair of two-loop skeletons. It suffices to know $L(\varepsilon, 0)$ and $L(0, \delta)$, i.e. the analytic continuation to zero loops, and the Borel limit $\varepsilon \to 0$, with $\delta = n\varepsilon$ fixed.

2.2 Analysis of vector resummation

In the case of the correlator of the vector current, the analytical problem was solved in closed form by the first author, who obtained [14]

$$L(\varepsilon, 0) = \frac{(1 + \varepsilon)(1 - 2\varepsilon)(1 - 2\varepsilon/3)}{B(2 - \varepsilon, 2 - \varepsilon)\Gamma(3 - \varepsilon)\Gamma(1 + \varepsilon)}, \tag{8}$$

$$L(0, \delta) = \left(\frac{\mu^2 e^{5/3}}{Q^2}\right)^\delta \frac{32}{2 - \delta} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - (1 - \delta)^2)^2}. \tag{9}$$

The simple Γ-function result (8) was long since known from [44]. The all-orders result (9) of [14] is in agreement with the independently calculated perturbative expansion of Ref.[45]. At two loops,

$$L(0, 0) = 16 \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - 1)^2} = 3 \tag{10}$$

gives the Jost-Luttinger [46] singularity, when $b = \alpha/3\pi$.

Before explaining how to solve the demanding case of the scalar correlator, we need to explain how the far easier vector case was handled. At n-loops, there are two diagrams: in the first a chain of $n-2$ fermion loops dresses a fermion line; in the second, it is exchanged between fermion and antifermion. The first case is easy: only Γ functions occur; the second is hard: there is an $F_{3,2}$ hypergeometric series of the form given in [47]. However, this series is needed only in the Borel limit, where it gives a trigamma function. The series expansion in δ of the 4-dimensional two-point two-loop scalar diagram, with a modification $(1/k^2)^{1+\delta}$ of the momentum dependence of the totally internal propagator, is [47]

$$I(\delta) = 8\sum_{l>0} \zeta(2l+1)l(1-4^{-l})\delta^{2l-2} \tag{11}$$

with $I(0) = 6\zeta(3)$ giving the familiar unmodified result (see e.g.[48]). The renormalon series in (9) is obtained from the powerful trigamma identity

$$4(1-\delta)\sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2-(1-\delta)^2)^2} = \frac{1}{(1-\delta)^2} - \frac{1}{(2-\delta)^2} - \frac{\delta}{2}I(\delta) \tag{12}$$

which cancels the singularities of (11) at both $\delta = 1$ and $\delta = 2$. The vector result (9) has double poles at the negative integers, and at positive integers greater than 2. Yet it has no pole at $\delta = 1$ and merely a single pole at $\delta = 2$. In the QCD case, the pole at $\delta = 2$ signals the need to absorb long-distance effects into a nonperturbative gluon condensate, at order $1/Q^4$. No pole can appear at $\delta = 1$ since there is no dimension-2 gauge-invariant operator in the massless theory that could produce nonperturbative effects of order $1/Q^2$. The appearance of $e^{5/3}$ in (9) is easy to understand: in $d \equiv 4-2\varepsilon$ dimensions, each of the $n-2$ one-loop insertions brings with it a factor

$$f(\varepsilon) \equiv \frac{3}{4}\left(\frac{1}{d-1}+\frac{1}{d-3}\right) = 1 + \frac{5\varepsilon}{3} + O(\varepsilon^2) \tag{13}$$

giving $[f(\varepsilon)]^{n-2} \to \exp(5\delta/3)$, in the Borel limit. To suppress it, one may set $\mu^2 = Q^2\exp(-5/3)$, in the $\overline{\mathrm{MS}}$ scheme, corresponding to the perhaps more physical procedure of subtracting the one-loop photon propagator at Q^2, in the MOM-scheme in QED or V-scheme in QCD (for its 2-loop definition see Ref.[49]).

2.3   Combinatoric and analytic complexity in the scalar case

In the case of the correlator of the scalar current $m\overline{\psi}\psi$ both the combinatorics and the analysis are more demanding. The behaviour at large Q^2 and large N_f is given by

$$\Pi_S = \Pi_1(\varepsilon)\left(1 + 2C_F\frac{I_A(b,\varepsilon)+I_B(b,\varepsilon)}{T_F N_f} + O(1/Q^2) + O(1/N_f^2)\right) \tag{14}$$

$$\Pi_1(\varepsilon) \equiv -2[m(\mu^2)]^2 Q^2 d_F \frac{(\mu^2/Q^2)^\varepsilon}{\varepsilon(1-2\varepsilon)} \tag{15}$$

$$I_A(b,\varepsilon) \equiv \int_0^b \frac{g(x)}{\varepsilon-x}\,\mathrm{d}x \tag{16}$$

$$I_B(b,\varepsilon) \equiv \sum_{n>1}\left(\frac{b}{b-\varepsilon}\right)^{n-1}\frac{G(\varepsilon,n\varepsilon)}{n(n-1)} \tag{17}$$

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with $b \equiv T_F N_f \alpha_s(\mu^2)/3\pi$ giving the large-N_f contribution to the beta function. Here N_f is the number of fermion flavours, C_F is the value of quadratic Casimir operator in the fundamental fermion representation, d_F is the dimensionality of this representation, and T_F specifies the normalization of the corresponding matrices. Concretely, $C_F = 4/3$, $d_F = 3$, and $T_F = 1/2$, in QCD. In the abelian case of QED, we simply set $C_F = d_F = T_F = 1$.

The stumbling block is the $\overline{\text{MS}}$-renormalized mass $m(\mu^2)$ in the one-loop term (15). Renormalization of the scalar vertices is mandatory: without it the discontinuity of Π_S in the physical (i.e. minkowski) region $-Q^2 = s > 0$ would be infinite at $\varepsilon = 0$. At large N_f, the multiplicative renormalization $Z_m = m_0/m(\mu^2)$ may be considered as additive, with a vertex counterterm $Z_m - 1$ giving integral (16). The numerator g of the integrand is the all-orders contribution to the anomalous mass dimension

$$\gamma_m(\alpha_s) = \frac{d \log m(\mu^2)}{d \log(\mu^2)} = -\frac{C_F b}{T_F N_f} g(b) + O(1/N_f^2) \qquad (18)$$

at large N_f. The one-loop value $g(0) = 9/4$ gives $\gamma_m(\alpha_s) = -\alpha_s/\pi + O(\alpha_s^2)$ in QCD.

It is straightforward to insert a chain of fermion loops in the one-loop diagram for the scalar vertex and obtain the critical exponent

$$g(\varepsilon) = \frac{(d-1)^2}{d} \frac{\Gamma(d-2)}{[\Gamma(d/2)]^2} \frac{\sin \pi\varepsilon}{\pi\varepsilon} \qquad (19)$$

at large N_f in $d \equiv 4 - 2\varepsilon$ dimensions, giving the all-orders result for $g(b)$ in (18), by virtue of the fixed point at $b = \varepsilon$, to leading order in $1/N_f$. We note, for future reference, that $g(\varepsilon)$ is finite for all $d > -1$, and hence that $g(b)$ is finite for $b < 5/2$. The validity of (19), for all $d > -1$, is the true origin of the one-loop result $g(0) = 9/4$, at $d = 4$.

Anomalous dimensions at $O(1/N_f)$ are given by functions that differ from (19) only by multiplication of a rational function of d. The $O(1/N_f^2)$ term in (18) was found in [50] in the case of QED, and very recently in [51] for the yet more demanding nonabelian case of QCD. There one finds derivatives of Γ functions, which still give zeta values. At $O(1/N_f^3)$, hypergeometric series [47] give multiple zeta values (MZVs), such as $\zeta(5,3) = \sum_{m>n>0} 1/m^5 n^3$, in anomalous dimensions. This irreducible MZV already occurs at $O(1/N_f)$ in the ε-expansion of the multiloop generator $G(\varepsilon, n\varepsilon)$ in (17).

Even the handling of the vertex renormalization (16), at large n, is nontrivial. For the n-loop term in the scalar correlator, one needs to expand the Γ functions and rationals of (19) to order ε^{n-2}, and then make a further Laurent expansion of the integrand. After integration, this three-fold series gets multiplied by $\sum_{k\geq 0} 2^k \varepsilon^{k-1}$, from the one-loop term. The fifth and final series is the most potent: one must multiply by $(\mu^2/Q^2)^\varepsilon = \sum_{k\geq 0} (L\varepsilon)^k/k!$ with the resulting complicated dependence on $L \equiv \log(\mu^2/Q^2)$ required to cancel nonlocal terms in the Laurent expansion of the multiloop diagrams. Thus the innocent-looking integral (16) generates a combinatoric plethora of products of rationals, zeta-values, powers of $1/\varepsilon$, powers of the coupling, and powers of logs. And thus far we speak only of the analytically tractable term, free of the MZVs that occur in true multiloop diagrams.

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Next one sees, in I_B, the astounding interconnectedness of perturbative quantum field theory: there is a vast conspiracy, on a scale that would be ludicrous in human affairs, between analytically nontrivial n-loop integrals in (17), where $G(\varepsilon, n\varepsilon)$ entails the $F_{3,2}$ hypergeometric series of [47], and the 5-fold series from the intricate combinatorical processing of $g(\varepsilon)$ by vertex renormalization. Everyone knows what this conspiracy must achieve: total cancellation of logs from the singular terms, as required by the locality of counterterms. How, one asks, is this conspiracy coordinated? Its success is not in doubt: field theory cannot fail. Yet nothing, prior to this work, appeared to indicate the analytical mechanism.

Clearly the analysis of the vector case cannot solve this problem. Both of its key assumptions are now vitiated. First, the n-loop terms in (17) have a form that does not conform to (5); secondly, Laurent expansion of either (16) or (17) generates a $1/\varepsilon^n$ singularity at n loops, in defiance of the restriction $k < n$ in (5). It was the Ward identity $Z_1 = Z_2$ of QED, and hence of QCD at large N_f, that protected us from these eventualities in the vector case, by giving one less factor of $1/\varepsilon$, and by allowing an Ansatz (5), with no singularity at $n = 1$.

2.4   Reconciliation in the scalar case

How is progress possible in this complex scalar case? By virtue of our recent finding that

$$G(\varepsilon, \varepsilon) = g(\varepsilon) \tag{20}$$

which expresses the remarkable fact that analytic continuation to $n = 1$, of the hypergeometric result for inserting chains of $n-2$ one-loop diagrams in a pair of two-loop skeletons, gives the anomalous dimension, to all orders in the coupling at large N_f. To prove (20), one must first show that the irreducibly hypergeometric terms in $G(\varepsilon, \delta)$ vanish at $\delta = \varepsilon$, hence cancelling the new singularity at $n = 1$ in (17), which was not encountered in the Ward-protected vector analysis, based on (2). Thanks to the systematic hypergeometric methods of [47] this is now possible. The surviving terms in the analytic continuation to $n = 1$ then give Γ functions, multiplied by a very complicated rational function of ε and δ. At $\delta = \varepsilon$, the match of $G(\varepsilon, \varepsilon)$ to the critical exponent $g(\varepsilon)$ is perfect.

This enables us to organize both the combinatorics and the analysis, by writing

$$G(\varepsilon, \delta) = g(\varepsilon) \frac{G_2(\varepsilon, \delta)}{G_1(\varepsilon, \delta)} \left\{ 1 + (\delta - \varepsilon) \left[G_E(\varepsilon) + G_D(\delta) + \varepsilon \delta G_3(\varepsilon, \delta) \right] \right\} \tag{21}$$

with a prefactor specified by

$$G_1(\varepsilon, \delta) \equiv \left(\frac{\mu^2}{Q^2} \left(f(\varepsilon) \right)^{1/\varepsilon} \right)^{\varepsilon - \delta} \tag{22}$$

$$G_2(\varepsilon, \delta) \equiv \frac{\Gamma(1 + \delta)}{\Gamma(1 + \delta - 2\varepsilon)} \frac{\Gamma(1 - \delta + \varepsilon)}{\Gamma(1 - \delta - \varepsilon)} \frac{\Gamma(1 - 2\varepsilon)\Gamma(1 - \varepsilon)}{\Gamma(1 + \varepsilon)} \tag{23}$$

and functions of ε and δ that we eventually obtain, by hypergeometric analysis, as

$$G_E(\varepsilon) \;=\; \varepsilon(1 - 2\varepsilon) \tag{24}$$

$$G_D(\delta) \;=\; \frac{2}{1 - \delta} - \frac{1}{2 - \delta} + \frac{8(1 - \delta)}{3} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - (1 - \delta)^2)^2} \tag{25}$$

$$\;=\; \sum_{k>0} \frac{k + 3}{3}(2 - 2^{-k})\delta^{k-1} - \frac{8}{3} \sum_{l>0} \zeta(2l + 1)l(1 - 4^{-l})\delta^{2l-1} \tag{26}$$

with (25) showing the renormalon structure and (26) giving the Taylor expansion about $\delta = 0$, thanks to (11,12). As in the vector case [14, 18] no even zeta value can occur in the expansion of the renormalon contributions, since $I(\delta) = I(-\delta)$ is conformally invariant. In the new scalar result (26) one has the further simplification that odd zeta values occur only in odd Taylor coefficients. The vastly more complicated residuum, G_3, which we also determined completely, makes no contribution to either the anomalous dimension or the finite part, since it is multiplied by a factor that vanishes at $\delta = \varepsilon$, $\delta = 0$, and $\varepsilon = 0$, corresponding to $n = 1$, $n = 0$, and $n \to \infty$. After extracting unity, G_E, and G_D, one may simply throw away whatever remains in G_3. We did this the hard way, by calculating everything, exactly.

The highly coordinated conspiracy (20) is signalled by the leading unity in the braces of (21), and the triviality of $G_1(\varepsilon, \varepsilon) = 1$. Moreover, the remarkable combination of Γ functions in (23) gives $G_2(\varepsilon, \varepsilon) = G_2(\varepsilon, 0) = G_2(0, \delta) = 1$, which means that we may set G_2 to unity, without the slightest effect on the outcome, for the same reason that we may set G_3 to zero. Next we note that (24) precisely cancels the rational denominator of the one-loop term (15): this renormalon-free term is as simple as one could ever hope. Inspecting the analytic structure of the nontrivial renormalon contribution (25) we see single poles at $\delta = 1$ and $\delta = 2$. The pole at $\delta = 2$ corresponds to the appearance of a gluon-condensate contribution of order $m^2 Q^2 \langle G_{\mu,\nu} G^{\mu,\nu} \rangle / Q^4$ in Π_S. This was to be expected, by analogy with the vector case.

The *totally new feature* is the pole at $\delta = 1$, reflecting the ambiguity of the constant term in the correlator, which is infinite, though formally proportional to m^4, in perturbation theory, while current algebra relates it to $\langle m\overline{\psi}\psi \rangle$. It is wonderful that an analysis of finite parts of massless diagrams at large loop numbers can so powerfully remind one of what one knows from massive diagrams [3] at low loop numbers, namely that the UV physics of the second subtraction in the dispersion relation is as profound as that of the first. It makes no sense to say that at large Q^2 we can forget about the second UV subtraction, because m^2/Q^2 is small. If we make believe that we can, the $\delta = 1$ infrared (IR) renormalon of massless diagrams will remind us of the ultraviolet physics of massive diagrams. It was, of course, the Ward identity that protected us from this consideration in the vector case.

Proceeding, we see that the closed forms (24,25) lead to renormalized contributions that are speedily found, to very high orders, by the combinatorics (6,7) that served in the vector analysis of [14], where results were obtained up to 20 loops, analytically, and up to 100 loops, numerically. It remains, however, to resum the leading term in (21). We may set $G_2 = 1$, and write $1/G_1 = (1 - G_1)/G_1 + 1$, with only the final unity requiring further

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attention. That is now feasible, since the formal transformation

$$\sum_{n>1} \left(\frac{b}{b-\varepsilon}\right)^{n-1} \frac{1}{n(n-1)} = 1 + \frac{\varepsilon}{b} \log\left(1 - \frac{b}{\varepsilon}\right) = -\sum_{n>1} \left(\frac{b}{\varepsilon}\right)^{n-1} \frac{1}{n} \qquad (27)$$

gives a rather simple Laurent series, when multiplied by $g(\varepsilon) = \sum_{n>0} g_n \varepsilon^{n-1}$. Finally, we expand $1/(\varepsilon - x)$ in x/ε and formally integrate (16).

Tidying up, we see that $I_A + I_B$ is equivalent to $J_A + J_B$, as far as pole terms and finite parts are concerned, where

$$J_A(b,\varepsilon) \equiv \sum_{n>1} b^{n-1} \left\{ \sum_{j=1}^{n-1} \frac{g_{n-j}\varepsilon^{-j}}{n(n-1)} - \sum_{k=0}^{\infty} \frac{g_{n+k}\varepsilon^k}{n} \right\} \qquad (28)$$

combines I_A with the recalcitrant leading term from I_B, and the remainder of I_B is equivalent to

$$J_B(b,\varepsilon) \equiv \varepsilon g(\varepsilon) \sum_{n>1} \left(\frac{b}{b-\varepsilon}\right)^{n-1} \frac{L_B(\varepsilon, n\varepsilon)}{G_1(\varepsilon, n\varepsilon)} \frac{1}{n} \qquad (29)$$

where

$$L_B(\varepsilon, \delta) \equiv \frac{G_1(\varepsilon, \delta) - 1}{\varepsilon - \delta} + \varepsilon(1 - 2\varepsilon) + G_D(\delta) \qquad (30)$$

is easily processed by (6,7), using (19,22,25). It is important to note that J_A is no longer minimal: the first sum combines pure pole terms of I_A, with weight $1/(n-1)$, and the pole terms from multiplication of $g(\varepsilon)$ by (27), with weight $-1/n$. Then a regular part appears, in the second series of J_A. The overall $1/\varepsilon$ singularity, from the one-loop diagram, means that the $k = 0$ term in (28) becomes singular, and the $k = 1$ term becomes finite. This will be seen to be crucial, in the following analysis.

2.5 Critical behaviour of the scalar correlator

Now we analyze the so-called scalar-scalar anomalous dimension, γ_{SS}, defined by [1]

$$\left(\frac{\partial}{\partial \log \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \log \alpha_s} + 2\gamma_m(\alpha_s)\right) \Pi_S = [m(\mu^2)]^2 Q^2 \left\{\gamma_{SS}(\alpha_s) + O\left(\frac{m^2}{Q^2}\right)\right\} \qquad (31)$$

The presence of an $O(m^2/Q^2)$ term in the braces on the r.h.s. of the renormalization-group equation (31) reminds us that *two* subtractions [42] are required for the scalar correlator. Here we are concerned with the first, which produces a scale dependence described by γ_{SS}. If one supposes that the second might be forgotten at large Q^2, the unity of field theory soon corrects one: we have already seen that the finite parts of *massless* diagrams at large loop numbers are profoundly aware of the $O(m^2/Q^2)$ UV physics in (31), via the $\delta = 1$ IR renormalon in (25), at $m^2/Q^2 = 0$.

Working to leading order in $1/Q^2$, and next-to-leading order in $1/N_f$, we write

$$\Pi_S = [m(\mu^2)]^2 Q^2 d_F \left(-2L - 4 + \frac{C_F b}{T_F N_f} H(L, b) + O(1/Q^2) + O(1/N_f^2)\right) \qquad (32)$$

$$\gamma_{SS} = d_F \left(-2 + \frac{C_F b}{T_F N_f} h(b) + O(1/N_f^2)\right) \qquad (33)$$

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where $b \equiv T_F N_f \alpha_s / 3\pi$ is the leading term of the 4-dimensional beta function at large N_f, i.e. the value of ϵ such that the critical dimension is $4 - 2\varepsilon$. Hence the renormalization-group equation (31) simplifies to

$$\left(\frac{\partial}{\partial L} + \varepsilon^2 \frac{\partial}{\partial \varepsilon} \right) \varepsilon H(L, \varepsilon) + 4(L+2)\varepsilon g(\varepsilon) = \varepsilon h(\varepsilon) \tag{34}$$

where $L \equiv \log(\mu^2/Q^2)$. Our aim is to determine $h(\varepsilon)$ to all orders. Then the dependence of $H(L, b)$ on L is completely determined by $H(0, b)$.

We stress the underlying principle of this work by using the argument ε in (34), where a reader more used to 4-dimensional perturbation theory might reasonably expect to see us choose $b \equiv T_F N_f \alpha_s / 3\pi$. The reason should be clear: the large-N_f beta function is $b - \varepsilon$ in $d \equiv 4 - 2\varepsilon$ dimensions, where ε need not be small. By working near the critical point, $b = \varepsilon$, without assuming that d is near 4, we bypass perturbation theory.

We have remarked that true anomalous dimensions differ from (19) only by rational functions of d, at order $1/N_f$. Of course the beta function is not such an object, since it vanishes at the fixed point, by definition. The reason why the $O(1/N_f)$ corrections to the 4-dimensional QED beta function are given by the integral of (8) is clear: the physics resides in the critical exponent that is the derivative of this integral. In the scalar case, the critical exponent is the derivative of $\varepsilon h(\varepsilon) + 4g(\varepsilon)$. With some ingenuity, one may obtain it by careful parsing of (28,30), which yield

$$\frac{1}{g(\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\frac{\varepsilon h(\varepsilon) + 4g(\varepsilon)}{4} \right) = \frac{(d-3)^2}{3} - 2 \,. \tag{35}$$

Here, in short order, is the proof of this fine parabola:

$$\frac{1}{d-3} \left\{ 2 - \frac{(f(\varepsilon) - 1)/\varepsilon + \varepsilon(1 - 2\varepsilon) + 2}{f(\varepsilon)} \right\} = \frac{1 - f(\varepsilon) - \varepsilon^2}{\varepsilon f(\varepsilon)} = \frac{(d-3)^2}{3} - 2 \tag{36}$$

with a seemingly troublesome singularity at $d = 3$ turning out to give a harmless minimum. The prefactor $1/(d-3) = 1/(1-2\varepsilon)$ comes from the one-loop result (15). The first term in braces comes from the first series in J_A; the remainder from setting $\delta = 0$ in L_B. The weight of the series is doubled by the shift $1/\varepsilon(1 - 2\varepsilon) = 1/\varepsilon + 2/(1 - 2\varepsilon)$; the fourth term follows from using (10) in (25), which gives $G_D(0) = 2$. By taking the derivative of $\varepsilon h(\varepsilon) + 4g(\varepsilon)$, we remove the nonminimal series in J_A, whose weight, $1/n$, is incommensurate with (6), which we use for the J_B terms. The rational function $f(\varepsilon)$ enters via (22). Taking the exact one-loop vector result from (13), we obtain parabola (35).

Recall what makes this possible: the circumstance (20) that analytic continuation of the hypergeometric series of [47] to $n = 1$, i.e. to *minus* one insertions, reproduces the large-N_f anomalous mass dimension to all orders in the coupling. We began with two bad problems, for which the vector analysis gave no preparation: first we had an extra singularity, vitiating the combinatorics; secondly we had incommensurate weights for mass renormalization and those terms that we could handle. The beauty of (20) is that it enabled us to solve both problems at the same time: transferring the combinatorically

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recalcitrant term to join the mass renormalization in J_A, we obtained the desired weight as $1/(n-1) - 1/n = 1/n(n-1)$, in conformity with (6). There was a price to pay: this transfer took a nonminimal term with it. But that was no problem: we knew from critical phenomena that we must take a derivative, so as to obtain a physically significant exponent. In that derivative we must include the precise multiple of $g(\varepsilon)$ that kills the nonminimal terms.

The road from analysis of the Saalschutzian $F_{3,2}$ series of multiloop [47] diagrams, to the simple parabola (35), was a long one. Along it there was a narrow bridge: $G(\varepsilon, \varepsilon) = g(\varepsilon)$. We have found no other route.

2.6 Analytical results at large N_f

Recalling that $b \equiv T_F N_f \alpha_s / 3\pi$, we obtain

$$\gamma_{SS}(\alpha_s) = d_F \left(-2 + \frac{C_F \alpha_s}{3\pi} \sum_{n>1} \left(T_F N_f \frac{\alpha_s}{3\pi} \right)^{n-2} \{ h_n + O(1/N_f) \} \right) \tag{37}$$

where $h(\varepsilon) = \sum_{n>1} h_n \varepsilon^{n-2}$. Working to merely 4 loops, we immediately find

$$h_2 = -\frac{15}{2}, \quad h_3 = 9, \quad h_4 = -18\zeta(3) + \frac{1625}{72}, \tag{38}$$

in agreement with [1]. The development

$$
\begin{aligned}
h_5 &= -27\zeta(4) + \frac{15}{2}\zeta(3) + \frac{1625}{96} \\
h_6 &= -54\zeta(5) + \frac{27}{2}\zeta(4) + \frac{177}{10}\zeta(3) + \frac{8923}{480} \\
h_7 &= -90\zeta(6) + 30\zeta(5) + \frac{53}{2}\zeta(4) - 18[\zeta(3)]^2 + \frac{593}{18}\zeta(3) + \frac{1955}{96} \\
h_8 &= -162\zeta(7) + \frac{375}{7}\zeta(6) + \frac{741}{14}\zeta(5) - 54\zeta(4)\zeta(3) + \frac{2621}{56}\zeta(4) \\
&\quad + \frac{75}{7}[\zeta(3)]^2 + \frac{715}{24}\zeta(3) + \frac{59693}{2688}
\end{aligned}
$$

was obtained , using

$$\frac{h_{n+1} + 4g_{n+1}}{4} = \frac{4g_{n-2} - 4g_{n-1} - 5g_n}{3n} \tag{39}$$

which solves (35), with

$$\sum_n g_n \varepsilon^{n-1} = \left[4 - \sum_{n>1} \left(\frac{3}{2^n} + \frac{n}{2} \right) \varepsilon^{n-2} \right] \exp\left(\sum_{l>2} \frac{2^l - 3 - (-1)^l}{l} \zeta(l) \varepsilon^l \right) \tag{40}$$

obtained from (19). By this means analytical results to 20 loops, and numerical results to 100 loops, are readily obtainable. Since $g(\varepsilon)$ is finite for $\varepsilon < 5/2$, so is $h(\varepsilon)$. Hence

the coefficients decrease rather rapidly, with $h_n = O((2/5)^n)$. For example, we found that $h_{16} \approx 1.867 \times 10^{-5}$.

This convergence is in marked contrast with the behaviour of the finite parts. Writing $H(L, b) = \sum_{n>1} H_n(L) b^{n-2}$, we used (28,30) to derive the all-orders solution

$$n(n-1)H_n(L) = n(h_{n+1} - 4(L+2)g_n) + 4g_{n+1} - 9(-1)^n D_n(L) \qquad (41)$$

$$\sum_n D_n(L)\delta^n/n! = \{1 + \delta G_D(\delta)\} \exp((L+5/3)\delta) \qquad (42)$$

of the renormalization-group equation (34). Two good checks are provided by the vanishing of the r.h.s. of (41) at $n = 0$ and $n = 1$, using $g_1 = 9/4$, $h_2 = 4g_2 = -15/2$ and $G_D(0) = 2$. Two stronger checks are provided by known results at two [3] and three [1] loops. We obtain from (41) the entirety of the former and the large-N_f terms of the latter, for all μ^2/Q^2. At $\mu^2 = Q^2$, we write $H_n \equiv H_n(0)$. In [1], we find

$$H_2 = 3\left[-\frac{131}{8} + 6\zeta(3)\right], \quad H_3 = 9\left[\frac{511}{36} - 4\zeta(3)\right], \qquad (43)$$

at $\mu^2 = Q^2$.

In order to extend explicit results to high loop numbers, and to analyze their UV and IR renormalon content, we separate $G_D(\delta) = G_-(\delta) + G_+(\delta)$ into

$$G_-(\delta) = -\frac{2}{3}\sum_{k>0}\frac{(-1)^k}{(k+\delta)^2} \qquad (44)$$

$$G_+(\delta) = \frac{2}{1-\delta} - \frac{1}{2-\delta} + \frac{2}{3}\sum_{k>2}\frac{(-1)^k}{(k-\delta)^2} \qquad (45)$$

and expand in δ. (Note however that the Taylor expansion of the *total* renormalon contribution is already given with great economy by (26), with odd zeta values in odd Taylor coefficients.) Computation of analytical results to 20 loops takes seconds, for any value of $L \equiv \log(\mu^2/Q^2)$. The analytical development to merely 8 loops is given by

$$H_4 = 90\zeta(5) - \frac{27}{4}\zeta(4) + \frac{157}{4}\zeta(3) - \frac{499069}{1728}$$

$$H_5 = -\frac{2304}{5}\zeta(5) - \frac{45}{4}\zeta(4) - \frac{4337}{60}\zeta(3) + \frac{1976311}{1728}$$

$$H_6 = 1701\zeta(7) - 15\zeta(6) + \frac{14829}{10}\zeta(5) + \frac{537}{40}\zeta(4) - 3[\zeta(3)]^2 + \frac{54643}{360}\zeta(3) - \frac{840309103}{155520}$$

$$H_7 = -\frac{99387}{7}\zeta(7) - \frac{160}{7}\zeta(6) - \frac{87026}{21}\zeta(5) - \frac{54}{7}\zeta(4)\zeta(3) + \frac{332}{21}\zeta(4)$$
$$- \frac{32}{7}[\zeta(3)]^2 - \frac{41614}{189}\zeta(3) + \frac{1275753995}{40824}$$

$$H_8 = 68850\zeta(9) - \frac{1323}{32}\zeta(8) + \frac{3967083}{56}\zeta(7) + \frac{3635}{112}\zeta(6) - \frac{27}{2}\zeta(5)\zeta(3) + \frac{7017943}{672}\zeta(5)$$
$$- \frac{639}{56}\zeta(4)\zeta(3) + \frac{12973}{896}\zeta(4) + \frac{727}{112}[\zeta(3)]^2 + \frac{28819423}{72576}\zeta(3) - \frac{315995418895}{1492992}$$

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2.7 The asymptotics of naive nonabelianization

We reiterate the often rehearsed (yet never properly justified) argument for Borel resummability of large-N_f singularities at $\delta < 0$, in the case of QCD. At truly large N_f, they give sign-constant series. Now we imagine that, in some vague[1] sense, the gluon loops "follow" the quark loops. Hence we perform the naive nonabelianization [15] $N_f \to N_f - \frac{11N_c}{2}$ ($N_c = 3$), which gives $b \to -\beta_0 \alpha_s(\mu^2)/\pi$, with $\beta_0 = (11N_c - 2N_f)/12$ giving the relative contributions of gluons and quarks in the one-loop beta function of QCD. In the real word, with $N_f \leq 6$ active quarks, we have $\beta_0 > 0$ and $\overline{b} \equiv \beta_0 \alpha_s(\mu)/\pi \approx 1/\log(Q^2/\Lambda^2)$, if we suppress large logarithms by renormalizing at $\mu^2 = Q^2$. By this sleight of hand, the singularities at $\delta < 0$ now give a sign-alternating asymptotic series that is resummable by Laplace transformation. A singularity at $\delta > 0$ leads to an intrinsic ambiguity of order $\exp(-\delta/\overline{b}) = O(Q^{-2\delta})$ in a dimensionless correlator. In the vector case (25) the dominant ambiguity, at $\delta = 2$, is associated with the gluon condensate. Here, in the scalar case with $\Pi_S = O(Q^2)$, a singularity at $\delta > 0$ leads to an ambiguity of order $Q^{2-2\delta}$. Hence the renormalon at $\delta = 1$ is in accord with the fact that the constant term in Π_S is inaccessible to perturbation theory. Interestingly, our analysis does not distinguish the scalar from the pseudoscalar channel. In the latter case, current algebra relates the constant term in the correlator to the quark condensate $\langle m\overline{\psi}\psi \rangle$.

At any given order $n > 1$, we may separate H_n into 6 parts:

$$H_n = H_n^{\overline{\text{MS}}} + H_n^{(0)} + H_n^{(-)} + H_n^{(1)} + H_n^{(2)} + H_n^{(+)} \tag{46}$$

where

$$H_n^{\overline{\text{MS}}} = \frac{n(h_{n+1} - 4(L+2)g_n) + 4g_{n+1}}{n(n-1)} \tag{47}$$

is the highly convergent part from $\overline{\text{MS}}$ renormalization and $H_n^{(0)}$ comes from the leading unity in the braces of (42). The next term comes from (44), with resummable UV renormalon singularities at $\delta < 0$; final three terms come from the IR renormalons in (45).

Table 1 shows the numerics of this breakdown, again at $\mu^2 = Q^2$. We comment on each contribution in turn.

1. The $\overline{\text{MS}}$-specific contribution $H_n^{\overline{\text{MS}}} = O((2/5)^n)$ is negligible at large n. This is because the critical exponent (19) is finite at all dimensions $d > -1$. The numerator of (8) shows that the same applies to the critical exponent in the vector case.

2. The modest growth of $H_n^{(0)}$ comes from the choice $\mu^2 = Q^2$, made so as to compare (43) with [1]. At $\mu^2 = Q^2 \exp(-5/3)$, corresponding to QED MOM- or V-schemes subtraction of quark-loop insertions, these terms fall off like $1/n^2$.

3. The series coming from $H_n^{(-)}$ is now regarded as infrared-safe: one may resum it by Laplace transformation of the Borel transform (44), obtaining

$$\sum_{n>1} H_n^{(-)}(L)[-\overline{b}]^{n-1} = 9 \int_0^\infty \frac{G_-(\delta)\exp((5/3+L)\delta) - G_-(0)}{\delta} \exp(-\delta/\overline{b})\,\mathrm{d}\delta \tag{48}$$

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[1]It *must* be vague. Unlike quark loops, gluon loops are not gauge invariant.

where $\bar{b} \approx 1/\log(\mu^2/\Lambda^2)$ replaces $-b$, by naive nonabelianization.

4. The total is dominated by the factorial growth of $H_n^{(1)} = O((n-2)!)$. Indeed this dominant $\delta = 1$ IR renormalon even gives a reasonable account at very low orders. For example, it gives a fraction $73.0/84.5 = 86\%$ of the 3-loop large-N_f result of [1]. Clearly perturbation theory would be in bad shape, if this renormalon entered phenomenology. Fortunately it does not. Rather it is a reminder, from the perturbative sector, of the long-distance physics in $\Pi_S(0)$, which current algebra relates to quark condensates. In the sum-rule analysis of [9], it was nullified by taking a twice-subtracted dispersion relation. In Higgs decay and spectral function of the QCD sum rules, it is nullified by taking the imaginary part, in the physical region $s = -Q^2 > 0$.

    As previously remarked, one must set aside the temptation to divide Π_S by Q^2 and then differentiate w.r.t. Q^2, so as to remove γ_{SS} from (31). That would *not* remove the $\delta = 1$ renormalon, since it would leave the kinematic singularity $\Pi_S(0)/Q^4$. Then long-distance physics would make the asymptotic perturbation series expire sooner than needs be, since it is always faithful to the motto *dulce et decorum est pro Wilson mori*: it's OK to explode in accordance with Wilson's OPE. Indeed, it often signals impending doom at the 2-loop level [15].

5. More long-distance physics resides in $H_n^{(2)} = O((n-2)!/2^n)$, which signals the presence of the gluon condensate in Wilson's scheme of things, as articulated in QCD by [26]. This $\delta = 2$ IR renormalon is also absent from the imaginary part, to leading order in $1/s$. When taking an imaginary part, in the physical region, one kills any single-pole renormalon, and turns a double-pole into a single pole, since the imaginary part of (42) acquires a factor $\sin(\pi\delta)$. The quark and gluon dimension-4 operators appear in the energy momentum tensor and are hence renormalization-group invariants, to leading order. Thus they are absent from the imaginary part, at order $1/\beta_0$, in the high-energy limit. Since there is no matrix element to absorb renormalons at $\delta = 1$ or $\delta = 2$, these, and only these, appear as single poles in (25).

6. The only IR (i.e. unresummable) large-β_0 renormalons that appear in the imaginary part at large s, are those in $H_n^{(+)}$, at $\delta > 2$. These correspond to long-distance physics in matrix elements of operators O_k with dimensions $d_k \geq 6$ in the OPE. The resultant ambiguity in Im Π_S is of order $m^2 \langle O_k \rangle / s^{d_k - 1}$.

2.8   Analysis of the imaginary part at large N_f

Table 1 might appear alarming. A far happier picture emerges in Table 2, where we analyze a *physical* quantity, namely the imaginary part Im $\Pi_S = 2\pi s R_S(1 + O(1/s))$ at $-Q^2 = s \equiv w^2$. For $w = M_H$, this contains the radiative corrections to decay of a Higgs boson of mass M_H into a quark-antiquark pair, ignoring terms of order $(m(M_H^2)/M_H)^2$. Now we are dealing with a multiplicatively renormalized quantity: the explicit dependence

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of R_S on μ^2 is cancelled by its implicit dependence, via $\alpha_s(\mu^2)$ and $m(\mu^2)$, giving

$$\left(\frac{\partial}{\partial \log \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \log \alpha_s} + 2\gamma_m(\alpha_s)\right) R_S = 0 \tag{49}$$

with $2\gamma_m$ appearing because R_S contains two powers of the renormalized mass in the Born approximation. We should, however, set $\mu^2 = O(s)$, to suppress large logarithms. Here, we set $\mu^2 = s \equiv w^2$ and obtain

$$R_S = 3[m(w)]^2 \left(1 + \frac{\alpha_s}{\pi} \sum_{n>1} b_S^{n-2} \{S_n + O(1/N_f)\}\right) \tag{50}$$

with $b_S \equiv -N_f \alpha_s(w^2)/6\pi$, which is replaced by $\beta_0 \alpha_s(w^2)/\pi \approx 1/\log(w^2/\Lambda^2)$ in naive nonabelianization. We have $S_2 = 17/3$, with the large-N_f term giving the total radiative correction at 2 loops. At $n \geq 2$ loops, we obtain the leading term at large-N_f as

$$S_n = A_n + \Delta_n \tag{51}$$
$$A_n = \frac{2(-1)^n}{n-1}\frac{g_n}{g_1} \tag{52}$$
$$\sum_{n\geq 2} \frac{\delta^{n-1}\Delta_n}{(n-2)!} = 2\exp(5\delta/3)\{1 + \delta G_-(\delta) + \delta G_+(\delta)\}\frac{\sin(\pi\delta)}{\pi\delta} - 2 \tag{53}$$

where g_n is the n-loop term in the large-N_f result (40) for the anomalous quark-mass dimension. The separation $S_n = A_n + \Delta_n$ into anomalous-dimension and renormalon contributions will be used in Sec. 3, where we shall retain only the latter in large-β_0 approximations. Here, we retain both terms, so as to present the exact analytical results at large-N_f.

We note that the high-energy imaginary part R_S receives powers of π^2 from two sources: from the even zeta values $\{\zeta(2k) \mid k > 1\}$ of euclidean analysis, and also from the analytic continuation of logarithms to the physical region. At large-N_f, the separation of these two effects is particularly clean: even zeta values occur only in the anomalous-dimension contributions A_n; powers of π^2 from analytic continuation result only from the factor $\sin(\pi\delta)/\pi\delta$ in the Borel transform (53) of the renormalon contributions Δ_n. We write the A_n terms in braces in the following large-N_f results. Up to 4 loops we obtain

$$S_2 = \left\{-\frac{5}{3}\right\} + \frac{22}{3} = \frac{17}{3} \tag{54}$$
$$S_3 = \left\{\frac{35}{36}\right\} - 4\zeta(3) - \frac{1}{3}\pi^2 + \frac{275}{18} = -4\zeta(3) - \frac{1}{3}\pi^2 + \frac{65}{4} \tag{55}$$
$$S_4 = \left\{\frac{4}{3}\zeta(3) - \frac{83}{108}\right\} - \frac{40}{3}\zeta(3) - \frac{22}{9}\pi^2 + \frac{3940}{81} = -12\zeta(3) - \frac{22}{9}\pi^2 + \frac{15511}{324} \tag{56}$$

with totals in agreement the with the large-N_f 3-loop [4] and 4-loop [1] terms. At 5 and 6 loops our new results

$$S_5 = A_5 + \Delta_5 = \left\{-\frac{3}{2}\zeta(4) + \frac{5}{6}\zeta(3) + \frac{65}{96}\right\}$$

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$$-60\zeta(5) + 4\zeta(3)\pi^2 - \frac{100}{3}\zeta(3) + \frac{1}{10}\pi^4 - \frac{275}{18}\pi^2 + \frac{64877}{324} \tag{57}$$

$$S_6 = A_6 + \Delta_6 = \left\{ \frac{12}{15}\zeta(5) - \zeta(4) - \frac{7}{9}\zeta(3) - \frac{451}{720} \right\}$$

$$-400\zeta(5) + \frac{80}{3}\zeta(3)\pi^2 - \frac{2000}{27}\zeta(3) + \frac{22}{15}\pi^4 - \frac{7880}{81}\pi^2 + \frac{244871}{243} \tag{58}$$

were found by expanding (40), to obtain A_n in the braces, and the new all-orders renormalon results (44,45), to obtain Δ_n via (53). This method may easily be continued up to 20 loops, analytically, and up to 100 loops, numerically.

The corresponding vector quantities in electron-positron annihilation, at large N_f, come from the old result (9) of [14], which immediately gives

$$\sum_{n>1} \frac{\delta^{n-2}V_n}{(n-2)!} = \frac{8\exp(5\delta/3)}{3(1-\delta)(2-\delta)} \left(-\sum_{k>0} \frac{(-1)^k}{(k+\delta)^2} + \sum_{k>2} \frac{(-1)^k}{(k-\delta)^2} \right) \frac{\sin(\pi\delta)}{\pi\delta} \tag{59}$$

At 6 loops the vector coefficient has grown by an order of magnitude and changed sign, with $V_6/V_2 \approx -11$. By contrast, $S_6/S_2 \approx 0.5$ in the scalar channel. This remarkable postponement of factorial growth, at large-N_f, does not depend on a cancellation between anomalous-dimension and renormalon terms; rather it reflects cancellations between the renormalons themselves, with $\Delta_6/\Delta_2 \approx 0.4$ showing no sign of the growth that had become clear at 6 loops in the vector channel. In this respect, the (pseudo-)scalar channel is better behaved than the (axial-)vector channel, at high energy and large N_f, despite warnings [52] that might suggest an opposite situation.

2.9 Postponed factorial growth

Considerable interest attaches to the numerics of Table 2, where it will be seen than S_n is amazingly well-behaved for $n < 7$, while H_n in Table 1 had already gone haywire at $n = 3$. We have discovered a plateau of tranquility at loop numbers $n = 2, 3, 4, 5, 6$, in the large-N_f terms of the imaginary part of the scalar correlator.

The behaviour at $n > 7$ in Table 2 is fairly clear.

1. The $\overline{\text{MS}}$ term $S_n^{\overline{\text{MS}}}$ falls of like $(2/5)^n$.

2. The leading unity of the braces of (53) gives power growth of $S_n^{(0)}$, modulated by a sine.

3. Eventually, the UV renormalons at $\delta < 0$ take over, giving an alternating series for $S_n^{(-)}$, now that we have naively nonabelianized. But they takes ages to get going: even at $n = 8$ they have not yet overtaken the humble $\delta = 0$ term.

4. The term $S_n^{(1)}$ from $\delta = 1$ is no longer a renormalon; the single pole has been cancelled by $\sin(\pi\delta)$.

5. The same applies to the $\delta = 2$ term $S_n^{(2)}$, except that it is, on average, smaller than the $\delta = 1$ term, by a factor of order $1/2^n$, after allowing for the sinusoidal oscillations.

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6. The unsummmable series with the coefficients $S_n^{(+)}$ from the IR renormalons at $\delta \geq 3$ grows factorially, eventually. However it is suppressed by a factor of order $1/3^n$, in comparison with the UV renormalons.

The staggering feature is the tally, S_n. For $n > 7$, it behaves like the wild animal that it truly is; for $n < 7$, all is sweetness and light. Presented with only the analytical expressions for S_n at $n = 2, 3, 4, 5, 6$, one would not have the slightest inkling of what is in store. Conversely, one may say that the large-N_f renormalons show themselves mercifully late, in this physical quantity.

Note that these statements are entirely dependent on the scheme chosen (for a discussion of the scheme-dependence of renormalon contributions see e.g. Refs.[53, 20]). As will be clear from the discussions of Sec. 3.5 and Table 5, the V-scheme results are in fact a much better indicator of the eventual asymptotics of the coefficients. Nonetheless given the widespread use of the $\overline{\text{MS}}$ scheme in the literature we initially focus here on the asymptotic behaviour in that scheme.

Now consider the case where $s = w^2$ with $w \approx 2$ GeV, as in strange-quark mass extraction [9, 54]. With $\beta_0 = 9/4$, the expansion parameter $\beta_0 \alpha_s(w)/\pi \approx 1/\log(w^2/\mu^2) \approx 0.2$ is now uncomfortably large. Hence the tranquil plateau at $n < 7$ loops is good news for the later analysis in [9], where 4-loop perturbative QCD was used, on both sides of the sum rule. The result was significantly different from an earlier 3-loop analysis in [54], by the same group, using a different truncation procedure (compare with the similar 3-loop studies of Ref.[55]).

The villains, which might have been waiting just round the corner, were the renormalons. Might a large 5-loop term significantly change the 4-loop result?

To date, we know of only one analytical technique for estimating such effects on the basis of genuinely new calculation, instead of reshuffling old input: naive nonabelianization of the large-N_f terms, which we have computed in the demanding scalar channel, at some cost of labour. The good news is that we found nothing alarming at $n = 5$ loops. The even better news is that all seems well at $n = 6$ loops. Only at $n = 7$ does the inevitable growth show signs of commencing. Hence the best indicator that we can compute suggests that the perturbative part of the strange-quark-mass extraction in [9] is in fine shape.

Indeed, the results of Table 2 indicate that the contribution of the 5-loop coefficient might be not crucial and that the corresponding perturbative series should be truncated at the 6-loop level in accordance with the common practice in treating the predictions of asymptotic perturbative expansions, which presumes their truncation at the minimal term.

This fits nicely with the claim in [9] that the condensate contributions are also under control. Had these contributions been substantial, our discovery of perturbative tranquility at large N_f would have been rather puzzling; now it may be taken as gratifying evidence of the depth to which the OPE connects ultraviolet and infrared physics.

2.10 Euclidean analysis at large N_f

In the vector channel we may express the results of high-energy perturbation theory in two ways: either in terms of $R(s)$, in the physical region, or in terms the Adler function

$$D(Q^2) \equiv Q^2 \int_0^\infty \frac{R(s)ds}{(s+Q^2)^2} \tag{60}$$

in the euclidean region. Here, $R(s) = 1 + \alpha_s/\pi + O(\alpha_s^2)$ gives the high-energy radiative corrections to the parton model in electron-positron annihilation and hence $D(Q^2)$ gives the corresponding radiative corrections to the derivative of the polarization function of the vector correlator.

In both cases, we ignore quark masses, so the transformation between one set of radiative corrections and the other is generated, to all orders, by the following integral

$$Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} \left(\frac{\mu^2}{s}\right)^\delta = \frac{\pi\delta}{\sin(\pi\delta)} \left(\frac{\mu^2}{Q^2}\right)^\delta \tag{61}$$

with the expansion of

$$\frac{\pi\delta}{\sin(\pi\delta)} = 1 + \sum_{k>0} \left(2 - 4^{1-k}\right) \zeta(2k)\delta^{2k} \tag{62}$$

telling one precisely how to remove from the imaginary part all and only those powers of π^2 that came from analytic continuation of logarithms.

In the scalar channel, we are confronted by a choice of euclideanizations of the high-energy imaginary part. The most prudent choice would appear to be the dispersion relation for the *second* derivative of the scalar correlator [42], which is multiplicatively renormalized for all values of m^2/Q^2 and is hence free of the IR renormalon at $\delta = 1$ in (45). Ignoring terms of order m^2/Q^2, this amounts to the euclideanization

$$\overline{D}_S(Q^2) = 2Q^2 \int_0^\infty \frac{sR_S(s)ds}{(s+Q^2)^3} = 3[m(Q^2)]^2 \left(1 + \frac{11}{3}\frac{\alpha_s(Q^2)}{\pi} + O(\alpha_s^2)\right) \tag{63}$$

of the radiative corrections to the imaginary part at high energy. However, a merely mathematical analogy with the vector case might lead one to consider the construct

$$\widetilde{D}_S(Q^2) = Q^2 \int_0^\infty \frac{R_S(s)ds}{(s+Q^2)^2} = 3[m(Q^2)]^2 \left(1 + \frac{17}{3}\frac{\alpha_s(Q^2)}{\pi} + O(\alpha_s^2)\right) \tag{64}$$

corresponding to a dispersion relation for the first derivative of $\Pi_S(Q^2)/Q^2$. Here one expects the asymptotic perturbation series to destroy itself earlier, leaving an ambiguity of order Λ^2/Q^2 that reflects the failure to remove the infinities in $\Pi_S(0)$. Therefore, this ambiguity has a perturbative origin.

We shall show that at large-N_f the perturbative series for the nonstandard euclideanization $\widetilde{D}_S$ is indeed worse behaved than the twice-differentiated euclideanization $\overline{D}_S$, in accord with the expectation from the OPE. It might therefore be expected that we shall

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proceed in Sec. 3 only with the safer alternative $\overline{D}_S$, as used in QCD sum-rules [42, 9]. In fact, we shall need both constructs, so as to study the logarithmic derivative of $\widetilde{D}_S$, which is the euclidean analog of the considered in the case of Higgs decay quantity (see Refs.[6, 12, 56]). In our case it can be defined as

$$R_D(Q^2) \equiv -\frac{1}{2}\frac{d\log\widetilde{D}_S(Q^2)}{d\log Q^2} = \frac{\widetilde{D}_S(Q^2) - \overline{D}_S(Q^2)}{2\widetilde{D}_S(Q^2)} = \frac{\alpha_s(Q^2)}{\pi} + O(\alpha_s^2) \qquad (65)$$

which satisfies a renormalization-group equation

$$\left(\frac{\partial}{\partial\log\mu^2} + \beta(\alpha_s)\frac{\partial}{\partial\log\alpha_s}\right)R_D = 0 \qquad (66)$$

that is free of the anomalous quark-mass dimension and hence suitable for analysis by the method of effective charges [57] (see also Ref.[58]), scheme-invariant perturbation theory [59], commensurate scale relations [60] and the standard PMS approach [61].

Here we assemble everything that is known about the $\overline{\text{MS}}$ perturbation series of the euclidean constructs (63,64,65). It is convenient to begin with

$$\widetilde{D}_S(Q^2) = 3[m(Q^2)]^2\left(1 + \sum_{n>0} d_n\left(\frac{\alpha_s(Q^2)}{\pi}\right)^n\right) \qquad (67)$$

$$d_1 = \frac{17}{3} \qquad (68)$$

$$d_2 = \frac{10801}{144} - \frac{39}{2}\zeta(3) - \left(\frac{65}{24} - \frac{2}{3}\zeta(3)\right)N_f \qquad (69)$$

$$\begin{aligned}
d_3 = {} & \frac{6163613}{5184} - \frac{109735}{216}\zeta(3) + \frac{815}{12}\zeta(5) \\
& - \left(\frac{46147}{486} - \frac{262}{9}\zeta(3) + \frac{5}{6}\zeta(4) + \frac{25}{9}\zeta(5)\right)N_f + \left(\frac{15511}{11664} - \frac{1}{3}\zeta(3)\right)N_f^2 \quad (70)
\end{aligned}$$

obtained by removing terms involving π^2 in the expansion

$$R_S(w^2) = 3[m(w^2)]^2\left(1 + \sum_{n>0} s_n\left(\frac{\alpha_s(w)}{\pi}\right)^n\right) \qquad (71)$$

of the imaginary part, given to 4 loops in [1]. To effect the inverse transformation, to 5 loops, one may use the fixed-order perturbative expansion in the minkowskian region

$$s_1 = d_1 \qquad (72)$$

$$s_2 = d_2 - \gamma_0(\beta_0 + 2\gamma_0)\pi^2/3 \qquad (73)$$

$$s_3 = d_3 - [d_1(\beta_0 + \gamma_0)(\beta_0 + 2\gamma_0) + \beta_1\gamma_0 + 2\gamma_1(\beta_0 + 2\gamma_0)]\pi^2/3 \qquad (74)$$

$$\begin{aligned}
s_4 = {} & d_4 - [d_2(\beta_0 + \gamma_0)(3\beta_0 + 2\gamma_0) + d_1\beta_1(5\beta_0 + 6\gamma_0)/2 + 4d_1\gamma_1(\beta_0 + \gamma_0) \\
& + \beta_2\gamma_0 + 2\gamma_1(\beta_1 + \gamma_1) + \gamma_2(3\beta_0 + 4\gamma_0)]\pi^2/3 \\
& + \gamma_0(\beta_0 + \gamma_0)(\beta_0 + 2\gamma_0)(3\beta_0 + 2\gamma_0)\pi^4/30 \qquad (75)
\end{aligned}$$

where the relation between s_4 and d_4 was derived in Ref.[23] and

$$\gamma_m(\alpha_s) \equiv \frac{d \log m}{d \log \mu^2} = -\sum_{n \geq 0} \gamma_n \left(\frac{\alpha_s}{\pi}\right)^{n+1} \tag{76}$$

gives the expansion of the anomalous quark-mass dimension, in the same manner that (1) gives the expansion of the beta function for the scale dependence of the coupling. Both expansions are known to 4 loops. The coefficients of γ_m are [11, 12]:

$$\gamma_0 = 1 \tag{77}$$

$$\gamma_1 = \frac{1}{16}\left[\frac{202}{3} - \frac{20}{9}N_f\right] \tag{78}$$

$$\gamma_2 = \frac{1}{64}\left[1249 - \left(\frac{2216}{27} + \frac{160}{3}\zeta(3)\right)N_f - \frac{140}{81}N_f^2\right] \tag{79}$$

$$\begin{aligned}
\gamma_3 = \frac{1}{256}\Bigg[& \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\zeta(5) \\
& - \left(\frac{91723}{27} + \frac{34192}{9}\zeta(3) - 880\zeta(4) - \frac{18400}{9}\zeta(5)\right)N_f \\
& + \left(\frac{5242}{243} + \frac{800}{9}\zeta(5) - \frac{160}{3}\zeta(4)\right)N_f^2 - \left(\frac{332}{243} - \frac{64}{27}\zeta(3)\right)N_f^3\Bigg]
\end{aligned} \tag{80}$$

while those of beta function are [13]:

$$\beta_0 = \frac{1}{4}\left[11 - \frac{2}{3}N_f\right] \tag{81}$$

$$\beta_1 = \frac{1}{16}\left[102 - \frac{38}{3}N_f\right] \tag{82}$$

$$\beta_2 = \frac{1}{64}\left[\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\right] \tag{83}$$

$$\begin{aligned}
\beta_3 = \frac{1}{256}\Bigg[& \frac{149753}{6} + 3564\zeta(3) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta(3)\right)N_f \\
& + \left(\frac{50065}{162} + \frac{6472}{81}\zeta(3)\right)N_f^2 + \frac{1093}{729}N_f^3\Bigg]
\end{aligned} \tag{84}$$

which will likewise be needed in our analysis.

The results at large-N_f are

$$d_{n-1} = (-N_f/6)^{n-2}\left(A_n + \widetilde{\Delta}_n + O(1/N_f)\right) \tag{85}$$

$$\sum_{n \geq 2} \frac{\delta^{n-1}\widetilde{\Delta}_n}{(n-2)!} = 2\exp(5\delta/3)\left\{1 + \delta G_-(\delta) + \delta G_+(\delta)\right\} - 2 \tag{86}$$

with the Borel transform (86) giving

$$\widetilde{\Delta}_5 = -60\zeta(5) - \frac{100}{3}\zeta(3) + \frac{64877}{324} \tag{87}$$

$$\widetilde{\Delta}_6 = -400\zeta(5) - \frac{2000}{27}\zeta(3) + \frac{244871}{243} \tag{88}$$

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at 5 and 6 loops.

We expect the factorial growth of (85), in $\widetilde{D}_S$, to be more drastic than that of

$$s_{n-1} = (-N_f/6)^{n-2} \left(A_n + \Delta_n + O(1/N_f) \right) \tag{89}$$

in the imaginary part R_S, since the latter is free of the spurious $\delta = 1$ renormalon that afflicts the former. Table 3 emphatically confirms this expectation at large N_f, where one sees that $\widetilde{\Delta}_6/\widetilde{\Delta}_2 \approx 69$ is two orders of magnitude larger than $\Delta_6/\Delta_2 \approx 0.40$.

In general, one expects that the asymptotic structure of perturbation theory expansions will differ for the physical quantity $R_S(s)$ and the euclideanization $\widetilde{D}_S(Q^2)$. To construct the latter from the former one takes the renormalization-group determined powers of the minkowski logarithm $\log(\mu^2/w^2)$ and performs the transformation

$$\log^{2k}\left(\frac{\mu^2}{w^2}\right) \to (2k)! \left(2 - 4^{1-k}\right) \zeta(2k) \tag{90}$$

on even powers. If the imaginary part is fairly well behaved, as at large N_f, it is unlikely that its euclideanization will be so. Indeed, the factorial growth of the r.h.s. of Eq.(90) should restore the factorial growth of the perturbative series for the euclidean quantity $\widetilde{D}_S$, expected from general grounds of quantum field theory. The large-N_f analysis of Table 3 suggests that the imaginary part R_S is rather well behaved for $n < 7$ loops, with the far worse behaviour of $\widetilde{D}_S$ (and thus $\widetilde{\Delta}_n$) resulting from its renormalon at $\delta = 1$, which was suppressed by the sine function in (53).

Next we consider the more prudent euclideanization (63), with a perturbation series

$$\overline{D}_S(Q^2) = 3[m(Q^2)]^2 \left(1 + \sum_{n>0} \overline{d}_n \left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \right) \tag{91}$$

$$\overline{d}_n = d_n - 2\gamma_{n-1} - \sum_{k=1}^{n-1} (k\beta_{n-k-1} + 2\gamma_{n-k-1})d_k \tag{92}$$

The large-N_f results for $\overline{D}_S$ are obtained from

$$\overline{d}_{n-1} = (-N_f/6)^{n-2} \left(A_n + \overline{\Delta}_n + O(1/N_f) \right) \tag{93}$$

$$\overline{\Delta}_n = \widetilde{\Delta}_n - (n-2)\widetilde{\Delta}_{n-1} \tag{94}$$

at $n > 2$ loops, with the $\delta = 1$ renormalon removed by the combination (94). At 2 loops, we have $\overline{\Delta}_2 = \widetilde{\Delta}_n - 2 = 16/3$. Table 3 shows that the factorial growth in $\overline{D}_S$ at large N_f is milder than in $\widetilde{D}_S$, but more severe than in the imaginary part R_S.

From (65,92) we obtain the first 5 terms in the expansion

$$R_D(Q^2) = \sum_{n\geq 0} r_n \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{n+1} \tag{95}$$

$$r_0 = \gamma_0 = 1 \tag{96}$$

$$r_1 = \gamma_1 + \frac{1}{2}\beta_0 d_1 \tag{97}$$

$$r_2 = \gamma_2 + \beta_0 d_2 - \frac{1}{2}\beta_0 d_1^2 + \frac{1}{2}\beta_1 d_1 \tag{98}$$

$$r_3 = \gamma_3 + \frac{3}{2}\beta_0 d_3 + \frac{1}{2}\beta_0 d_1^3 - \frac{3}{2}\beta_0 d_1 d_2 + \beta_1 d_2 - \frac{1}{2}\beta_1 d_1^2 + \frac{1}{2}\beta_2 d_1 \tag{99}$$

$$r_4 = \gamma_4 + 2\beta_0 d_4 + 2\beta_0 d_1^2 d_2 - 2\beta_0 d_1 d_3 - \frac{1}{2}\beta_0 d_1^4 - \beta_0 d_2^2$$
$$+ \frac{3}{2}\beta_1 d_3 + \frac{1}{2}\beta_1 d_1^3 - \frac{3}{2}\beta_1 d_1 d_2 - \frac{1}{2}\beta_2 d_1^2 + \beta_2 d_2 + \frac{1}{2}\beta_3 d_1 \tag{100}$$

where the 5-loop coefficient d_4 is unknown, while the estimates for the 5-loop anomalous quark mass dimension are known from the results of application of the Padé resummation method [25] (note, however, that the analytical calculation of order $O(1/N_f^2)$-corrections to $\gamma_m(\alpha_s)$ [51] indicate that the latter ones should be refined). At large N_f we simply obtain

$$r_n = (-N_f/6)^n \left(\frac{n}{2}\widetilde{\Delta}_{n+1} + O(1/N_f) \right) \tag{101}$$

for $n > 0$, with no contribution from the anomalous quark-mass dimension beyond the one-loop result $r_0 = \gamma_0 = 1$. It follow that at large-N_f the perturbation series for R_D explodes as violently as that for $\widetilde{D}_S$.

3   Subtleties of the naive nonabelianization

In the previous section we mainly concentrated on the analysis of large N_f perturbative results, for the different theoretical quantities, related to the correlator of quark scalar currents. However, as was already explained in Sec.2.7, it is of definite interest to study the truncated and Borel-resummed perturbative series within the framework of the NNA Ansatz, which is postulated by applying the substitution $N_f \rightarrow N_f - 11N_c/2 = -6\beta_0$ (where β_0 is the first coefficient of the QCD β-function, defined by Eq.(1)), supplemented by retaining the leading terms in powers of β_0 in the reorganized perturbative series. This procedure enables one to transform large-N_f results, which are related to QED (note that in QED β_0 is proportional to N_f), to the nonabelian case of QCD.

In this section we shall study a number of theoretical issues related to the application of the NNA approximation in the scalar channel. In particular, we shall concentrate on obtaining estimates of uncalculated higher-order terms in perturbative series for quantities, related to the correlator of quark scalar currents both in the euclidean and minkowskian regions. We shall also formulate different procedures for the resummation of the large minkowskian π^2-terms, within the framework of the NNA approach.

3.1   Estimates of the higher order corrections in the $\overline{\text{MS}}$-scheme.

We begin by considering the expressions

$$d_n^{\text{NNA}} = \beta_0^{n-1}\widetilde{\Delta}_{n+1} \tag{102}$$

$$s_n^{\text{NNA}} = \beta_0^{n-1}\Delta_{n+1} \tag{103}$$

$$\overline{d}_n^{\text{NNA}} = \beta_0^{n-1}\overline{\Delta}_{n+1} \tag{104}$$

for the coefficients d_n, s_n and $\overline{d}_n$ in $\widetilde{D}_S$, R_S and $\overline{D}_S$ at $n+1$ loops in the $\overline{\text{MS}}$-scheme. These are obtained by the naive nonabelianization of the terms in (85,89,93) that do not involve the large-N_f anomalous quark-mass dimension contribution A_n, which is known to be unamenable to naive nonabelianization in general (for detailed discussions, related to the deep-inelastic scattering anomalous dimensions see the works of Ref.[62]), and in any case is small at 5 loops, since it falls off like $(2/5)^n$.

Table 4 shows that all three NNA estimators give the correct sign and order of magnitude at 3 and 4 loops. At 4 loops, the success of $\overline{d}_3^{\text{NNA}}$ is rather remarkable, since we are using only the $(N_f - 33/2)^2$ approximation to a quadratic to estimate the full result. In particular, at $N_f = 5$, as in Higgs decay, $\overline{d}_3$ differs from the NNA estimate by only 8%. It is significant that this success of NNA occurs in the safer euclideanization $\overline{D}_S$, which includes neither the π^2 terms of R_S nor the spurious renormalon of $\widetilde{D}_S$. Accordingly we take

$$\overline{d}_4^{\text{NNA}} = \beta_0^3\left(\frac{17597}{324} + \frac{20}{3}\zeta(3) - 60\zeta(5)\right) \tag{105}$$

as our favoured NNA estimator.

Note, that even if we choose an overall factor of two as the conservative uncertainty in the estimating power of the NNA procedure (which is motivated by inspecting the related numbers of Table 4 for $N_f = 3$), we arrive to the conclusion that the 5-loop perturbative approximation for $\overline{D}_S$ is really well-behaved. Indeed, taking $N_f = 3$ and $\alpha_s \approx 0.3$ we obtain the following series

$$\begin{aligned}
\overline{D}_S &= 1 + 3.67\left(\frac{\alpha_s}{\pi}\right) + 14.17\left(\frac{\alpha_s}{\pi}\right)^2 + 77.36\left(\frac{\alpha_s}{\pi}\right)^3 + 2 \times 1.26\left(\frac{\alpha_s}{\pi}\right)^4 \\
&= 1 + 0.350 + 0.129 + 0.067 + 0.0002
\end{aligned} \tag{106}$$

with a rather small 5-loop term. Thus, the 4-loop extractions of the running mass m_s of Ref.[9], which is based on the consideration of the $\overline{D}_S$-function, indeed contains rather small perturbative QCD uncertainties due to the truncation of the corresponding series at the 4-loop level.

Let us now turn to the study of the NNA predictions for the coefficients of the perturbative series for R_S in the minkowskian region. As in the case of $\overline{D}_S$ and $\widetilde{D}_S$-functions, this procedure gives the correct sign and order of magnitude at the 3- and 4-loop levels (see Table 4). Taking into account the large N_f-result for Δ_5 (see Eq.(57)) we get the following NNA prediction for the 5-loop term in R_S

$$s_4^{\text{NNA}} = \beta_0^3\left(\frac{64877}{324} - \frac{100}{3}\zeta(3) - 60\zeta(5) - \frac{275}{18}\pi^2 + 4\zeta(3)\pi^2 + \frac{1}{10}\pi^4\right) \tag{107}$$

which gives small and positive numbers

$$s_4^{\text{NNA}}(N_f = 3) \approx 49; \quad s_4^{\text{NNA}}(N_f = 4) \approx 39; \quad s_4^{\text{NNA}}(N_f = 5) \approx 31 \tag{108}$$

Following now the conservative pattern of fixing the uncertainty of the NNA approximation of $\overline{d}_n$-terms by an overall factor of 2, we present the NNA-inspired estimates of s_4-term in the following form: $s_4 \approx 2s_4^{\text{NNA}}$. Keeping this in mind, we arrive at the following numerical estimates of s_4 for different numbers of N_f:

$$s_4(N_f = 3) \sim 98; \quad s_4(N_f = 4) \sim 78; \quad s_4(N_f = 5) \sim 62 \qquad (109)$$

For $N_f = 5$ these estimates are quite in accord with the result of applying the [2/2] asymptotic Padé-approximation method, namely $s_4^{APAP}(N_f = 5) \approx 67$ [24]. However, for $N_f = 3$ the Padé estimate of Ref.[24], namely $s_4^{APAP}(N_f = 3) \approx 251$, is over 2.6 times larger than the related NNA inspired estimate of Eq.(109).

Note, that contrary to what was found in the application of the Padé-resummation method to the e^+e^- annihilation R-ratio [22], the variant of asymptotic Padé estimates used in Ref.[24], which is performed in the minkowskian region directly, does not allow one to separate the effects of analytical continuation proportional to π^2. In the NNA approach the contributions of the π^2-effects leading in β_0 are taken into account explicitly and can be resummed to all-orders of perturbation theory. We shall consider this technical problem in Sec.3.3. Another observation is that the estimates of Eq.(109) differ in both sign and order of magnitude from the ones obtained in Ref.[23] using a variant of the effective-charges procedure [21, 63]. In Sec.3.6 we shall return to more detailed considerations of the problems related to the application of the effective charges method in the scalar channel.

We conclude this subsection by demonstrating the behaviour of the $\overline{\text{MS}}$- perturbative series for the physical quantity R_S of Eq.(71) in the cases of $N_f = 3$ and $N_f = 5$, which are relevant to the spectral function of the QCD sum rules and the hadronic decay width of the Higgs boson. Taking $\alpha_s \approx 0.3$ in the first case and $\alpha_s \approx 0.114$ in the latter one we have

$$N_f = 3 \; : \; R_S \; \sim \; 1 + 5.667\left(\frac{\alpha_s}{\pi}\right) + 31.864\left(\frac{\alpha_s}{\pi}\right)^2 + 89.156\left(\frac{\alpha_s}{\pi}\right)^3 + 98\left(\frac{\alpha_s}{\pi}\right)^4 \qquad (110)$$
$$= \; 1 + 0.541 + 0.291 + 0.078 + 0.008$$

$$N_f = 5 \; : \; R_S \; \sim \; 1 + 5.667\left(\frac{\alpha_s}{\pi}\right) + 29.147\left(\frac{\alpha_s}{\pi}\right)^2 + 41.758\left(\frac{\alpha_s}{\pi}\right)^3 + 62\left(\frac{\alpha_s}{\pi}\right)^4 \qquad (111)$$
$$= \; 1 + 0.206 + 0.0384 + 0.0021 + 0.00014$$

One can see, that in both cases the perturbative series are rather well behaved and that the NNA-inspired estimates of 5-loop terms are over 10 times smaller than the 4-loop ones explicitly calculated in Ref.[1] . In view of this we conclude that the 4-loop phenomenological studies, based on the 4-loop series of (110,111) are in rather good shape, and that the manifestation of the asymptotic growth of these perturbative series is postponed. Indeed, in accordance with the results of Tables 2,3 this feature can manifest itself starting from $n = 7$ loops. In its turn, this means that in the process of concrete phenomenological applications of the perturbative results for R_S in the energy region where $\alpha_s \le 0.3$ one can restrict oneself to a consideration of the partial sums of the truncated perturbative series with $n \le 6$ loops, estimating roughly the remaining perturbative uncertainty in the $\overline{\text{MS}}$-scheme by the value of the smallest term taken into account. More detailed numerical studies are performed in Sec. 3.5.

3.2 Large N_f versus N_c -theoretical motivation for NNA

It is interesting to consider the motivation for the NNA approximation from the theoretical point of view. In the vector case the NNA term can be proved to have some very special properties by analyzing the operators that build the leading UV renormalon singularity [64, 65]. It is convenient to consider a 'planar approximation' where at each order in the N_f expansion only terms leading in N_c are retained. In this way one obtains an expansion in multinomials of N_f and N_c. So for the Adler-function coefficients in the vector case one can write, after extracting an overall factor of $(3/4)C_F$ [18, 66],

$$d_n = d_n^{[n]}N_f^n + d_n^{[n-1]}N_f^{n-1}N_c + d_n^{[n-2]}N_f^{n-2}N_c^2 + \ldots + d_n^{[1]}N_f N_c^{n-1} + d_n^{[0]}N_c^n , \qquad (112)$$

so that the large-N_f expansion runs from left-to-right, and the large-N_c from right-to-left. One can now formulate two versions of NNA. The standard one is derived by replacing N_f by $(11N_c - 12\beta_0)/2$ to arrive at

$$d_n = d_n^{(n)}\beta_0{}^n + d_n^{(n-1)}\beta_0{}^{n-1}N_c + d_n^{(n-2)}\beta_0{}^{n-2}N_c^2 + \ldots + d_n^{(1)}\beta_0 N_c^{n-1} + d_n^{(0)}N_c^n . \qquad (113)$$

One can, however , define a 'dual NNA' by replacing N_c by $(12\beta_0 + 2N_f)/11$, to obtain an expansion in β_0 with different coefficients. The standard NNA is exact in the large-N_f limit, and the 'dual NNA' is exact in the large-N_c limit. Of course it is the standard NNA that is of practical use since we have all-orders large-N_f results. If one extracts the NNA term d_n^{NNA} of the standard expansion and re-expands it one can obtain an expansion in N_f and N_c with coefficients $\tilde{d}_n^{[n-r]}$. By construction the leading-N_f term is reproduced, so $d_n^{[n]} = \tilde{d}_n^{[n]}$, but the sub-leading terms will not be reproduced. Nonetheless by making use of the operator analysis of [64] one can show that [65],

$$d_n^{[n-r]} \approx \tilde{d}_n^{[n-r]}[1 + O(1/n)] , \qquad (114)$$

so that for fixed-r and large orders of perturbation theory the re-expansion of the NNA term approximates the sub-leading in N_f terms with asymptotic accuracy $O(1/n)$. For the dual NNA term one can prove an exactly similar result, where sub-leading in N_c terms are reproduced to asymptotic accuracy $O(1/n)$ on re-expansion of the dual NNA term [65]. Such weak asymptotic results about the NNA terms will hold provided that in the large-N_f, and large-N_c limits the leading UV renormalon asymptotics is controlled by a *single* operator contribution. This is the case for the vector Adler function of Eq.(60). In planar approximation there are two relevant four-fermion operators, $\mathcal{O}_+$ and $\mathcal{O}_-$ of [65], but $\mathcal{O}_-$ is scalar after Fierzing and decouples. These operators are defined as [65]

$$\mathcal{O}_\pm = \mathcal{O}_V \pm \mathcal{O}_A$$

$$\mathcal{O}_V = \left(\overline{\psi}\gamma_\mu T^A \psi\right)\left(\overline{\psi}\gamma^\mu T^A \psi\right) \quad , \quad \mathcal{O}_A = \left(\overline{\psi}\gamma_\mu\gamma_5 T^A \psi\right)\left(\overline{\psi}\gamma^\mu\gamma_5 T^A \psi\right)$$

where T^A denotes the colour matrices. The remaining four-fermion operator $\mathcal{O}_+$ gives the leading asymptotics in the large-N_c limit , and in the large-N_f limit the operator corresponding to the single renormalon chains involved in NNA ($\mathcal{O}_1$ of [65]) dominates the asymptotics [64]. This operator is defined as [65]

$$\mathcal{O}_1 = (1/g^4)\partial_\nu F^{\nu\mu}\partial^\rho F_{\rho\mu} \quad .$$

Let us see how these weak asymptotic results work by re-expanding the vector d_1, d_2, explicitly. We shall denote the dual NNA term by d_n^{NNA*}, the $\overline{MS}$ scheme with $\mu^2 = Q^2$ is assumed,

$$
\begin{aligned}
d_1 &= -0.115 N_f + 0.655 N_c \\
d_1^{NNA} &= -0.115 N_f + 0.643 N_c \\
d_1^{NNA*} &= -0.119 N_f + 0.655 N_c \ ,
\end{aligned}
\tag{115}
$$

and

$$
\begin{aligned}
d_2 &= 0.086 N_f^2 - 1.40 N_f N_c + 2.10 N_c^2 \\
d_2^{NNA} &= 0.086 N_f^2 - 0.948 N_f N_c + 2.61 N_c^2 \\
d_2^{NNA*} &= 0.069 N_f^2 - 0.763 N_f N_c + 2.10 N_c^2 \ .
\end{aligned}
\tag{116}
$$

So we see that the weak asymptotic property holds, and the two versions of NNA give surprisingly good approximations for the sub-leading in N_f and N_c coefficients, given that this is only supposed to be an asymptotic result.

Now let us repeat this analysis for the scalar $\widetilde{D}_S$-function of Eq.(67). We find

$$
\begin{aligned}
d_2 &= -1.91 N_f + 17.19 N_c \\
d_2^{NNA} &= -1.91 N_f + 10.49 N_c \\
d_2^{NNA*} &= -3.13 N_f + 17.19 N_c \ ,
\end{aligned}
\tag{117}
$$

and

$$
\begin{aligned}
d_3 &= 0.93 N_f^2 - 21.25 N_f N_c + 72.08 N_c^2 \\
d_3^{NNA} &= 0.93 N_f^2 - 10.22 N_f N_c + 28.10 N_c^2 \\
d_3^{NNA*} &= 2.38 N_f^2 - 26.21 N_f N_c + 72.08 N_c^2
\end{aligned}
\tag{118}
$$

So we see that whilst the dual NNA term works reasonably well, the standard NNA in which we are interested yields the subleading in N_f coefficients with correct sign, but significantly reduced accuracy. For the coefficients of more physically-interesting quantity R_S of Eq.(71), which as we saw from the analysis of Sec.2.8, is perturbatively better behaved, we find

$$
\begin{aligned}
s_2 &= -1.36 N_f + 11.98 N_c \\
s_2^{NNA} &= -1.36 N_f + 7.47 N_c \\
s_2^{NNA*} &= -2.18 N_f + 11.98 N_c \ ,
\end{aligned}
\tag{119}
$$

and

$$
\begin{aligned}
s_3 &= 0.26 N_f^2 - 8.59 N_f N_c + 18.24 N_c^2 \\
s_3^{NNA} &= 0.26 N_f^2 - 2.85 N_f N_c + 7.83 N_c^2 \\
s_3^{NNA*} &= 0.60 N_f^2 - 6.63 N_f N_c + 18.24 N_c^2 \ .
\end{aligned}
\tag{120}
$$

We see that, again, the dual NNA term works quite well, but the standard version performs less satisfactorily. As in the vector case, this can be understood in terms of the operators involved [67]. In the scalar case it is the four-fermion operator $\mathcal{O}_+$ which previously dominated the large-N_c asymptotics which is vector after Fierzing and decouples. The remaining four-fermion operator $\mathcal{O}_-$ will dominate the large-N_c asymptotics, underwriting the success of the dual NNA. In the large-N_f limit, however, it turns out that $\mathcal{O}_-$ *and* the one-chain operator in the scalar case are *both* involved in determining the asymptotics, and so standard NNA will not satisfy the weak asymptotic result that sub-leading in N_f terms are reproduced. Whilst the special property of NNA which holds for the vector case will not be true for the scalar, the numerical accuracy of the approximation is not so bad (see Table 4) despite the less satisfactory performance evident from the results of Eqs.(117)-(120).

3.3   Analytic continuation of fractional powers of α_s

In this section we shall study the analytical continuation of the euclidean construct $\widetilde{D}_S(Q^2)$ introduced in (64). This is related to the quantity R_S by an analytical continuation to the minkowskian region. The continuation is essentially the same as that involved in the vector case for the analytical continuation of the Adler D-function to the QCD R-ratio. The effects of analytical continuation have been much studied [29]-[40] in attempts to improve the convergence of the QCD perturbation series by resumming an infinite subset of analytical continuation terms at each order in perturbation theory. Such resummation may be accomplished conveniently by representing the continuation as a contour integral around a circle in the complex $-Q^2$ plane (for one of the first discussion of this realization of the contour-improved technique see Ref.[34]). One can then perform the contour integration numerically , at some given order of perturbation theory. In the process one resums an infinite subset of potentially large analytical continuation terms involving powers of π^2, which arise in the running of the coupling around the circular contour. Such an expansion is termed "contour-improved" . For the case of a one-loop coupling an explicit closed-form result can be given for the contour integral. We would like to generalize these results to the present case where the mass anomalous dimension gives rise to fractional powers of α_s (recent analogous independent considerations were given in Ref.[43]). Using the NNA all-orders results for $\widetilde{D}_S$ and R_S we shall then perform various numerical studies on the performance of fixed-order perturbation theory, and its "contour-improved" version.

The analytical continuation between $\widetilde{D}$ and R of (67) and (71) can be written in the form

$$R_S(w^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \ 3[m(e^{i\theta}w^2)]^2 \left(1 + \sum_{n>0} d_n \left(\frac{\alpha_s(e^{i\theta}w^2)}{\pi}\right)^n\right) , \qquad (121)$$

involving a contour integral around a circle in the complex $w^2 = -Q^2$ plane, as mentioned above.

We can relate the running mass $m(Q^2)$ to the renormalization scheme invariant mass

$\hat{m}$ as follows (see e.g. [41, 42]):

$$m(Q^2) = \hat{m} \, \exp\left[-\int^{\alpha_s(Q^2)} \frac{\gamma_m(x)}{\beta(x)} dx + \frac{\gamma_0}{\beta_0} ln(2\beta_0) \right] \; , \tag{122}$$

where the second term in the exponent is the commonly used normalization of the definition of the invariant mass $\hat{m}$. Since we shall be working within the NNA procedure, we shall set $\gamma_i = 0, (i > 0)$ and $\beta_i = 0 (i > 0)$. In this approximation one has

$$[m(Q^2)]^2 = \hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0} \left(\frac{\alpha_s(Q^2)}{\pi} \right)^{2\gamma_0/\beta_0} \; . \tag{123}$$

Inserting this expression for the running mass into (121) one arrives at

$$R_S(w^2) = 3\hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left(\frac{\alpha_s(e^{i\theta}w^2)}{\pi} \right)^{2\gamma_0/\beta_0} \left(1 + \sum_{n>0} d_n \left(\frac{\alpha_s(e^{i\theta}w^2)}{\pi} \right)^n \right) \; . \tag{124}$$

The contour-improved expansion is obtained by performing the integration term-by-term. For a one-loop beta-function, appropriate for the NNA approximation, one has

$$\alpha_s(e^{i\theta}w^2) = \frac{\alpha_s(w^2)}{[1 + i\beta_0 \theta \alpha_s(w^2)/\pi]} \; , \tag{125}$$

and so d_n in the "contour-improved" NNA expansion will be multiplied by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{[\alpha_s(w^2)/\pi]^{2\gamma_0/\beta_0 + n}}{[1 + i\beta_0 \theta \alpha_s(w^2)/\pi]^{2\gamma_0/\beta_0 + n}} \equiv \left(\frac{\alpha_s(w^2)}{\pi} \right)^{2\gamma_0/\beta_0} A_n(\alpha_s(w^2)) \; . \tag{126}$$

The function $A_n(\alpha_s)$ is given in closed form by

$$A_n(\alpha_s) = \frac{1}{\beta_0 \delta_n \pi} \left(1 + \beta_0^2 \alpha_s^2 \right)^{\delta_n/2} \left(\frac{\alpha_s}{\pi} \right)^{n-1} \sin\left[\delta_n \arctan\left(\beta_0 \alpha_s\right)\right] \; , \tag{127}$$

where $\delta_n \equiv 1 - n - 2\gamma_0/\beta_0$. For $\delta_n \to 0$ this reproduces the well-known factor $(1/\pi\beta_0)\arctan(\beta_0\alpha_s)$ obtained by resumming all the analytical continuation terms only involving β_0 for the e^+e^- R-ratio, while for $n = 0$ the expansion of the r.h.s. of Eq.(127) to first order in $\alpha_s = \pi/(\beta_0 ln(w^2/\Lambda^2))$ coincides with the result previously obtained in [4]. Finally we can write down two expansions for R with NNA,

$$R_S = 3\hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0} \left(\frac{\alpha_s(w^2)}{\pi} \right)^{2\gamma_0/\beta_0} \left(1 + \sum_{n>0} s_n^{NNA} \left(\frac{\alpha_s(w^2)}{\pi} \right)^n \right) \; , \tag{128}$$

or, alternatively, the "contour-improved" NNA expansion,

$$R_S = 3\hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0} \left(\frac{\alpha_s(Q)}{\pi} \right)^{2\gamma_0/\beta_0} \left(A_0^{NNA}(\alpha_s(w^2)) + \sum_{n>0} d_n^{NNA} A_n^{NNA}(\alpha_s(w^2)) \right) \; . \tag{129}$$

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The A_n^{NNA} for $n > 1$ are defined from (127) on setting $\delta_n = 1 - n$. For $n = 0$ one needs to be careful. The NNA terms are of the form $\beta_0^{i-1}\alpha_s^i$, with $n = 0$ one needs to isolate the terms linear in γ_0 that arise on expanding (127) in powers of α_s. One finds

$$A_0^{NNA}(\alpha_s) = 1 - \frac{\gamma_0}{\beta_0}\ln(1 + {\beta_0}^2{\alpha_s}^2) - \frac{2\gamma_0}{{\beta_0}^2\alpha_s}\arctan(\beta_0\alpha_s) + \frac{2\gamma_0}{\beta_0} . \tag{130}$$

Note that this term contains *all* contributions depending on the anomalous dimension γ_0. The remaining A_n^{NNA} for $n > 0$ are precisely the same as the functions which arise in the case of the e^+e^- R-ratio. The $n = 1$ case corresponds to $\delta_n \to 0$ and so one has the well-known arctan alluded to earlier,

$$A_1^{NNA}(\alpha_s) = \frac{1}{\pi\beta_0}\arctan(\beta_0\alpha_s) . \tag{131}$$

For $n > 1$ the A_n^{NNA} are in fact simple rational functions of α_s/π. One has, for instance,

$$A_2^{NNA}(\alpha_s) = \frac{(\alpha_s/\pi)^2}{1 + \beta_0^2\alpha_s^2}$$

$$A_3^{NNA}(\alpha_s) = \frac{(\alpha_s/\pi)^3}{(1 + \beta_0^2\alpha_s^2)^2}$$

$$A_4^{NNA}(\alpha_s) = \left(\left(\frac{\alpha_s}{\pi}\right)^4 - \frac{\pi^2\beta_0^2}{3}\left(\frac{\alpha_s}{\pi}\right)^6\right)/(1 + \beta_0^2\alpha_s^2)^3 . \tag{132}$$

3.4 Scheme dependence of NNA results

Before proceeding to some numerical studies we need to confront one further important subtlety. As we have defined them the NNA expansions in (128) and (129) are scheme-dependent. Of course, we expect the partial sums to be scheme-dependent. The problem is that the *all-orders* sum for R_S will depend on our choice of renormalization scale. Since R_S is a physical quantity this is clearly undesirable. Following other similar analyses for the vector correlator [36, 66, 69, 70] we will use in the next section a Borel sum of the divergent series to define the all-orders sum, using regulation to cope with the IR renormalon contributions. The all-orders sum so defined can then be compared with fixed-order perturbation theory partial sums to obtain an estimate of the likely effect of uncalculated higher-order corrections. The problem is that the all-orders (Borel) sum of the series in (128) combined with the fractional power of $(\alpha_s/\pi)^{2\gamma_0/\beta_0}$ depends on the renormalization scale used for α_s. The difficulty is the fractional power of α_s involving $1/\beta_0$. For illustrative purposes suppose that we used the so-called V-scheme (see e.g. Ref.[49]) corresponding to $\overline{MS}$ with $\mu^2 = e^{-5/3}w^2$ rather than $\mu^2 = w^2$. Writing α_s^V and $\alpha_s^{\overline{MS}}$ for the two scale choices, we have, assuming a one-loop beta-function,

$$\left(\frac{\alpha_s^{\overline{MS}}}{\pi}\right)^{2\gamma_0/\beta_0} = \left(\frac{\alpha_s^V}{\pi}\right)^{2\gamma_0/\beta_0}\left[1 + \frac{5}{3}\beta_0\frac{\alpha_s^V}{\pi}\right]^{-2\gamma_0/\beta_0}$$

$$= \left(\frac{\alpha_s^V}{\pi}\right)^{2\gamma_0/\beta_0}\left[1 - \frac{10}{3}\gamma_0\left(\frac{\alpha_s^V}{\pi}\right) + \frac{2\gamma_0}{\beta_0}\left(\frac{2\gamma_0}{\beta_0} + 1\right)\frac{25}{18}{\beta_0}^2\left(\frac{\alpha_s^V}{\pi}\right)^2 + \ldots\right] . \tag{133}$$

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Recalling that the NNA terms have the structure $\beta_0^{i-1}\alpha_s^i$ it is clear that only the terms linear in γ_0 in the expansion in the second line, appear in NNA. Thus, since not all terms involved in the change of scheme are retained, the resummed NNA expansions will be scheme-dependent. The resolution of this problem is to avoid powers of α_s involving $1/\beta_0$. This can be accomplished by identifying an effective charge $\hat{R}$ related to R by [57, 68],

$$R_S = 3\hat{m}^2 [2\beta_0 \hat{R}]^{2\gamma_0/\beta_0} . \tag{134}$$

$\hat{R}$ will have the perturbative expansion

$$\hat{R} = \left(\frac{\alpha_s(w^2)}{\pi}\right)\left(1 + \sum_{n>0} \hat{s}_n \left(\frac{\alpha_s(w^2)}{\pi}\right)^n\right) . \tag{135}$$

Since this only involves integer powers of α_s all terms involved in a change of scheme at the one-loop level now contribute to the NNA result, and the resummed NNA expansion will be scheme-independent.

$$R_S = 3\hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0}\left[\left(\frac{\alpha_s(w^2)}{\pi}\right)\left(1 + \sum_{n>0} \hat{s}_n^{NNA} \left(\frac{\alpha_s(w^2)}{\pi}\right)^n\right)\right]^{2\gamma_0/\beta_0} , \tag{136}$$

is a reformulation of NNA for R_S in which the resummed series is scheme-independent. The unsatisfactory "scheme-dependent" version in (128) follows from it if only the terms linear in γ_0 are retained in expanding the series in $\hat{s}_i^{NNA}$. Writing $S^{2\gamma_0/\beta_0} = \exp[(2\gamma_0/\beta_0)\ln(S)]$ and expanding the exp to $O(\gamma_0)$ one arrives at

$$1 + \sum_{n>0} s_n^{NNA} \left(\frac{\alpha_s(w^2)}{\pi}\right)^n = 1 + \frac{2\gamma_0}{\beta_0}\ln\left[1 + \sum_{n>0} \hat{s}_n^{NNA} \left(\frac{\alpha_s(w^2)}{\pi}\right)^n\right] , \tag{137}$$

which relates the two versions of NNA for R_S. Using this result one can rewrite the reformulated expansion of (136) in terms of the s_i^{NNA}, (128) is replaced by,

$$R_S = 3\hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0}\left(\frac{\alpha_s(w^2)}{\pi}\right)^{2\gamma_0/\beta_0} \exp\left[\sum_{n>0} s_n^{NNA} \left(\frac{\alpha_s(w^2)}{\pi}\right)^n\right] . \tag{138}$$

We can immediately write a contour-improved version which replaces (129),

$$R_S = 3\hat{m}^2 (2\beta_0)^{2\gamma_0/\beta_0}\left(\frac{\alpha_s(w^2)}{\pi}\right)^{2\gamma_0/\beta_0} \exp\left[A_0^{NNA}(\alpha_s(w^2)) - 1 + \sum_{n>0} d_n^{NNA} A_n^{NNA}(\alpha_s(w^2))\right] . \tag{139}$$

Using the Borel Sum to resum the series in the exponent in (138) and combining with the fractional power of α_s we will now obtain an all-orders result for R_S which is independent of renormalization scale, as required since R_S is a physical quantity.

3.5 Numerical studies on the convergence of the NNA results

We shall now perform some numerical studies on the reformulated NNA expansions of (138) and (139). We shall consider the partial sums $R_{\overline{\text{MS}}}^{(n)}$ and $R_V^{(n)}$ obtained by summing the series in the exponent in (138) up to and including the s_n^{NNA} term, in the $\overline{\text{MS}}$ and V schemes, respectively. The prefactor $3\hat{m}^2(2\beta_0)^{2\gamma_0/\beta_0}$ is set to unity. We shall also consider the analogous partial sums of the contour-improved expansion in (139), $R_{\overline{MS}}^{(n)CI}$ and $R_V^{(n)CI}$. In Table 5 we begin by displaying the s_n^{NNA} coefficients in the $\overline{\text{MS}}$ scheme, and the V-scheme. We assume $N_f = 5$ active flavours of quarks. We shall use these two schemes to illustrate some of the scheme-dependence subtleties discussed in the last Section. In contrast to the "plateau of tranquility" evident in the limited growth of the $\overline{\text{MS}}$ scheme coefficients for $n < 7$, which was alluded to in Sec. 2.9, we see that the corresponding V-scheme coefficients grow much more rapidly. Alternating sign growth is evident even in low orders reflecting the asymptotic alternating factorial growth contributed by the leading UV renormalon singularity at $\delta = -1$ contained in the Borel transform $G_-(\delta)$ in (44). In the $\overline{\text{MS}}$ scheme this behaviour is temporarily screened in low orders by the $\exp(5\delta/3)$ factor in (53), which is absent in the V-scheme. In Table 6 we show the partial sums for the choice of coupling $\alpha_s^{\overline{\text{MS}}} = 0.114$, ($\alpha_s^V = 0.12895$) appropriate for Higgs width determination ($N_f = 5$ is assumed). As can be seen convergence is rapid for all the expansions. Of course, this is only temporary since the series is asymptotic, and for sufficiently large orders the alternating factorial growth of coefficients due to the leading ultra-violet renormalon will be evident. The V-scheme leads to slightly faster convergence than $\overline{\text{MS}}$. We show the s_n^{NNA} coefficients in the $\overline{\text{MS}}$ and V-schemes up to $n = 10$. Further note that for $n > 5$ the partial sums in the two schemes are in complete agreement to the number of significant figures quoted, emphasising the scheme-invariance of the resummed expansions in (138) and (139). In Table 7 the partial sums for $\alpha_s^{\overline{MS}} = 0.3$, ($\alpha_s^V = 0.46736$) appropriate for the strange quark mass determination are recorded ($N_f = 3$ is assumed). As one would anticipate the convergence is much less impressive. Further the analytic continuation terms are much more important at this larger value of the coupling, and the agreement of the results in the two schemes, and the apparent convergence is much more evident for the contour-improved expansion (139). This fact supports the application of the contour-improved NNLO expansions for the extraction of the s-quark mass value from the Cabibbo suppressed τ-decay mode directly in the $\overline{\text{MS}}$-scheme [71] and within a realization of the effective charges approach [43]. To emphasise the scheme-dependence of the all-orders sum of the "conventional" NNA expansions in (128) and (129) we tabulate the corresponding partial sums in Table 8, for $\alpha_s^{\overline{\text{MS}}} = 0.114$. The partial sums in the two schemes are clearly converging towards two different results, 0.04134 in the $\overline{\text{MS}}$ scheme and 0.04039 in the V-scheme. The difference ≈ 0.001 is of order $(\alpha_s/\pi)^2$, as one would anticipate from (133).

We can compare the partial sums in Tables 6-8 with the all-orders Borel sum of the series based on the Borel transform in (53). IR renormalons require regulation since they contribute singularities on the positive real axis in the Borel plane. In common with similar numerical studies on the vector correlator [36, 66, 69, 70] we shall take a Cauchy

Principal Value (PV). In the V-scheme we then have

$$\sum_{n>0} s_n^{NNA,V}\left(\frac{\alpha_s^V}{\pi}\right)^n \equiv PV\int_0^\infty d\delta\; e^{-\delta\pi/(\beta_0\alpha_s^V)}\left[\frac{\sin\pi\delta}{\pi\delta}\,2\left(G_+(\delta)+G_-(\delta)+\frac{\gamma_0}{\delta}\right)-\frac{2\gamma_0}{\delta}\right],$$
(140)

where the UV and IR renormalon contributions $G_-(\delta)$ and $G_+(\delta)$ are given by (44) and (45) respectively. Writing the 'sin' as a sum of complex exponentials and using partial fractions the separate UV and IR renormalon contributions can be expressed [66] in terms of generalized exponential integral functions $Ei(n,w)$, defined for $Re\,w > 0$ by

$$Ei(n,w) = \int_1^\infty dt\frac{e^{-wt}}{t^n}\;.$$
(141)

One finds

$$\begin{aligned}
\sum_{n>0} s_n^{NNA,V}\left(\frac{\alpha_s^V}{\pi}\right)^n &=& A_0^{NNA}(\alpha_s^V)-1+4A_1^{NNA}(\alpha_s^V)\\[2mm]
&& -\;\frac{4}{3\pi\beta_0}\sum_{n>0}\frac{(-1)^n}{n^2}[\phi_+(1,n)+\phi_+(2,n)-\phi_-(1,n)-\phi_-(2,n)]\\[2mm]
&& +\;\frac{1}{\pi\beta_0}\left[\frac{16}{3}\phi_-(1,1)-\frac{7}{3}\phi_-(1,2)+\frac{4}{3}\phi_-(2,1)-\frac{4}{3}\phi_-(2,2)\right],
\end{aligned}$$
(142)

where ϕ_+ and ϕ_- are defined by [66]

$$\begin{aligned}
\phi_+(p,q) &=& e^{q\pi/(\beta_0\alpha_s^V)}(-1)^q\,Im\left[Ei\left(p,\frac{q\pi}{\beta_0\alpha_s^V}+i\pi\right)\right]\\[2mm]
\phi_-(p,q) &=& e^{-q\pi/(\beta_0\alpha_s^V)}(-1)^q\,Im\left[Ei\left(p,-\frac{q\pi}{\beta_0\alpha_s^V}-i\pi\right)\right]\\[2mm]
&& -\;\frac{e^{-q\pi/(\beta_0\alpha_s^V)}(-1)^q\pi}{(p-1)!}\left(\frac{q}{\beta_0}\right)^{p-1}Re\left[\left(\frac{\pi}{\alpha_s^V}+i\pi\beta_0\right)^{p-1}\right].
\end{aligned}$$
(143)

Including 1000 terms in the sum over ϕ_+ and ϕ_- in (142) gives a result accurate to five significant figures. Exponentiating and evaluating R from (138) then yields the values indicated "PV" in the last row of Tables 6 and 7. The all-orders Borel sum in (140) is scheme-dependent by itself, but on exponentiating it and combining with the factor of $(\alpha_s/\pi)^{2\gamma_0/\beta_0}$ in (138) one obtains a scheme-independent result by construction. In the $\overline{\text{MS}}$ scheme the Borel transform in (140) has an extra factor $e^{5\delta/3}$. At the smaller value of the coupling $\alpha_s^{\overline{\text{MS}}}=0.114$ in Table 6 one sees that for $n>4$ the partial sums are all in good agreement with the exponentiated PV Borel result. The partial sums are in fact stable to four significant figures up to $n\approx40$ where violent oscillations due to the leading UV renormalon will be evident. The situation is somewhat less stable at the larger value of $\alpha_s^{\overline{\text{MS}}}=0.3$ in Table 7, with oscillations due to the leading UV result clearly visible in the V-scheme. For $n>4$ the $\overline{\text{MS}}$ result is in reasonable agreement with the exponentiated PV Borel sum, oscillations become evident for $n>9$, the V-scheme results break down for $n>7$. The contour-improved expansion results are stable but somewhat smaller

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than those obtained with the standard perturbative expansion. On the evidence of these numerical comparisons one would anticipate that, even at the larger value of the coupling, fixed-order perturbation theory at NNLO ($n = 3$), the level of exact calculation at present, would give a reasonable approximation. Finally, in Table 8 we give in the last row the *PV* Borel sums in the $\overline{\text{MS}}$ and V schemes. We see that these values are in good agreement with the respective scheme-dependent "conventional NNA" partial sums for $n > 4$.

3.6 Effective charges from the scalar correlator

In Sec.3.1 we discussed the problem of estimates of the higher-order QCD corrections to the spectral function of QCD sum rules in the scalar channel and to the decay width of a Standard Electroweak Model Higgs boson to quark-antiquark pairs within the NNA approach. In fact this problem was already analysed a few years ago [23] using a variant of the procedure developed in Refs.[21, 63], which is based on application of the effective charges approach of Ref.[57]. Usually this approach is applied to the renormalization scheme-dependent expansions of the quantities, which satisfy the renormalization group equations without anomalous dimension terms (see e.g. Eq.(66)). However, the quantities of Eqs.(67),(71),(91) we are interested in obey the renormalization group equations with anomalous mass dimension function (see e.g. Eq.(49)), which arises due to the factor of two powers of the running quark mass appearing in the Born approximation of their expansions. The appearance of the anomalous dimension function in the corresponding renormalization group equations reflects the scale-scheme dependence of the running quark mass and it generates the additional scheme-dependence of the perturbative series under investigation (in contrast to the familiar e^+e^- annihilation R-ratio, the scheme-dependence of R_S is starts from the α_s/π-term). This additional scheme-dependence is analogous to the factorization scale-scheme dependence of the moments of deep-inelastic scattering structure functions (for some related discussions see Sec.3.4). In the process of the effective-charges motivated studies of Ref.[23] the careful treatment of this "factorization-like" scale-scheme ambiguity of the definition of running mass was overlooked. To overcome this shortcoming one should define the related effective charges either using the representation of Eq.(123) of Sec.3.3 [57, 68] or defining the logarithmic derivative of the quantities under consideration [57].

In this section we shall follow the latter prescription and consider the euclidean construct R_D in (65) which is the unique combination of first and second derivatives of the high-energy scalar correlator that satisfies a renormalization-group equation of the form (66) and gives $R_D = \alpha_s/\pi + O(\alpha_s^2)$. Thus it provides the simplest way of defining an effective coupling

$$\frac{\widetilde{\alpha}_s(Q^2)}{\pi} \equiv R_D(Q^2) \equiv -\frac{1}{2} \frac{d \log \widetilde{D}_S(Q^2)}{d \log Q^2} = \sum_{n \geq 0} r_n \left(\frac{\alpha_s(Q^2)}{\pi} \right)^{n+1} \tag{144}$$

in the scalar channel. This leads to an effective beta function

$$\widetilde{\beta}(\widetilde{\alpha}_s(Q)) \equiv \frac{d \log R_D(Q^2)}{d \log Q^2} = -\sum_{n \geq 0} \widetilde{\beta}_n \left(\frac{\widetilde{\alpha}_s(Q^2)}{\pi} \right)^{n+1} \tag{145}$$

33

whose coefficients $\widetilde{\beta}_n$ are scheme-independent combinations of the coefficients β_n of the $\overline{\text{MS}}$ beta function and the coefficients r_n of the $\overline{\text{MS}}$ expansion of R_D. Clearly $\widetilde{\beta}_n$ may differ from β_n only for $n > 1$. To simplify the presentation, we define $c_n \equiv \beta_n / \beta_0$. Then for $n > 1$ we have

$$\frac{\widetilde{\beta}_n - \beta_n}{n-1} = \beta_0 (r_n - \Omega_n) \tag{146}$$

where Ω_n is determined by products of elements of $\{r_k, c_k \mid k < n\}$ with weights summing to n. In particular

$$\Omega_2 = r_1(c_1 + r_1) \tag{147}$$

$$\Omega_3 = r_1\left(c_2 - \frac{1}{2}c_1 r_1 - 2r_1^2 + 3r_2\right) \tag{148}$$

$$\Omega_4 = r_1\left(c_3 - \frac{4}{3}c_2 r_1 + \frac{2}{3}c_1 r_2 + \frac{14}{3}r_1^3 - \frac{28}{3}r_1 r_2 + 4r_3\right)$$
$$+ \frac{1}{3}\left(c_2 r_2 - c_1 r_3 + 5r_2^2\right) \tag{149}$$

to 5 loops (see [21, 63]).

The estimation method based on this effective charge is $r_4 \approx \Omega_4$. It was motivated by the assumption that in electron-positron annihilation and deep-inelastic scattering sum rules such scheme-independent process-dependent effective β functions will behave in a way that is broadly similar to the $\overline{\text{MS}}$ β function, with $\widetilde{\beta}_n$ having the same sign and magnitude as β_n at 3 and 4 loops. The calculation of Ref.[13] confirmed this assumption (for more detailed comparison of the behaviour of the effective charges β-functions for these processes with the $\overline{\text{MS}}$ β-function at the 4-loop level see Ref.[56]).

Note however that the effective coupling $R_D(Q^2)$ contains the spurious IR renormalon at $\delta = 1$, because it was constructed from the first and second derivatives of the correlator, with the first requiring UV subtraction for finite m^2/Q^2. The only way that massless perturbation theory can tell us that we did something that would be illegal in the massive theory is to make the massless perturbation explode factorially, leaving an intrinsic ambiguity of order Λ^2/Q^2 at the point where the sign-constant asymptotic series becomes senseless. We emphasize that this Λ^2/Q^2 effect is profoundly perturbative and should not be confused with suggestions of Λ^2/Q^2 effects beyond those expected from the OPE [72]. In our present case we know precisely why $\widetilde{D}_S(Q^2)$ is sick: the affliction results from perturbative malpractice, in failing to remove UV infinities at finite mass. If one follows the good practice of [42], by taking the second derivative, then the spurious $\delta = 1$ renormalon disappears.

It is therefore instructive to compare the performance of the Ω estimator with its obvious alternative $\overline{r}_4 \approx \overline{\Omega}_4$ based on the effective charge

$$\overline{R}_D(Q^2) \equiv -\frac{1}{2}\frac{d \log \overline{D}_S(Q^2)}{d \log Q^2} = \sum_{n \geq 0} \overline{r}_n \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{n+1} \tag{150}$$

where $\overline{r}_n$ is obtained by replacing d_n in (96)-(100) by $\overline{d}_n$ from (91). This effective charge is the unique choice formed from the second and third derivatives of the correlator and is

hence the simplest that is free from the spurious renormalon. To compute $\overline{\Omega}_n$ exactly one merely replaces r_n by $\overline{r}_n$ in (147)–(149).

In Tables 9 and 10 we compare the estimators Ω and $\overline{\Omega}$ with exact results at 3 and 4 loops. In Table 9 we also give the naive nonabelianizations of r_n and Ω_n. Those for $\overline{r}_n$ and $\overline{\Omega}_n$ are in Table 10. Here

$$r_n^{\text{NNA}} = \frac{n}{2}\beta_0^n\widetilde{\Delta}_{n+1} \tag{151}$$

$$\overline{r}_n^{\text{NNA}} = \frac{n}{2}\beta_0^n\overline{\Delta}_{n+1} \tag{152}$$

are obtained by $N_f \to N_f - 33/2$ in (101) and its corresponding version for the effective charge $\overline{R}_D(Q^2)$. The NNA expressions for the estimators Ω are

$$\Omega_2^{NNA} = \left(r_1^{NNA}\right)^2 \tag{153}$$

$$\Omega_4^{NNA} = 3r_2^{NNA}r_1^{NNA} - 2\left(r_1^{NNA}\right)^3$$

$$\Omega_4^{NNA} = 4r_3^{NNA}r_1^{NNA} - \frac{28}{3}r_2^{NNA}\left(r_1^{NNA}\right)^4 + \frac{14}{3}\left(r_1^{NNA}\right)^4 + \frac{5}{3}\left(r_2^{NNA}\right)^2$$

or

$$\Omega_2^{NNA} = \frac{1}{4}\beta_0^2\widetilde{\Delta}_2^2 \tag{154}$$

$$\Omega_3^{NNA} = \frac{1}{4}\beta_0^3\left(6\widetilde{\Delta}_3\widetilde{\Delta}_2 - \widetilde{\Delta}_2^3\right)$$

$$\Omega_4^{NNA} = \beta_0^4\left(3\widetilde{\Delta}_4\widetilde{\Delta}_2 - \frac{7}{3}\widetilde{\Delta}_3\widetilde{\Delta}_2^2 + \frac{7}{24}\widetilde{\Delta}_2^4 + \frac{5}{3}\widetilde{\Delta}_3^2\right)$$

while the ones for $\overline{\Omega}$ can be obtained by r_i^{NNA} to $\overline{r}_i^{NNA}$ in Eqs.(153) or $\widetilde{\Delta}_n$ to $\overline{\Delta}_n$ in Eqs.(154). For example, the analytical form of the NNA expressions for estimators Ω_4 and $\overline{\Omega}_4$ read

$$\Omega_4^{\text{NNA}} = \beta_0^4\left(\frac{80}{3}[\zeta(3)]^2 + \frac{44}{9}\zeta(3) + \frac{124927}{324}\right) \tag{155}$$

$$\overline{\Omega}_4^{\text{NNA}} = \beta_0^4\left(\frac{80}{3}[\zeta(3)]^2 + \frac{668}{9}\zeta(3) + \frac{33463}{324}\right) \tag{156}$$

They result from neglect of c_k in (149), since $c_k = O(N_f^{k-1})$, at large N_f, while r_k and $\overline{r}_k$ are $O(N_f^k)$.

In Table 9 we present results using the Ω estimator. The comparison with exact results at 3 and 4 loops is not so good as in the QCD and QED studies of Refs.[21, 63] correspondingly. At $N_f = 5$, for example, we have $r_2/\Omega_2 \approx 0.702$, while $r_3/\Omega_3 \approx 2.34$, so the exact growth factor r_3/r_2 is $2.34/0.702 \approx 3.33$ times that estimated by Ω. Considering the NNA approximations for r_n and Ω_n we get the similar results. Thus effective-charge analysis, based on both Ω and Ω^{NNA}, give the results over factor 3 larger than the growth factor estimated by (151).

At $N_f = 5$, this gives $r_2^{\text{NNA}}/r_2 \approx 0.60$ and $r_3^{\text{NNA}}/r_3 \approx 0.59$, with an actual growth factor r_3/r_2 only 2% greater than that indicated at large-β_0. This is in accord with the global pattern of Table 4, which shows that NNA gives a reasonable account of growth from 3 to 4 loops in all 3 quantities that were considered there. Now we turn attention to the final 3 rows of Table 9. The values of Ω_4 in the final row are exact, since by definition they entail only input from lower orders of perturbation theory. The value of r_4 is quite unknown: the Ω method takes Ω_4 as its estimate. The smaller values of r_4^{NNA} are simply obtained by replacing N_f by $N_f - 33/2$ in the exact large-N_f result for r_4. The intermediate values of Ω_4^{NNA} come from (155). The N_f-independent ratio $r_4^{\text{NNA}}/\Omega_4^{\text{NNA}} \approx 0.46$ is a precise measure of the uncertainty of the Ω estimator at large N_f. The N_f-dependent ratio $\Omega_4^{\text{NNA}}/\Omega_4 \approx 0.5$ is a precise measure of the uncertainty of NNA for $N_f = 3, 4, 5$. These two effects lead to a factor of 4 difference between the estimators r_4^{NNA} and Ω_4, as rival candidates for r_4.

Now we turn to Table 10, where the $\overline{\Omega}$ method fails to get the sign right for r_3, at 4 loops. Inspecting the final 3 rows, we see that the $\overline{\Omega}$ estimator is 10^3 times its target at large N_f. On the other hand, the NNA estimate (152) is very successful. In view of the these failings of $\overline{\Omega}$, we proceed only with the Ω estimator.

To convert a prediction of r_4 into one for the 5-loop term s_4 in the physically relevant imaginary part, we use the known terms in the quartic

$$
\begin{aligned}
r_4 - \gamma_4 - 2\beta_0 s_4 &= 37136.85285 - 7810.216455 N_f + 575.3282994 N_f^2 \\
&\quad - 16.32062026 N_f^3 + 0.1444281007 N_f^4
\end{aligned}
\tag{157}
$$

and parametrize errors in the Ω estimator by $r_n/\Omega_n = 1 + \delta_n$. Then we obtain

$$
\begin{aligned}
s_4 &= \tilde{s}_4 - \gamma_4/2\beta_0 & (158) \\
\tilde{s}_4(N_f = 3) &= 472 + 4574\delta_4 & (159) \\
\tilde{s}_4(N_f = 4) &= 146 + 3529\delta_4 & (160) \\
\tilde{s}_4(N_f = 5) &= -129 + 2615\delta_4 & (161)
\end{aligned}
$$

with central values that are small compared with a realistic estimate of the uncertainty, bearing in mind that $\delta_2 \approx -0.3$ and $\delta_3 \approx 1.3$, at $N_f = 5$.

As an indication of the problem at 3 and 4 loops, we give the 4-loop effective beta function in terms of $\tilde{a}_s \equiv R_D$ at $N_f = 3, 4, 5$:

$$
\begin{aligned}
\tilde{\beta}(N_f = 3) &= -2.250\tilde{a}_s - 4.000\tilde{a}_s^2 + 58.920\tilde{a}_s^3 - 2148.503\tilde{a}_s^4 & (162) \\
\tilde{\beta}(N_f = 4) &= -2.083\tilde{a}_s - 3.208\tilde{a}_s^2 + 53.852\tilde{a}_s^3 - 1687.191\tilde{a}_s^4 & (163) \\
\tilde{\beta}(N_f = 5) &= -1.917\tilde{a}_s - 2.417\tilde{a}_s^2 + 49.356\tilde{a}_s^3 - 1303.490\tilde{a}_s^4 & (164)
\end{aligned}
$$

with a sign change at 3 loops as a result of the appearance of a large and positive 3-loop coefficient. This pattern was already observed to occur in the effective beta-function for the minkowskian analog of R_D in Ref.[6], where it was considered as an indication of the existence of the spurious perturbative infrared fixed point. Indeed, this zero is compensated by the 4-loop terms, which remove the spurious fixed point (for the demonstration of the existence of a similar feature in the minkowskian region see Ref.[12]).

They are however $O(50)$ times larger than those in the $\overline{\text{MS}}$-scheme beta functions

$$\begin{align}
\beta(N_f = 3) &= -2.250a_s - 4.000a_s^2 - 10.060a_s^3 - 47.228a_s^4 \tag{165} \\
\beta(N_f = 4) &= -2.083a_s - 3.208a_s^2 - 6.349a_s^3 - 31.387a_s^4 \tag{166} \\
\beta(N_f = 5) &= -1.917a_s - 2.417a_s^3 - 2.827a_s^3 - 18.852a_s^4 \tag{167}
\end{align}$$

with $a_s \equiv \alpha_s/\pi$.

From this perspective, it is unsurprising that the Ω estimator performs less reliably at 3 and 4 loops than in the cases considered in Refs.[21, 63]: at 3 loops it is bound to overestimate r_2, because of the sign change of $\widetilde{\beta}_2$; at 4 loops it is bound to underestimate r_3, because $\widetilde{\beta}_3/\beta_3^{\overline{\text{MS}}} \approx 50$, whereas the procedure assumes that it is $O(1)$.

We conclude that the application of Ω estimator in the scalar channel allows values of the 5-loop coefficient s_4 between -10^3 and $+10^3$, if the accuracy is no better than $\delta_4 = 0 \pm 0.3$, i.e. no better than at 3 loops. If it is no better than at 4 loops, the range widens further, by a factor of about 3.

The appearance of large and negative estimates for s_4 in Ref.[23], which contradict the results of application of the NNA procedure, described in detail in Sec.3.1, is a reflection of the similar problem encountered in Ref.[23] in a simplified variant of the effective-charges approach in the same scalar channel.

4   Conclusions

In this paper we have extended the existing large-N_f analysis of the vector correlator [14] to the previously uninvestigated case of the scalar correlator. Because of the absence of the Ward identity $Z_1 = Z_2$ present in the vector case, and the inevitable involvement of the quark mass anomalous dimension, the analysis amd combinatorics was considerably more complicated. An all-orders large-N_f result for the anomalous mass dimension γ_m was obtained in (39,40), and thanks to the remarkable identity (20) relating insertions into two-loop skeleton diagrams to the anomalous dimension, it was possible to obtain the all-orders large-N_f result for the coefficient function of R_S in (51)-(53). As in the vector case the Borel transform of (44) and (45) contained UV and IR renormalons. A new feature of the scalar analysis was the presence of a leading IR renormalon at $\delta = 1$ in (45). This is not present in the physical quantity R_S thanks to the analytical continuation factor $\sin(\pi\delta)/\pi\delta$ in (53), and the leading singularity is the UV renormalon at $\delta = -1$. It is, however, present in the quantity $\widetilde{D}_S(Q^2)$ of (64), which would correspond to the obvious generalization of the vector Adler function , related by a singly subtracted dispersion relation to the scalar vacuum polarization Π_S. The presence of this leading IR renormalon is connected with the fact that the undetermined constant $\Pi_S(0)$ in Π_S is *infinite*, a circumstance which did not occur in the vector case thanks to the $Z_1 = Z_2$ Ward identity. A more satisfactory choice for the scalar Adler function is therefore the twice subtracted construct $\overline{D}_S$ defined in (63). An initial survey of the growth of the coefficients S_n in (50) was given in Table 2, where it is seen that in the $\overline{\text{MS}}$ scheme for $n < 7$ there is rather stable behaviour, with rapid growth corresponding to the leading UV renormalon evident for $n > 7$. The

perturbation series for $\overline{D}_S, \widetilde{D}_S$ and its logarithmic derivative R_D are studied in Sec. 2.10. In Sec.3 we moved to a study of so-called "naive non-abelianization" (NNA) , where N_f in the large-N_f result is replaced by $N_f - 11N_c/2 = -6\beta_0$, and the pieces of perturbative coefficients containing the leading power of β_0 are resummed to all-orders. For the quantities $\overline{D}_S, \widetilde{D}_S$ and R_S the use of NNA to estimate the known three and four-loop coefficients was found to give the correct sign and order of magnitude. Moreover, the NNA-inspired estimates predict that at the five-loop level the numerical values of the corresponding coefficients are positive and not very large. In the case of the imaginary part of the scalar correlator, related to the Higgs boson decay width to quark-antiquark pairs, and to the QCD Sum Rules spectral functions, we have $s_4^{NNA}(N_f = 3) \approx 49$, $s_4^{NNA}(N_f = 4) \approx 39$, $s_4^{NNA}(N_f = 5) \approx 31$. Using the conservative estimate of an overall factor of two for the uncertainty of the NNA estimates in the scalar channel (suggested by careful comparison of the three and four-loop NNA estimates to the results of explicit calculation, see Table 4) we conclude that the NNA-inspired estimate $s_4 \approx 2s_4^{NNA}$ gives us the following numbers $s_4(N_f = 3) \sim 98$, $s_4(N_f = 4) \sim 78$, $s_4(N_f = 5) \sim 62$. For $N_f = 5$ our estimate is in good agreement with the result of previous studies, based on application of the [2/2] asymptotic Padé estimation technique [24]. However, for $N_f = 3$ the method of Ref.[24] agrees with our estimates only in sign and is over 2.6 times larger than the estimates proposed by us. In Sec. 3.2 we noted that for the vector correlator the NNA term does have some special properties, deriving from an analysis of the operators that build the leading UV renormalon singularity. On re-expansion it reproduces the sub-leading in N_f contributions to asymptotic accuracy $O(1/n)$ in n^{th} order perturbation theory. A similar result holds for a "dual-NNA" , exact in the large-N_c limit. For the scalar case a corresponding weak asymptotic result was not expected to hold, but the "dual NNA" term should still provide a good approximation. In Sec. 3.3 we considered the analytical continuation from $\widetilde{D}_S$ to R_S . We recast the running mass $m(Q^2)$ in terms of the RG-invariant mass $\hat{m}$. The analytical continuation was similar to the much studied Euclidean to Minkowskian continuation for the e^+e^- R-ratio [29-40]. The presence of an anomalous dimension complicated the analysis slightly. We arrived at the "contour-improved" expansion for R_S in (129), in which a subset of analytical continuation terms involving π^2 are resummed to all-orders, at each order of the expansion. A further subtlety due to the presence of an $\alpha_s^{2\gamma_0/\beta_0}$ term involving the anomalous dimension, was that the straightforward NNA expansions in (128) and (129) have all-orders sums which are RS-dependent. We showed how this could be remedied in Sec. 3.4 , and obtained the reformulated expansions in (138) and (139) whose all-orders sums were scheme-independent. In 3.5 we performed comparisons of fixed-order perturbation theory with the all-orders sum defined using a Cauchy principal value of the Borel sum, based on the Borel transform of (44),(45),(Tables 6-8). Two values of the coupling, $\alpha_s^{\overline{\text{MS}}} = 0.114$ and $\alpha_s^{\overline{\text{MS}}} = 0.3$, appropriate for the calculation of the Higgs decay width to a quark-antiquark pair, and for the strange quark mass determination from QCD Sum Rules, respectively, were considered. The $\overline{\text{MS}}$ scheme and V-scheme were used to illustrate the scheme-dependence issue. Even at the larger value of the coupling it seemed that satisfactory accuracy was achieved at order $n = 3$, the highest order at which exact calculations are so far available. However, from the analysis of Ref.[73] one can conclude that the uncertainties of the existing m_s extractions from the QCD sum rules for the scalar correlator might be underestimated. In view of our analysis we conclude, that these

possible additional uncertainties are not coming from the uncalculated 5-loop perturbative QCD contributions, but are mainly related to the uncertainty of the experimental model for the spectral function of the QCD sum rules in the scalar channel.

In overall conclusion we have provided a framework for extending the numerous published investigations of higher-order perturbative behaviour, analytical continuation, approximate all-orders resummation, and estimates of higher-order corrections, for the vector correlator and its derived quantities, to the much less studied case of the scalar correlator. We hope that our results may be of use in further phenomenological investigations.

5   Note Added in Proof

After this work was submitted for publication we were informed that the 3-loop corrections to the correlator of scalar currents have been calculated up to $(m^2/Q^2)^4$-terms [74]. The application of Padé resummation technique [75, 76] allowed the authors of Ref.[77] to specify the excat mass-dependence of the scalar-scalar correlator at 3-loops in a semi-analytical way. The analytical mass dependence of the 3-loop double-bubble contribution to the scalar correlator was studied in Ref.[78]. It should be stressed, however, that none of these calculations affect the results reported in this our work.

Another, more closely related investigation, was performed in Ref.[79], where the effects of the UV-renormalons to the IR-safe Adler function of Eq.(63) of the scalar correlator were studied both in the 1-renormalon and 2-renormalon chain approximations. However, this analysis was based not on the exact calculations of the renormalon chain diagrams in the large N_f-limit, but on the latge N_c-analysis of the contributing 4-fermion operators. Moreover, in Ref.[79] the analysis of the IR renormalon contributions was not considered. In view of these differencies, it would be of interest to compare our results with those of Ref.[79] in more detail.

6   Acknowledgements

additional works related to the content of our paper, after its submission for publication.

Table 1: Renormalons in (46)

n	$H_n^{\overline{\text{MS}}}$	$H_n^{(0)}$	$H_n^{(-)}$	$H_n^{(1)}$	$H_n^{(2)}$	$H_n^{(+)}$	H_n
2	19.625	-12.500	2.594	-48.000	9.750	1.043	-27.488
3	11.093	6.944	5.870	73.000	-11.125	-1.306	84.476
4	-8.684	-5.787	12.276	-173.778	18.069	2.289	-155.615
5	-0.056	5.787	45.579	556.056	-35.785	-4.764	566.817
6	1.882	-6.430	207.742	-2270.519	83.144	11.212	-1972.969
7	-0.509	7.655	1177.991	11416.893	-223.934	-29.266	12348.830
8	-0.142	-9.569	7883.559	-68593.216	694.766	83.893	-59940.709
9	0.095	12.404	60911.356	480286.469	-2465.171	-262.531	538482.621

Table 2: Renormalons in (51)

n	$S_n^{\overline{\text{MS}}}$	$S_n^{(0)}$	$S_n^{(-)}$	$S_n^{(1)}$	$S_n^{(2)}$	$S_n^{(+)}$	S_n
2	-1.667	3.333	1.097	4.000	-1.000	-0.097	5.667
3	0.972	-0.512	-0.576	10.667	-2.167	-0.232	8.152
4	0.834	-7.880	-0.999	19.285	-1.655	-0.263	9.323
5	0.055	-13.817	-2.495	10.576	9.338	0.762	4.419
6	-0.155	9.160	10.385	-68.229	46.309	5.344	2.813
7	-0.056	105.021	-17.734	-249.544	92.873	15.175	-54.266
8	0.006	200.490	183.066	-237.016	-36.445	13.543	123.645
9	0.008	-255.017	-1342.424	1147.753	-829.274	-87.712	-1366.667
10	0.001	-2257.800	9978.990	5101.747	-2297.027	-516.006	10009.905
11	0.000	-4046.531	-91303.959	5275.323	-176.520	-1500.459	-91752.148

Table 3: Contributions to (85,89,93)

n	A_n	Δ_n	Δ_n	$\overline{\Delta}_n$
2	-1.66667	7.33333	7.33333	5.33333
3	0.97222	10.4696	7.17968	3.13622
4	0.83422	32.6145	8.48885	11.6754
5	0.05531	97.9534	4.36402	0.10978
6	-0.15502	503.887	2.96849	112.074
7	-0.05583	2194.28	-54.2101	-325.157
8	0.00581	16465.8	123.639	3300.11

Table 4: Ratios of NNA estimates to exact results

	$N_f = 3$	$N_f = 4$	$N_f = 5$
d_2^{NNA}/d_2	0.514	0.496	0.477
s_2^{NNA}/s_2	0.507	0.490	0.472
$\overline{d}_2^{\text{NNA}}/\overline{d}_2$	0.498	0.484	0.469
d_3^{NNA}/d_3	0.354	0.346	0.339
s_3^{NNA}/s_3	0.482	0.565	0.747
$\overline{d}_3^{\text{NNA}}/\overline{d}_3$	0.764	0.871	1.081

Table 5: s_n^{NNA} coefficients in the $\overline{\text{MS}}$ and V-schemes

n	$\overline{\text{MS}}$	V
1	7.3333	4
2	13.761	-4.3408
3	31.184	6.7614
4	30.727	-58.738
5	40.061	456.674
6	-1402.21	-4161.77
7	6129.65	47635.6
8	-129864.7	-640093.5
9	$1.8231\ 10^6$	$9.8022\ 10^6$
10	$-3.2028\ 10^7$	$-1.6900\ 10^8$

Table 6: Partial sums of (137,138) for $\alpha_s^{\overline{\text{MS}}} = 0.114$

n	$R_{\overline{\text{MS}}}^{(n)}$	$R_V^{(n)}$	$R_{\overline{\text{MS}}}^{(n)CI}$	$R_V^{(n)CI}$
1	0.04099	0.04210	0.04049	0.04153
2	0.04174	0.04179	0.04152	0.04166
3	0.04180	0.04181	0.04180	0.04174
4	0.04181	0.04180	0.04180	0.04179
5	0.04181	0.04181	0.04180	0.04181
6	0.04181	0.04181	0.04180	0.04180
7	0.04181	0.04181	0.04181	0.04181
8	0.04181	0.04181	0.04181	0.04181
9	0.04181	0.04181	0.04181	0.04181
10	0.04181	0.04181	0.04181	0.04181
.	....	....	....	....
PV	0.04179	0.04179	0.04179	0.04179

Table 7: Partial sums of (137,138) for $\alpha_s^{\overline{\text{MS}}} = 0.3$

n	$R_{\overline{\text{MS}}}^{(n)}$	$R_V^{(n)}$	$R_{\overline{\text{MS}}}^{(n)CI}$	$R_V^{(n)CI}$
1	0.24971	0.33333	0.21619	0.25609
2	0.28934	0.29778	0.25056	0.26237
3	0.30037	0.30706	0.26816	0.27759
4	0.30162	0.29310	0.27509	0.27676
5	0.30180	0.31222	0.27853	0.27606
6	0.30109	0.28234	0.27906	0.27800
7	0.30144	0.34521	0.27765	0.26881
8	0.30061	0.21538	0.27433	0.27246
9	0.30191	0.76056	0.26774	0.27747
10	0.29934	0.01704	0.26192	0.25527
.	....	....	....	....
PV	0.30138	0.30138	0.30138	0.30138

Table 8: Partial sums of the "conventional" NNA (128,129) for $\alpha_s^{\overline{\text{MS}}} = 0.114$

n	$R_{\overline{\text{MS}}}^{(n)}$	$R_V^{(n)}$	$R_{\overline{\text{MS}}}^{(n)CI}$	$R_V^{(n)CI}$
1	0.03977	0.04159	0.03939	0.04110
2	0.04034	0.04133	0.04018	0.04121
3	0.04039	0.04135	0.04034	0.04133
4	0.04039	0.04134	0.04038	0.04133
5	0.04039	0.04134	0.04039	0.04134
6	0.04039	0.04134	0.04039	0.04134
7	0.04039	0.04134	0.04039	0.04134
8	0.04039	0.04134	0.04039	0.04134
9	0.04039	0.04134	0.04039	0.04134
10	0.04039	0.04134	0.04039	0.04134
.	....	....	....	....
PV	0.04038	0.04133	0.04038	0.04133

Table 9: Exact values of r_n and Ω_n, with NNA estimates of r_n and Ω_n

	$N_f = 3$	$N_f = 4$	$N_f = 5$
r_1	10.17	9.56	8.94
r_1^{NNA}	8.25	7.64	7.03
r_2	90.78	77.13	64.06
r_2^{NNA}	53.00	45.44	38.46
Ω_2	121.44	106.02	91.28
Ω_2^{NNA}	68.06	58.35	49.39
r_3	1091.55	822.19	585.53
r_3^{NNA}	557.25	442.36	344.46
Ω_3	620.75	424.80	250.41
Ω_3^{NNA}	188.77	149.85	116.69
r_4^{NNA}	5020.88	3690.50	2643.84
Ω_4^{NNA}	11020.06	8100.07	5802.84
Ω_4	20583.50	14704.33	10024.21

Table 10: Exact values of $\overline{r}_n$ and $\overline{\Omega}_n$, with NNA estimates of $\overline{r}_n$ and $\overline{\Omega}_n$

	$N_f = 3$	$N_f = 4$	$N_f = 5$
$\overline{r}_1$	7.92	7.47	7.03
$\overline{r}_1^{\text{NNA}}$	6.00	5.56	5.11
$\overline{r}_2$	36.53	29.94	23.52
$\overline{r}_2^{\text{NNA}}$	15.88	13.61	11.52
$\overline{\Omega}_2$	76.75	67.34	58.25
$\overline{\Omega}_2^{\text{NNA}}$	36.00	30.86	26.12
$\overline{r}_3$	233.13	138.29	55.79
$\overline{r}_3^{\text{NNA}}$	199.49	158.36	123.31
$\overline{\Omega}_3$	-145.01	-183.52	-219.13
$\overline{\Omega}_3^{\text{NNA}}$	-146.21	-116.07	-90.38
$\overline{r}_4^{\text{NNA}}$	5.63	4.14	2.96
$\overline{\Omega}_4^{\text{NNA}}$	5921.10	4352.18	3117.87
$\overline{\Omega}_4$	6619.34	4648.70	3131.62

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