On the Discontinuous Galerkin Method for Friedrichs Systems in Graph Spaces

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Abstract. Solutions of Friedrichs systems are in general not of Sobolev regularity and may possess discontinuities along the characteristics of the differential operator. We state a setting in which the well-posedness of Friedrichs systems on polyhedral domains is ensured, while still allowing changes in the inertial type of the boundary. In this framework the discontinuous Galerkin method converges in the energy norm under h- and p-refinement to the exact solution.

1 Introduction

Friedrichs systems are first-order linear boundary value problems which allow the study of a wide range of hyperbolic, parabolic and elliptic differential equations in a unified framework [1]. Because of this unifying approach, Friedrichs systems provide tools for the study of mixed-type problems, i.e. boundary value problems, which change their type depending on the position in the domain. For instance, equations in compressible gas dynamics can be transformed into Friedrichs systems, where regions of supersonic flow correspond to a locally hyperbolic differential operator, while subsonic regions correspond to a local model of elliptic type [2].

In general, the solution of a Friedrichs systems is not contained in a Sobolev space. Instead it belongs to the associated graph space, i.e. it is weakly differentiable along the charactistics of the differential operator. Functions in the graph space may be discontinuous. In addition, poles in the solution, due to type-changes of the differential operator or the boundary conditions, may lead to a loss of the integration-by-parts rule in its classical sense [3]. This is in close connection to the question of well-posedness of Friedrichs systems [4–10].

In 1973 Reed and Hill [11, 12] introduced the discontinuous Galerkin method (DGFEM) to solve the neutron transport equation. Already in this paper, numerical experiments make the good approximation and stability properties of the DGFEM for boundary value problems with discontinuous solution apparent.

Assuming shape-regularity, LeSaint and Raviart prove in [13] for meshes with triangular and quadrilateral elements the suboptimal $L^2(\Omega)$-error bound

$$\|u - u_{\mathrm{DG}}\|_{L^2(\Omega)} \leq C\, h^p\, \|u\|_{W^{p+1,2}(\Omega)}, \qquad C > 0,$$

for solutions u in $W^{p+1,2}(\Omega)$ and DG solutions u_{DG}. Johnson and his coworkers [14, 15] show for equations with non-constant coefficients and certain Friedrichs systems an improved $\mathcal{O}(h^p)$ bound in the DG energy norm. Bey and Oden [16] extend the analysis to non-uniform p. In the framework by Houston, Schwab and Süli [17, 18] the exact solution u is only required to be elementwise of Sobolev regularity. Thus u may be discontinuous along element edges.

The afore-mentioned publications have in common that their a priori analysis is restricted to solutions which are of elementwise or global Sobolev regularity. Solutions with discontinuities across elements are not covered, for which already Reed and Hill and also others [16, 18] highlighted the competitive performance of the DG method with numerical experiments.

In this publication we address the convergence of the discontinuous Galerkin method in graph spaces. We base our analysis on Friedrichs systems which allow typical changes in the inertial type of the boundary conditions such as between in- and outflow components, but which at the same time satisfy basic requirements such as the integration-by-parts formula.

2 Friedrichs Systems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $m \in \mathbb{N}$. Given a tensor $B \in W^{1,\infty}(\Omega)^{m\times m\times n}$ and a matrix $C \in L^\infty(\Omega)^{m\times m}$, we consider the differential operator

$$\mathcal{L} : v \mapsto \partial_k(B_{ijk}\, v_j) + C_{ij} v_j,$$

making use of the Einstein summation convention and assuming that v and the coefficients B and C are real-valued. We denote by ν the unit outward normal of Ω and by $B(\nu)$ the matrix $B(\nu)_{ij} = B_{ijk}\,\nu_k$. With $D_{ij} := C_{ij} + \frac{1}{2}\,\partial_k B_{ijk}$, the symmetry condition

$$B_{ijk} = B_{jik}, \qquad i, j \in \{1, \ldots, m\},\ k \in \{1, \ldots, n\}, \tag{1}$$

implies for $v, w \in H^1(\Omega)^m$

$$\langle \mathcal{L}v, w\rangle_\Omega + \langle v, \mathcal{L}w\rangle_\Omega = 2\langle Dv, w\rangle_\Omega + \langle B(\nu)\, v, w\rangle_{\partial\Omega}, \tag{2}$$

where $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ are the L^2-scalar products on Ω and $\partial\Omega$. If (1) is satisfied and there is a constant $\gamma > 0$ such that $v \cdot D\, v - \gamma v \cdot v$ is positive on Ω, we call $\mathcal{L}$ accretive. Notice that $\langle \mathcal{L}v, v\rangle_\Omega \geq \gamma\, \|v\|_{L^2(\Omega)^m}$ for $v \in H^1_0(\Omega)^m$.

Boundary operators $J : \partial\Omega \to \mathbb{R}^{m\times m}$ are semi-admissible if $R := J + \frac{1}{2}\, B(\nu)$ is positive semi-definite [1]. Then, due to (2),

$$\langle \mathcal{L}v, v\rangle_\Omega + \langle Jv, v\rangle_{\partial\Omega} = \langle Dv, v\rangle_\Omega + \langle Rv, v\rangle_{\partial\Omega} \geq \gamma\, \|v\|_{L^2(\Omega)^m} \tag{3}$$

for $v \in H^1(\Omega)^m$. Let J^T be the transpose of J. Given the formal adjoint operator $\mathcal{L}' : v \mapsto -B_{jik}\partial_k v_j + C_{ji} v_j$, the adjoint boundary operator $J' = J^\mathsf{T} + B(\nu)$ satisfies

$$\langle \mathcal{L}v, w\rangle_\Omega + \langle Jv, w\rangle_{\partial\Omega} = \langle v, \mathcal{L}'w\rangle_\Omega + \langle v, J'w\rangle_{\partial\Omega}, \qquad v, w \in H^1(\Omega)^m.$$

Let $f \in L^2(\Omega)^m$. One says that a function $u \in L^2(\Omega)^m$ solves the boundary value problem $\mathcal{L}u = f$, $Ju = 0$ weakly if for all $v \in C^1(\overline{\Omega})^m$ with $J'v = 0$ the identity

$$\langle f, v\rangle_\Omega = \langle u, \mathcal{L}'v\rangle_\Omega \tag{4}$$

holds.

Friedrichs [1] proves that a weak solution always exits if $\mathcal{L}$ is accretive and J is semi-admissible. Clearly, $\mathcal{L}u$ is equal to f in the sense of distributions. Therefore the solution u belongs to the graph space of $\mathcal{L}$. That is the set

$$H(\mathcal{L}, \Omega) := \{v \in L^2(\Omega)^m : \mathcal{L}v \in L^2(\Omega)^m\},$$

which is normed by

$$\|v\|_{\mathcal{L}}^2 := \|v\|_{L^2(\Omega)^m}^2 + \|\mathcal{L}v\|_{L^2(\Omega)^m}^2.$$

In what sense u satisfies the boundary conditions is more intricate. We assume initially that u belongs to $C^0(\overline{\Omega})^m$ to delay the definition of a trace operator. Because the test functions in (4) are contained in $\ker J'$, the test space may be too small to ensure that $Ju = 0$. We call $P_J, P_{J'} \in \mathbb{R}^{m\times m}$ a pair of projections if

$$P_J + P_{J'} = I \qquad \text{and} \qquad P_J P_{J'} = P_{J'} P_J = 0.$$

Semi-admissible boundary operators J are called admissible if for each $x \in \partial\Omega$ there is a pair of projections $P_J, P_{J'} \in \mathbb{R}^{m\times m}$ such that

$$J(x) = -P_J(x)^\mathsf{T} B(\nu, x) \quad \text{and} \quad J'(x) = B(\nu, x) P_{J'}(x).$$

Under sufficient regularity, e.g. for each $v \in C^1(\overline{\Omega})^m$ there is a $\dot{v} \in C^1(\overline{\Omega})^m$ such that $P_J v = \dot{v}$ on $\partial\Omega$, admissibility of J guarantees $Ju = 0$. Boundary value problems consisting of an accretive differential operator and admissible boundary operators are called Friedrichs systems.

The following example, which is an adaptation of [3], shows that for weak solutions the integration-by-parts formula is in general not valid.

Example 1. Let $\Omega = (0,1)^2$ and

$$\mathcal{L}v := \mathcal{L}_{\mathrm{CR}} v + v, \qquad \mathcal{L}_{\mathrm{CR}} v := \begin{pmatrix} -\partial_x & \partial_y \\ \partial_y & \partial_x \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The boundary conditions

$$J|_{x=1} := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, J|_{x=0} := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, J|_{y=1} := \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, J|_{y=0} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

are admissible. Since $\mathcal{L}_{\mathrm{CR}}$ is the Cauchy-Riemann operator and the function $v(\phi, r) := r^{-1/2}(\cos\phi/2, -\sin\phi/2)$ in polar coordinates (ϕ, r) represents the holomorphic function $z^{-1/2}$, it follows that $\mathcal{L}_{\mathrm{CR}} v = 0$ and $\mathcal{L}v = v$. Let $\psi \in$

$C^1(\overline{\Omega})$ be a radially symmetric function with support in the unit ball and which is equal to 1 in a neighbourhood of the origin. Then $u := \psi\, v \in L^2(\Omega)^m$ satisfies pointwise and weakly the homogeneous boundary conditions $Ju = 0$ on $\partial\Omega$ and $\mathcal{L}u = f$ with $f := (\psi(r) - \psi'(r))v \in L^2(\Omega)^m$.

For bounded smooth functions w which satisfy the homogeneous boundary conditions, the operator $i\mathcal{L}_{\mathrm{CR}}$ is self-adjoint and therefore $\int_\Omega \mathcal{L}_{\mathrm{CR}} w \cdot w\, dx = 0$. In contrast, $\int_\Omega \mathcal{L}_{\mathrm{CR}} u \cdot u\, dx = \pi/4$. Consequently, formula (2) is not valid for $u \in H(\mathcal{L}, \Omega)$.

The loss of the integration-by-parts formula has far-reaching implications on the analysis of the discontinuous Galerkin method. It is, for instance, used for the definition of the energy norm.

Insight why the formula fails is given by the trace operator of $H(\mathcal{L}, \Omega)$. We report relevant properties of the operator, but refer for details to [19]. The trace operator

$$\mathcal{T} : H(\mathcal{L}, \Omega) \to H^{-1/2}(\partial\Omega)^m, v \mapsto \langle B(\nu)v, \cdot\rangle$$

is bounded, but in general not surjective. We equip the trace space $H(\mathcal{T}, \partial\Omega) := \operatorname{im} \mathcal{T}$ with the norm

$$\|v\|_{\mathcal{T}} := \inf\{\|w\|_{\mathcal{L}} : w \in H(\mathcal{L}, \Omega) \text{ and } \mathcal{T}w = v\}.$$

The terminology trace operator and trace space for $\mathcal{T}$ and $H(\mathcal{T}, \partial\Omega)$ is justified by the observation that all mappings $\mathcal{J} : H(\mathcal{L}, \Omega) \to V$ which vanish on $H^1_0(\Omega)^m$ can be factorised in the form $\mathcal{J} = \dot{\mathcal{J}} \circ \mathcal{T}$ where $\dot{\mathcal{J}} : H(\mathcal{T}, \partial\Omega) \to V$ is continuous and V is any abstract normed vector space. Up to homeomorphy only $H(\mathcal{T}, \partial\Omega)$ has this property.

Example 2. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : y > |x| \text{ and } y < 1\}$ and $\mathcal{L}v = \partial_x v$. Then $y^{-1/2} \in H(\mathcal{L}, \Omega)$. The only admissible boundary conditions J with respect to $\mathcal{L}$ are inflow conditions. Yet for them, $\langle Jv, v\rangle_{\partial\Omega}$ diverges. Thus $(v, w) \mapsto \langle Jv, w\rangle_{\partial\Omega}$ is not continuous on $H(\mathcal{L}, \Omega) \times H(\mathcal{L}, \Omega)$ and (3) cannot be continuously extended from $H^1(\Omega)^m$ to $H(\mathcal{L}, \Omega)$.

The properties of the trace space are connected to the eigenvalues of $B(\nu)$. Due to (1), for each $x \in \partial\Omega$ there is an orthogonal transformation X and a diagonal matrix Λ such that $B(\nu) = X^\mathsf{T} \Lambda X$. Substituting in Λ negative entries by 0 gives Λ_+. The positive and negative semi-definite components of $B(\nu)$ are $B_+(\nu) := X^\mathsf{T} \Lambda_+ X$ and $B_-(\nu) := B(\nu) - B_+(\nu)$, respectively. The absolute part is $|B|(\nu) := B_+(\nu) - B_-(\nu)$. A change in the rank of $B_+(\nu)$ or $B_-(\nu)$ is termed a change in the inertial type of $B(\nu)$.

The space $L^2_B(\partial\Omega)$ is the set of all integrable functions $v : \partial\Omega \to \mathbb{R}^m$ for which the norm

$$\|v\|_B^2 := \int_{\partial\Omega} v \cdot |B|(\nu) v\, dx$$

is finite. The space $L^\infty_B(\partial\Omega)^{m\times m}$ consists of the matrices in $L^\infty(\partial\Omega)^{m\times m}$ which define an endomorphism on $L^2_B(\partial\Omega)$.

Traces $\langle B(\nu)v, \cdot\rangle$ contained in $H^{-1/2}(\partial\Omega)^m \setminus L^2(\partial\Omega)^m$ can arise through a coupling of in- and outflow components. Pointwise we understand under in- and outflow components the eigenspaces of $B(\nu)$ associated to negative and positive eigenvalues, respectively. For instance, in Example 2 in- and outflow boundary are coupled through the sign change of $B(\nu)$ at the origin. Traces in $H^{-1/2}(\partial\Omega)^m \setminus L^2(\partial\Omega)^m$ are not limited to domains with corners. Coupling with tangential components of $\mathcal{L}$ has comparable effects.

Example 3. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and

$$\mathcal{L}v = \partial_x \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v + \frac{\partial_y}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} v.$$

The trace space of $u \in H(\mathcal{L}, \Omega)$ is equal to

$$\{(v_1 - v_2, v_1 + v_2, 0) : v_1 \in H^{1/2}(\partial\Omega)^m \text{ and } v_2 \in H^{-1/2}(\partial\Omega)^m\}$$

with the intrinsic norm $(\|v_1\|^2_{H^{1/2}(\partial\Omega)^m} + \|v_2\|^2_{H^{-1/2}(\partial\Omega)^m})^{1/2}$.

To provide a basis for definition of the discontinuous Galerkin method for Friedrichs systems, we introduce the additional condition that there is a factorisation of $B(\nu)$ of the form $B(\nu) = (R + R^\top)F$. Then one can ensure that the solution of the Friedrichs system is unique and contained in the closure $H(\mathcal{L}, B, \Omega)$ of $H^1(\Omega)^m$ in the norm

$$(\|v\|^2_{\mathcal{L}} + \|v\|^2_B)^{1/2}, \qquad v \in H^1(\Omega)^m.$$

Functions in $H(\mathcal{L}, B, \Omega)$ have a trace in $L^2_B(\partial\Omega)$ and satisfy formula (2).

Theorem 1. *Let $\mathcal{L}$ be an accretive operator and J be semi-admissible. Suppose that there are two projections $P_1, P_2 \in L^\infty_B(\partial\Omega)^{m\times m}$ such that $J = -B(\nu)P_1$ and $J' = B(\nu)P_2$. We also adopt the hypothesis that there is an $F \in L^\infty_B(\partial\Omega)^{m\times m}$ such that $B(\nu) = (R + R^\top)F$. Then for each $f \in L^2(\Omega)^m$ and $g \in L^2_B(\partial\Omega)$ there is a unique function $u \in H(\mathcal{L}, B, \Omega)$ which solves $\mathcal{L}u = f$ and $Ju = Jg$. Furthermore, u depends continuously on f and g.*

For details we refer to [19]. We remark that P_1 and P_2 are not necessarily a pair of projections. Also note that we assume $g \in L^2_B(\partial\Omega)$ and not $g \in H(\mathcal{T}, \partial\Omega)$.

Example 4. Let H be the Heaviside function. Selecting

$$(P_1)_{ij} := X_{ki}H(\Lambda_{kk})X_{kj}, \quad (P_2)_{ij} := X_{ki}H(-\Lambda_{kk})X_{kj}, \quad F := P_2 - P_1$$

shows that inflow boundary conditions satisfy the requirements set in Theorem 1.

3 The Discontinuous Galerkin Method

Let $T = \{\kappa_1, \kappa_2, \ldots, \kappa_N\}$ be a decomposition of Ω into polyhedral elements κ_i. Suppose that all $\kappa \in T$ are an affine image of a fixed master element $\hat{\kappa}$, i.e. $\kappa = F_\kappa(\hat{\kappa})$ for all $\kappa \in T$, where F_κ is an injective affine mapping and where $\hat{\kappa}$ is either the open unit simplex or the open unit hypercube in $\mathbb{R}^n$. We denote by $\mathcal{P}_k$ the space of polynomials on $\hat{\kappa}$ with total degree less or equal k. If $\hat{\kappa}$ is the hypercube then we also consider the space $\mathcal{Q}_k$ of tensor-polynomials on $\hat{\kappa}$ with degree less or equal k in each coordinate direction. Let $p = (p_1, p_2, \ldots, p_N)$ be a vector which associates to each element κ_i the polynomial degree p_i. We consider the finite element spaces

$$S(T,p) = \{v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}\},$$

where $\mathcal{R}_k$ is either $\mathcal{P}_k$ or $\mathcal{Q}_k$.

The finite element spaces $S(T,p)$ are contained in the broken graph space

$$H(\mathcal{L}, B, T) := \bigoplus_{\kappa \in T} H(\mathcal{L}, B, \kappa).$$

At the boundary $\partial\kappa_i \cap \partial\kappa_j$ between the element κ_i and a neighbour κ_j, a member v of $H(\mathcal{L}, B, T)$ has, in general, two distinct traces: one from the restriction $v|_{\kappa_i}$ and one from $v|_{\kappa_j}$. We denote the internal trace $(v|_{\kappa_i})|_{\partial\kappa_i}$ of κ_i by v^+ and the external trace $(v|_{\kappa_j})|_{\partial\kappa_i \cap \partial\kappa_j}$ of κ_i by v^-. Altogether the external trace v^- is composed from the traces of all elements neighbouring κ_i. The difference $v^+ - v^-$ is denoted by $[v]$.

For $v, w \in H(\mathcal{L}, B, T)$ let

$$\begin{aligned} B_{\mathrm{DG}}(v,w) &= \langle \mathcal{L}v, w\rangle_\Omega + \langle Jv, w\rangle_{\partial\Omega} + \sum_{\kappa \in T} \langle B_-(\nu)[v], w^+\rangle_{\partial\kappa\setminus\partial\Omega}, \\ \ell_{\mathrm{DG}}(w) &= \langle f, w\rangle_\Omega + \langle Jg, w\rangle_{\partial\Omega}. \end{aligned}$$

The integration-by-parts formula

$$B_{\mathrm{DG}}(v,v) = \langle Dv, v\rangle_\Omega + \langle Rv, v\rangle_{\partial\Omega} + \tfrac{1}{2}\textstyle\sum_{\kappa \in T} \langle |B|(\nu)[v], [v]\rangle_{\partial\kappa\setminus\partial\Omega}$$

induces the energy norm $\|v\|_{\mathrm{DG}} := \sqrt{B_{\mathrm{DG}}(v,v)}$ on $H(\mathcal{L}, B, T)$. We remark that $H(\mathcal{L}, B, T)$ is not complete in this norm.

Let T' be another finite element mesh on Ω and suppose that $v \in H(\mathcal{L}, B, T) \cap H(\mathcal{L}, B, T')$. The energy norm is mesh-independent in the sense that

$$\begin{aligned} &\langle Dv, v\rangle_\Omega + \langle Rv, v\rangle_{\partial\Omega} + \tfrac{1}{2}\textstyle\sum_{\kappa \in T} \langle |B|(\nu)[v], [v]\rangle_{\partial\kappa\setminus\partial\Omega} \\ &= \langle Dv, v\rangle_\Omega + \langle Rv, v\rangle_{\partial\Omega} + \tfrac{1}{2}\textstyle\sum_{\kappa \in T'} \langle |B|(\nu)[v], [v]\rangle_{\partial\kappa\setminus\partial\Omega}. \end{aligned}$$

The positive definiteness of B_{DG} implies that there is a unique discontinuous Galerkin solution $u_{\mathrm{DG}} \in S(T,p)$ to

$$B_{\mathrm{DG}}(u_{\mathrm{DG}}, w) = \ell_{\mathrm{DG}}(w) \qquad \forall\, w \in S(T,p).$$

The solution satisfies the stability estimate

$$\|u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq \gamma^{-1}\|f\|_{L^2(\Omega)^m} + C\|g\|_B.$$

The constant $C > 0$ depends on the boundary operator J but not on the approximation space [19].

The next theorem shows that the discontinuous Galerkin method converges in the energy norm under h- and p-refinement.

Theorem 2. *The discontinuous Galerkin solution satisfies the bound*

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq C \inf\{\|u - v\|_{\mathcal{L}} : v \in S(T, p) \cap H(\mathcal{L}, \Omega)\}.$$

The constant $C > 0$ is independent of p and T.

The proof relies on Galerkin orthogonality and on the factorisation of the boundary conditions with F, cf. [19].

References

1. Friedrichs, K.O.: Symmetric positive linear differential equations. Comm. Pure Appl. Math. **11** (1958) 333–418
2. Dautray, R., Lions, J.L.: Mathematical analysis and numerical methods for science and technology. Vol. 3. Springer-Verlag, Berlin (1990)
3. Moyer, R.D.: On the nonidentity of weak and strong extensions of differential operators. Proc. Amer. Math. Soc. **19** (1968) 487–488
4. Lax, P.D., Phillips, R.S.: Local boundary conditions for dissipative symmetric linear differential operators. Comm. Pure Appl. Math. **13** (1960) 427–455
5. Sarason, L.: On weak and strong solutions of boundary value problems. Comm. Pure Appl. Math. **15** (1962) 237–288
6. Phillips, R.S., Sarason, L.: Singular symmetric positive first order differential operators. J. Math. Mech. **15** (1966) 235–271
7. Rauch, J.: Symmetric positive systems with boundary characteristic of constant multiplicity. Trans. Amer. Math. Soc. **291** (1985) 167–187
8. Rauch, J.: Boundary value problems with nonuniformly characteristic boundary. J. Math. Pures Appl. (9) **73** (1994) 347–353
9. Secchi, P.: A symmetric positive system with nonuniformly characteristic boundary. Differential Integral Equations **11** (1998) 605–621
10. Secchi, P.: Full regularity of solutions to a nonuniformly characteristic boundary value problem for symmetric positive systems. Adv. Math. Sci. Appl. **10** (2000) 39–55
11. Reed, W.H., Hill, T.R.: Triangular mesh methods for the neutron transport equation. Proceedings of the 1973 Conference on Mathematical Models and Computational Techniques of Nuclear Systems, Fifth biennial topical meeting of the mathematical and computational division of the American Nuclear Society, vol. 1, session 1 (1973) 10–31
12. Reed, W.H., Hill, T.R.: Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479 (1973)

13. LeSaint, P., Raviart, P.A.: On a finite element method for solving the neutron transport equation. In: Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974). Academic Press, New York (1974) 89–123. Publication No. 33
14. Johnson, C., Pitkäranta, J.: Convergence of a fully discrete scheme for two-dimensional neutron transport. SIAM J. Numer. Anal. **20** (1983) 951–966
15. Johnson, C., Nävert, U., Pitkäranta, J.: Finite element methods for linear hyperbolic problems. Comput. Methods Appl. Mech. Engrg. **45** (1984) 285–312
16. Bey, K.S., Oden, J.T.: *hp*-version discontinuous Galerkin methods for hyperbolic conservation laws. Comput. Methods Appl. Mech. Engrg. **133** (1996) 259–286
17. Houston, P., Schwab, C., Süli, E.: Stabilized *hp*- finite element methods for first-order hyperbolic problems. SIAM J. Numer. Anal. **37** (2000) 1618–1643
18. Houston, P., Süli, E.: *hp*-adaptive discontinuous Galerkin finite element methods for first-order hyperbolic problems. SIAM J. Sci. Comput. **23** (2001) 1226–1252
19. Jensen, M.: Discontinuous Galerkin methods for Friedrichs systems with irregular solutions. PhD thesis, Oxford University (2004)