CYCLES IN HYPERBOLIC MANIFOLDS OF NON-COMPACT TYPE AND FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS

JENS FUNKE* AND JOHN MILLSON**

Abstract. Using the theta correspondence, we study a lift from (not necessarily rapidly decreasing) closed differential $(p-n)$-forms on a non-compact arithmetic quotient of hyperbolic p-space to Siegel modular forms of degree n. This generalizes earlier work of Kudla and the second named author (in the case of hyperbolic space). We give a cohomological interpretation of the lift and analyze its Fourier expansion in terms of periods over certain cycles. For Riemann surfaces, i.e., the case $p=2$, we obtain a complete description using the theory of Eisenstein cohomology.

1. Introduction

Throughout the 1980's, Kudla and the second named author studied integral transforms Λ from closed differential forms on arithmetic quotients of the symmetric spaces of orthogonal and unitary groups to spaces of classical Siegel and Hermitian modular forms ([11, 12, 13, 14]). These transforms came from the theory of dual reductive pairs and the theta correspondence.

In [14] they computed the Fourier expansion of $\Lambda(\eta)$ in terms of periods of η over certain totally geodesic cycles under the assumption that η was rapidly decreasing. This also gave rise to the realization of intersection numbers of these 'special' cycles with cycles with compact support as Fourier coefficients of modular forms.

It is clear from [7],[4] and [6] that the situation is far more complicated when the hypothesis of rapid decay is dropped. The purpose of this paper is to initiate a systematic study of this transform for non rapidly decreasing differential forms η by considering the case for the finite volume quotients of hyperbolic space coming from unit groups of isotropic quadratic forms over $\mathbb{Q}$. We expect that many of the techniques and features of this case will carry over to the more general situation.

We now give a more precise description of this paper. Let $V(\mathbb{Q})$ be a rational vector space of dimension $m=p+1$ with a symmetric bilinear form $(\ ,\ )$ of signature $(p,1)$ and put $G(\mathbb{Q})=SO(V(\mathbb{Q}))$. We let L be an integral lattice in $V(\mathbb{Q})$ and $\Gamma(\mathbb{Q})$ be a torsion-free subgroup of the stabilizer of L in $G(\mathbb{Q})$. We denote by B the associated symmetric space to $G(\mathbb{R})$, and we assume that the hyperbolic manifold $M=\Gamma\backslash B$ is non-compact.

Kudla and the second named author ([11, 12]) constructed a certain theta function $\theta(\tau,Z)$ for $\tau\in\mathbb{H}_n$, the Siegel upper half space, and $Z\in B$, which is a *non-holomorphic*

*Fellow of the European PostDoc Institute (EPDI) 99-01.
**Partially supported by NSF-grant DMS 98-03520.

Siegel modular form of weight $\frac{m}{2}$ with values in the closed differential n-forms of M. For η a rapidly decreasing closed differential $(p-n)$-form in M, they then defined the transform

$$\Lambda(\eta)(\tau) = \int_M \eta \wedge \theta(\tau, Z). \tag{1.1}$$

They showed that $\Lambda(\eta)(\tau)$ is a holomorphic cusp form, see [14]. Moreover, the Fourier coefficients are given as periods of η over certain geometrically defined composite, in general non-compact, 'special' cycles C_β in M attached to positive definite $\beta \in Sym_n(\mathbb{Q})$, i.e.,

$$\Lambda(\eta)(\tau) = \sum_{\beta>0} \left(\int_{C_\beta} \eta \right) e^{2\pi i tr(\beta\tau)}. \tag{1.2}$$

The lift factors through the cohomology $H_c^{p-n}(M,\mathbb{C})$ with compact support, and the period $\int_{C_\beta} \eta$ is the evaluation of the pairing of $[\eta] \in H_c^{p-n}(M,\mathbb{C})$ with the relative cycle $C_\beta \in H_{p-n}(M, \partial M, \mathbb{Z})$. The key point is here that the Fourier coefficients θ_β of $\theta(\tau)$ are the Poincaré-dual forms of the cycles C_β.

In the case of $p = 2$ and $n = 1$, this lift is closely related to the work of Shintani [15] on the inverse of the Shimura lift.

The Borel-Serre compactification makes M a compact manifold with boundary $\overline{M}$. Here each boundary component is a $(p-1)$-torus at the various cusps of M. We develop a machinery to determine the growth of $\theta(\tau, Z)$ and show

Theorem 1.1.
$\theta(\tau, Z)$ extends to a smooth differential form on $\overline{M}$. Moreover, the coefficients of the restriction of $\theta(\tau, Z)$ to each boundary component are given by a linear combination of holomorphic Siegel cusp forms of weight $\frac{m}{2}$ coming from the orthogonal group $O(p-1)$.

We can therefore extend the theta integral (1.1) to $(p-n)$-forms η on $\overline{M}$. For the special case $n = p$ and $\eta = 1$, the theta integral was already studied by Kudla ([9, 10]).

Theorem 1.2.
Let η be a closed differential $(p-n)$-form on $\overline{M}$. Then $\Lambda(\eta)(\tau)$ is a holomorphic *Siegel modular form of weight $\frac{m}{2}$ for a suitable congruence subgroup of $Sp(n,\mathbb{Z})$.*

The key point is here that there exists another, *rapidly decreasing* theta function $\Xi(\tau, Z)$ such that

$$\bar{\partial}\, \theta(\tau, Z) = d\, \Xi(\tau, Z). \tag{1.3}$$

Here $\bar{\partial}$ operates on the τ-variable and d on the Z-variable. This, together with Stokes' theorem, implies that $\Lambda(\eta)(\tau)$ satisfies the Cauchy-Riemann equations.

The form Ξ exists in general but it is not necessarily rapidly decreasing. Thus the problem of when $\Lambda(\eta)$ is holomorphic is rather delicate. In fact, in [6] it was shown that in the case of signature $(p, 2)$ analogous theta integrals are in general non-holomorphic modular forms.

We call the space of holomorphic Siegel cusp forms of weight $\frac{m}{2}$ and degree n coming from theta series attached to $O(p-1)$ the space of unstable cusp forms and denote it by $\Theta^{(n)}(p-1)$. (For $p=2$ and $n=1$, these cusp forms correspond to Eisenstein series of weight 2 under the Shimura correspondence).

By Theorem 1.1 the image of exact forms lies in the space of unstable cusp forms. Denoting the space of holomorphic Siegel modular forms of weight $\frac{m}{2}$ and degree n by $M^{(n)}_{m/2}$, we therefore obtain

Theorem 1.3.

The transform Λ factors through the cohomology $H^{p-n}(\overline{M},\mathbb{C}) \simeq H^{p-n}(M,\mathbb{C})$ modulo unstable Siegel cusp forms, i.e., Λ defines a map

$$\Lambda : H^{p-n}(\overline{M},\mathbb{C}) \longrightarrow M^{(n)}_{m/2}/\Theta^{(n)}(p-1).$$

By Theorem 1.2 we see by the Koecher principle that the Fourier expansion of $\Lambda(\eta)(\tau)$ is given by

$$\Lambda(\eta)(\tau) = \sum_{\beta \geq 0} a_\beta(\eta) e^{2\pi i tr(\beta\tau)} \tag{1.4}$$

with

$$a_\beta(\eta) = \int_M \eta \wedge \theta_\beta(\tau). \tag{1.5}$$

(For $n=1$, the vanishing of the negative coefficients follows from a direct calculation which we omit).

For the singular coefficients, the $\theta_\beta(\tau)$ turn out to be rapidly decreasing, and we have

Theorem 1.4.

$$a_\beta(\eta) = \begin{cases} 0 & \textit{if} \quad rk(\beta) < n-1 \\ (-1)^n \int_{C^s_\beta} \eta & \textit{if} \quad rk(\beta) = n-1. \end{cases}$$

In particular, we see that $\Lambda(\eta)(\tau)$ is in general no longer a cusp form. Here, for β positive semi-definite of rank $n-1$, the 'singular' cycles C^s_β are linear combinations of embedded $(p-n)$-subtori at each component of the Borel-Serre boundary of M. The coefficients are values of Dirichlet series attached to the boundary components. Note that the C^s_β can be considered as absolute cycles in M and therefore the period of η over C^s_β is cohomological.

The calculation of the singular Fourier coefficients uses extensively ideas from [10], where the case of $n=p$ was considered. However, through a careful growth analysis of the theta series involved we are able to greatly simplify the concept of the calculations, avoiding the usage of a wave packet attached to Eisenstein series. This observation should also be very helpful for extending the much more general results of [14].

The situation for the positive definite coefficients is considerably more complicated as now θ_β is nonzero at the boundary and therefore homotopy- and Stokes-type arguments for the computation of (1.5) are no longer available. In particular, the calculation for η rapidly decreasing (see [13]) does not extend to arbitrary η. This corresponds to the fact that the period $\int_{C_\beta}\eta$ (where C_β is the (in general relative) cycle mentioned above) no longer has a (co)homological interpretation.

In fact, if η is an exact form which extends to the boundary, the equation

$$a_\beta(\eta) \stackrel{?}{=} \int_{C_\beta} \eta \tag{1.6}$$

is in general no longer valid! We define the 'defect' $\delta_\beta(\eta) = a_\beta(\eta) - \int_{C_\beta}\eta$ and show that δ_β descends to a function on $Z^{p-n}(\overline{M})/Z^{p-n}(\overline{M},\partial\overline{M})$, where $Z^*(\overline{M})$ is the space of closed differential forms on $\overline{M}$ and $Z^*(\overline{M},\partial\overline{M})$ the subspace of forms which vanish at the boundary. Moreover, we show that the defect can be non-zero on the subspace of exact $(p-n)$-forms supported near $\partial\overline{M}$.

For the case of a Riemann surface, i.e., for the case of $SO(2,1)$ and $n=1$, we have a complete picture:

Theorem 1.5. *Let $p=2$ and $n=1$. Then each class in $H^1(\overline{M},\mathbb{C})$ has a representative η such that*

$$\Lambda(\eta)(\tau) = \left(\int_{C_0^s}\eta\right) + \sum_{\beta>0}\left(\int_{C_\beta}\eta\right)e^{2\pi i\beta\tau}. \tag{1.7}$$

Hence (1.7) holds in $M_{3/2}^{(1)}/\Theta^{(1)}(1)$ for all closed 1-forms η in $\overline{M}$.

The point is here that via the theory of Eisenstein cohomology $H^1(\overline{M},\mathbb{C})$ splits into its cuspidal (or L_2) cohomology and a part defined by Eisenstein series coming from cohomology classes at the boundary. We are able to directly compute (1.5) for forms defined by cusp forms and Eisenstein series, thus verifying (1.6).

Furthermore, we can consider the 'truncated' part $\theta^c(\tau)$ of the form $\theta(\tau)$, which is obtained by subtracting the Eisenstein form of the restriction of $\theta(\tau)$ to the boundary from $\theta(\tau)$ itself. $\theta^c(\tau)$ is again a modular form of weight $3/2$ with values now in the rapidly decreasing differential 1-forms of the Riemann surface M.

For $\beta>0$, we define C_β^c to be the homology class dual to the β-th Fourier coefficient of $\theta^c(\tau)$. This definition and the following result is completely analogous to the one by Hirzebruch-Zagier for Hilbert modular surfaces ([7]):

Theorem 1.6. *Let $p=2$ and $n=1$. The map*

$$\eta \mapsto \int_M \eta\wedge\theta^c(\tau)$$

factors through $H^1(\overline{M},\mathbb{C})$, and if C is the homology class dual to $[\eta]$, we have that

$$\int_M \eta\wedge\theta^c(\tau) = -[C_0^s.C] + \sum_{\beta>0}[C_\beta^c.C]e^{2\pi i\beta\tau}$$

is a holomorphic modular form of weight $3/2$. Here $[\,.\,]$ denotes the cohomological intersection product.

It seems natural to expect that this generalizes to $SO(p,1)$ (at least when the Eisenstein classes involved are not residual), and we hope to come back to this issue in the near future.

We can also define in the general case

$$\Lambda(C)(\tau) = \int_C \theta(\tau) \tag{1.8}$$

for C being a special cycle of complementary dimension n. For this lift, we have complete control over the Fourier coefficients:

Theorem 1.7. *$\Lambda(C)(\tau)$ is a holomorphic Siegel modular form of weight $\frac{m}{2}$ and degree n and*

$$\Lambda(C)(\tau) = \sum_{\beta>0} [C.C_\beta]_{tr} e^{2\pi i tr(\beta\tau)} \ + \ (-1)^n \sum_{\substack{\beta\geq 0 \\ rk(\beta)=n-1}} [C.C_\beta^s] e^{2\pi i tr(\beta\tau)}.$$

Here $[C.C_\beta]_{tr}$ denotes the transversal intersection number of C and C_β in M, i.e., the sum of the transversal intersections counted with multiplicities.

We would like to thank Steve Kudla for many crucial discussions and his encouragement. The first named author would like to thank the Max-Planck-Institut für Mathematik in Bonn and the Department of Algebra and Geometry at the University of Barcelona for their hospitality where major work for this paper was done.

2. Preliminaries

Let $V(\mathbb{Q})$ be a rational vector space of dimension $m = p+1$ and let $(\ ,\ )$ be a non-degenerate symmetric bilinear form on $V(\mathbb{Q})$ with signature $(p,1)$. Let $L \subset V(\mathbb{Q})$ be an integral $\mathbb{Z}$-lattice of full rank, i.e., $L \subset L^{\#}$, the dual lattice. We let $G(\mathbb{Q}) = SO(V(\mathbb{Q}))$ viewed as an algebraic group over $\mathbb{Q}$. We denote by $\Gamma(L)$ the stabilizer of the lattice L and fix a neat subgroup Γ of finite index in $\Gamma(L) \cap G_0(\mathbb{R})$, which acts trivially on $L^{\#}/L$. Here $G_0(\mathbb{R})$ is the connected component of the identity of $G(\mathbb{R})$.

Let B be the real hyperbolic space of dimension p and realize B as one component of the two-sheeted hyperboloid of vectors of length -1:

$$B = \{Z \in V(\mathbb{R}) : (Z,Z) = -1\}^0. \tag{2.1}$$

Fix a base point $Z_0 \in B$ and let K be the stabilizer of Z_0 in $G_0(\mathbb{R})$. Then $K \simeq SO(p)$ is a maximal compact subgroup of $G_0(\mathbb{R})$, and we have

$$B \simeq G_0(\mathbb{R})/K. \tag{2.2}$$

Note that we can identify B as the set of negative lines in $V(\mathbb{R})$ and therefore also as the space of minimal majorants of $(\ ,\ )$ by defining, for $Z \in B$, the majorant

$$(\ ,\ )_Z = \begin{cases} (\ ,\ ) & \text{on } Z^\perp; \\ -(\ ,\ ) & \text{on } \mathbb{R}\, Z. \end{cases} \tag{2.3}$$

For the tangent space $T_Z(B)$ we have the standard canonical identification

$$T_Z(B) \simeq Z^\perp. \tag{2.4}$$

We fix an orientation on V, and this induces an orientation of B by requiring that, for every properly oriented basis $\{w_1, ..., w_p\}$ for $T_Z(B) \simeq Z^\perp$, the basis $\{w_1, ..., w_p, Z\}$ is properly oriented for V. Note that the action of $G_0(\mathbb{R})$ on B preserves this orientation.

We assume that the hyperbolic manifold $M = \Gamma\backslash B$ is non-compact. It is well known [1] that this is the case if and only if $V(\mathbb{Q})$ has an isotropic vector. Then Γ acts with finitely many orbits on the set of isotropic lines in $V(\mathbb{Q})$, the cusps of M. We choose cusp representatives $\ell_0, \ell_1, ..., \ell_r$ and primitive vectors $u_j \in L$ such that

$$\ell_j = \mathbb{Q}\, u_j \qquad \text{and} \qquad (u_j, Z) < 0 \tag{2.5}$$

for all $Z \in B$. We will express this second condition by saying u_i is forward pointing. We note that every null line has a canonical orientation given by the class of a forward pointing vector. We also choose $g_j \in G_0(\mathbb{Q}) = G(\mathbb{Q}) \cap G_0(\mathbb{R})$ such that

$$g_j\, u_0 = u_j \tag{2.6}$$

and with $g_0 = 1$. Pick another isotropic vector $u_0' \in V(\mathbb{Q})$ such that $(u_0, u_0') = -1/2$. This gives an isomorphism

$$\ell_0^\perp/\ell_0 \simeq W(\mathbb{Q}) := [u_0, u_0']^\perp \tag{2.7}$$

Note that W is positive definite of dimension $p-1$. We choose a basis $\{w_1, ..., w_{p-1}\}$ of W such that $u_0, w_1, ..., w_{p-1}, u_0'$ is a positively oriented basis for $V(\mathbb{Q})$ and call such a basis a Witt basis for $V(\mathbb{Q})$. Note that this also gives rise to an orientation of $\ell_0^\perp/\ell_0$. With respect to this basis, $(\ ,\ )$ is of the form

$$(\ ,\ ) \sim \begin{pmatrix} & & -1/2 \\ & S & \\ -1/2 & & \end{pmatrix}, \tag{2.8}$$

where S is the matrix of the bilinear form restricted to W.

We can assume that the base point Z_0 is rational and contained in the hyperbolic plane $[u_0, u_0']$. Since we assumed $(Z, Z) = -1$ and $(Z, u_0) < 0$, we see that $Z_0 = u_0 + u_0'$, i.e., in coordinates:

$$Z_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{2.9}$$

Note that majorant $(\ ,\ )_{Z_0} =: (\ ,\ )_0$ associated to the base point Z_0 is given by

$$(\ ,\ )_0 \sim \begin{pmatrix} 1/2 & & \\ & S & \\ & & 1/2 \end{pmatrix}. \tag{2.10}$$

We pick another basis for $V(\mathbb{R})$ as follows. We let

$$e_1 = u_0 - u_0' \qquad \text{and} \qquad e_{p+1} = u_0 + u_0' = Z_0. \tag{2.11}$$

We have $(e_1, e_1) = 1$ and $e_1 \perp Z_0$ and extend e_1 to an orthonormal basis $\{e_1, \cdots, e_p\}$ for $Z_0^\perp$. With respect to this basis $\{e_1, \cdots, e_{p+1}\}$ the bilinear form has the matrix

$$(\, , \,) \sim \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{2.12}$$

Let $\mathfrak{g}$ be the Lie algebra of $G_0(\mathbb{R})$ and $\mathfrak{k}$ be that of K. We then have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{2.13}$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. We identify $\mathfrak{p}$ with $Z_0^\perp$ in an $SO(p)$-equivariant way via

$$\begin{array}{ccc} Z_0^\perp & \xrightarrow{\sim} & \mathfrak{p} \\ v & \longmapsto & v \wedge Z_0, \end{array} \tag{2.14}$$

where $w \wedge w' \in \bigwedge^2 V$ is identified with an element of $\mathfrak{g}$ given by

$$(w \wedge w')(v) = (w, v)w' - (w', v)w. \tag{2.15}$$

We identify the basis $\{e_1, \cdots, e_p\}$ for $Z_0^\perp$ with a basis of $\mathfrak{p}$. With respect to this basis we have

$$\mathfrak{p} \simeq \left\{ \begin{pmatrix} 0 & v \\ {}^t v & 0 \end{pmatrix} : v \in Z_0^\perp \right\}. \tag{2.16}$$

We let $\{\omega_1, \cdots, \omega_p\}$ be the dual basis of $\mathfrak{p}^*$ corresponding to this basis.

We will denote coordinates with respect to the Witt basis $\{u_0, w_1, ..., w_{p-1}, u_0'\}$ with y_{ij} and coordinates with respect to the basis $\{e_i\}$ with x_{ij}.

Let P be the $\mathbb{Q}$-parabolic subgroup of G defined by

$$P(\mathbb{Q}) = \{g \in G(\mathbb{Q}) : g\ell_0 = \ell_0\}. \tag{2.17}$$

Then for the unipotent radical $N(\mathbb{Q})$ of $P(\mathbb{Q})$, we have

$$N(\mathbb{Q}) \simeq W(\mathbb{Q}), \tag{2.18}$$

and the isomorphism is explicitly given by

$$N(\mathbb{Q}) \simeq \left\{ n(w) = \begin{pmatrix} 1 & 2(\cdot, w) & (w, w) \\ & 1_W & w \\ & & 1 \end{pmatrix} : \ w \in W(\mathbb{Q}) \right\}. \tag{2.19}$$

The maximal $\mathbb{Q}$-split torus $A(\mathbb{Q})$ is given by

$$A(\mathbb{Q}) \simeq \left\{ a(t) = \begin{pmatrix} t & & \\ & 1_W & \\ & & t^{-1} \end{pmatrix} : \ t \in \mathbb{Q} \right\}. \tag{2.20}$$

We define

$$M = P(\mathbb{R}) \cap K \simeq SO(W(\mathbb{R})) \tag{2.21}$$

and have the standard decompositions

$$G_0(\mathbb{R}) = N(\mathbb{R})\, A_0(\mathbb{R})\, K \tag{2.22}$$

and

$$P_0(\mathbb{R}) = N(\mathbb{R})\, A_0(\mathbb{R})\, M, \tag{2.23}$$

where $P_0(\mathbb{R}) = P(\mathbb{R}) \cap G_0(\mathbb{R})$ and $A_0(\mathbb{R}) = A(\mathbb{R}) \cap G_0(\mathbb{R}) \simeq \mathbb{R}_+$.

For $t \in \mathbb{R}_+$, let

$$A_t = \{a(t') \in A_0(\mathbb{R}) : t' > t\}, \tag{2.24}$$

and for an open relatively compact subset $\omega \subset N(\mathbb{R})$, define the Siegel set

$$\mathfrak{S}_t = \omega A_t K \subset G_0(\mathbb{R}). \tag{2.25}$$

Then by [1] there exists a Siegel set $\mathfrak{S} \subset G_0(\mathbb{R})$ such that

$$G_0(\mathbb{R}) = \bigcup_j \Gamma g_j \mathfrak{S} \tag{2.26}$$

and

$$B = \bigcup_j \Gamma g_j \mathfrak{S}', \tag{2.27}$$

where $\mathfrak{S}' = \mathfrak{S} \cdot Z_0$.

Let N_j, $0 \leq j \leq r$, be the point-wise stabilizer of the cusps $\ell_j = \mathbb{Q}\, u_j$ in N and $\Gamma_j = N_j \cap \Gamma$. We have

$$N_j = g_j N g_j^{-1}. \tag{2.28}$$

There exist lattices $\Lambda_j \subset W(\mathbb{Q})$ such that

$$\Gamma_j = \{g_j n(\lambda) g_j^{-1} : \lambda \in \Lambda_j\}. \tag{2.29}$$

Recall that by adding for each cusp ℓ_j the torus $\Gamma_j \backslash N_j$ to the manifold $M = \Gamma \backslash B$ we obtain (with the appropriate topology) the compact manifold with boundary $\bar{M}$. This is the Borel-Serre compactification, see [3]. We have

$$\overline{M} = M \coprod_{j=0}^{r} \Gamma_j \backslash N_j. \tag{2.30}$$

We introduce upper-half space coordinates on B associated to an isotropic line, which we take to be ℓ_0. We consider the map

$$\sigma : A_0(\mathbb{R}) \times N(\mathbb{R}) \longrightarrow B \tag{2.31}$$

given by

$$\sigma(a, n) = n\, a\, Z_0 \tag{2.32}$$

Via the parametrization of $A_0(\mathbb{R}) \times N(\mathbb{R})$ by $\mathbb{R}_+ \times \mathbb{R}^{p-1}$ we obtain coordinates on B by

$$(t, b) \longmapsto Z(t, b) := n(b)\, a(t)\, Z_0. \tag{2.33}$$

We have

$$Z(t,b) = \begin{pmatrix} t + t^{-1}(b,b) \\ t^{-1}b \\ t^{-1} \end{pmatrix}, \tag{2.34}$$

where we identified $\mathbb{R}^{p-1}$ with $W(\mathbb{R}) \simeq N(\mathbb{R})$. We observe that in $\mathbb{P}(V)$ we have

$$\lim_{t\to\infty} Z(t,b) = \ell_0, \tag{2.35}$$

whereas in the Borel-Serre enlargement of B we have $\lim_{t\to\infty} Z(t,b) = b \in \ell_0^{\perp}/\ell_0$.

We extend σ to $N \times A \times K \longrightarrow G$ by $\sigma(n,a,k) = nak$, and this induces an isomorphism between the left-invariant forms on NA and the horizontal left-invariant forms on G which we identify with $\mathfrak{p}^*$. It is easily seen that a basis for the left-invariant forms on NA is given (in terms of the left-invariant forms $\frac{dt}{t}$ on A and db_i, $1 \leq i \leq p-1$ on N) by $\{\nu_1, \nu_2, \cdots, \nu_p\}$, where

$$\nu_1 = \frac{dt}{t} \qquad \text{and} \qquad \nu_i = \frac{db_{i-1}}{t} \qquad \text{for} \qquad 2 \leq i \leq p. \tag{2.36}$$

We have

Lemma 2.1.

$$\sigma^* \, \omega_i = \nu_i \qquad \textit{for} \quad 1 \leq i \leq p.$$

Proof. We only have to prove this at the identity. Then the basis $\{\nu_1, \cdots, \nu_p\}$ for $\mathfrak{a}^* + \mathfrak{n}^*$ is dual to the basis $\{2u_0 \wedge u_0', 2e_j \wedge u_0, 2 \leq j \leq p\}$. The image of this basis under $d\sigma|_e$ when projected onto $\mathfrak{p}$ (the horizontal Maurer-Cartan forms annihilate $\mathfrak{k}$) is the basis $\{e_1 \wedge e_{p+1}, e_2 \wedge e_{p+1}, \cdots, e_p \wedge e_{p+1}\}$. But this basis is dual to $\{\omega_i\}$ per definitionem. $\square$

We will need a refinement of these coordinates associated to positive semi-definite subspace U of $V(\mathbb{Q})$ of dimension n such that the radical

$$R(U) = \{u \in U : (u,U) = 0\} \tag{2.37}$$

is non-zero. In this case we see by signature considerations that there exists a rational isotropic line ℓ such that

$$R(U) = \ell. \tag{2.38}$$

We may choose the above Witt decomposition such that

$$U = \ell_0 \; + U \cap W, \tag{2.39}$$

i.e., $\ell = \ell_0$. We write $U' = U \cap W$ and let U'' be the orthogonal complement of U' in W, hence

$$W = U' \oplus U'', \tag{2.40}$$

with the summands orthogonal for both $(\ ,\ )$ and $(\ ,\ )_0$.

We define subgroups N' and N'' of N with

$$N' \simeq U' \qquad \text{and} \qquad N'' \simeq U'' \tag{2.41}$$

under the isomorphism from W to N. We also define

$$N_U = \{n \in N : n|_U = id\} = \{n \in N : n|_{U'} = id\}.$$

We observe that

$$N_U = N''. \tag{2.42}$$

Indeed, for $w, w' \in W$, we have $n(w)w' = w' + (w, w')u_0$, whence $N_U = (U')^{\perp} = U''$.

We can write

$$n(b) = n(b')n(b'') \tag{2.43}$$

with $n(b') \in N'$ and $n(b'') \in N''$; so $b' \in \mathbb{R}^{n-1} \simeq U'$ and $b'' \in \mathbb{R}^{p-n} \simeq U''$. We obtain a product decomposition

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \mathbb{R}^{p-n} \longrightarrow B \tag{2.44}$$

with

$$\sigma(t, b', b'') = Z(t, b', b'') := n(b')n(b'')a(t)Z_0. \tag{2.45}$$

3. Special Cycles

We define special cycles in B as follows: Let U be a positive definite subspace of $V(\mathbb{R})$ of dimension $n \leq p$, and define

$$B_U = \{Z \in B : Z \perp U\}. \tag{3.1}$$

Note that B_U is a totally geodesic submanifold, isomorphic to the hyperbolic space of dimension $p-n$. If $U = span_{\mathbb{R}} X$ for an n-frame $X = (x_1, \cdots, x_n)$ in $V(\mathbb{R})$, we also write B_X for B_U. An orientation on U (say, coming from X) induces one on B_U as follows. We have a canonical isomorphism

$$T_Z(B_U) \simeq Z^{\perp} \cap U^{\perp}. \tag{3.2}$$

Then $T_Z(B_U)$ receives an orientation by the rule that the orientation of $T_Z(B_U)$ followed by the orientation of $U = U \cap Z^{\perp}$ is the orientation of $T_Z(B) \simeq Z^{\perp}$.

Let G_U be the point-wise stabilizer of U in G and put $\Gamma_U = \Gamma \cap G_U$. We then define $C_U = \Gamma_U \backslash B_U$; the image of B_U in M.

For $\beta \in Sym_n(\mathbb{Q})$, we consider the corresponding hyperboloid

$$\Omega_\beta = \{X \in V(\mathbb{Q}) : \frac{1}{2}(X, X) = \beta\}, \tag{3.3}$$

with $(X, X)_{ij} = (x_i, x_j)$.

We fix a congruence condition $h \in \left(L^{\#}\right)^n$ once and for all.

If β is positive definite, then Γ acts on $\Omega_\beta \cap (L^n + h)$ with finitely many orbits, and we define the composite cycle

$$C_\beta = \sum_{\Gamma \backslash \Omega_\beta \cap (L^n + h)} C_X. \tag{3.4}$$

We now construct special cycles on the Borel-Serre boundary of $\overline{M}$. Let U be a positive semidefinite subspace of $V(\mathbb{Q})$ of dimension n with nonzero radical $R(U) = \ell$. We denote the unipotent radical of the parabolic associated to ℓ by $N_\ell \simeq \ell_{\perp}/\ell$ and write $\Gamma_\ell = \Gamma \cap N_\ell$. The boundary component corresponding to the cusp ℓ is the $(p-1)$-torus $\Gamma_\ell \backslash N_\ell$ with universal cover $\ell^\perp/\ell$. We then define the $(p-n)$ cycle B_U at the boundary by

$$B_U = \{w \in \ell^\perp/\ell : (U, w) = 0\}. \tag{3.5}$$

We write $C_U = \Gamma_U \backslash B_U$ with $\Gamma_U = N_U \cap \Gamma_\ell$ and note that in the Borel-Serre compactification this cycle only depends on the equivalence class of the cusp ℓ; i.e., we have $C_U = C_{\gamma U}$ with $\gamma \in \Gamma$ such that $\gamma\ell = \ell_i$ for some i.

An orientation for U gives one for C_U in the following way:

Pick any null line $\ell' = \mathbb{Q}u'$ as above such that ℓ and ℓ' span a hyperbolic plane whose orthogonal complement in V we denote by W. Recall that the forward pointing vectors (see (2.5)) give an orientation for ℓ and ℓ' respectively. The orientation of U induces one for $U' = U \cap W$ by requiring that the orientation of ℓ followed by the one of U' gives the orientation for U. B_U is isomorphic to the orthogonal complement of U in $\ell_0 \perp W$, and we require that the orientation of ℓ followed by the ones of $B_U = T_Z(B_U)$, U' and finally of ℓ_0' gives the orientation of V.

A fixed orientation for U defines a sign character $\epsilon(X)$ for X a rational n-frame with $span_{\mathbb{Q}}(X) = U$, by setting $\epsilon(X) = 1$ if X defines the same orientation on U and $\epsilon(X) = -1$ otherwise. So $C_X = \epsilon(X) C_U$.

Remark 3.1. When working with coordinates for B adopted to U (see Section 2) one obtains a different orientation for C_U which differs from the given one by a factor of $(-1)^{(n-1)(p-n)}$.

The construction of a composite cycle in this situation is more complicated:

Let $\beta \in Sym_n(\mathbb{Q})$ positive semidefinite and of rank $n-1$. We define

$$\Omega^s_\beta = \{X \in V^n : \frac{1}{2}(X, X) = \beta \quad \text{and } \mathrm{rank}(X) = n\}, \tag{3.6}$$

the 'singular' part of the hyperboloid Ω_β. Since β is singular, the radical $R(X)$ of the span of $X \in \Omega^s_\beta$ is nonzero, i.e., $R(X) = \ell = \ell_X$ for some rational isotropic line ℓ. For such a line, we define

$$\Omega_{\beta,\ell} = \{X \in V(\mathbb{Q}) : (X, X) = \beta \quad \text{and} \quad R(X) = \ell\} \subset \ell^\perp. \tag{3.7}$$

We then have

$$\Omega^s_\beta = \coprod_{j=0}^{r} \coprod_{\gamma \in \Gamma_j \backslash \Gamma} \Omega_{\beta, \gamma^{-1}\ell_j}, \tag{3.8}$$

where $\ell_0, \cdots, \ell_r$ are the cusp representatives of the Γ-orbits of rational isotropic lines.

We also write $\Omega_{\beta,j}$ for Ω_{β,ℓ_j} and

$$\mathcal{L}_{\beta,j} = \Omega_{\beta,j} \cap (L^n + h). \tag{3.9}$$

Lemma 3.2. *Let ℓ be a rational isotropic line. There is a finite number of rational n-dimensional subspaces $U_1, \cdots, U_a$ of $\ell^\perp$ such that*

$$\{U_1, \cdots, U_a\} = \{span(X) : X \in \Omega_{\beta,\ell} \cap (L^n + h)\}. \tag{3.10}$$

Proof. Indeed, we consider the quadratic space $\ell^\perp/\ell$ which is positive definite. Then there are only finitely many $\bar{X} \in (\Omega_{\beta,\ell} \cap (L^n + h))/\ell$ such that $\frac{1}{2}(\bar{X}, \bar{X}) = \beta$. Pulling back to $\ell^\perp$ then gives the lemma. $\square$

For each cusp ℓ_j, find a collection $U_{ij}, i = 1, \ldots, a_j$, of n-dimensional subspaces of $\ell_j^\perp$ as in the lemma. We will write

$$\Omega_{\beta,i,j} = \{X \in \Omega_{\beta,j} : \mathit{span}(X) = U_{ij}\} \qquad \text{and} \qquad \mathcal{L}_{\beta,i,j} = \Omega_{\beta,i,j} \cap (L^n + h), \tag{3.11}$$

so that

$$\Omega_{\beta,j} = \coprod_{i=1}^{a_j} \Omega_{\beta,i,j} \qquad \text{and} \qquad \mathcal{L}_{\beta,j} = \coprod_{i=1}^{a_j} \mathcal{L}_{\beta,i,j}. \tag{3.12}$$

Lemma 3.3. *The action of Γ_j on $\Omega_{\beta,j}$ induces a free action of $\Gamma_{U_{ij}} \backslash \Gamma_j$ on $\Omega_{\beta,i,j}$. Here $\Gamma_{U_{ij}} = N_{U_{ij}} \cap \Gamma_j$.*

Proof. We show that the action of Γ_j on $\Omega_{\beta,j}$ carries $\Omega_{\beta,i,j}$ into itself and that the induced action of $\Gamma_{U_{ij}}$ is trivial.

Indeed, an element $\gamma \in \Gamma_j$ operates on an element $x \in \ell_j + W = \ell_j^\perp$ by adding a multiple of u_j to x. Thus γ leaves stable any subspace of $(\ell_j)^\perp$ containing ℓ_j, whence γ leaves $\Omega_{\beta,i,j}$ stable. Consequently, Γ_j leaves $\Omega_{\beta,i,j}$ stable. Also, $\Gamma_{U_{ij}}$ acts trivially on U_{ij} whence it acts trivially on $\Omega_{\beta,i,j}$.

Finally, if $\gamma \in \Gamma_j$ satisfies $\gamma X = X$, then, since X spans U_{ij}, necessarily $\gamma|_{U_{ij}} = 1$. $\square$

Let $\mathcal{C}_{\beta,i,j}$ be a set of coset representatives of this action on $\mathcal{L}_{\beta,i,j}$, i.e.,

$$\mathcal{C}_{\beta,i,j} = \left(\Gamma_j/\Gamma_{U_{ij}}\right) \backslash \left(\Omega_{\beta,i,j} \cap (L^n + h)\right). \tag{3.13}$$

We will see below that $\mathcal{C}_{\beta,i,j}$ is infinite. It is clear that the collection of the $\mathcal{C}_{\beta,i,j}$ provides a set of representatives for $\Gamma \backslash \Omega_\beta \cap (L^n + h)$.

Pick $a \in \mathbb{Q}^n$ in the radical of β. Then for all $X \in \Omega_\beta^s$,

$$X \cdot a \in \ell_X = \mathbb{Q} u_X, \tag{3.14}$$

with $u_X \in \ell_X$ as in (2.5). We can take a nonzero and primitive in $\mathbb{Z}^n$. With this condition, X determines a up to ± 1, and we write $X \cdot a = \nu(X) u_X$, where $\nu(X)$ is determined up to a sign. Following [10] we call X *reduced* if with such a choice of a we have

$$X \cdot a = \nu(X) u_X \tag{3.15}$$

with $\nu(X) \in [0, 1)$. Note that if X is reduced so is γX with $\gamma \in \Gamma$ and $\nu(X) = \nu(\gamma X)$. We write Ω_β^{red} for the set of reduced elements in Ω_β^s.

Lemma 3.4. Γ *acts with finitely many orbits on the reduced elements in $\Gamma \backslash \Omega_\beta^s \cap (L^n + h)$, and $\mathcal{C}_{\beta,i,j}^{red} := \mathcal{C}_{\beta,i,j} \cap \Omega_\beta^{red}$ forms a set of representatives.*

Proof. It is enough to show that for each pair i, j, $\mathcal{C}_{\beta,i,j}^{red}$ consists of only finitely many elements. We write $\mathcal{C}_{\beta,U_{ij},h}$ for $\mathcal{C}_{\beta,i,j}$.

Choose $m \in SL_n(\mathbb{Z})$ such that $me_0 = a$, where $e_0 = {}^t(1, 0, \cdots, 0)$. We put $\beta_0 = {}^tm\beta m$ and $k = hm$. Then right multiplication by m gives a bijection from $\mathcal{C}_{\beta,U_{ij},h}$ to $\mathcal{C}_{\beta_0,U_{ij},k}$.

As the vector e_0 is a primitive integral vector in the radical of β_0, we have

$$\beta_0 = \begin{pmatrix} 0 & 0 \\ 0 & \beta_0' \end{pmatrix}. \tag{3.16}$$

where β_0' is a positive definite $(n-1)$ by $(n-1)$ matrix. We write $U_{ij} = \ell_j \perp U'$ with U' positive definite. Picking an appropriate basis for U' we can assume that the n by n matrix $g(Y)$ for $Y \in \mathcal{C}_{\beta_0,U_{ij},k}$ is of the form

$$g(Y) = \begin{pmatrix} & y_0 \\ 0 & Y_1' \end{pmatrix} = \begin{pmatrix} y_{01} & y_0' \\ 0 & Y_1' \end{pmatrix}, \tag{3.17}$$

where $y_0 = (y_{01}, y_0')$ is a row vector of size n and Y_1' is an invertible $(n-1)$ by $(n-1)$ matrix. Similarly, the congruence condition k is of the form

$$g(k) = \begin{pmatrix} k_{01} & k_0' \\ 0 & k_1' \end{pmatrix}. \tag{3.18}$$

Also, since $y_1 \equiv k_1 \mod \mathbb{Z}u_0$ we have $y_{01} \equiv k_{01} \mod \mathbb{Z}$. Since β_0' is positive definite, there are only finitely many Y_1' which represent β_0'. We have

$$g(n(u')Y) = \begin{pmatrix} y_{01} & y_0' + 2(u', Y_1') \\ 0 & Y_1' \end{pmatrix} \tag{3.19}$$

for $n(u') \in \Gamma_j$. Hence there are only finitely many y_0', but y_{01} runs through the set $\{k_{01} + n : n \in \mathbb{Z}, n \neq -k_{01}\}$. Assuming $k_{01} \in [0, 1)$ we observe $\nu(Y) = k_{01}$ for Y reduced. This proves the assertion.

□

The proof of the lemma shows that the representatives of $\mathcal{C}_{\beta,i,j}$ come in natural $\mathbb{Z}$-classes: If $X \in \Omega_{\beta,\ell}$ and $X \cdot a \in \ell_X$ as above, then $\{\tilde{X} = X + u_X{}^ta' : k \in \mathbb{Z}\}$ with $a' \in \mathbb{Z}^n$ and ${}^ta'a = 1$ defines the $\mathbb{Z}$-class. From this we see that each class contains exactly two reduced frames. If X is reduced with respect to a, then $\tilde{X} = X - u_X{}^ta'$ with $a' \in \mathbb{Z}^n$ and ${}^ta'a = 1$ is reduced with respect to $-a$. Moreover, $\nu(\tilde{X}) = 1 - \nu(X)$.

Recall that the first periodic Bernoulli polynomial is defined by

$$\mathbf{B}_1(\alpha) = \begin{cases} \alpha - \frac{1}{2} & \text{if} \quad \alpha \in (0,1) \\ 0 & \text{if} \quad \alpha = 0. \end{cases} \tag{3.20}$$

We readily check

$$\mathbf{B}_1(\nu(X))\epsilon(X) = \mathbf{B}_1(\nu(\tilde{X}))\epsilon(\tilde{X}). \tag{3.21}$$

We are finally ready to define the singular weighted composite cycle C_β^s by

$$C_\beta^s = \sum_{X \in \Gamma \backslash \Omega_\beta^{red} \cap (L^n + h)} \frac{1}{2}\mathbf{B}_1(\nu(X))C_X. \tag{3.22}$$

Remark 3.5. We could also define for a complex parameter s the cycle

$$C_{\beta,s} = \frac{1}{2} \sum_{X \in \Gamma \backslash \Omega_\beta \cap (L^n + h)} |\nu(X)|^{-s} C_X. \tag{3.23}$$

Then the arguments of the previous lemma show that $C_{\beta,s}$ converges for $Re(s) > 1$ and has a meromorphic continuation to the whole complex plane. For the value at $s = 0$ we have

$$C_{\beta,0} = C^s_\beta. \tag{3.24}$$

(see also the proof of Prop. 6.7.)

4. A Cohomology Class for the Weil Representation

Recall that the metaplectic cover $G' = Mp(n,\mathbb{R})$ of the symplectic group $Sp(n,\mathbb{R})$ is a central group extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp(n,\mathbb{R}) \longrightarrow Sp(n,\mathbb{R}) \longrightarrow 1 \tag{4.1}$$

of $Sp(n,\mathbb{R})$. Here $\mathbb{C}^1 = \{z \in \mathbb{C} : |z| = 1\}$. We fix a splitting $Mp(n,\mathbb{R}) = Sp(n,\mathbb{R}) \times \mathbb{C}^1$ and denote by $K' \subset Mp(n,\mathbb{R})$ the inverse image of the standard maximal compact subgroup

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \ : \ a + ib \in U(n) \right\} \tag{4.2}$$

of $Sp(n,\mathbb{R})$. Then K' admits a character $\det^{1/2}$; i.e., its square descends to the determinant character of $U(n)$.

$G \times G'$ acts on the Schwartz space $S(V(\mathbb{R})^n)$ via (the restriction of) the Weil representation $\omega = \omega_{V(\mathbb{R})}$ associated to the additive character $t \longmapsto e(t) := \exp(2\pi i t)$, see for example [16]. Recall that the action of G' on $\psi \in S(V(\mathbb{R})^n)$ is characterized by the formulae

$$\omega\left(\left(\begin{smallmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{smallmatrix}\right)\right)\psi(X) = (\det a)^{m/2}\psi(Xa) \tag{4.3}$$

for $a \in GL_n^+(\mathbb{R})$;

$$\omega\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right)\psi(X) = e^{\pi i tr(b(X,X))}\psi(X) \tag{4.4}$$

for $b \in Sym_n(\mathbb{R})$;

$$\omega\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\psi(X) = \gamma\hat{\psi}(X), \tag{4.5}$$

where $\hat{\psi}$ is the Fourier transform of ψ and γ an eighth root of unity.

The central $\mathbb{C}^1$ acts by

$$\omega((1,t))\psi = \begin{cases} t\psi & \text{if } m \text{ is odd} \\ \psi & \text{if } m \text{ is even} \end{cases} \tag{4.6}$$

for all $t \in \mathbb{C}^1$.

The group G acts on $S(V(\mathbb{R})^n)$ via

$$\omega(g)\psi(X) = \psi(g^{-1}X), \tag{4.7}$$

which commutes with the action G'.

For $Z \in B$, we define the corresponding Gaussian by

$$\varphi_0(X,Z) = \exp(-\pi tr(X,X)_Z) \tag{4.8}$$

and put $\varphi_0(X) = \varphi_0(X, Z_0)$. Note that $\varphi_0(X,Z)$ is G-invariant; i.e.,

$$\varphi_0(gX, gZ) = \varphi_0(X,Z). \tag{4.9}$$

The space of differential n-forms on B is

$$\mathcal{A}^n(B) \simeq \left[C^\infty(G) \otimes \bigwedge^n(\mathfrak{p}^*) \right]^K, \tag{4.10}$$

where the isomorphism is given by evaluating at Z_0.

The main result of [11] (cf. also [14]), specialized to our situation, is the construction of a certain differential n-form of B with values in the Schwartz space $S(V(\mathbb{R})^n)$.

Theorem 4.1 ([11])**.** *For each n with $0 \le n \le p$, there is a nonzero Schwartz form*

$$\varphi_n \in [S(V(\mathbb{R})^n) \otimes \mathcal{A}^n(B)]^G \simeq \left[S(V(\mathbb{R}))^n \otimes \bigwedge^n(\mathfrak{p}^*) \right]^K, \tag{4.11}$$

such that

(i)
$$d\varphi_n = 0;$$
i.e., for each $X \in V(\mathbb{R})^n$, $\varphi_n(X)$ is a closed n-form on B which is G_X-invariant:
$$g^*\varphi_n(X) = \varphi_n(X)$$
for $g \in G_X$, the stabilizer of X in G.

(ii) *The forms are compatible with the wedge product:*
$$\varphi_{n_1} \wedge \varphi_{n_2} = \varphi_{n_1+n_2},$$
where $\varphi_n = 0$ for $n > p$.

(iii) *Assume $U = U(X)$ for a linear independent, positive definite n-frame X in $V(\mathbb{R})$. Then a Poincaré dual of $C_U = \Gamma_U \backslash B_U$ is given by*
$$\left[e^{\pi(X,X)} \sum_{\gamma \in \Gamma_U \backslash \Gamma} \gamma^* \varphi_n(X) \right].$$

In [11, 12] Poincaré dual form means the following: Let $C \subset M = \Gamma \backslash B$ be a cycle of dimension n. The η is a Poincaré dual form of C if

$$\int_C \omega = \int_M \omega \wedge \eta \tag{4.12}$$

holds for all *compactly supported* (or rapidly decreasing) closed n-forms ω.

We now give some explicit formulae for the forms $\varphi_n \in [S(V(\mathbb{R}))^n \otimes \bigwedge^n(\mathfrak{p}^*)]^K$.

Via the basis $\{e_1, \cdots, e_p\}$ for $Z_0^\perp$ we identify $\mathfrak{p}$ with $\mathbb{R}^p$. Then ω_i becomes the functional on $\mathfrak{p}$ which picks out the i-th coordinate. For $X = (x_1, ..., x_n) \in V(\mathbb{R})^n \simeq$

$M_{m,n}(\mathbb{R})$ (w.r.t. the basis $\{e_1,\dots,e_{p+1}\}$), $m = p+1$, and for $1 \le s \le n$, we then define the 1-form

$$\omega(s,X) = \sum_{i=1}^{p} x_{is}\omega_i. \tag{4.13}$$

Note that $\omega(s,X)$ only depends on the s-th column vector x_s of X: $\omega(s,X) = \omega(s,x_s)$. We set

$$2^{-n/2}\varphi_n(X) = \left(\bigwedge_{s=1}^{n} \omega(s,X)\right) \cdot \varphi_0(X) \tag{4.14}$$

$$= \varphi_1(x_1) \wedge \cdots \wedge \varphi_1(x_n) \tag{4.15}$$

with $\varphi_0(X) = \exp(-\pi tr(X,X)_0)$, as before.

Note that this differs from the corresponding quantity in [11] by a factor of $2^{n/2}$.

We easily see

$$\varphi_n(X) = 2^{n/2} \sum_{1\le j_1<\cdots<j_n\le p} P_{j_1,\cdots,j_n}(X)\ \exp(-\pi tr(X,X)_0)\ \otimes \omega_{j_1} \wedge \cdots \wedge \omega_{j_n}, \tag{4.16}$$

where $P_{j_1\cdots j_n}(X)$ is the determinant of the n by n matrix obtained from X by removing all rows except the $j_1,\cdots,j_n$. Occasionally we will write $\hat{X}$ for this matrix, suppressing the coordinates. We write $\varphi_{j_1,\cdots,j_n}(X) = P_{j_1,\cdots,j_n}(X)\ \exp(-\pi tr(X,X)_0)$.

Then it is easy to see that we have

$$\varphi_n(X)(W) = 2^{n/2}\det(X,W)\ \exp(-\pi tr(X,X)_0) \tag{4.17}$$

for $W \in T_{Z_0}(B)^n \simeq \mathfrak{p}^n \simeq (Z_0^{\perp})^n$. Lemma 2.1 gives

Corollary 4.2.

$$\sigma^*\varphi_n(X) = 2^{n/2} \sum_{1\le j_1<\cdots<j_n\le p} P_{j_1,\cdots,j_n}(X) e^{-\pi(X,X)_0} \otimes \nu_{j_1} \wedge \cdots \wedge \nu_{j_n}.$$

We write $\varphi_n(X,Z)$ for the corresponding n-form on B; for $g \in G_0(\mathbb{R})$, we have per construction

$$\varphi_n(gX,gZ) = \varphi_n(X,Z), \tag{4.18}$$

which also implies Th. 4.1 (i).

Fundamental for the relationship to modular forms is

Theorem 4.3 ([11, 12]).
φ_n is an eigenvector of the maximal compact $K' \subset Mp(n,\mathbb{R})$ under the action of the Weil representation. We have

$$\omega(k')\,\varphi_n \;=\; \det(k')^{m/2}\varphi_n$$

for $k' \in K'$.

We denote by $\mathcal{L}_m$ the G'-homogeneous line bundle over G'/K' to the character $\det^{-m/2}$ of K'. Then the previous theorem can reformulated as

$$\varphi_n \in [\mathcal{L}_m \otimes S(V(\mathbb{R})^n) \otimes \mathcal{A}^n(B)]^{G\times G'} \tag{4.19}$$

$$\simeq \left[\mathbb{C}_{\chi_m} \otimes S(V(\mathbb{R})^n) \otimes \bigwedge^n(\mathfrak{p}^*)\right]^{K\times K'}, \tag{4.20}$$

where $\mathbb{C}_{\chi_m}$ is the one-dimensional module on which K' acts via the character $\det^{-m/2}$.

For $\tau = u + iv \in \mathbb{H}_n = \{\tau \in Sym_n(\mathbb{C}) : Im(\tau) > 0\} \simeq G'/K'$, the Siegel space of genus n, we define in the usual way

$$\varphi_n(\tau, X, Z) = \det(v)^{-m/4}\omega(g'_\tau)\varphi_n(X, Z). \tag{4.21}$$

Here $g'_\tau \in Sp_n(\mathbb{R})$ is a standard element carrying the base point $iI_n \in \mathbb{H}_n$ to τ; i.e.,

$$g'_\tau = \begin{pmatrix} v^{\frac{1}{2}} & v^{-\frac{1}{2}}u \\ 0 & v^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}. \tag{4.22}$$

This is well defined, and we obtain

Proposition 4.4.

$$\varphi_n(\tau, X, Z)(W) = 2^{n/2}\det(v)^{1/2}\det(X, W)e^{\pi i tr(X,X)_{\tau,Z}}$$

for $W \in (T_Z(B))^n \simeq \left(Z^\perp\right)^n$ *and with* $(X, X)_{\tau,Z} = u(X, X) + iv(X, X)_Z$.

For a congruence condition $h \in (L^\#)^n$, we define the theta series $\theta(\tau)$ with values in the differential n-forms of B by

$$\theta(\tau, Z) = \sum_{X\in(L^n+h)} \varphi_n(\tau, X, Z). \tag{4.23}$$

By the standard machinery of the theta correspondence (Poisson summation formula), we get

Theorem 4.5 ([11, 12])**.**
$\theta(\tau, Z)$ is a non-holomorphic Siegel modular form of weight $m/2$ with values in the Γ-invariant differential forms of B for some suitable congruence subgroup of $Sp(n, \mathbb{Z})$.

In [14] it was shown that $\bar{\partial}\varphi_n$ (with respect to the symplectic variable $\tau \in \mathbb{H}$) is *exact* in the orthogonal variable $Z \in B$; i.e., there exists

$$\psi_{n-1} \in [\mathcal{L}_m \otimes S(V(\mathbb{R}))^n \otimes \mathcal{A}^{n-1}(B) \otimes \mathcal{A}^{0,1}(\mathbb{H}_n)]^{G\times G'} \tag{4.24}$$

such that

$$\bar{\partial}\varphi_n = d\psi_{n-1}. \tag{4.25}$$

Defining the analogous theta series

$$\Xi(\tau, Z) = \theta_\psi(\tau, Z) = \sum_{X\in(L^n+h)} \psi(\tau, X, Z) \tag{4.26}$$

we obtain

$$\bar{\partial}\,\theta(\tau,Z) = d\,\Xi(\tau,Z). \tag{4.27}$$

We now give a concrete formula for ψ_{n-1}. Consider the double complex

$$[\mathcal{L}_m \otimes S(V(\mathbb{R}))^n \otimes \mathcal{A}^i(B) \otimes \mathcal{A}^{0,j}(\mathbb{H}_n)]^{G\times G'} \tag{4.28}$$

with maps $d, \bar{\partial}$. The Lie-algebra version of this complex is the following. Let $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ be the complexified Cartan decomposition of $\mathfrak{sp}_n$. We can identify $\mathfrak{p}'$ with the complex tangent space of $\mathbb{H}_n \simeq G'/K'$ at the base point $i1_n$, and the Harish-Chandra decomposition $\mathfrak{p}' = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ gives the splitting of $\mathfrak{p}'$ into the holomorphic and antiholomorphic tangent spaces. We let $\nu_{jk}, 1 \leq j \leq k \leq n$ be dual to the standard basis of $\mathfrak{p}_- \subset Sym_n(\mathbb{C})$. Evaluation at the base points gives an isomorphism of the above complex with

$$C^{i,j} = [\mathbb{C}_{\chi_m} \otimes S(V(\mathbb{R}))^n \otimes \bigwedge^i \mathfrak{p}^* \otimes \bigwedge^j \mathfrak{p}_-^*]^{K\times K'}. \tag{4.29}$$

We define $d, \bar{\partial}$ on $C^{i,j}$ via transport of structure, for explicit formulae see [14]. Note that $\varphi_n \in C^{n,0}$ and $\psi_{n-1} \in C^{n-1,1}$.

We put (in coordinates for $\{e_i\}$)

$$A_{jk}(X) = (-1)^{k-1} x_{m,j} e^{-\frac{1}{2}\pi(x_k,x_k)_0} \varphi_1(x_1) \wedge \cdots \wedge \widehat{\varphi_1(x_k)} \wedge \cdots \wedge \varphi_1(x_n), \tag{4.30}$$

where $\widehat{\ }$ over a term denotes that this term is omitted in the product. We have

$$A_{jk} = (-1)^{k-1} \sum_{1\leq\alpha_1<\cdots\alpha_{n-1}\leq p} x_{mj}\, P^{(k)}_{\alpha_1,\cdots,\alpha_{n-1}}(X)\varphi_0(X)\, \otimes \omega_{\alpha_1} \wedge \cdots \wedge \omega_{\alpha_{n-1}} \tag{4.31}$$

Here $P^{(k)}_{\alpha_1,\cdots,\alpha_{n-1}}(X)$ is the following polynomial. Let $X^{(k)}$ denote the m by $(n-1)$ submatrix of X obtained by deleting the k-th column. Then $P^{(k)}_{\alpha_1,\cdots,\alpha_{n-1}}(X) = P^{(k)}_{\alpha_1,\cdots,\alpha_{n-1}}(X^{(k)})$ is the minor obtained from $X^{(k)}$ using the rows $\alpha_1, \cdots, \alpha_{n-1}$.

We now define

$$\psi_{n-1} = i2^{n/2}\left[\sum_{1\leq j\leq n} A_{jj} \otimes \nu_{jj}\nu_{j_1} \wedge \cdots \wedge \nu_{j_n}; + \frac{3}{4}\sum_{1\leq j<k\leq n}(A_{jk}+A_{kj}) \otimes \nu_{jk}\right]. \tag{4.32}$$

Then

Theorem 4.6 ([14])**.**

$$\bar{\partial}\,\varphi_n = d\,\psi_{n-1}.$$

We write $\psi_{jj;\alpha_1,\cdots,\alpha_n}$ and $\psi_{jk;\alpha_1,\cdots,\alpha_n}$ for the coefficient of $\omega_{\alpha_1} \wedge \cdots \wedge \omega_{\alpha_{n-1}}$ in A_{jj} and $A_{jk} + A_{kj}$ respectively.

5. The Growth of $\theta(\tau, Z)$ and $\Xi(\tau, Z)$

In this section we prove that $\theta(\tau, Z)$ extends to the Borel-Serre boundary and that $\Xi(\tau, Z)$ is rapidly decreasing on $\Gamma\backslash B$.

Since $\varphi_n(X, g^{-1}Z) = \varphi_n(gX, Z)$, it suffices to prove the required estimates on the fixed Siegel set $\mathfrak{S}'$. By some standard arguments we can also assume that the lattice L is of the form

$$L = L \cap \ell_0 \;+\; L \cap W \;+\; L \cap \ell_0'. \tag{5.1}$$

We first consider an arbitrary n-form $\varphi \in [S(V(\mathbb{R})^n \otimes \bigwedge^n \mathfrak{p}^*]^K$ in the polynomial Fock space, that is, the space of Schwartz functions of the form $p(X)\varphi_0(X)$ with p a polynomial function on $V(\mathbb{R})^n$. (In the Fock model of the Weil representation these become polynomials on $\mathbb{C}^{nm}$.)

We extend our basis $\omega_1, \cdots, \omega_n$ of $\mathfrak{p}$ to a frame field $V_1(Z), \ldots, V_p(Z)$. We then have

$$\begin{aligned}\theta_\varphi(\tau, Z(t,b))\,&(V_{i_1}(Z(t,b)), \cdots, V_{i_n}(Z(t,b)))\\ &= \sum_{X \in (L^n + h)} \varphi(\tau, Z_0, a(t)^{-1}n(b)^{-1}X)(w_{i_1}, \cdots, w_{i_n}).\end{aligned} \tag{5.2}$$

Thus the problem of estimating a form of the above type on $\mathfrak{S}'$ reduces to estimating an expression of the following type

$$\theta(t,b,R) = \sum_{X \in (L^n+h)} p\left(a(t)^{-1}n(b)^{-1}X\right) \exp\left(-\pi R(a(t)^{-1}n(b)^{-1}X)\right), \tag{5.3}$$

where $p(X)$ is a homogeneous polynomial function on V^n and R is a complex-valued quadratic function on V^n with positive definite real part. We now make some elementary observations concerning the growth of such expressions in t. We define $\theta^*(t,b,R)$ by

$$\theta^*(t,b,R) = \sum_{X \in (L^n+h)} \left| p\left(a(t)^{-1}n(b)^{-1}X\right) \exp\left(-\pi R(a(t)^{-1}n(b)^{-1}X)\right)\right|. \tag{5.4}$$

Via $V(\mathbb{R})^n \simeq M_{m,n}(\mathbb{R})$ we think of p as a polynomial in some coordinates of $V(\mathbb{R})^n$.

From now on we use coordinates y_{ij} with respect to a Witt basis, see Section 2. Writing $X = \left(\begin{smallmatrix} y_1 \\ Y' \\ y_m \end{smallmatrix}\right)$ we have

$$a(t)^{-1}n(b)^{-1}X = \begin{pmatrix} t^{-1}\left(y_1 - 2(Y', b) + (b,b)y_m\right) \\ Y' - b \cdot y_m \\ t y_m \end{pmatrix}. \tag{5.5}$$

As a warm up we note

Lemma 5.1. *Suppose $p(y_{ij})$ is in the ideal in $\mathbb{C}[y_{ij}]$ generated by $\{y_{mj} : 1 \le j \le n\}$, the ideal of polynomial functions which vanish on $(\ell_0^\perp)^n$. Then $\theta^*(t,b,R)$ is exponentially decreasing on $\mathfrak{S}'$.*

Proof. We may replace R by $c\sum_{i,j} y_{ij}^2$ for a suitable constant c (since we are taking absolute values of the terms in the sum). Under the hypothesis of the lemma the only terms that appear in the sum have $y_{mj}(X) \neq 0$ for some j. But these terms appear in the exponential multiplied by t^2, and the lemma follows. □

Remark 5.2. The forms ψ_{n-1} and φ_n are *not* of this form.

We have an isomorphism

$$S(V(\mathbb{R})^n) \longrightarrow S((\ell_0)^n) \otimes S(W(\mathbb{R})^n) \otimes S((\ell_0')^n) \tag{5.6}$$

given by the partial Fourier transform operator

$$\mathcal{F}_0(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \widehat{\varphi_1} \otimes \varphi_2 \otimes \varphi_3. \tag{5.7}$$

Here $\widehat{\varphi_1}$ is the usual Fourier transform on $(\ell_0)^n$. The right hand side is sometimes referred to as the mixed model of the Weil representation.

We will need some formulae relating the action of ω and $\mathcal{F}_0$ on $S(V(\mathbb{R})^n)$. Identifying $(\ell_0)^n$ with $M\mathbb{Z}^n$ for some $M \in \mathbb{Q}$ we denote the Fourier transform variable (dual to y_1) by $\xi \in \mathbb{R}^n$.

Lemma 5.3. *Let* $(\xi, w, y_m) \in (\mathbb{R})^n \times W(\mathbb{R})^n \times (\ell_0'(\mathbb{R}))^n$.

(i) *For* $n(b) \in N(\mathbb{R})$ *with* $b \in W$,

$$\mathcal{F}_0(n(b)\varphi)(\xi, w, y_m) = e\left(\xi^t(-2(b,w) + (b,b)y_m)\right) \mathcal{F}_0\varphi(\xi, w - by_m, y_m);$$

(ii) *For* $a(t) \in A(\mathbb{R})$,

$$\mathcal{F}_0(a(t)\varphi)(\xi, w, y_m) = t^n \mathcal{F}_0\varphi(t\xi, w, ty_m);$$

(iii) *For* $a'(v) = \left(\begin{smallmatrix} v & 0 \\ 0 & {}^tv^{-1} \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$ *with* $v \in GL_n(\mathbb{R})$,

$$\mathcal{F}_0(a'(v)\varphi)(\xi, w, y_m) = (\det v)^{\frac{m}{2}-1} \mathcal{F}_0\varphi(\xi^t v^{-1}, wv, y_m v)$$

(iv) *For* $n'(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$ *with* $u \in Sym_n(\mathbb{R})$,

$$\mathcal{F}_0(n'(u)\varphi)(\xi, w, y_m) = e\left(tr(u\frac{(w,w)}{2})\right) \mathcal{F}_0\varphi(\xi + \frac{1}{2}y_m u, w, y_m).$$

Proof. This is an easy exercise which we omit. □

We introduce the following notation

$$a^\times(v)\varphi(\xi, w, y_m) = \varphi(\xi^t v^{-1}, wv, y_m v), \tag{5.8}$$

$$\phi(b, \xi, w, y_m) = e\left(\xi^t(-2(b,w) + (b,b)y_m)\right). \tag{5.9}$$

Note $|\phi(b, \xi, w, y_m)| = 1$.

Let $I \subset S(V^n)$ be the ideal of Schwartz functions in the polynomial Fock space that vanish on the linear subspace W^n of V^n. Note that W^n is defined by the equations

$$y_{1j} = 0 \qquad \textit{and} \qquad y_{mj} = 0 \tag{5.10}$$

for $j = 1, \cdots, n$, i.e., $I = < y_{1j}, y_{mj} >$. We observe that if $\mathcal{F}_0\varphi \in I$ then also $\mathcal{F}_0(a'(v)\varphi)$ and $\mathcal{F}_0(n'(u)\varphi) \in I$.

Lemma 5.4. *Suppose $\mathcal{F}_0\varphi$ is in the ideal I. Then $\theta(t,b)$ is exponentially decreasing on $\mathfrak{S}'$.*

Proof. We may write

$$\theta(t,b) =< \Theta_{h+L^n}, n(b)a(t)\varphi > . \tag{5.11}$$

Here Θ_{h+L^n} is the sum of Dirac deltas (placed at the points of $h+L^n$) and $< \, , \, >$ denotes the Kronecker pairing.

We write $h = h_1 + h'$ with $h_1 \in (\ell_0)^n$. Then there is a constant C such that

$$\mathcal{F}_0\Theta_{h+L^n} = Ce(\xi^t h_1)\Theta_{h'+L^n} \tag{5.12}$$

by Poisson summation. Using the formulas from the previous lemma we obtain

$$\mathcal{F}_0((n(b)a(t))\varphi)(\xi, w, y_m) = \phi(b,\xi,w,y_m)\mathcal{F}_0\varphi(t\xi, w, ty_m). \tag{5.13}$$

Hence

$$\theta(t,b) = Ct^n \sum_{\substack{\xi \in M^{-1}\mathbb{Z}^n \\ (w,y_m)\in W\times(\ell_0')^n+h'}} \phi(b,\xi,w,y_m)e(\xi^{\,t}h_1)\mathcal{F}_0\varphi(t\xi,w,ty_m). \tag{5.14}$$

The lemma now follows from an argument analogous to that of Lemma 5.1. □

Note however that one cannot conclude from the lemma that $\theta^*(t,b)$ is rapidly decreasing.

Lemma 5.5. *Suppose $p(y_{ij})$ is divisible by y_{1j} for some j but no higher power of y_{1j}. Then $\mathcal{F}_0\varphi$ is in the ideal I. Moreover, for every $v \in GL_n(\mathbb{R})$ and $u \in Sym_n(\mathbb{R})$ the function $\mathcal{F}_0(n'(u)a'(v)\varphi)$ is in the ideal I.*

Proof. The first statement is clear for we may write

$$\varphi(X) = y_{1j}e^{-\pi y_{1j}^2}\psi(X). \tag{5.15}$$

where $\psi(X)$ does not involve y_{1j}. Now taking $\mathcal{F}_0$ does not change the function since $y_{1j}e^{-\pi y_{1j}^2}$ is its own Fourier transform (up to the constant $-i$).

The second statement follows from the first one, the formulae (iii) and (iv) of Lemma 5.3 and the observation that if $\mathcal{F}\varphi \in I$, then $a^\times(v)\mathcal{F}\varphi \in I$ and $e\left(tr(u\frac{(w,w)}{2})\right)\mathcal{F}_0\varphi(\xi + \frac{1}{2}y_m u, w, y_m) \in I$. □

Corollary 5.6. *If φ satisfies the hypothesis of the lemma, then for every $\tau \in \mathbb{H}_n$, $\theta(t,b,\tau)$ decays exponentially on $\mathfrak{S}'$, where*

$$\theta(t,b,\tau) = \sum_{X\in(L^n+h)} p(Xv^{\frac{1}{2}})e^{-\pi(X,X)_{\tau,Z(t,b)}}. \tag{5.16}$$

We now check that the form ψ_{n-1} satisfies the hypothesis of the previous lemma. It is enough to do this for the individual components $\psi_{jk;\alpha_1,\cdots,\alpha_{n-1}}$

Lemma 5.7.

(i) $\mathcal{F}_0\psi_{jj;\alpha_1,\cdots,\alpha_{n-1}} \in I$ *for all* $1 \le \alpha_1 < \cdots < \alpha_{n-1} \le p$
(ii) $\mathcal{F}_0\psi_{jk;\alpha_1,\cdots,\alpha_{n-1}} \in I$ *for all* $1 \le \alpha_1 < \cdots < \alpha_{n-1} \le p$

Proof. (i) follows immediately from Lemma 5.5 and the explicit formulae for ψ_{n-1}: We have

$$\psi_{jj;\alpha_1,\cdots,\alpha_{n-1}}(X) = x_{mj}P^{(j)}_{\alpha_1,\cdots,\alpha_{n-1}}(X)\varphi_0(X) \tag{5.17}$$

$$= (y_{1j}-y_{mj})P^{(j)}_{\alpha_1,\cdots,\alpha_{n-1}}(X)\varphi_0(X). \tag{5.18}$$

Now observe that $P^{(j)}_{\alpha_1,\cdots,\alpha_{n-1}}(X)$ is a polynomial which does not involve y_{1j}.

(ii) is more complicated. By an argument similar to the previous one we find that $\mathcal{F}_0\psi_{jk;\alpha_1,\cdots,\alpha_{n-1}} \in I$ provided $\alpha_1 \neq 1$. However, for $\alpha_1 = 1$ it is no longer true. We have, assuming $j<k$,

$$\psi_{jk;\alpha_1,\cdots,\alpha_{n-1}} = \left[(-1)^{k-1}x_{mj}P^{(k)}_{1,\cdots,\alpha_{n-1}}(X) + (-1)^{j-1}x_{mk}P^{(j)}_{1,\cdots,\alpha_{n-1}}(X)\right]\varphi_0(X). \tag{5.19}$$

We expand $P^{(k)}_{1,\cdots,\alpha_{n-1}}(X)$ by the first row and obtain

$$P^{(k)}_{1,\cdots,\alpha_{n-1}}(X) = (-1)^{j-1}x_{1j}P^{(j,k)}_{\alpha_2,\cdots,\alpha_{n-1}}(X) + R_k, \tag{5.20}$$

where R_k is a polynomial not involving x_{1j}. Similarly,

$$P^{(j)}_{1,\cdots,\alpha_{n-1}}(X) = (-1)^{k-2}x_{1k}P^{(j,k)}_{\alpha_2,\cdots,\alpha_{n-1}}(X) + R_j. \tag{5.21}$$

We obtain

$$\psi_{jk}(X) \equiv 2(-1)^{j+k}(y_{1j}^2 - y_{1k}^2)P^{(j,k)}_{\alpha_2,\cdots,\alpha_{n-1}}(X)\varphi_0(X) \mod \mathcal{F}_0^{-1}I. \tag{5.22}$$

Taking $\mathcal{F}_0$ now shows that the right hand side is in $\mathcal{F}_0^{-1}I$. This boils down to the fact that the Fourier transform of $x^2e^{-\pi x^2}$ is $(\frac{1}{2\pi}-x^2)e^{-\pi x^2}$. □

We conclude

Theorem 5.8.
$\Xi(\tau,Z)$ is rapidly decreasing.

We now determine the growth of $\theta(\tau,Z)=\theta_{\varphi_n}(\tau,Z)$ on $\mathfrak{S}'$. Recall

$$\varphi_{\alpha_1,\cdots,\alpha_n}(X) = 2^{n/2}P_{\alpha_1,\cdots,\alpha_n}(X)\varphi_0(X) \tag{5.23}$$

and put $\theta_J(\tau,Z) = \sum_{X\in L^n+h}\varphi_{\alpha_1,\cdots,\alpha_n}(X)$ with $J=\{\alpha_1,\cdots,\alpha_n\}$.

We write

$$L^n + h = (\ell_0)^n + h_0) + (L^n\cap W^n + h_W) + (\ell_0')^n + h_0') \tag{5.24}$$

according to the decomposition (5.1).

The following lemma gives the growth of the components $\theta_J(\tau,Z)$ of $\theta(\tau,Z)$.

Lemma 5.9.

(i) *$\varphi_{\alpha_1,\cdots,\alpha_n}\in\mathcal{F}_0^{-1}I$ if and only if $\alpha_1=1$.*
(ii) *If $\alpha_1=1$, then $\theta_J(\tau,Z)$ has exponential decay on $\mathfrak{S}'$.*
(iii) *If $\alpha_1\neq 1$, then* $\theta_J(\tau,Z) = \begin{cases} O(t^n) & \textit{if} \quad h_0'\in(\ell_0')^n \\ O(t^ne^{-Ct^2}) & \textit{if} \quad h_0'\notin(\ell_0')^n. \end{cases}$
as $t\to\infty$.
(iv) *$\theta_J^*(\tau,Z)=O(t^n)$ if $h_0'\in(\ell_0')^n$.*

Proof. For (i) develop $P_{\alpha_1,\cdots,\alpha_n}(X)$ after the first row and proceed as in the proof of Lemma 5.7. (ii) follows from (i) and Lemma 5.4. For (iii), we first write $\varphi_{\alpha_1,\cdots,\alpha_n}(a(t)^{-1}n(b)^{-1}X) = 2^{n/2}P_{\alpha_1,\cdots,\alpha_n}(X)e^{-2\pi t^{-2}(\sum y_{1k}^2)-\pi tr(X',X')-2\pi t^2(\sum y_{mk}^2)}$ with $n(b)^{-1}X = (y_1, X', y_m)$. The assertion now follows from $\sum_{k\in\mathbb{Z}+h} e^{-\pi(k/t)^2} = O(t)$ as $t \to \infty$, which can be most easily seen by taking the Fourier transform, and $\sum_{k\in\mathbb{Z}+h'} e^{-\pi(kt)^2} = O(e^{-Ct^2})$ if and only if $h' \notin \mathbb{Z}$. This also implies (iv) in the case of $\alpha_1 \neq 1$. If $\alpha_1 = 1$, then (iv) reduces to $\sum_{k\in\mathbb{Z}+h} |\frac{k}{t}|e^{-\pi(k/t)^2} = O(t)$, which is an easy calculus exercise. □

Note that the condition $h_0' \in (\ell_0')^n$ certainly is equivalent to $h \in (\ell_0^\perp)^n$. Following ([10]) we call the congruence condition $h \in (L^\#)^n$ *non-singular* if for all frames $X \in h + L^n$ of rank n, the radical $R(X)$ is empty. Otherwise we call h singular.

Near the cusp given by ℓ_0 we can change the upper-half space coordinates (t, b) to (s, b) with $s = 1/t$. Then the restriction of a differential form on to the boundary component coming from ℓ_0 is given by setting $s = 0$ (and corresponds to $t \to \infty$).

Theorem 5.10. *(Theorem 1.1)*

(i) *$\theta(\tau)$ extends to the Borel-Serre boundary of M; i.e., defines a closed differential form on $\overline{M}$.*
(ii) *If h is non-singular or $n = p$, then $\theta(\tau)$ is rapidly decreasing on M; hence $\theta(\tau)|_{\partial\overline{M}} = 0$.*
(iii) *If h is singular, then the restriction of θ_φ to the component of the Borel-Serre boundary e_P coming from the parabolic P is the restriction of the theta series to the positive definite subspace W of V. More precisely, under the assumption (5.24),*

$$\theta_\varphi|_{e_P}(\tau, Z(b)) = \begin{cases} \sum\limits_{X\in L^n\cap(W^n+h_W)} \varphi(\tau, X, Z(b)) & \text{if} \quad h \in (\ell_0^\perp)^n \\ 0 & \text{if} \quad h \notin (\ell_0^\perp)^n. \end{cases}$$

Here

$$\varphi(\tau, X, Z(b)) = \sum_{2\leq\alpha_1<\cdots<\alpha_n\leq p} P_{\alpha_1,\cdots,\alpha_n}(X)\exp(-\pi tr(X,X)\tau)db_{\alpha_1}\wedge\cdots\wedge db_{\alpha_n}$$

for $X \in W^n$.

Proof. Everything follows from Corollary 4.2, Lemma 2.1, Lemma 5.9 and

$$\lim_{t\to\infty} t^{-n}\theta_{\alpha_1,\cdots,\alpha_n}(\tau, Z) = P_{\alpha_1,\cdots,\alpha_n}(X)\exp(-\pi tr(X,X)\tau)db_{\alpha_1}\wedge\cdots\wedge db_{\alpha_n}, \tag{5.25}$$

which is seen by taking the operator $\mathcal{F}_0$, Lemma 5.3 and Poisson summation. □

Remark 5.11. Theorem 5.10 (iii) also shows a nice functorial property of the Weil representation. We have

$$\left(\omega_{V(\mathbb{R})}(g'(\tau)\theta_\varphi)\right)|_{e_P}(Z(b)) = \omega_{W(\mathbb{R})}(g'(\tau))\left(\theta_\varphi|_W\right)(Z(b)),$$

where $\omega_{W(\mathbb{R})}$ is the Weil representation attached to the positive definite space $W(\mathbb{R})$ and $\theta_\varphi|_W$ is the theta series restricted to W.

By Theorem 5.10 we can now define for a closed differential $(p-n)$-form η on $\overline{M}$,

$$\Lambda(\eta)(\tau) = \int_M \eta(Z) \wedge \theta(\tau, Z). \tag{5.26}$$

This extends the lift considered in [14] to forms which do not vanish at the boundary.

Theorem 5.12. *(Theorem 1.2)*
$\Lambda(\eta)(\tau)$ *is a* <u>holomorphic</u> *Siegel modular form of weight* $m/2$.

Proof. This will now follow from Theorem 5.8 and the following calculation, see [14]:

$$\begin{aligned}\bar{\partial}\Lambda(\eta)(\tau) &= \bar{\partial}\int_M \eta(Z) \wedge \theta_{\varphi}(\tau, Z) = \int_M \eta(Z) \wedge \theta_{\bar{\partial}\varphi}(\tau, Z) \\ &= \int_M \eta(Z) \wedge \theta_{d\psi}(\tau, Z) = \int_M d(\eta(Z) \wedge \theta_{\psi}(\tau, Z)) = 0.\end{aligned} \tag{5.27}$$

The last equation is Stokes' Theorem. Here we need θ_ψ rapidly decreasing. □

Remark 5.13. In the analogous situation of locally symmetric spaces associated to orthogonal groups of arbitrary signature θ_ψ is not rapidly decreasing and the above argument breaks down. The theta integral is non-holomorphic in general, see [6]. In [14] it was assumed that η was rapidly decreasing and the above argument showed the holomorphicity of the theta integral.

6. The Singular Fourier Coefficients

In this section we compute the singular Fourier coefficients of the theta integral $\Lambda(\eta)(\tau)$.

For the β-th Fourier coefficient, we have

$$a_\beta(\eta) = \int_M \eta \wedge \sum_{X \in \Omega_\beta \cap (L^n + h)} \varphi(iv, Z, X) e^{-2\pi tr(\beta v)}. \tag{6.1}$$

First note that Prop. 4.4 implies that only for rank $(X) = n$ we have $\varphi(X) \neq 0$. Therefore $a_\beta = 0$ unless $\beta = \frac{1}{2}(X, X)$ is singular and positive semidefinite with $\operatorname{rank}(X) = n$ (or β positive definite).

Theorem 6.1. *(Theorem 1.4) Assume that β is positive semi-definite of rank $n-1$. Then*

$$a_\beta(\eta) = (-1)^n \int_{C_\beta} \eta.$$

Proof. With the notation of Section 3 we have

$$e^{2\pi tr(\beta v)} a_\beta = \int_{\Gamma\backslash B} \eta \wedge \sum_{X\in\Omega_\beta^s\cap(L^n+h)} \varphi(iv,Z,X) \tag{6.2}$$

$$= \int_{\Gamma\backslash B} \eta \wedge \sum_{j=0}^{r} \sum_{\gamma\in\Gamma_j\backslash\Gamma} \sum_{X\in\mathcal{L}_{\beta,j}} \gamma^*\varphi(iv,Z,X) \tag{6.3}$$

$$= \sum_{j=0}^{r} \int_{\Gamma\backslash B} \eta \wedge \sum_{\gamma\in\Gamma_j\backslash\Gamma} \sum_{X\in\mathcal{L}_{\beta,j}} \gamma^*\varphi(iv,Z,X) \tag{6.4}$$

$$= \sum_{j=0}^{r} \sum_{i=1}^{a_j} \int_{\Gamma\backslash B} \eta \wedge \sum_{\gamma\in\Gamma_j\backslash\Gamma} \sum_{X\in\mathcal{L}_{\beta,i,j}} \gamma^*\varphi(iv,Z,X). \tag{6.5}$$

Proposition 6.2.

$$\int_{\Gamma\backslash B} \eta \wedge \sum_{\gamma\in\Gamma_j\backslash\Gamma} \sum_{X\in\mathcal{L}_{\beta,i,j}} \gamma^*\varphi(iv,Z,X) = \int_{\Gamma_j\backslash B} \eta \wedge \sum_{X\in\mathcal{L}_{\beta,i,j}} \varphi(iv,Z,X).$$

Proof. The considerations in Section 5 imply that it is enough to show that $\sum_{X\in\mathcal{L}_{\beta,i,j}} \varphi(iv, a(t)^{-1}n(b)^{-1}X)$ is rapidly decreasing for $t\to\infty$ and $t\to 0$. Taking $m\in SL_n(\mathbb{Z})$ as in the proof of Lemma 3.4 we find via Prop. 4.4 that

$$\sum_{X\in\mathcal{L}_{\beta,U_{ij},h}} \varphi(iv, a(t)^{-1}n(b)^{-1}X) = \sum_{Y\in\mathcal{L}_{\beta_0,U_{ij},k}} \varphi(iv', a(t)^{-1}n(b)^{-1}Y) \tag{6.6}$$

with $v' = {}^t m^{-1} v m^{-1}$ and β_0, $k = hm$ and β as in (3.16). But now Lemma 6.4 and Lemma 5.9(i) show that we have $\varphi|_{\mathcal{L}_{\beta_0,U_{ij},k}} \in \mathcal{F}_0^{-1} I$ (in the notation of Section 5). As in (3.17), $Y\in\mathcal{L}_{\beta_0,U_{ij},k}$ is of the form $\left(\begin{smallmatrix} 0 & y_0 \\ & Y_1' \end{smallmatrix}\right)$ with $y_0\in(\ell_0)^n$ and finitely many possibilities for Y_1'. So we can apply $\mathcal{F}_0$ to the sum over $\mathcal{L}_{\beta_0,U_{i,j},k}$ and Lemma 5.5 gives the rapid decay as $t\to\infty$. The decay as $t\to 0$ is clear. □

Remark 6.3. In [10] and [14] unfolding was not attempted in the above situation. This led to considerable complications. In [10], the case $n = p$, Kudla introduced a wave packet attached to the standard Eisenstein series for $O(p,1)$ to compute the integral. The method employed in the following is conceptually much simpler (even though the actual calculations are quite similar). Moreover, it should be immediately available in the more general situation of [14] for not rapidly decreasing η.

We define a smooth differential n-form $\theta_{i,j}(Z)$ on $\Gamma_j\backslash B$ by

$$\theta_{i,j}(Z) = \sum_{X\in\mathcal{L}_{\beta,i,j}} \varphi(Z,X). \tag{6.7}$$

and put

$$\theta(\eta,\beta,U_{ij}) = \int_{\Gamma_j\backslash B} \eta\wedge\theta_{i,j}(Z). \tag{6.8}$$

Hence

$$e^{2\pi tr(\beta v)}a_\beta = \sum_{j=0}^{r}\sum_{i=1}^{a_j}\theta(\eta,\beta,U_{ij}). \tag{6.9}$$

We also define a function $\Phi(X,Z)$ via

$$\eta\wedge\varphi(Z,X) = \Phi(X,Z)d\mu, \tag{6.10}$$

where $d\mu = t^{-p}dt\wedge db_1\wedge\cdots\wedge db_{p-1}$ is the Riemannian volume form, and set

$$\Phi_{i,j}(Z) = \sum_{X\in\Omega_{\beta,i,j}}\Phi(Z,X). \tag{6.11}$$

Picking the standard fundamental domain for $\Gamma_j\backslash B$ we obtain

$$\theta(\eta,\beta,U_{ij}) = (-1)^{p-1}\int_0^\infty\left(\int_{\mathbb{R}^{p-1}/\Lambda_j}\Phi_{i,j}(Z(t,b))db\right)t^{-p}dt. \tag{6.12}$$

Recall (2.29) that the lattice $\Lambda_j\subset W\simeq N_j\simeq\mathbb{R}^{p-1}$ is given by $\Lambda_j\simeq\Gamma_j$. Here and from now on $N_j = N_j(\mathbb{R})$ and $W = W(\mathbb{R})$. We denote the torus $\Lambda_j\backslash W$ by $\mathbb{T}_j$.

We write $A_{i,j}(t)$ for the inner integral of (6.12). We have

$$A_{i,j}(t) = \int_{\Gamma_j\backslash N_j}\Phi_{i,j}(Z(t,b))db \tag{6.13}$$

$$= \int_{\Gamma_j\backslash N_j}\sum_{X\in\mathcal{L}_{\beta,i,j}}\Phi(Z(t,b),X)db \tag{6.14}$$

$$= \sum_{X\in\mathcal{L}_{\beta,i,j}}\int_{\Gamma_j\backslash N_j}\Phi(Z(t,b),X)db. \tag{6.15}$$

We also get a splitting

$$W = U_{ij}\cap W + (U_{ij}^\perp\cap W). \tag{6.16}$$

Now note that the right hand side of (6.15) is multiplicative under finite coverings! We pass to a subgroup $\tilde{\Gamma}_j\simeq\tilde{\Lambda}_j$ of finite index κ in Γ_j given by

$$\tilde{\Lambda}_j = (U_{ij}\cap\tilde{\Lambda}_j) + (U_{ij}^\perp\cap\tilde{\Lambda}_j) =: \Lambda' + \Lambda'' \tag{6.17}$$

and obtain a degree κ covering

$$\tilde{\mathbb{T}}_j = \mathbb{T}'\times\mathbb{T}_{U_{ij}}\to\mathbb{T}_j \tag{6.18}$$

with $\mathbb{T}' = \Lambda'\backslash U_{ij}$ and $\mathbb{T}_{U_{ij}} = \Lambda''\backslash U_{ij}^\perp$. We obtain

$$\kappa A_{i,j}(t) = \sum_{X\in\mathcal{L}_{\beta,i,j}}\int_{\mathbb{T}'\times\mathbb{T}_{U_{ij}}}\Phi(Z(t,b),X)db. \tag{6.19}$$

We will use the horospherical coordinates adopted to U_{ij}. We may choose a Witt basis $u,w_1,\dots,w_{p-1},u'$ with $u,w_1,\dots,w_{n-1}\in U_{ij}$. Then the horospherical coordinates $(t,b_1,\cdots,b_{p-1})$ are adopted to U_{ij}. The decomposition

$$Z(t,b) = Z(t,b',b'') \tag{6.20}$$

corresponds to the splitting (6.16). We write η in terms of these coordinates as

$$\eta(t,b) = f(t,b)db_n \wedge \cdots \wedge db_{p-1} + \eta'(t,b), \tag{6.21}$$

where $\eta'(t,b)$ is in the ideal of forms on $\Gamma_j \backslash B$ generated by $\{dt, db_1, \cdots, db_{n-1}\}$.

For $\varphi_n(Z(t,b',b''),X)$, we have

Lemma 6.4. *Suppose $U := span(X) = span\{u_0, w_1, \cdots, w_{n-1}\}$. Then*

$$\varphi_n(Z(t,b',b''),X) = 2^{n/2}\frac{1}{2}\det g(X)e^{-\pi(X,X)_{Z(t,b',0)}}\, t^{-n-1}dt \wedge db_1 \wedge \cdots \wedge db_{n-1},$$

where $g(X)$ is the matrix expressing the basis X for U in terms of the basis $\{u_0, w_1, \cdots, w_{n-1}\}$ for U; i.e.,

$$(x_1, \cdots, x_n) = (u_0, w_1, \cdots, w_{n-1})g(X).$$

Proof. We have $\hat{X} = (x_{ij})$ with

$$x_{ij} = (x_j, e_i) \qquad \text{for} \quad 1 \leq i \leq p \qquad \text{and} \qquad x_{p+1,j} = -(x_j, e_{p+1}). \tag{6.22}$$

Moreover $u_0 = \frac{1}{2}(e_1 + e_{p+1})$. But by assumption $(x_j, u_0) = 0$ for all j and $(x_j, e_i) = 0$ for $i \geq n$. It follows immediately that the only vanishing Plücker coordinate $P_{j_1,\cdots,j_n}(X)$, which is non-zero, is $P_{1,2,\cdots,n}$, and this has value $\frac{1}{2}\det g(X)$.

We next observe

$$\varphi_n(Z(t,b),X) = \varphi_n(Z_0, a(t)^{-1}n(b)^{-1}X). \tag{6.23}$$

Now we have (using the previous formulas)

$$\det g\left(a(t)^{-1}n(b)^{-1}X\right) = t^{-1}\det g(X), \tag{6.24}$$

$$\exp(-\pi(X,X)_{Z(t,b)}) = \exp(-\pi(X,X)_{Z(t,b',0)}), \tag{6.25}$$

and

$$\nu_1 \wedge \cdots \wedge v_n = t^{-n}dt \wedge db_1 \wedge \cdots \wedge db_{n-1}. \tag{6.26}$$

The lemma now follows from Corollary 4.2

□

Writing $h(X,t,b') = \det(v)^{1/2}2^{n/2}\frac{1}{2}\det g(X)t^{-1}\varphi_0(iv, a(t)^{-1}n(b')^{-1}X)$, Lemma 6.4 and (6.21) give

(6.27)
$$\eta \wedge \varphi(iv, Z(t,b',b'')) = (-1)^{(p-n)n}t^{-n}h(X,t,b')f(t,b',b'')dt \wedge db_1 \wedge \cdots \wedge db_{p-1}$$

and

$$\Phi(X, Z(t,b',b'')) = (-1)^{(p-n)n}t^{p-n}h(X,t,b')f(t,b',b''). \tag{6.28}$$

Thus the inner integral in (6.12) is given by

$$\kappa A_{i,j}(t) = (-1)^{(p-n)n}t^{p-n}\sum_{X \in \mathcal{L}_{\beta,i,j}}\int_{\mathbb{T}'} h(X,t,b')\left(\int_{\mathbb{T}_{U_{ij}}} f(t,b',b'')db''\right)db'. \tag{6.29}$$

But the inner integral is equal to the period of the differential form η over the closed cycle $C_{U_{ij}}(t,b') \subset \Gamma_j \backslash B$ given by

$$C_{U_{ij}}(t,b') = (-1)^{(n-1)(p-n)} n(b')a(t)\mathbb{T}_{U_{ij}}. \tag{6.30}$$

(For the sign, see Remark 3.1). Since η is closed, the period is independent of b' and t, and we obtain

$$\begin{aligned}\kappa A_{i,j}(t) = 2^{n/2}\det(v)^{\frac{1}{2}}(-1)^{p-n}t^{p-n-1}\left(\int_{C_{U_{ij}}}\eta\right) \\ \times \sum_{X\in\mathcal{L}_{\beta,i,j}} \det g(X)\int_{\mathbb{T}'}\varphi_0(iv,a(t)^{-1}n(b')^{-1}X)db'.\end{aligned} \tag{6.31}$$

We now unfold

$$I = \sum_{X\in\mathcal{L}_{\beta,i,j}} \int_{\mathbb{T}'} \det g(X)\varphi_0(iv,a(t)^{-1}n(b')^{-1}X)db'. \tag{6.32}$$

We observe $\mathbb{T}' \simeq \tilde{\mathbb{T}}_j/\mathbb{T}_{U_{ij}} \to \mathbb{T}_j/\mathbb{T}_{U_{ij}} \simeq \frac{N/N_{U_{ij}}}{\Gamma_j/\Gamma_{U_{ij}}}$ is a covering of degree κ.

We let $\mathcal{D}''$ be a fundamental domain for $\Gamma_j/\Gamma_{U_{ij}}$ in $N/N_{U_{ij}}$. By Lemma 3.3 we have

$$I = \kappa \sum_{X\in\mathcal{C}_{\beta,i,j}} \det g(X) \sum_{\gamma\in\Gamma_j/\Gamma_{U_{ij}}} \int_{\mathcal{D}''}\varphi_0(a(t)^{-1}n(b)^{-1}\gamma^{-1}X)db \tag{6.33}$$

$$= \kappa \sum_{X\in\mathcal{C}_{\beta,i,j}} \det g(X) \int_{N/N_{U_{ij}}}\varphi_0(a(t)^{-1}n(b)^{-1}X)db. \tag{6.34}$$

So we have proved

Proposition 6.5.

$$\begin{aligned}A_{i,j}(t) = 2^{n/2}\det(v)^{\frac{1}{2}}(-1)^{(p-n)}t^{p-n-1}\left(\int_{C_{U_{ij}}}\eta\right) \\ \times \sum_{X\in\mathcal{C}_{\beta,i,j}} \det g(X)\int_{N'}\varphi_0(iv,a(t)^{-1}n(b')^{-1}X)db'.\end{aligned}$$

For the integral, we write

$$I(t,X) = \int_{N'} \exp(-\pi tr(X,X)_{iv,Z(t,b')})db', \tag{6.35}$$

and it is not hard to see ([10] Lemma 5.3)

Lemma 6.6.

$$I(t,X) = 2^{-(n-1)/2}t^{n-1}e(\frac{1}{2}tr(i\beta v))(\det v)^{-\frac{1}{2}}|\det g(X)|^{-1}\xi^{-\frac{1}{2}}\exp(-\frac{\pi}{2}t^{-2}\xi^{-1}),$$

where

$$\xi = \xi(X) = \frac{\det v[{}^t g_1(X)]}{\det v\ \det g(X)^2}.$$

Here $g(X) = \left(\begin{smallmatrix} g_0(X) \\ g_1(X) \end{smallmatrix}\right)$, *where* $g_0(X)$ *is the first row and* $g_1(X)$ *an* $(n-1)$ *by* n *matrix.*

We are now ready to compute $\theta(\eta, \beta, U_{ij})$. We have

$$\theta(\eta, \beta, U_{ij}) = \frac{(-1)^{(n-1)}}{2^{1/2}} \left(\int_{C_{U_{ij}}} \eta \right) e\left(\frac{1}{2} tr(i\beta v)\right) \times \sum_{X \in \mathcal{C}_{\beta,i,j}} \int_0^\infty sgn \det(g(X)) \xi^{-\frac{1}{2}} \exp\left(-\frac{\pi}{2} t^{-2} \xi^{-1}\right) t^{-2} dt \tag{6.36}$$

At this point interchanging of summation and integration in (6.36) is not allowed. Instead, we define for $s \in \mathbb{C}$,

$$I(s) = \int_0^\infty \sum_{X \in \mathcal{C}_{\beta,i,j}} sgn \det(g(X)) \xi^{-\frac{1}{2}} \exp\left(-\frac{\pi}{2} t^{-2} \xi^{-1}\right) t^{-2-s} dt. \tag{6.37}$$

Via a similar argument as in Prop. 6.2, the sum is rapidly decreasing as $t \to \infty$ so that $I(s)$ is entire and for $Re(s) > 1$ we can interchange integration and summation by an argument similar to Lemma 5.9(iv); see also the proof of Prop.6.7 below. Noting $sgn \det(g(X)) = \epsilon(X)$ in the notation of Section 3, we obtain

$$I(s) = \sum_{X \in \mathcal{C}_{\beta,i,j}} \epsilon(X) \xi^{-\frac{1}{2}} \int_0^\infty \exp\left(-\frac{\pi}{2} t^{-2} \xi^{-1}\right) t^{-2-s} dt \tag{6.38}$$

$$= 2^{(s-1)/2} \pi^{-(s+1)/2} \Gamma\left(\frac{1+s}{2}\right) \sum_{X \in \mathcal{C}_{\beta,i,j}} \epsilon(X) \xi^{\frac{s}{2}}. \tag{6.39}$$

The above series is (up to the factor $\det(v)^{-s}$) the Dirichlet series

$$\Omega(s, v, \beta) = \sum_{X \in \mathcal{C}_{\beta,i,j}} f(s, v, g(X)), \tag{6.40}$$

where $f : \mathbb{C} \times \mathbb{P}_n \times GL_n(\mathbb{R}) \longrightarrow \mathbb{C}$ is given by

$$f(s, v, g) = \frac{sgn \det g}{|\det g|^s} \det v[{}^t g_1]^s \tag{6.41}$$

and $g = \left(\begin{smallmatrix} g_0 \\ g_1 \end{smallmatrix}\right)$ as in Lemma 6.6.

Proposition 6.7 ([10]). $\Omega(s, v, \beta)$ *has an analytic continuation into the entire complex plane and*

$$\Omega(0, v, \beta) = -\sum_{X \in \mathcal{C}^{red}_{\beta,i,j}} \mathbf{B}_1(\nu(X)) sgn \det(g(X)). \tag{6.42}$$

Proof. Again we take $m \in SL_n(\mathbb{Z})$ as in the proof of Lemma 3.4 and obtain

$$\Omega(s,v,\beta) = \sum_{X \in \mathcal{C}_{\beta,U_{ij},h}} f(s,v,g(X)) \tag{6.43}$$

$$= \sum_{Y \in \mathcal{C}_{\beta_0,U_{ij},k}} f(s,v',g(Y)). \tag{6.44}$$

in the notation of the proof of Lemma 3.4 with $v' = {}^t m^{-1} v m^{-1}$. Note $g(Y) = y_{01} \det(Y_1')$. Then

$$\sum_{Y \in \mathcal{C}_{\beta_0,U_{ij},k}} f(s,v',g(Y)) = \sum_{y_{01} \equiv k_{01}} sgn(y_{01}) |y_{01}|^{-s} \sum_{Y_1'} sgn \det(Y_1') \frac{\det v'[{}^t Y'_1]^s}{|\det Y_1'|^s}. \tag{6.45}$$

The sum over Y_1' is finite and can be evaluated for $s = 0$, while the first is equal to $H(k_{01}, s) - H(1-k_{01}, s)$, where $H(x,s) = \sum_{n=0}^{\infty} (x+n)^{-s}$ is the Hurwitz ζ-function. The series converges for $Re(s) > 1$ and has an analytic continuation to the whole complex plane. Observing $H(x,0) = \frac{1}{2} - x = \mathbf{B}_1(x)$ for $x \in [0,1)$ finishes the proof of the proposition. Note here that the two Hurwitz ζ-functions correspond to the two reduced elements in one $\mathbb{Z}$-class in $\mathcal{C}_{\beta_0,U_{ij},k}$. □

Hence

$$I(0) = -2^{-1/2} \sum_{X \in \mathcal{C}^{red}_{\beta,i,j}} \mathbf{B}_1(\nu(X)) sgn \det(g(X)) \tag{6.46}$$

and therefore

$$\theta(\eta,\beta,U_{ij}) = (-1)^n \sum_{X \in \mathcal{C}^{red}_{\beta,i,j}} \frac{1}{2} \mathbf{B}_1(\nu(X)) \epsilon(X) e^{-2\pi tr(\beta v)}. \tag{6.47}$$

Considering (6.9) in conjunction with the definition of the cycle C_β from Section 3 concludes the proof of Theorem 6.1!

□

7. The Positive Definite Fourier Coefficients

7.1. **The defect $\delta_\beta(\eta)$.** For $\beta > 0$, the main point of [11, 12, 13] (in much greater generality) is that the Fourier coefficient

$$\theta_\beta = \sum_{X \in \Omega_\beta \cap (L^n + h)} \varphi(iv, Z, X) e^{-2\pi tr(\beta v)} \tag{7.1}$$

$$= \sum_{X \in \Gamma \backslash \Omega_\beta \cap (L^n+h)} \sum_{\gamma \in \Gamma_X \backslash \Gamma} \gamma^* \varphi(iv, Z, X) e^{-2\pi tr(\beta v)} \tag{7.2}$$

is a Poincaré dual form for the composite cycle C_β, i.e.,

$$a_\beta(\eta) = \int_M \eta \wedge \theta_\beta = \int_{C_\beta} \eta \tag{7.3}$$

for all $\eta \in Z^k_{rd}(M)$, the rapidly decreasing closed k-forms in M, and $k = p - n$. (Actually, the case $n = p - 1$ is not treated there, but for $p = 2$ and $n = 1$ we will show below that this is still true).

Furthermore, by Stokes' Theorem, (7.3) also holds on the space of relative coboundaries $B^k(\overline{M}, \partial\overline{M})$. Slightly more general we have

Lemma 7.1. *If η is an exact k-form vanishing on $\partial\overline{M}$, then (7.3) holds.*

Proof. We consider the inclusion $i : M \longrightarrow \overline{M}$ and note that as a consequence of the relationship between duality on $H^*(\overline{M})$ and duality on $H^*(\partial\overline{M})$, we obtain

$$[i^*\theta_\beta] = PD[\partial_*[C_\beta]], \tag{7.4}$$

see e.g. [2], Th. 9.2, p. 357. We write $\eta = d\omega$, whence the restriction of ω to $\partial\overline{M}$ is closed. Then

$$\int_M \eta \wedge \theta_\beta = \int_{\partial\overline{M}} \omega \wedge \theta_\beta = \int_{\partial C_\beta} \omega = \int_{C_\beta} \eta. \tag{7.5}$$

Here the second equality follows from (7.4) and that ω is closed on $\partial\overline{M}$. ☐

However, (7.3) will not hold for all $\eta \in Z^k(\overline{M})$ unless C_β is compact (which can only happen for $k = p - n \leq 4$).

In fact, we define the defect

$$\delta_\beta(\eta) = a_\beta\,(\eta) - \int_{C_\beta} \eta \tag{7.6}$$

for $\eta \in Z^k(\overline{M})$. By the above discussion, δ_β factors through

$$Z^k_{rd}(M) + B^k(\overline{M}, \partial\overline{M}) = Z^k(\overline{M}, \partial\overline{M}), \tag{7.7}$$

the closed forms vanishing at the boundary. The equality in (7.7) follows from the fact that $H^k(\overline{M}, \partial\overline{M}, \mathbb{C}) \simeq H^k_c(M, \mathbb{C})$ has a system of representatives consisting of rapidly decreasing forms. So we proved

Lemma 7.2. *For $n < p - 1$ or $p = 2$ and $n = 1$, δ_β descends to a map*

$$\delta_\beta : \frac{Z^k(\overline{M})}{Z^k(\overline{M}, \partial\overline{M})} \longrightarrow \mathbb{C}. \tag{7.8}$$

We take a neighborhood U of the boundary of $\overline{M}$ such that $\partial\overline{M}$ is a deformation retract of U and obtain a projection map $\pi : U \longrightarrow \partial\overline{M}$. We pick a smooth 'bump' function ρ on M supported in U with $\rho|_V = 1$ for another neighborhood $V \subset U$ and define a map

$$\iota : A^{k-1}(\partial\overline{M}) \longrightarrow Z^k(\overline{M}) \tag{7.9}$$

by $\iota(\omega) = d\,(\rho\pi^*(\omega))$ on U and $\iota(\omega) = 0$ elsewhere. Note that $\iota(\omega)|_{\partial\overline{M}} = d\omega$.

Lemma 7.3. *We have the following exact sequence*

$$0 \longrightarrow \frac{A^{k-1}(\partial\overline{M})}{Z^{k-1}(\partial\overline{M})} \xrightarrow{\bar{\iota}} \frac{Z^k(\overline{M})}{Z^k(\overline{M}, \partial\overline{M})} \xrightarrow{\bar{r}} H^k(\partial\overline{M}, \mathbb{C}). \tag{7.10}$$

Here $\bar{r}$ is the quotient of the restriction map $r : Z^k(\overline{M}) \longrightarrow Z^k(\partial\overline{M})$ to the boundary; in general this is not surjective. Also note that $\bar{\iota}$ is independent of the choices involved, so that (7.10) is intrinsic to the situation.

We can therefore study the map (7.8) via the exact sequence (7.10).

Proposition 7.4. *δ_β is not identically zero on (the image of) $\frac{A^{k-1}(\partial\overline{M})}{Z^{k-1}(\partial\overline{M})}$.*

Proof. Let $\omega \in A^{k-1}(\partial\overline{M})$. Then the calculation (7.5) for $\iota(\omega)$ is no longer valid (unless ω is closed) - and it is clear that there are examples so that (7.5) does not hold, i.e., $\delta_\beta(\iota(\omega)) \neq 0$. For Riemann surfaces, we make this more explicit in the next section. □

It is very tempting to investigate the other piece coming from $H^k(\partial\overline{M}, \mathbb{C})$ using Eisenstein cohomology. We carry this out for the Riemann surface case.

7.2. **The defect for Riemann surfaces.** For the remainder of this section we consider the special case of $SO_0(2,1)$. In particular, we prove the Theorems 1.5 and 1.6.

There is a double covering $SL_2(\mathbb{R}) \longrightarrow SO_0(2,1)$ and the symmetric space D is just the upper half plane $\mathbb{H}$. We therefore work with SL_2 in this section. Accordingly, we change notation and write $z = x + iy \in D \simeq \mathbb{H}$ for the orthogonal variable. We write dx_i for the basic differential form of the boundary component $\mathbb{T}_i$ of the Borel-Serre compactification corresponding to the cusp ℓ_i. Hence $dx_i = (g_i^{-1})^* dx$. Finally, for convenience we assume that $\Gamma = \Gamma(N)$, the principal congruence subgroup. Hence the width of all cusps is equal to N.

We first illustrate that the defect is not identically zero on $\frac{A^0(\partial\overline{M})}{Z^0(\partial\overline{M})}$.

By Theorem 5.10, the restriction of $\theta(\tau, z)$ to a boundary component $\mathbb{T}_i$ is given by

$$\theta(\tau, z)|_{\mathbb{T}_i} = \left(\sum_{X \in W_i \cap (L+h)} P_2(X) e^{\pi i (X,X)\tau} \right) dx_i =: \theta_i(\tau) dx_i \tag{7.11}$$

(For the isotropic line ℓ_i defining a cusp, $W_i = \ell_i^\perp / \ell_i$ is one-dimensional, and identifying $W_i(\mathbb{R})$ with $\mathbb{R}$ we have $P_2(X)e^{\pi i (X,X)} = X e^{\pi i X^2}$.)

Pick a function $f \in A^0(\partial M) = \oplus_i A^0(\mathbb{T}_i)$ only supported at the cusp ℓ_0. Then

$$\int_M \iota(f) \wedge \theta(\tau, z) = \left(\int_{\mathbb{T}_0} f(x) dx \right) \theta_0(\tau); \tag{7.12}$$

i.e., $A_\beta(\iota(f))$ is up to a factor the integral of f over the whole boundary circle. On the other hand,

$$\int_{C_\beta} \iota(f) = \int_{\partial C_\beta \cap \mathbb{T}_0} f, \tag{7.13}$$

which is the oriented sum of the evaluations of f at the boundary points of C_β. It is certainly easy to find f, where these two terms are not the same; i.e., $\delta_\beta(\iota(f)) \neq 0$.

We briefly review the relevant facts for Eisenstein series and Eisenstein cohomology needed.

We introduce the tangential Eisenstein series for the cusp ℓ_i,

$$E_i^T(s,z) = \sum_{\gamma\in\Gamma_i\backslash\Gamma} Im(g_i^{-1}\gamma)^*(y^s dx) \tag{7.14}$$

with $s\in\mathbb{C}$. We easily see

$$E_i^T(s,z) = \frac{1}{2y}\left(E_i(s+1,z)_{-2}dz + E_i(s+1,z)_2 d\bar{z}\right) \tag{7.15}$$

with

$$E_i(s,z)_n = \sum_{\gamma\in\Gamma_i\backslash\Gamma} Im(g_i^{-1}\gamma z)^s \lambda(g_i^{-1}\gamma,z)^n, \tag{7.16}$$

where $\lambda(g,z) = \frac{cz+d}{|cz+d|}$ for $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$.

The following theorem is well known, convenient references are [8] and [5].

Theorem 7.5. (i) *The series $E_i(s,z)_{\pm 2}$ converge for $s>1$ and have a meromorphic continuation to $\mathbb{C}$. At $s=1$, $E_i(s,z)_{\pm 2}$ are holomorphic, and the Fourier expansions $E_{ij}(1,z)_{\pm 2}$ at a cusp ℓ_j are given by*

$$\frac{1}{y}E_{ij}(1,z)_{-2} = \left(\delta_{ij} + \frac{1}{y}a_{ij}(0)\right) + \sum_{m=1}^{\infty} a_{ij}(m)e^{2\pi i m z/N}, \tag{7.17}$$

$$\frac{1}{y}E_{ij}(1,z)_{2} = \left(\delta_{ij} + \frac{1}{y}a_{ij}(0)\right) + \sum_{m=1}^{\infty} \overline{a_{ij}(m)}e^{2\pi i m \bar{z}/N}. \tag{7.18}$$

(ii) *The tangential Eisenstein series $E_i^T(s,z)$ is holomorphic at $s=0$ and defines a harmonic 1-form on M, which extends to the boundary. For two different cusps i and j, the difference*

$$E_i^T(0,z) - E_j^T(0,z) \tag{7.19}$$

is closed, and its restriction to the boundary is $dx_i - dx_j \in Z^1(\partial M) = \oplus_k Z^1(\mathbb{T}_k)$. We call the space of all linear combination of tangential Eisenstein series consisting of closed forms $\mathcal{E}_0$.

(iii) *The cohomology $H^1(M,\mathbb{C})$ splits as*

$$H^1(M,\mathbb{C}) = H^1_{(2)}(M,\mathbb{C}) \oplus H^1_{Eis}(M,\mathbb{C}),$$

where $H^1_{Eis}(M,\mathbb{C})$ is the image of $\mathcal{E}_0$ in $H^1(M,\mathbb{C})$, while $H^1_{(2)}(M,\mathbb{C})$ is the L_2-cohomology. Its classes can be represented by weight-2 cusp forms. Note $H^1_{(2)}(M,\mathbb{C}) \simeq H^1_!(M,\mathbb{C}) := Im\,(H^1_c(M,\mathbb{C}) \to H^1(M,\mathbb{C}))$.

Theorem 1.5 now will follow from the vanishing of the defect δ_β for tangential Eisenstein series and weight-2 cusp forms. Via (7.1) we have to show

$$\int_{\Gamma\backslash B} \eta \wedge \sum_{\gamma\in\Gamma_X\backslash\Gamma} \gamma^*\varphi(iv,z,X) = e^{-\pi(X,X)}\int_{C_X}\eta. \tag{7.20}$$

for $(X,X)>0$. $X^\perp$ has signature $(1,1)$ and therefore the stabilizer Γ_X is either infinitely cyclic or trivial. In the first case, the cycle C_X is a closed geodesic and (7.20) holds for any 1-form η. When the stabilizer is trivial, the cycle C_X is an infinite geodesic joining two cusps.

Theorem 7.6. *Assume C_X is an infinite geodesic. Then*

$$\int_{\Gamma\backslash B} E_i^T(0,z)\wedge\sum_{\gamma\in\Gamma}\gamma^*\varphi(z,X) = \; e^{-\pi(X,X)}\int_{C_X} E_i^T(0,z). \tag{7.21}$$

Proof. First note that unfolding in (7.21) is not allowed. Recall we have a Witt basis u_0, w, u_0' for V, and we can assume that $X = 2au_0 + bw$ with $a\in\mathbb{Q}$ and $b\in\mathbb{Q}_+$ so that C_X is the geodesic joining the cusps ∞ and $\frac{a}{b}\in\mathbb{Q}$. The stabilizer of the cusp ∞ is $\Gamma_\infty=\Gamma_\infty(N)=\{\left(\begin{smallmatrix}1&k\\0&1\end{smallmatrix}\right): k\in N\mathbb{Z}\}$.

We have

$$\int_{\Gamma\backslash B} E_i^T(0,z)\wedge\sum_{\gamma\in\Gamma}\gamma^*\varphi(z,X) = \int_{\Gamma\backslash B} E_i^T(0,z)\wedge\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{k\in N\mathbb{Z}}\gamma^*\varphi(z,X+2kbu_0). \tag{7.22}$$

We introduce a holomorphic function $I(s)$ for $s\in\mathbb{C}$ by

$$I(s) = \int_{\Gamma\backslash B} E_i^T(0,z)\wedge\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{k\in N\mathbb{Z}}\gamma^*\Big(y^s\varphi(z,X+2kbu_0)\Big) \tag{7.23}$$

and unfold

$$I(s) = \int_{\Gamma_\infty\backslash B} E_i^T(0,z)\wedge\sum_{k\in N\mathbb{Z}} y^s\varphi(z,X+2kbu_0). \tag{7.24}$$

To justify this we first need some explicit formulae for φ. We have

$$\varphi(z,X+2kbu_0) = \sqrt{2}be^{-\pi(2\frac{(a-xb+kb)^2}{y^2}+b^2)}\frac{dx}{y} + \sqrt{2}\left(\frac{a-xb+kb}{y}\right)e^{-\pi(2\frac{(a-xb+kb)^2}{y^2}+b^2)}\frac{dy}{y} \tag{7.25}$$

$$=: \varphi_1(k,X)dx+\varphi_2(k,X)dy, \tag{7.26}$$

so that the Fourier transform with respect to k is given by

$$\widehat{\varphi}(z,X+2kbu_0) = \widehat{\varphi_1}(k,X)dx+\widehat{\varphi}_2(k,X)dy \tag{7.27}$$

with

$$\widehat{\varphi_1}(k,X) = e^{-\pi b^2}e^{-\pi\frac{(yk)^2}{2b^2}}e^{-2\pi ikx}e^{2\pi i\frac{a}{b}k}, \tag{7.28}$$

$$\widehat{\varphi}_2(k,X) = -i\frac{ky}{2b^2}e^{-\pi b^2}e^{-\pi\frac{(yk)^2}{2b^2}}e^{-2\pi ikx}e^{2\pi i\frac{a}{b}k}. \tag{7.29}$$

By Poisson summation and (7.15) we obtain

$$I(s) = \int_{\Gamma_\infty\backslash B} \frac{1}{2y}(E_i(1,z)_{-2} + E_i(1,z)_2)\left(\frac{1}{N}\sum_{k\in\frac{1}{N}\mathbb{Z}} \widehat{\varphi_2}(k,X)\right) y^s \tag{7.30}$$

$$+ \frac{-i}{2y}(E_i(1,z)_{-2} - E_i(1,z)_2)\left(\frac{1}{N}\sum_{k\in\frac{1}{N}\mathbb{Z}} \widehat{\varphi_1}(k,X)\right) y^s dxdy, \tag{7.31}$$

and this is rapidly decreasing as $y \to \infty$ and of moderate growth as $y \to 0$. Hence unfolding is valid for $Re(s)$ sufficiently large. We pick the standard fundamental domain for $\Gamma_\infty\backslash B$ and integrate w.r.t. x. This picks out the 0-th Fourier coefficient:

$$I(s) = \frac{-iy}{4b^2N}e^{-\pi b^2}\int_0^\infty \sum_{m=1}^\infty (\chi(m)a_i(m) - \overline{\chi(m)a_i(m)})me^{-\pi\frac{(my)^2}{2(bN)^2}-2\pi\frac{my}{N}}y^s dy \tag{7.32}$$

$$-\frac{i}{2}e^{-\pi b^2}\int_0^\infty \sum_{m=1}^\infty (\chi(m)a_i(m) - \overline{\chi(m)a_i(m)})e^{-\pi\frac{(my)^2}{2(bN)^2}-2\pi\frac{my}{N}}y^s dy, \tag{7.33}$$

where $\chi(m) = e^{2\pi i\frac{a}{b}m/N}$. Hence

$$I(s) = \frac{-ie^{-\pi b^2/2}}{2\sqrt{2}b}\sum_{m=1}^\infty (\chi(m)a_i(m) - \overline{\chi(m)a_i(m)}) \tag{7.34}$$

$$\times \int_0^\infty \left(\frac{my}{\sqrt{2}bN} + \sqrt{2}b\right)e^{-\pi\left(\frac{my}{\sqrt{2}bN}+\sqrt{2}b\right)^2}y^s dy$$

$$= \frac{-iNe^{-\pi b^2/2}}{4}(\sqrt{2}bN)^s\left(L(E_i(1,z)_{-2},\chi,s+1) - L(E_i(1,z)_2,\overline{\chi},s+1)\right) \tag{7.35}$$

$$\times \int_{b^2/2}^\infty \left(\sqrt{t}-\sqrt{2}b\right)^s e^{-\pi t}dt,$$

where $L(E_i(1,z)_{\pm 2},\dots,s)$ are the (twisted) L-functions attached to $E_i(1,z)_{-2}$ and $E_i(1,z)_2$. Specializing to $s = 0$ finally gives

$$\int_M E_i^T(0,z)\wedge\sum_{\gamma\in\Gamma}\gamma^*\varphi(Z,X) = \frac{-iN}{4\pi}(L(E_i(1,z)_{-2},\chi,1) - L(E_i(1,z)_2,\overline{\chi},1))e^{-\pi b^2}. \tag{7.36}$$

But now one easily checks that

$$\int_{C_X} E_i^T(0,z) = \frac{-iN}{4\pi}(L(E_i(1,z)_{-2},\chi,1) - L(E_i(1,z)_2,\overline{\chi},1)). \tag{7.37}$$

This proves the theorem. □

Remark 7.7. The given proof (or a slightly simpler version of it) also works for $\eta = f(z)dz$ with $f(z)$ a weight-2 cusp form. This is important, since for C_X an infinite geodesic, the proof of the basic identities (7.20) and (7.3) for η rapidly decreasing is actually not included in [13],[14].

Because of Stokes' theorem we have $\int_{\partial \overline{M}} \theta(\tau, z) = 0$ and therefore $\sum_i \theta_i(\tau) = 0$. Thus

$$Eis(\theta)(\tau, z) := \sum_i \theta_i(\tau) E_i^T(0, z) \tag{7.38}$$

defines a closed differential form in $\overline{M}$ with values in the holomorphic cusp forms of weight 3/2, and we define the truncated theta function

$$\theta^c(\tau, z) = \theta(\tau, z) - Eis(\theta, z). \tag{7.39}$$

So $\theta^c(\tau)$ is per construction a rapidly decreasing closed differential 1-form in M with values in the non-holomorphic modular forms of weight 3/2. We write

$$\theta^c(\tau, z) = \sum_\beta \theta^c_\beta(v, z) e^{2\pi i \beta \tau} \tag{7.40}$$

for the Fourier expansion. For $\eta = f(z)dz$ with $f(z)$ a weight-2 cusp form we still have

$$\int_M \eta \wedge \theta^c_\beta = \int_M \eta \wedge \theta_\beta = \int_{C_\beta} \eta, \tag{7.41}$$

as cusp forms are orthogonal to Eisenstein series. (By Theorem 7.6, (7.41) actually also holds for tangential Eisenstein since it is not too hard to show that the integral of the wedge of two tangential Eisenstein series vanishes.) This justifies the

Definition 7.8. We define C^c_β to be the homology class dual to the Fourier coefficient θ^c_β.

C^c_β does not depend on v since (7.41) and Th. 7.6 show that $\int_M \eta \wedge \theta^c_\beta$ indeed does not depend on v.

This discussion proves Theorem 1.6.

8. The Theta Integral over Special Cycles

We can also define a lift

$$\Lambda(\tau, C_U) = \int_{C_U} \theta_{\varphi_n}(\tau, Z), \tag{8.1}$$

where C_U is the special cycle coming from a positive definite subspace U of dimension $p-n$ in V. Note that C_U has dimension n.

We write $L_U = L \cap U$ and $L_{U^\perp} = L \cap U^\perp$ and obtain a decomposition

$$L^n + h = \sum_{i=1}^s (L^n_U + h'_i) \ + \ (L^n_{U^\perp} + h''_i) \tag{8.2}$$

with $h'_i \in \left(L^\#_u\right)^n$ and $h''_i \in \left(L^\#_{U^\perp}\right)^n$. By $\theta_{C_U}(\tau, L_{U^\perp} + h''_i)$ we denote the top degree theta integral $\Lambda(\tau, 1) = \int_{C_U} \sum_{X \in (L^n_{U^\perp} + h''_i)} \varphi_n(\tau, X)$ for the hyperbolic space C_U. Note that the top degree lift $\Lambda(\tau, 1)$ was computed in [10].

Proposition 8.1. *With the above notation, we have*

$$\Lambda(\tau, C_U) = \sum_{i=1}^{s} \vartheta(\tau, L_U + h'_i)\theta_{C_U}(\tau, L_{U^\perp} + h''_i)$$

where $\vartheta(\tau, L_U + h'_i) = \sum_{X\in(L_U^n+h'_i)} e^{\pi i tr((X,X)\tau)}$ *is the standard theta series of degree* n *for the positive definite space* U.

Proof. Using the explicit formula for $\varphi_n = \varphi_{n,V}$ from Section 4 one easily checks that under the pullback $i_U^* : \mathcal{A}^n(B) \longrightarrow \mathcal{A}^n(B_U)$

$$i_U^*\varphi_{n,V} = \varphi_{0,U} \otimes \varphi_{n,U^\perp}, \tag{8.3}$$

where $\varphi_{0,U}$ is just the standard Gaussian for the positive definite space U. From this the proposition easily follows. □

Theorem 8.2. *(Theorem 1.6)*

$$\Lambda(\tau, C_U) = \sum_{\beta>0}[C_U.C_\beta]_{tr}e^{2\pi i tr(\beta\tau)} \;+\; (-1)^n \sum_{\substack{\beta\geq 0\\ rk(\beta)=n-1}} [C_U.C^s_\beta]e^{2\pi i tr(\beta\tau)},$$

where $[C_U.C_\beta]_{tr}$ *is the transversal intersection number of* C_U *and* C_β *(i.e., the sum of the transverse intersections counted with multiplicities* $+1$ *and* -1*) and* $[C_U.C^s_\beta]$ *is the evaluation of the cohomological intersection product.*

Proof. First assume for simplicity that in (8.2) we have $s = 1$ and write $h' = h'_1$ and $h'' = h''_1$. Let $\beta \in Sym_n(\mathbb{Q})$ be positive definite. It is easy to see that a $(p-n)$-cycle D_Y with $\frac{1}{2}(Y,Y) = \beta$ intersects D_U transversely if and only if the orthogonal projection of Y onto $U^\perp$ spans has rank n. From that we conclude that the transversal intersection number $[C_U.C_\beta]_{tr}$ is given by

$$[C_U.C_\beta]_{tr} = \sum_{\substack{\alpha_1\geq 0\\ \alpha_2>0\\ \alpha_1+\alpha_2=\beta}} r(\alpha_1, U)\deg(C_{\alpha_2,C_U}), \tag{8.4}$$

where $r(\alpha_1, U)$ is the representation number of α_1 in $L_U^n + h'$ and

$$\deg(C_{\alpha_2,C_U}) = \sum_{X\in\Gamma_U\backslash\Omega_{\alpha_2}\cap(L^n_{U^\perp}+h'')} \epsilon(X) \tag{8.5}$$

is the (weighted) degree of the 0-cycle C_{α_2} in the space C_U defined by α_2. But the right hand side of (8.4) is exactly the β-th Fourier coefficient of $\vartheta(\tau, L_U + h')$ times the positive definite part of $\theta_{C_U}(\tau, L_{U^\perp} + h'')$, which is given by (8.5), see [10].

The statement for the singular coefficients is clear as Th. 1.4 shows that the β-th coefficient of θ is the Poincaré dual of the absolute cycle C^s_β. But one can also see in the same combinatorial way as above that the Fourier coefficient attached to a semidefinite β represents the intersection numbers at the Borel-Serre boundary of the singular cycle C^s_β and the boundary of C_U. □

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Department of Mathematics, Rawles Hall, Indiana University, Bloomington, IN 47405, USA

E-mail address: `jefunke@indiana.edu`

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

E-mail address: `jjm@math.umd.edu`