

# THE SMOOTH REPRESENTATIONS OF $\mathrm{GL}_2(\mathcal{O})$

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ABSTRACT. We present some unpublished results of Kutzko together with results of Hill, giving a classification of the smooth (complex) representations of  $\mathrm{GL}_2(\mathcal{O})$ , where  $\mathcal{O}$  is the ring of integers in a local field with finite residue field.

## 1. INTRODUCTION

Let  $\mathcal{O}$  be the ring of integers in a local field with finite residue field, and let  $\mathfrak{p}$  be its maximal ideal. The smooth representations of  $\mathrm{GL}_2(\mathcal{O})$ , or equivalently, the representations of the finite groups  $\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^r)$ ,  $r \geq 1$ , have in one form or another been known to some mathematicians since the late 70s. There has been two different approaches to this problem. On the one hand, there is the Weil representation approach of Nobs and Wolfart (applied to the case  $\mathcal{O} = \mathbb{Z}_p$  in [13]). On the other hand, there is the approach via orbits and Clifford theory due to Kutzko (unpublished), and independently to Nagornyj [11].

The related case of  $\mathrm{SL}_2(\mathbb{Z}_p)$ ,  $p \neq 2$ , has been studied by Kloosterman [9, 10], Tanaka [17, 18], Kutzko (unpublished), and Shalika (for general  $\mathcal{O}$  and  $p \neq 2$ ) [15]. Another description of the representations of  $\mathrm{SL}_2(\mathbb{Z}_p)$  (including the case  $p = 2$ ) was given by Nobs and Wolfart [12, 14], using Weil representations. The case  $\mathrm{PGL}_2(\mathcal{O})$  with odd residue characteristic has been treated by Silberger [16].

The series of papers by Hill [4, 5, 6, 7] use the method of orbits and Clifford theory as a basis for dealing with the general case of  $\mathrm{GL}_n(\mathcal{O})$ , and contain several general results, albeit a complete classification only in certain cases. In particular, Hill's work gives a construction (up to a description of orbits) of the so called *regular* representations (resp. *split regular* in the odd conductor case).

Despite all these works, the literature on the representations of  $\mathrm{GL}_2(\mathcal{O})$  remains in a somewhat unsatisfactory state: The Weil representation approach of Nobs and Wolfart works only for  $\mathrm{GL}_2$  and closely related groups of small rank, and Nagornyj's paper, in addition to restricting to the case  $\mathcal{O} = \mathbb{Z}_p$  with  $p$  odd, omits certain details and uses an ad hoc argument in the cuspidal case. Moreover, although Hill's construction of regular representations covers most of the representations of  $\mathrm{GL}_2(\mathcal{O})$ , it omits a discussion of the orbits, and does not work for cuspidal representations in the odd conductor case. Hill does have a construction of so called strongly semisimple representations which include the cuspids in the odd conductor case, but this involves a certain implicit step (see Subsect. 4.2), and a more direct construction of the cuspidal representations is desirable.

The purpose of this paper is to give a complete and uniform construction of all the irreducible representations of the groups  $\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^r)$ , for  $r \geq 2$  (the case  $r = 1$  being well-known classically), via the approach of orbits and Clifford theory. This amounts to a reconstruction of Kutzko's unpublished results, using whenever

possible, the general results of Hill in order to fit the present construction into a more general picture.

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## 2. ORBITS AND CLIFFORD THEORY

Let  $F$  be a local field with finite residue field  $k$  of arbitrary characteristic. Denote by  $\mathcal{O}$  its ring of integers, by  $\mathfrak{p}$  its maximal ideal, and by  $\varpi$  a prime element. For any integer  $r \geq 1$  we write  $\mathcal{O}_r$  for the finite ring  $\mathcal{O}/\mathfrak{p}^r$ . Let  $G_r = GL_2(\mathcal{O}_r)$ ,  $K_i = \{g \in G_r \mid g \equiv 1 \pmod{\mathfrak{p}^i}\}$ , for  $1 \leq i \leq r-1$ . From now on let  $i \geq r/2$ ; then  $x \mapsto 1 + \varpi^i x$  induces an isomorphism

$$M_2(\mathcal{O}_{r-i}) \xrightarrow{\sim} K_i.$$

The group  $G_r$  acts on  $M_2(\mathcal{O}_{r-i})$  by conjugation, via its quotient  $G_{r-i}$ . This action is transformed by the above isomorphism into the action of  $G_r$  on the normal subgroup  $K_i$ .

Fix an additive character  $\psi : \mathcal{O} \rightarrow \mathbb{C}^\times$  with conductor  $\mathfrak{p}^r$  and define for any  $\beta \in M_2(\mathcal{O}_{r-i})$ , a character  $\psi_\beta : K_i \rightarrow \mathbb{C}^\times$  by

$$\psi_\beta(x) = \psi(\mathrm{Tr}(\beta(x-1))).$$

Then  $\beta \mapsto \psi_\beta$  gives an isomorphism

$$M_2(\mathcal{O}_{r-i}) \cong \mathrm{Hom}(K_i, \mathbb{C}^\times).$$

For  $g \in G_r$  we have

$$\psi_{g\beta g^{-1}}(x) = \psi(\mathrm{Tr}(g\beta g^{-1}(x-1))) = \psi(\mathrm{Tr}(\beta g^{-1}(x-1)g)) = \psi_\beta(g^{-1}xg).$$

Thus, the above isomorphism transforms the action of  $G_r$  on  $M_2(\mathcal{O}_{r-i})$  into (the inverse) conjugation of characters.

We will make use of the following well-known results of Clifford theory.

**Theorem 2.1.** *Let  $G$  be a finite group, and  $N$  a normal subgroup. For any irreducible representation  $\rho$  of  $N$ , define the stabilizer,  $T(\rho) = \{g \in G \mid \rho^g \cong \rho\}$ , of  $\rho$ . Then the following hold.*

- i) *If  $\pi$  is an irreducible representation of  $G$ , then  $\pi|_N = e(\bigoplus_{\rho \in \Omega} \rho)$ , where  $\Omega$  is an orbit of irreducible representations of  $N$  under the action of  $G$  by conjugation, and  $e$  is a positive integer.*
- ii) *Suppose that  $\rho$  is an irreducible representation of  $N$ , and let*

$$A = \{\theta \in \mathrm{Irr}(T(\rho)) \mid \langle \mathrm{Res}_N^{T(\rho)}(\theta), \rho \rangle \neq 0\},$$

$$B = \{\pi \in \mathrm{Irr}(G) \mid \langle \mathrm{Res}_N^G(\pi), \rho \rangle \neq 0\}.$$

*Then  $\theta \mapsto \mathrm{Ind}_{T(\rho)}^G(\theta)$  is a bijection of  $A$  onto  $B$ .*

- iii) *Let  $H$  be a subgroup of  $G$  containing  $N$ , and suppose that  $\rho$  is an irreducible representation of  $N$  which has an extension  $\tilde{\rho}$  to  $H$  (i.e.  $\tilde{\rho}|_N = \rho$ ). Then*

$$\mathrm{Ind}_N^H(\rho) = \bigoplus_{\chi \in \mathrm{Irr}(H/N)} \chi \tilde{\rho},$$

where each  $\chi\tilde{\rho}$  is irreducible, and where we have identified  $\text{Irr}(H/N)$  with representations of  $H$  that are trivial on  $N$ .

iv) If  $\rho$  is an irreducible representation of  $N$  and  $T(\rho)/N$  is cyclic, then there exists an extension of  $\rho$  to  $T(\rho)$ .

For proofs of the above, see for example [8], 6.2, 6.11, 6.17, and 11.22, respectively. The above results *i*) and *ii*) show that in order to obtain a classification of the representations of  $G_r$ , it is enough to classify the orbits of irreducible representations  $\psi_\beta$  of a normal subgroup  $K_i$ , and to describe all the irreducible representations of the stabilizers  $T(\psi_\beta)$  which contain  $\psi_\beta$  (when restricted to  $K_i$ ), that is, to decompose  $\text{Ind}_{K_i}^{T(\psi_\beta)}(\psi_\beta)$  into irreducible representations. This is what we will do in the following.

**2.1. Orbits.** Set  $l = [\frac{r+1}{2}]$ ,  $l' = [\frac{r}{2}]$ ; thus  $l + l' = r$ . Then  $K_l$  is the largest abelian group among the kernels  $K_i$ , so we can describe its irreducible representations  $\psi_\beta$ , and at the same time it has the advantage of being sufficiently close to the stabilizers of its representations to enable us to describe the irreducible components of  $\text{Ind}_{K_i}^{T(\psi_\beta)}(\psi_\beta)$ .

We will consider  $G_r$ -orbits on  $M_2(\mathcal{O}_l) = M_2(\mathcal{O}_{r-l}) \cong \text{Hom}(K_l, \mathbb{C}^\times)$ . The Jordan and rational canonical forms imply that the orbits fall into four types, according to their reductions mod  $\mathfrak{p}$ . The following is a list of the possible orbits in  $M_2(k)$ :

- (1)  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ ,
- (2)  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ , where  $a \neq d$ ,
- (3)  $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$ , where  $x^2 - sx + \Delta$  is irreducible,
- (4)  $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ .

If  $x \in M_2(\mathcal{O}_i)$ , for some  $i \geq 1$ , we shall write  $\bar{x}$  for the image of  $x$  in  $M_2(k)$ . Let  $\beta \in M_2(\mathcal{O}_l)$ . Then  $\psi_\beta|_{K_{r-1}} = \psi_{\bar{\beta}}$ , and so if  $\beta'$  is another matrix in  $M_2(\mathcal{O}_l)$ , we have

$$\beta \equiv \beta' \pmod{\mathfrak{p}} \iff \psi_\beta|_{K_{r-1}} = \psi_{\beta'}|_{K_{r-1}}.$$

If  $aI_2 \in M_2(k)$  is a scalar matrix ( $I_2$  is the unit matrix), then  $\psi_{aI_2}(x) = \psi(a \text{Tr}(x - 1)) = \psi(a(\det(x) - 1))$ . Thus  $\psi_{aI_2} = \chi_a \circ \det$ , for some character  $\chi_a$  of the group  $\{g \in \mathcal{O}_r^\times \mid g \equiv 1 \pmod{\mathfrak{p}^{r-1}}\}$ . Since  $\mathcal{O}_r^\times$  is abelian,  $\chi_a$  has an extension to  $\mathcal{O}_r^\times$ . Consequently  $\psi_{aI_2}$  has an extension to  $G_r$ , which we denote by  $\tilde{\psi}_{aI_2}$ .

Suppose now that  $\pi$  is a representation of  $G_r$  such that  $\pi|_{K_l}$  contains  $\psi_\beta$ , with  $\beta$  a matrix of type (1) such that  $\bar{\beta} = aI_2$ . Then  $(\pi \otimes \tilde{\psi}_{-aI_2})|_{K_{r-1}} = \psi_0|_{K_{r-1}} = \mathbf{1}_{K_{r-1}}$ . Thus  $\pi$  factors through  $G_{r-1}$  after twisting by a one-dimensional character, i.e.  $\pi$  has the group  $K_{r-1} \cap \text{SL}_2(\mathcal{O}_r)$  in its kernel. Working inductively, we assume that the representations of  $G_{r-1}$  are known. Hence the representations corresponding to orbits of type (1) are described.

Similarly, if  $\pi$  is a representation of  $G_r$  corresponding to an orbit of type (4) containing an element  $\beta$  such that  $\bar{\beta} = aI_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , then  $\pi \otimes \tilde{\psi}_{-aI_2}$  corresponds to an orbit in  $M_2(\mathcal{O}_l)$  that can be represented by a matrix  $\beta'$  such that  $\beta' \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(mod  $\mathfrak{p}$ ). This implies that  $\beta' = \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}$  for some  $a, b, c, d \in \mathfrak{p}$ . Conjugating  $\beta'$  by  $\begin{pmatrix} 1 & 0 \\ a & 1+b \end{pmatrix}$ , we can assume that  $\beta'$  has the form  $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$  for some  $\Delta, s \in \mathfrak{p}$ .

If  $\beta$  represents an orbit of type (3), such that  $\beta \equiv \begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix} \pmod{\mathfrak{p}}$ , then  $\beta = \begin{pmatrix} a & 1+b \\ -\Delta+c & s+d \end{pmatrix}$  for some  $a, b, c, d \in \mathfrak{p}$ , and conjugating by  $\begin{pmatrix} 1 & 0 \\ a & 1+b \end{pmatrix}$ , we can assume that  $\beta$  has the form  $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$  for some  $\Delta, s \in \mathcal{O}_{l'}$ .

We have therefore reduced the problem of describing the representations of  $G_r$  to describing the representations corresponding to orbits in  $M_2(\mathcal{O}_{l'})$ , represented by elements of the following types

- (1')  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ , with  $a \not\equiv d \pmod{\mathfrak{p}}$ ,
- (2')  $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$ , where  $x^2 - sx + \Delta$  is irreducible mod  $\mathfrak{p}$ ,
- (3')  $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$ , where  $\Delta, s \in \mathfrak{p}$ .

All of these orbits are *regular* in the sense of Hill [6]. Indeed, by Theorem 3.6 in [6], an element  $\beta \in M_2(\mathcal{O}_{l'})$  is regular if and only if the centralizer  $C_{\mathbf{G}_1}(\beta)$  is abelian, where  $\mathbf{G}_1 = \mathrm{GL}_2(\bar{k})$ , and  $\bar{k}$  is an algebraic closure of  $k$ . Now simple calculations show that  $C_{\mathbf{G}_1}(\beta)$  is conjugate to the group

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \left\{ \begin{pmatrix} a & b \\ -\Delta b & a+\bar{s}b \end{pmatrix} \in \mathrm{GL}_2(\bar{k}) \right\}, \text{ or } \left\{ \begin{pmatrix} a & * \\ 0 & a \end{pmatrix} \mid a \in k^\times \right\},$$

when  $\beta$  is of type (1'), (2'), or (3'), respectively.

We observe that the representations corresponding to orbits of type (1') and (3') are *split regular* in the sense of Hill (cf. [6], 4.4). This means that the orbits are regular and that the characteristic polynomial splits completely mod  $\mathfrak{p}$  (not necessarily into distinct factors). The representations corresponding to orbits of type (2') are *cuspidal* (cf. [7], 4.2), i.e. the characteristic polynomial is irreducible mod  $\mathfrak{p}$ .

Let  $\beta \in M_2(\mathcal{O}_{l'})$  be an element in an orbit of one of the above three types. Since each of these orbits is regular, it follows from Corollary 3.7 in [6] that the stabilizer of  $\psi_\beta$  in  $G_r$  is given by

$$T(\psi_\beta) = \mathcal{O}_r[\hat{\beta}]^\times K_{l'},$$

where  $\hat{\beta}$  is a lift of  $\beta$  to an element in  $M_2(\mathcal{O}_r)$ .

To describe the representations of  $G_r$  corresponding to the above orbits, we shall proceed by considering two cases depending on whether  $r$  is even or odd.

### 3. THE CASE $r = 2l, l' = l$

Let  $\beta$  be a representative of one of the three orbits above, and choose a lift  $\hat{\beta} \in M_2(\mathcal{O}_r)$ . Let  $\theta \in \mathrm{Hom}(\mathcal{O}_r[\hat{\beta}]^\times, \mathbb{C}^\times)$  be a character such that  $\theta$  is equal to  $\psi_\beta$  when restricted to  $K_l \cap \mathcal{O}_r[\hat{\beta}]^\times$  (such a  $\theta$  exists since  $\mathcal{O}_r[\hat{\beta}]^\times$  is abelian). Then  $\theta\psi_\beta$  is a one-dimensional representation of  $T(\psi_\beta)$  such that  $\theta\psi_\beta(xy) = \theta(x)\psi_\beta(y)$ , for  $x \in \mathcal{O}_r[\hat{\beta}]^\times, y \in K_l$ .

Define the representation

$$\pi(\theta, \beta) = \mathrm{Ind}_{\mathcal{O}_r[\hat{\beta}]^\times K_l}^{G_r}(\theta\psi_\beta).$$

By Theorem 2.1 ii),  $\pi(\theta, \beta)$  is an irreducible representation,  $\pi(\theta, \beta) = \pi(\theta', \beta) \Rightarrow \theta = \theta'$ , and every irreducible representation of  $G_r$  corresponding to an orbit of one

of the above three types is obtained in this way. This is a special case of Theorem 4.1 in [6], which holds for any  $\mathrm{GL}_n$  and any regular orbit.

We observe that a different choice of representative  $\beta$  for the orbit, gives a conjugate character  $\psi_\beta$  and a conjugate stabilizer  $T(\psi_\beta)$ . The resulting induced representations are therefore isomorphic for any choice of representative  $\beta$ . Moreover, a different choice of  $\hat{\beta}$ , even though it may give a different group  $\mathcal{O}_r[\hat{\beta}]^\times$  and different characters  $\theta$ , will give a resulting set of representations  $\theta\psi_\beta$ , that equals the original one. This is because in both cases we obtain the set of irreducible components of  $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$ . Thus, the set of representations  $\pi(\theta, \beta)$  depends only on the orbit of  $\beta$ , and is independent of the choice of lift  $\hat{\beta}$ .

#### 4. THE CASE $r = 2l - 1$ , $l' = l - 1$

As before, Let  $\beta$  be a representative of one of the three orbits above, and choose a lift  $\hat{\beta} \in M_2(\mathcal{O}_r)$ . The thing that makes this case more difficult is that we start from a character  $\psi_\beta$  on  $K_l$ , while  $T(\psi_\beta) = \mathcal{O}_r[\hat{\beta}]^\times K_{l'}$ , and  $K_l \not\leq K_{l'}$ , so it is not as easy to decompose  $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$ . This the ‘‘odd conductor case’’ in turn falls into two subcases: the split case, and the cuspidal case.

**4.1. The split case.** Following Hill [6], Sect. 4, we assume that  $\beta$  lies in a split regular orbit, i.e. of type (1') or (3'). Let  $B_r$  be the subgroup of  $G_r$  of upper-triangular matrices. By the proof of Lemma 4.5 in [6], the group  $H_\beta = K_l(B_r \cap K_{l'})$  has the property that  $H_\beta/K_l$  is a maximal isotropic subspace of  $K_{l'}/K_l$  with respect to the form<sup>1</sup>

$$\langle \cdot, \cdot \rangle_{\bar{\beta}} : K_{l'}/K_l \times K_{l'}/K_l \longrightarrow k, \quad \langle xK_l, yK_l \rangle_{\bar{\beta}} = \mathrm{Tr}(\bar{\beta}(\overline{mn} - \overline{nm})),$$

where  $x = 1 + \varpi^{l'} m$  and  $y = 1 + \varpi^{l'} n$ . Moreover,  $H_\beta$  is normal in  $T(\psi_\beta)$ , and we have the following diagram of groups.

$$\begin{array}{ccc} & & T(\psi_\beta) \\ & \nearrow & | \\ K_{l'} & & \mathcal{O}_r[\hat{\beta}]^\times H_\beta \\ | & \nearrow & | \\ H_\beta & & \mathcal{O}_r[\hat{\beta}]^\times K_l \\ | & \nearrow & \\ K_l & & \end{array}$$

We now review the main steps of the proof of Theorem 4.6 in [6]. Let  $N = \ker(\psi_\beta)$  and write  $\bar{\psi}_\beta$  for  $\psi_\beta$  viewed as a character of  $K_l/N$ . One can show that  $H_\beta/N$  and  $\mathcal{O}_r[\hat{\beta}]^\times K_l/N$  are both abelian. Hence,  $\bar{\psi}_\beta$  can be extended to both these groups. Next it is shown that there exists exactly  $q^2$  ( $q^n$  in the case of  $\mathrm{GL}_n$ )  $\mathcal{O}_r[\hat{\beta}]^\times H_\beta/N$ -stable extensions of  $\bar{\psi}_\beta$  to  $H_\beta/N$ . Let  $\psi'_\beta$  be one of these. Furthermore, one can

<sup>1</sup>The form takes values in  $k$ , not in  $\mathbb{C}$  as stated by Hill.

construct an extension  $\psi''_\beta$  of  $\bar{\psi}_\beta$  to  $\mathcal{O}_r[\hat{\beta}]^\times K_l/N$ , such that  $\psi''_\beta$  and  $\psi'_\beta$  agree on  $(H_\beta/N) \cap (\mathcal{O}_r[\hat{\beta}]^\times K_l/N)$ . Then one can check that the map

$$\psi'''_\beta : \mathcal{O}_r[\hat{\beta}]^\times H_\beta/N \longrightarrow \mathbb{C}, \quad \psi'''_\beta(xhN) = \psi''_\beta(xN)\psi'_\beta(hN),$$

where  $x \in \mathcal{O}_r[\hat{\beta}]^\times, h \in H_\beta$ , is a well-defined homomorphism. Denote by  $\tilde{\psi}_\beta$  the composition of  $\psi'''_\beta$  with the natural homomorphism  $\mathcal{O}_r[\hat{\beta}]^\times H_\beta \rightarrow \mathcal{O}_r[\hat{\beta}]^\times H_\beta/N$ . Thus  $\tilde{\psi}_\beta$  is an extension of  $\psi_\beta$  to  $\mathcal{O}_r[\hat{\beta}]^\times H_\beta$ .

Now define the characters

$$\rho_\beta = \mathrm{Ind}_{H_\beta}^{K_{l'}}(\tilde{\psi}_\beta|_{H_\beta}), \text{ and } \zeta_\beta = \mathrm{Ind}_{\mathcal{O}_r[\hat{\beta}]^\times H_\beta}^{T(\psi_\beta)}(\tilde{\psi}_\beta).$$

It follows from [6], Prop. 4.2, that  $\rho_\beta$  is irreducible. Moreover, it is shown that  $\zeta_\beta$  is an extension of  $\rho_\beta$ . Frobenius reciprocity and counting degrees shows that the induced representation  $\mathrm{Ind}_{K_{l'}}^{T(\psi_\beta)}(\rho_\beta)$  decomposes as the sum of the representations  $\omega\zeta_\beta$ , where  $\omega$  runs through the linear characters of the abelian group  $T(\psi_\beta)/K_{l'}$ . Finally, it is shown that every irreducible component of  $\mathrm{Ind}_{K_l}^{K_{l'}}(\psi_\beta)$  has the form  $\mathrm{Ind}_{H_\beta}^{K_{l'}}(\psi'_\beta)$  for some  $\mathcal{O}_r[\hat{\beta}]^\times$ -stable extension  $\psi'_\beta$  of  $\psi_\beta$ . Thus, every irreducible component of  $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$  has the form  $\omega\zeta_\beta$ , where  $\zeta_\beta$  is induced from some  $\tilde{\psi}_\beta$ , and  $\omega$  is as above, and all the characters  $\omega\zeta_\beta$  are distinct for distinct  $\psi'_\beta$  and  $\omega$ .

Thus, every representation  $\omega\zeta_\beta$  is isomorphic to an induced representation

$$\mathrm{Ind}_{\mathcal{O}_r[\hat{\beta}]^\times H_\beta}^{T(\psi_\beta)}(\omega'\tilde{\psi}_\beta),$$

where  $\omega'$  is a linear character of  $\mathcal{O}_r[\hat{\beta}]^\times H_\beta/H_\beta$ , and distinct  $\omega'$  give distinct  $\omega\zeta_\beta$ .

**4.2. The cuspidal case.** We now assume that  $\beta$  is in a cuspidal orbit, i.e. of type (2'). In [7], Prop. 3.6 Hill gives a construction of so called strongly semisimple representations, which include the cuspidal ones as a special case. This construction is less explicit than ours because it involves the non-constructive existence of an irreducible constituent of  $\mathrm{Ind}_{K_l}^{K_{l'}}(\psi_\beta)$ . For the construction of the cuspidal representations, there are also several slightly different methods giving parametrizations in terms of certain characters of  $\mathcal{O}_{F_2}^\times$ , where  $F_2$  is the quadratic unramified extension of  $F$  (cf. [1] and its references). The method we give here is similar to a more general construction used by Bushnell and Kutzko (cf. [3]) in their description of supercuspidal representations of  $\mathrm{GL}_n(F)$ . In particular, the method can be adapted to the construction of cuspidal representations for any  $\mathrm{GL}_n(\mathcal{O})$ . We will make use of the following result from [2], 8.3.3.

**Proposition 4.1.** *Let  $G$  be a finite group and  $N$  a normal subgroup, such that  $G/N$  is an elementary abelian  $p$ -group. Thus  $G/N$  has a structure of  $\mathbb{F}_p$ -vector space. Let  $\chi$  be a one-dimensional representation of  $N$ , which is stabilized by  $G$ . Define an alternating bilinear form*

$$h_\chi : G/N \times G/N \longrightarrow \mathbb{C}^\times, \quad h_\chi(g_1N, g_2N) = \chi([g_1, g_2]) = \chi(g_1g_2g_1^{-1}g_2^{-1}).$$

*Assume that the form  $h_\chi$  is non-degenerate. Then there exists a unique up to isomorphism irreducible representation  $\rho_\chi$  of  $G$  such that  $\rho_\chi|_N$  contains  $\chi$ .*

Note that the form is well-defined<sup>2</sup>, so the non-trivial thing here is the uniqueness part.

Set  $Z^1 = \mathcal{O}_r[\hat{\beta}]^\times \cap K_1$ . The strategy to find the irreducible constituents of  $\text{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$  is as follows. First we extend  $\psi_\beta$  to a representation  $\tilde{\psi}_\beta$  of  $Z^1 K_l$ , as in the even conductor case. Next we apply the above proposition to the case where  $G = Z^1 K_{l'}$ ,  $N = Z^1 K_l$ , and  $\chi = \tilde{\psi}_\beta$ . We thus obtain a representation  $\eta_\beta$  containing  $\psi_\beta$ , and it is clear that every representation of  $Z^1 K_{l'}$  containing  $\psi_\beta$  is of this form. Finally, we use Theorem 2.1, *iv*) to lift  $\eta_\beta$  to a representation of  $T(\psi_\beta)$ . Again it is clear that we have in this way obtained every irreducible component of  $\text{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$ .

To carry out the above strategy, we need to establish several facts concerning the groups  $T(\psi_\beta)$ ,  $Z^1 K_{l'}$ ,  $Z^1 K_l$ , and their respective representations: First, to apply Proposition 4.1, we need show that  $Z^1 K_l$  is normal in  $Z^1 K_{l'}$ , that the quotient is elementary abelian, that  $Z^1 K_{l'}$  stabilizes the representation  $\tilde{\psi}_\beta$ , and that the form  $h_{\tilde{\psi}_\beta}$  is non-degenerate. Then, to apply Theorem 2.1, *iv*), we need to show that  $Z^1 K_{l'}$  is normal in  $T(\psi_\beta)$ , that  $T(\psi_\beta)$  stabilizes the representation  $\eta_\beta$ , and that the quotient  $T(\psi_\beta)/Z^1 K_{l'}$  is cyclic.

We prove all the above facts in one go. First we show that both  $Z^1 K_{l'}$  and  $Z^1 K_l$  are normal in  $T(\psi_\beta)$  (and thus  $Z^1 K_l$  is normal in  $Z^1 K_{l'}$ ). It is clear that  $Z^1 K_{l'}$  is normal in  $T(\psi_\beta)$  since  $Z^1 K_{l'} = T(\psi_\beta) \cap K_1$ , and  $K_1$  is normal in  $G_r$ . Now let  $g \in T(\psi_\beta)$  and  $x \in Z^1 K_l$ . Write  $g = ek$  with  $e \in \mathcal{O}_r[\hat{\beta}]^\times$  and  $k \in K_{l'}$ , and write  $x = zh$ , with  $z \in Z^1$  and  $h \in K_l$ . Then

$$g x g^{-1} = z \cdot e([z^{-1}, k](k h k^{-1}))e^{-1},$$

and the two factors in the bracket lie in  $K_l$  since  $[K_1, K_{l'}] \leq K_l$ . Since  $K_l$  is normal in  $G_r$ , the whole expression  $e([z^{-1}, k](k h k^{-1}))e^{-1}$  in fact lies in  $K_l$ , so  $g x g^{-1} \in Z^1 K_l$ , as required. Now

$$\begin{aligned} \tilde{\psi}_\beta(g x g^{-1}) &= \tilde{\psi}_\beta(z) \psi_\beta(e([z^{-1}, k](k h k^{-1}))e^{-1}) \\ &= \tilde{\psi}_\beta(z) \psi_\beta([z^{-1}, k]) \psi_\beta(h) \\ &= \tilde{\psi}_\beta(x) \psi_\beta([z^{-1}, k]), \end{aligned}$$

where the second equality follows since  $T(\psi_\beta)$  stabilizes  $\psi_\beta$ . Thus, to show that  $T(\psi_\beta)$  stabilizes  $\tilde{\psi}_\beta$ , we need only to show that  $\psi_\beta([z^{-1}, k]) = 1$ . For this, write  $k = 1 + y$ , with  $y \in \mathfrak{p}' M_2(\mathcal{O}_r)$  and notice that

$$[z^{-1}, k] = z(1 + y)z^{-1}(1 - y + y^2) = z(1 + y)z^{-1} - y,$$

(here we have used the fact that  $y^3 = 0$  and  $zy^2 = y^2$  in  $M_2(\mathcal{O}_r)$ ). Then

$$\psi_\beta([z^{-1}, k]) = \psi(\text{Tr}(\beta(zyz^{-1} - y))) = \psi(\text{Tr}(z(\beta y)z^{-1} - \beta y)) = 1,$$

where we have used  $z\hat{\beta} = \hat{\beta}z \Rightarrow z\beta = \beta z$ , and the fact that  $\text{Tr}$  is a conjugacy class function.

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<sup>2</sup> $\chi([g_1 n_1, g_1 n_2]) = \chi(g_1 g_2 (g_2^{-1} n_1 g_2 n_2 n_1^{-1} g_1^{-1} n_2^{-1} g_1) g_1^{-1} g_2^{-1})$   
 $= \chi(g_1 g_2 (g_2^{-1} n_1 g_2 n_2 n_1^{-1} g_1^{-1} n_2^{-1} g_1) (g_1 g_2)^{-1} [g_1, g_2])$   
 $= \chi(g_2^{-1} n_1 g_2 n_2 n_1^{-1} g_1^{-1} n_2^{-1} g_1) \chi([g_1, g_2]) = \chi(g_2^{-1} n_1 g_2) \chi(n_2 n_1^{-1}) \chi(g_1^{-1} n_2^{-1} g_1) \chi([g_1, g_2])$   
 $= \chi([g_1, g_2]).$

Next, note that since  $[K_{\nu'}, K_{\nu'}] \leq K_l$ , the bilinear form does not depend on the lift  $\tilde{\psi}_\beta$ . The form  $h_{\psi_\beta} = h_{\tilde{\psi}_\beta}$  is related to the above form  $\langle \cdot, \cdot \rangle_{\bar{\beta}}$  in the following way. First, there is a natural isomorphism

$$\frac{K_{\nu'}}{(Z^1 \cap K_{\nu'})K_l} \xrightarrow{\sim} \frac{Z^1 K_{\nu'}}{Z^1 K_l}, \quad k(Z^1 \cap K_{\nu'})K_l \longmapsto kZ^1 K_l, \text{ where } k \in K_{\nu'},$$

and moreover, it is shown in the proof of Prop. 4.2 of [6] that  $\langle xK_l, yK_l \rangle_{\bar{\beta}} = 0$  if and only if  $\psi_\beta([x, y]) = 1$ , so  $h_{\psi_\beta}(xZ^1 K_l, yZ^1 K_l) = 0$  if and only if  $\langle xK_l, yK_l \rangle_{\bar{\beta}} = 0$ . By [6], Corollary 4.3<sup>3</sup> the radical of the form  $\langle \cdot, \cdot \rangle_{\bar{\beta}}$  is equal to  $(\mathcal{O}_r[\hat{\beta}]^\times \cap K_{\nu'})K_l/K_l$ , so by the above isomorphism the radical of  $h_{\psi_\beta}$  is  $Z^1 K_l(\mathcal{O}_r[\hat{\beta}]^\times \cap K_{\nu'})/Z^1 K_l = Z^1 K_l/Z^1 K_l = 1$ . We thus conclude that the form  $h_{\psi_\beta}$  is non-degenerate. Since  $K_{\nu'}/K_l \cong M_2(k)$  is elementary abelian, the same is true for any of its quotients.

Finally, we show that  $T(\psi_\beta)/Z^1 K_{\nu'}$  is cyclic, and that  $T(\psi_\beta)$  stabilizes the representation  $\eta_\beta$ . The former is true because the quotient is isomorphic to  $\mathcal{O}_r[\hat{\beta}]^\times/Z^1 \cong k[\hat{\beta}]^\times$ , and this is the multiplicative group of a finite field (the residue field of the unramified extension generated by a lift of  $\beta$  to  $M_2(\mathcal{O}_r)$ ). The latter is true because for  $g \in T(\psi_\beta)$ , the conjugate representation  $\eta_\beta^g$  is another irreducible representation of  $Z^1 K_{\nu'}$  whose restriction to  $Z^1 K_l$  contains  $\tilde{\psi}_\beta^g = \tilde{\psi}_\beta$ , so the uniqueness in Prop. 4.1 implies that  $\eta_\beta^g$  is equivalent to  $\eta_\beta$ .

To conclude, we have shown that any irreducible component of  $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$  is of the form  $\tilde{\eta}_\beta$ , for some lift  $\tilde{\eta}_\beta$  of  $\eta_\beta$  to  $T(\psi_\beta)$ . It follows that any irreducible representation of  $G_r$  containing  $\psi_\beta$  is of the form

$$\mathrm{Ind}_{T(\psi_\beta)}^{G_r}(\tilde{\eta}_\beta),$$

for some  $\tilde{\eta}_\beta$ , and distinct inducing representations give distinct induced representations. Again the dependence on the choice of lift  $\hat{\beta}$  is irrelevant, since for any choice we obtain the set of irreducible components of  $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$ . The choice of representative  $\beta$  is irrelevant for the same reason as before.

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<sup>3</sup>Note that the second equality in the corollary is incorrect.



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