

# ISOMORPHISM RIGIDITY IN ENTROPY RANK TWO

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ABSTRACT. We study the rigidity properties of a class of algebraic  $\mathbb{Z}^3$ -actions with entropy rank two. For this class, conditions are found which force an invariant measure to be the Haar measure on an affine subset. This is applied to show isomorphism rigidity for such actions, and to provide examples of non-isomorphic  $\mathbb{Z}^3$ -actions with all their  $\mathbb{Z}^2$ -sub-actions isomorphic. The proofs use lexicographic half-space entropies and total ergodicity along critical directions.

## 1. INTRODUCTION

An **algebraic  $\mathbb{Z}^d$ -action** is a  $\mathbb{Z}^d$ -action on a compact abelian metrizable group by automorphisms. Rigidity for such actions is a circle of results that give explicit descriptions of all invariant measures for  $\alpha$ , or all measurable isomorphisms between such systems, under certain hypotheses for  $d > 1$ . The case of a  $\mathbb{Z}^d$ -action by toral automorphism is well studied (see [9] for isomorphism rigidity and [5], [8], [10] for measure rigidity); in the toral case individual elements of the action have finite entropy. For actions on zero-dimensional groups the case of a general irreducible action was first studied in [11], where again the individual elements of the action have finite entropy. The general case of mixing algebraic  $\mathbb{Z}^d$ -actions on zero-dimensional groups with zero entropy was studied in [1], [2] and [3]. Another type of rigidity — differences between apparently related zero-dimensional systems forcing them to be disjoint — is studied in [6] also using entropy methods.

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Our purpose is to study isomorphism rigidity for a particular class of algebraic  $\mathbb{Z}^3$ -actions on connected groups with the property that all the  $\mathbb{Z}^2$ -sub-actions have positive entropy. We follow the path taken in [3], and avoid repetition of certain technicalities by citing results from that paper. The special class of systems studied allows a transparent proof, but it is clear that the underlying rigidity phenomena is more extensive than what is shown here suggests.

The systems we study, *ET* systems, have two distinguished polynomial parameters. The first of these,  $g$ , is *Expanding*, and the second,  $f$ , is *Triangular*. The assumption on  $g$  is essential for the method used here, while the assumption on  $f$  does not seem to be essential but does significantly simplify the arguments. The paradigmatic example of an *ET* system is the so-called space helmet (cf. [4, Example 5.8]; Example 2).

Let  $\mathbf{X}_i = (X_i, \alpha_i)$  be an algebraic  $\mathbb{Z}^d$ -action for  $i = 1, 2$ . A factor map  $\varphi: \mathbf{X}_1 \rightarrow \mathbf{X}_2$  is a (Borel) measurable map from  $X_1$  onto  $X_2$  with  $\varphi \circ \alpha_1^{\mathbf{n}}(x) = \alpha_2^{\mathbf{n}} \circ \varphi(x)$  for a.e.  $x \in X_1$  and all  $\mathbf{n} \in \mathbb{Z}^d$ . A factor map  $\varphi: \mathbf{X}_1 \rightarrow \mathbf{X}_2$  is

- a **conjugacy** if it is invertible;
- **affine** if  $\varphi(x) = \varphi_g(x) + y$  for some  $y \in X_2$  and continuous group homomorphism  $\varphi_g: X_1 \rightarrow X_2$ ; and is
- an **algebraic isomorphism** if it is an isomorphism of the groups  $X_1$  and  $X_2$ .

We will always use  $\lambda_X$  to denote the Haar measure on a compact abelian group  $X$ .

A class of systems exhibits **isomorphism rigidity** if every conjugacy or factor map is affine, and exhibits **measure rigidity** if very mild additional assumptions on an invariant ergodic Borel measure (positive entropy, for example) force it to be a translate of the Haar measure of a closed subgroup.

We use a standard approach to the description of algebraic  $\mathbb{Z}^d$ -actions (see [15] for more background). Let  $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$  be the ring of Laurent polynomials in  $d$  commuting variables with integer coefficients, and write  $\mathbf{u}^{\mathbf{n}}$  for the monomial  $u_1^{n_1} \dots u_d^{n_d}$ . A polynomial  $f \in R_d$  is a sum

$$f(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{u}^{\mathbf{n}},$$

where  $f_{\mathbf{n}} \in \mathbb{Z}$  for all  $\mathbf{n} \in \mathbb{Z}^d$  and  $f_{\mathbf{n}} = 0$  for all but finitely many  $\mathbf{n}$ . Any  $R_d$ -module  $M$  defines an algebraic  $\mathbb{Z}^d$ -action  $\mathsf{X}_M = (X_M, \alpha_M)$  as follows. The compact group  $X_M = \widehat{M}$  is the (Pontryagin) character group of the additive group  $M$ ; the action  $\alpha_M$  is defined by  $\alpha_M^{\mathbf{n}}(x) = \widehat{\beta^{\mathbf{n}}}(x)$  where  $\beta^{\mathbf{n}}(m) = \mathbf{u}^{\mathbf{n}}m$ . By duality, any algebraic  $\mathbb{Z}^d$ -action  $\mathsf{X}$  similarly defines an  $R_d$ -module  $M_{\mathsf{X}}$ ; the module structure is given by defining the product of  $f \in R_d$  with  $a \in M_{\mathsf{X}}$  to be

$$fa = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha^{\mathbf{n}}} a.$$

This gives a one-to-one correspondence between algebraic  $\mathbb{Z}^d$ -actions on compact (metrizable) abelian groups and (countable)  $R_d$ -modules.

A system  $\mathsf{X}_M$  is called **prime** if  $M$  is a cyclic  $R_d$ -module of the form  $R_d/P$  for some prime ideal  $P \subset R_d$ . Notice that by duality, prime systems  $\mathsf{X}_M$  and  $\mathsf{X}_N$  are algebraically isomorphic if and only if their defining ideals are equal.

**Example 1.** The module  $M = R_2/\langle 1 + u_1 + u_2 \rangle$  defines an algebraic  $\mathbb{Z}^2$ -action as follows. The dual of  $M$  is

$$X_M = \{x \in \mathbb{T}^{\mathbb{Z}^2} : x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} = 0 \text{ for any } \mathbf{n} \in \mathbb{Z}^2\},$$

and the  $\mathbb{Z}^2$ -action  $\alpha_M$  is the restriction of the shift action

$$\alpha_M^{\mathbf{n}}(x)_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}} \text{ for any } \mathbf{n}, \mathbf{m} \in \mathbb{Z}^2$$

to  $X_M$ . By [13] (see also [15, Th. 20.8, Th. 23.1], [16])  $\alpha$  is isomorphic to a Bernoulli shift, so there are many non-affine conjugacies from  $\mathsf{X}_M$  to itself: the system does not exhibit isomorphism rigidity.

**Example 2.** The module  $N = R_3/\langle 1 + u_1 + u_2, u_3 - 2 \rangle$  defines an algebraic action as follows. Let

$$X_N = \{x \in \mathbb{T}^{\mathbb{Z}^3} : x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} = 0, x_{\mathbf{n}+\mathbf{e}_3} = 2x_{\mathbf{n}} \text{ for } \mathbf{n} \in \mathbb{Z}^3\},$$

and define the  $\mathbb{Z}^3$ -action  $\alpha_N$  to be the restriction of the usual shift action to  $X_N$ . This example and the behaviour of its lower-rank subactions was studied in [4, Ex. 5.8]; the structure of its non-expansive subdynamics has earned  $\mathsf{X}_N$  the sobriquet of ‘space helmet’.

The  $\mathbb{Z}^3$ -action  $\alpha_N$  in Example 2 can be obtained from the  $\mathbb{Z}^2$ -action  $\alpha_M$  in Example 1 by defining the third generator to be multiplication by 2 on each coordinate, and then passing to the invertible extension

of this non-invertible map. Thus a conjugacy from  $X_M$  to itself extends to a conjugacy from  $X_N$  to itself if and only if it commutes with multiplication by 2. Theorem 4 shows that very few maps can have this property; in particular it implies the following proposition.

**Proposition 3.** *Every conjugacy from  $X_N$  to itself is affine.*

In the next section we define the class  $ET$  and show how isomorphism rigidity may be deduced from the technical Proposition 6. Section 3 gives examples of algebraically non-isomorphic  $\mathbb{Z}^3$ -actions for which all  $\mathbb{Z}^2$ -sub-actions are conjugate. Section 4 contains the proof of Proposition 6, following the ideas in [3].

## 2. THE CLASS $ET$ AND ISOMORPHISM RIGIDITY

An algebraic  $\mathbb{Z}^3$ -action  $X_M$  is said to be of class  $ET$  if  $M$  is Noetherian and there are polynomials  $f \in \mathbb{Z}[u_1, u_2]$  and  $g \in \mathbb{Z}[u_3]$  with the following properties.

- Both  $f$  and  $g$  annihilate  $M$ .
- $g$  is monic, and every zero  $z$  of  $g$  has  $|z| > 1$  ( $g$  is *Expanding*).
- The Newton polygon of  $f$  (the convex hull of  $\{\mathbf{n}: f_{\mathbf{n}} \neq 0\}$ ) is a triangle with corners at  $(0, 0, 0)$ ,  $(a, 0, 0)$ ,  $(0, a, 0)$  for some  $a > 0$  ( $f$  is *Triangular*) and the coefficients corresponding to these corners are  $\pm 1$ .

Notice that property  $ET$  is stable in the following sense. If  $X$  and  $Y$  are  $ET$ , then so is  $X \times Y$ . If  $M$  is a Noetherian module with the property that  $X_{R_d/P}$  is  $ET$  for every prime ideal associated to  $M$ , then  $X_M$  is  $ET$ . In both cases the reason is that the product of two expanding polynomials is expanding, and the product of two triangular polynomials is triangular.

**Theorem 4.** *Let  $X_1$  and  $X_2$  be  $ET$  algebraic  $\mathbb{Z}^3$ -actions. Suppose  $\alpha_1$  is mixing and the  $\mathbb{Z}^2$ -sub-action generated by  $\alpha_1^{\mathbf{e}_2}$  and  $\alpha_1^{\mathbf{e}_3}$  has completely positive entropy. Then every factor map from  $X_1$  to  $X_2$  is affine.*

**Corollary 5.** *Let  $P_1 \neq P_2$  be prime ideals in  $R_3$ . Assume that  $X_{R_3/P_2}$  is  $ET$ ,  $X_{R_3/P_1}$  is  $ET$  and mixing, and the  $\mathbb{Z}^2$ -sub-action generated by  $\alpha_{R_3/P_1}^{\mathbf{e}_2}$  and  $\alpha_{R_3/P_1}^{\mathbf{e}_3}$  has completely positive entropy. Then  $X_{R_3/P_1}$  and  $X_{R_3/P_2}$  are not measurably conjugate.*

*Proof.* By Theorem 4, it is enough to show that the two actions are not algebraically isomorphic. If  $\varphi$  were an algebraic isomorphism between  $X_{R_3/P_1}$  and  $X_{R_3/P_2}$ , then the dual map  $\widehat{\varphi}: R_3/P_2 \rightarrow R_3/P_1$  would be a module isomorphism, which would imply that  $P_1 = P_2$ .  $\square$

Theorem 4 is proved using the restricted version of measure rigidity stated in Proposition 6. An  $\alpha_M$ -invariant measure  $\mu$  is **totally ergodic** if  $\alpha_M^n$  is an ergodic transformation of  $(X_M, \mu)$  for all  $\mathbf{n}$  in  $\mathbb{Z}^3 \setminus \{0\}$ .

**Proposition 6.** *Let  $X_M$  be an ET algebraic  $\mathbb{Z}^3$ -action. Suppose  $\mu$  is a totally ergodic  $\alpha_M$ -invariant measure. Then there exists a closed  $\alpha_M$ -invariant subgroup  $Y \subset X_M$  such that  $\mu$  is invariant under translation by elements of  $Y$ , and the action generated by  $\alpha_M^{\mathbf{e}_2}$  and  $\alpha_M^{\mathbf{e}_3}$  induced on the factor  $X_M/Y$  has zero entropy with respect to  $\mu$ .*

Theorem 4 follows by a well-known argument of Thouvenot from Proposition 6 (see [3] or [9] for more details). A sketch of this argument follows. As noted above, if  $X_1$  and  $X_2$  are ET, then so is  $X_1 \times X_2$ . A conjugacy  $\varphi$  from  $X_1$  to  $X_2$  defines a joining  $\mu$  supported on the graph  $G_\varphi$  of  $\varphi$ . The projection map  $\pi_1: X_1 \times X_2 \rightarrow X_1$  onto the first coordinate satisfies  $\mu(A) = \lambda_{X_1}(\pi_1(A \cap G_\varphi))$  for any  $A \subset X_1 \times X_2$ . The measure  $\mu$  satisfies the assumption of Proposition 6 since  $\alpha_1$  is mixing. By assumption, the sub-action generated by  $\alpha_1^{\mathbf{e}_2}$  and  $\alpha_1^{\mathbf{e}_3}$  has completely positive entropy. By Proposition 6, the entropy on the factor  $X_1 \times X_2/Y$  vanishes, so this factor must be trivial. It follows that  $\mu$  is the Haar measure of the affine subset  $x_0 + Y$  for some  $x_0 \in X_1 \times X_2$ , and the graph  $G_\varphi$  agrees with the affine subset  $x_0 + Y$  a.e. We conclude that  $\varphi$  is affine  $\lambda_{X_1}$ -a.e.

### 3. EXAMPLES

We give examples of algebraically — hence, by Theorem 4, measurably — non-isomorphic  $\mathbb{Z}^3$ -actions all of whose  $\mathbb{Z}^2$ -sub-actions are measurably conjugate.

To study the action of a subgroup  $\Lambda \subset \mathbb{Z}^3$ , it is useful to define the corresponding sub-ring

$$R_\Lambda = \mathbb{Z}[u^\mathbf{n}: \mathbf{n} \in \Lambda] \subset R_3.$$

Dynamical properties of the sub-action  $\alpha_M|_\Lambda$  are governed by the structure of the module  $M$  over the ring  $R_\Lambda$ . Notice that in the case of a prime system  $\mathbf{X}_{R_d/P}$ , the only prime ideal in  $R_\Lambda$  associated to  $R_d/P$  as a module over  $R_\Lambda$  is  $P \cap R_\Lambda$ .

**Example 7.** Let

$$P_1 = \langle 1 + u_1 + u_2, u_3 - 2 \rangle \text{ and } P_2 = \langle 1 + u_1 + u_2, u_3 + 2 \rangle.$$

We will show that the prime actions  $\mathbf{X}_{R_3/P_i}$  for  $i = 1, 2$  are mixing and have zero entropy. Furthermore, the sub-actions corresponding to subgroups of rank two have completely positive entropy, are isomorphic to Bernoulli shifts and are conjugate. By Corollary 5, the two actions are not conjugate.

The ideals are prime ideals since in each case

$$R_3/P_i = \mathbb{Z}[1/2, u_1^{\pm 1}, u_2^{\pm 1}]/\langle 1 + u_1 + u_2 \rangle.$$

The only difference between the two actions is that  $\alpha_{R_3/P_1}^{\mathbf{e}_3}$  acts by multiplication by 2 and  $\alpha_{R_3/P_2}^{\mathbf{e}_3}$  acts by multiplication by  $-2$ . This shows that  $\alpha_{R_3/P_1}^{2\mathbf{e}_3}$  and  $\alpha_{R_3/P_1}^{2\mathbf{e}_3}$  act identically, so the restrictions of the actions to the subgroup  $\mathbb{Z}^2 \times (2\mathbb{Z})$  are algebraically isomorphic, and the entropies for any rank two subgroup of  $\mathbb{Z}^2 \times (2\mathbb{Z})$  must therefore agree. Since the entropy of the  $\Lambda$ -sub-action and the entropy for the  $2\Lambda$ -sub-action determine each other, it follows that the sub-actions  $\alpha_{R_3/P_1}|_\Lambda, \alpha_{R_3/P_2}|_\Lambda$  induced by any subgroup  $\Lambda \subset \mathbb{Z}^3$  of rank two have equal entropy.

We claim that both  $\mathbf{X}_{R_3/P_1}$  and  $\mathbf{X}_{R_3/P_2}$  are mixing and have zero entropy. For mixing, we need to show that  $u^\mathbf{n} - 1 \notin P_i$  for every  $\mathbf{n}$  in  $\mathbb{Z}^3 \setminus \{\mathbf{0}\}$ . Assume that  $u^\mathbf{n} - 1 \in P_i$ , and let  $\zeta_3 = e^{2\pi i/3}$ . Then  $(\zeta_3, \zeta_3^2, \pm 2)$  belongs to the variety of  $P_i$ . This shows that  $\zeta_3^{n_1+2n_2}(\pm 2)^{n_3} = 1$ , so  $n_3 = 0$ . Going through the argument again using the points  $(1, -2, \pm 2), (-2, 1, \pm 2)$  on the variety shows that  $\mathbf{n} = \mathbf{0}$ . The action has zero entropy since it is a prime action whose prime ideal is not principal (see [12] or [15, Cor. 18.5]).

Let  $\Lambda \subset \mathbb{Z}^3$  be any subgroup isomorphic to  $\mathbb{Z}^2$ . We claim that the sub-action  $\alpha_{R_3/P_i}|_\Lambda$  is isomorphic to a two-dimensional Bernoulli shift for  $i = 1, 2$ . By [13] (see also [15, Th. 23.1]), an algebraic  $\mathbb{Z}^d$ -action is isomorphic to a Bernoulli shift if and only if it has completely positive entropy. By [12], (see also [15, Th. 20.8]) this is the case if and only

if every associated prime ideal is generated by a single polynomial which is not cyclotomic. Applying this to the sub-action  $\alpha_{R_3/P_i}|_\Lambda$ , it is enough to show that  $P_i \cap R_\Lambda$  is generated by a single polynomial which is not cyclotomic.

Assume first that  $P_i \cap R_\Lambda$  is trivial. In this case  $R_\Lambda$  is a sub-ring of  $R_3/P_i$ . This is impossible since  $R_\Lambda$  has transcendence degree two but  $R_3/P_i$  has transcendence degree one, so  $P_i \cap R_\Lambda$  cannot be trivial. Similar arguments show that  $P_i \cap R_\Lambda$  must be a principal ideal. By the above we already know that  $P_i$  does not contain any cyclotomic polynomial, so  $P_i \cap R_\Lambda$  is not generated by a cyclotomic polynomial.

In the remaining examples we will not repeat the arguments above showing that the action is a prime action, has zero entropy, is mixing and that every sub-action for a rank two subgroup is Bernoulli. In each case these use well-known algebraic methods and characterizations from [15]. We will just check the following property: systems  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are  $\mathbb{Z}^2$ -**entropy equivalent** if for every subgroup  $\Lambda \subset \mathbb{Z}^3$  of rank two,  $h(\alpha_1|_\Lambda) = h(\alpha_2|_\Lambda)$ .

**Example 8.** Let

$$M = R_3/\langle 1 + u_1 + u_2, u_3 - 4 \rangle \text{ and } N = R_3/\langle 1 + u_1^2 + u_2^2, u_3 - 2 \rangle.$$

As before,  $\mathbf{X}_M$  and  $\mathbf{X}_N$  are mixing, zero entropy prime actions with the property that every sub-action of a rank two subgroup has completely positive entropy. By Corollary 5, the two actions are not conjugate.

We claim the actions are  $\mathbb{Z}^2$ -entropy equivalent. To see this, consider the sub-action  $\alpha_N|_\Gamma$  for  $\Gamma = (2\mathbb{Z})^3$  and use the rescaled variables  $w_j = u_j^2$  and the ring  $S_3 = \mathbb{Z}[w_1^{\pm 1}, w_2^{\pm 1}, w_3^{\pm 1}]$  to study the sub-action (see [17] for other applications of rescaling). Define a homomorphism of  $S_3$ -modules by

$$\varphi: (S_3/\langle 1 + w_1 + w_2, w_3 - 4 \rangle)^4 \longrightarrow N$$

by

$$\varphi(a_1, a_2, a_3, a_4) = a_1 + u_1 a_2 + u_2 a_3 + u_1 u_2 a_4.$$

This is an isomorphism, which shows that the action  $\alpha_N|_\Gamma$  is algebraically isomorphic to four disjoint copies of  $\alpha_M$ . Now let  $\Lambda \subset \mathbb{Z}^3$  be a subgroup isomorphic to  $\mathbb{Z}^2$ . Clearly  $2\Lambda \subset \Gamma$ , and the algebraic isomorphism dual to  $\varphi$  carries  $\alpha_N|_{2\Lambda}$  to  $\alpha_M|_\Lambda$ . It follows that

$$h(\alpha_N|_\Lambda) = \frac{1}{4} h(\alpha_N|_{2\Lambda}) = h(\alpha_M|_\Lambda),$$

showing that  $X_M$  and  $X_N$  are  $\mathbb{Z}^2$ -entropy equivalent.

In Examples 7 and 8 the pairs of ideals were closely related. In the following the same triangular polynomial  $f(u_1, u_2) = 1 + u_1 + u_2$  is used, but the expanding polynomials are chosen with rationally independent roots.

By [12] or [15, Th. 18.1], for any non-zero polynomial  $h \in R_d$  the entropy of the action  $\alpha_{R_d/\langle h \rangle}$  is given by

$$h(\alpha_{R_d/\langle h \rangle}) = \log \mathbb{M}(h),$$

where the Mahler measure  $\mathbb{M}(h)$  of the polynomial  $h$  is defined by

$$\log \mathbb{M}(h) = \int_0^1 \cdots \int_0^1 \log |h(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})| dt_1 \cdots dt_d.$$

For  $d = 1$  the Mahler measure can be directly expressed in terms of expanding eigenvalues. Jensen's formula (see [7, Lemma 1.8] or [15, Prop. 16.1]) shows that

$$(1) \quad \int_0^1 \log |h(e^{2\pi i t})| dt = \sum_{j=1}^s \log^+ |\zeta_j| + \log |a_s|$$

for a polynomial  $h(u) = a_s \prod_{i=1}^s (u - \zeta_i)$ . This shows that for  $d = 1$  the Mahler measure of  $h$  only depends on the absolute values of the zeros of  $h$  and its leading coefficient.

**Example 9.** Let  $g_1(u_3) = u_3^2 + 2u_3 + 10$  and  $g_2(u_3) = u_3^2 + 4u_3 + 10$ . Eisenstein's criterion for  $p = 2$  shows that both polynomials are irreducible in  $\mathbb{Z}[u_3]$ . If  $g_i(u_3) = (u_3 - \zeta_{i,1})(u_3 - \zeta_{i,2})$  then  $\zeta_{i,j} \notin \mathbb{R}$  and  $|\zeta_{i,j}|^2 = 10$  for  $i, j = 1, 2$ .

Let  $M_1 = R_3/\langle 1+u_1+u_2, g_1(u_3) \rangle$  and  $M_2 = R_3/\langle 1+u_1+u_2, g_2(u_3) \rangle$ . As before  $X_{M_i}$  is a prime, mixing, zero entropy  $ET$  system for which every sub-action of a rank two subgroup has completely positive entropy for  $i = 1, 2$ . By Corollary 5, the two actions are not conjugate.

We show that  $X_{M_1}$  and  $X_{M_2}$  are  $\mathbb{Z}^2$ -entropy equivalent, so that the sub-actions for any subgroup of rank two are conjugate. For  $\Lambda = \ll \mathbf{e}_1, \mathbf{e}_2 \gg$ , this is clear since the associated prime ideal for this particular sub-action of  $X_{M_i}$  is

$$\langle 1 + u_1 + u_2, g_i(u_3) \rangle \cap R_2 = \langle 1 + u_1 + u_2 \rangle.$$



For a subgroup of rank two with  $\mathbf{e}_3$  as a generator, the associated prime ideals for the sub-action are generated by  $g_1$  and  $g_2$ , which are monic polynomials with two roots of absolute value  $\sqrt{10}$ . Therefore  $\mathbb{M}(g_1) = \mathbb{M}(g_2) = 10$ , and the entropy is a multiple of  $\log 10$ . The multiplicative factor depends only on the geometry of  $\Lambda$  and not on the polynomial  $g_i$  (see the general case below for more details). For a subgroup in general position, we need to argue that the Mahler measure of two (in general unknown) polynomials coincide, and that two (potentially inscrutable) multiplicities coincide.

To see what is creating the exact value of the entropy for a rank two subgroup, first consider the subgroup  $\Lambda = \langle\langle \mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_3 \rangle\rangle$ . Write  $v_1 = u_1$  and  $v_2 = u_2 u_3^{-1}$ . The structure of the subsystem is given by the structure of  $\widehat{\mathbf{X}}_{M_i}$  as a module over the ring  $S = \mathbb{Z}[v_1^{\pm 1}, v_2^{\pm 1}]$ . Let  $c_1 = -2$ ,  $c_2 = -4$  be the different traces in  $g_1, g_2$ . Since

$$1 + u_1 + u_2 = 1 + v_1 + v_2 u_3,$$

the relation

$$(2) \quad (1 + v_1 + v_2 \zeta_{i,1})(1 + v_1 + v_2 \zeta_{i,2}) = (1 + v_1)^2 + 10v_2^2 + c_i v_1 v_2 + c_i v_2 = 0$$

holds for  $i = 1, 2$ . Since this relation is irreducible over  $\mathbb{Z}$ , the polynomial in (2) generates the only principal prime ideal associated to  $M_i$  over  $S$ . This means the whole subsystem is an invertible extension of the system defined by the relation (2) in the plane spanned by  $\Lambda$ . Thus

$$\begin{aligned} h(\alpha_{M_i} | \Lambda) &= \log \mathbb{M}((1 + v_1)^2 + 10v_2^2 + c_i v_1 v_2 + c_i v_2) \\ &= \log \mathbb{M}(1 + v_1 + v_2 \zeta_{i,1}) + \log \mathbb{M}(1 + v_1 + v_2 \zeta_{i,2}) \\ &= \log |\zeta_{i,1}| + \int_0^1 \log^+ |(1 + e^{2\pi i t}) / \zeta_{i,1}| dt \\ &\quad + \log |\zeta_{i,2}| + \int_0^1 \log^+ |(1 + e^{2\pi i t}) / \zeta_{i,2}| dt \\ &= \log 10 \end{aligned}$$

by Jensen's formula (1). In particular, the entropy is the same for  $i = 1$  and  $2$ .

A similar argument for the subgroup  $\Lambda = \langle\langle \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3 \rangle\rangle$  will show that the entropy is governed by the relation

$$(1 + v_1 + v_2 \zeta_{i,1}^{-1})(1 + v_1 + v_2 \zeta_{i,2}^{-1}) = 0,$$

giving entropy

$$\begin{aligned}
h(\alpha_{M_i}|\Lambda) &= \log \mathbb{M}(1 + v_1 + v_2 \zeta_{i,1}^{-1}) + \log \mathbb{M}(1 + v_1 + v_2 \zeta_{i,2}^{-1}) \\
&= \log |\zeta_{i,1}^{-1}| + \int_0^1 \log^+ |\zeta_{i,1}(1 + e^{2\pi it})| dt \\
&\quad + \log |\zeta_{i,2}^{-1}| + \int_0^1 \log^+ |\zeta_{i,2}(1 + e^{2\pi it})| dt \\
&= \int_0^1 \log^+ |\zeta_{i,1}(1 + e^{2\pi it})| dt + \int_0^1 \log^+ |\zeta_{i,2}(1 + e^{2\pi it})| dt \\
&\quad - \log(10),
\end{aligned}$$

which is again independent of  $i$ .

For the general case, let  $\Lambda \subset \mathbb{Z}^3$  be any rank two subgroup of  $\mathbb{Z}^3$  not already considered. Find an element  $\mathbf{n} \in \Lambda \cap (\mathbb{Z}^2 \times \{0\})$  that is non-zero and not a non-trivial multiple of any other element of  $\Lambda$ , and choose  $\mathbf{m} \in \Lambda$  linearly independent to  $\mathbf{n}$  with  $m_3 > 0$ . The points  $\mathbf{n}$  and  $\mathbf{m}$  generate a finite-index subgroup of  $\Lambda$ , so we may assume without loss of generality that  $\Lambda = \langle\langle \mathbf{n}, \mathbf{m} \rangle\rangle$ . Let  $S = \mathbb{Z}[\mathbf{u}^{\pm \mathbf{n}}, \mathbf{u}^{\pm \mathbf{m}}]$  be the associated sub-ring. Consider first the projection  $(m_1, m_2, 0)$  of  $\mathbf{m}$  onto the  $u_1, u_2$ -plane: let  $w_1 = \mathbf{u}^{\mathbf{n}}$  and  $w_2 = u_1^{m_1} u_2^{m_2}$  and write  $S' = \mathbb{Z}[w_1^{\pm 1}, w_2^{\pm 1}]$ . The structure of  $M_i$  as an  $S'$ -module has a single associated principal prime ideal  $\langle \tilde{f} \rangle$  with multiplicity  $s(\Lambda)$ . As indicated in the notation, the point of projecting to the  $u_1, u_2$ -plane is to remove the variable  $u_3$  and ensure that the sub-module structure is the same for  $i = 1$  and  $2$ . Now write  $w_3 = \mathbf{u}^{\mathbf{m}}$  so that  $S = \mathbb{Z}[w_1^{\pm 1}, w_3^{\pm 1}]$ , and notice that  $w_3 = u_3^{m_3} w_2$ . The structure of  $M_i$  as an  $S$ -module is then determined by the relation

$$10^{2m_3} \tilde{f}(w_1, w_3 \zeta_{i,1}^{-m_3}) \cdot \tilde{f}(w_1, w_3 \zeta_{i,2}^{-m_3}) = 0$$

in  $s(\Lambda)$  copies of the skew plane  $\mathbb{T}^\Lambda$ . Since  $m_3 \neq 0$ , this relation is irreducible over  $\mathbb{Z}$ . The entropy of the action of  $\Lambda$  is given by

$$\begin{aligned}
&\frac{1}{s(\Lambda)} h(\alpha_{M_i}|\Lambda) \\
&= \int_0^1 \int_0^1 \log |10^{2m_3} \tilde{f}(e^{2\pi it}, e^{2\pi is} \zeta_{i,1}^{-m_3}) \cdot \tilde{f}(e^{2\pi it}, e^{2\pi is} \zeta_{i,2}^{-m_3})| ds dt
\end{aligned}$$

$$= 2m_3 \log 10 + \int_0^1 \sum \log^+ |\lambda_s \zeta_{i,1}^{m_3}| ds + \int_0^1 \sum \log^+ |\lambda_s \zeta_{i,2}^{m_3}| ds$$

where the summation is over the roots of  $\tilde{f}(\lambda_s, e^{2\pi i s}) = 0$ . It follows that the entropy is independent of  $i$ .

#### 4. PROOF OF PROPOSITION 6

Let  $M$  be a Noetherian  $R_3$ -module, so  $M \cong R_3^k/J$  for some  $R_3$ -submodule  $J$  of defining relations. The dual group to  $R_3^k$  is  $(\mathbb{T}^k)^{\mathbb{Z}^3}$ , so the dual group of  $M$  is the subgroup annihilating  $J$ . For  $x \in (\mathbb{T}^k)^{\mathbb{Z}^3}$  write  $x_{\mathbf{n}} = (x_{\mathbf{n}}^{(1)}, \dots, x_{\mathbf{n}}^{(k)}) \in \mathbb{T}^k$  for the coordinate corresponding to  $\mathbf{n} \in \mathbb{Z}^3$ .

The algebraic  $\mathbb{Z}^3$ -action  $\alpha_M$  can be realized as the usual shift action on the closed, shift-invariant subgroup of  $(\mathbb{T}^k)^{\mathbb{Z}^3}$  defined by

$$X_M = \{x \in (\mathbb{T}^k)^{\mathbb{Z}^3} : f_1(\sigma)(x^{(1)}) + \dots + f_k(\sigma)(x^{(k)}) = 0 \\ \text{for every } (f_1, \dots, f_k) \in J\},$$

where  $\sigma$  is the  $\mathbb{Z}^d$ -shift on  $(\mathbb{T}^k)^{\mathbb{Z}^3}$ , and  $f(\sigma)$  is the map obtained by substituting the shift into  $f$ .

Let  $\mathbf{X} = \mathbf{X}_M$  be a  $\mathbb{Z}^3$ -action with property *ET*, and write  $g(u_3)$  for the expanding polynomial relation and  $f(u_1, u_2)$  for the triangular relation. The last paragraph means that  $X$  can be thought of as a subshift with alphabet  $\mathbb{T}^k$  for some finite  $k \geq 1$ . Define

$$S = \{\mathbf{m} \in \mathbb{Z}^3 : m_2 \geq 0, m_3 \geq 1\} \cup \{\mathbf{x} \in \mathbb{Z}^3 : m_2 \geq 1\}$$

and

$$U = \mathbb{Z}\mathbf{e}_1.$$

The set  $S$  is to be thought of as a lexicographic ‘future’ for the  $\mathbb{Z}^2$ -action generated by  $u_2$  and  $u_3$ , and the tube  $U$  is the ‘present’. In our setting  $U$  is a copy of  $\mathbb{Z}$ ; in the more general setting of [3] the set  $U$  really is a tube. We will show later that these notions of ‘future’ and ‘present’ give rise to the correct entropy.

Choose a partition  $P$  of the alphabet  $\mathbb{T}^k$  with the following vertical generating property: for any  $\mathbf{n} \in \mathbb{Z}^3$ , the  $P$ -name of the coordinates  $x = (x_{\mathbf{n}})$  for  $m_1 = n_1, m_2 = n_2, m_3 > n_3$  determines  $x_{\mathbf{n}}$  completely. Such a partition exists by the expanding assumption on the relation  $g$ ; indeed any sufficiently fine partition will do. This is the only point

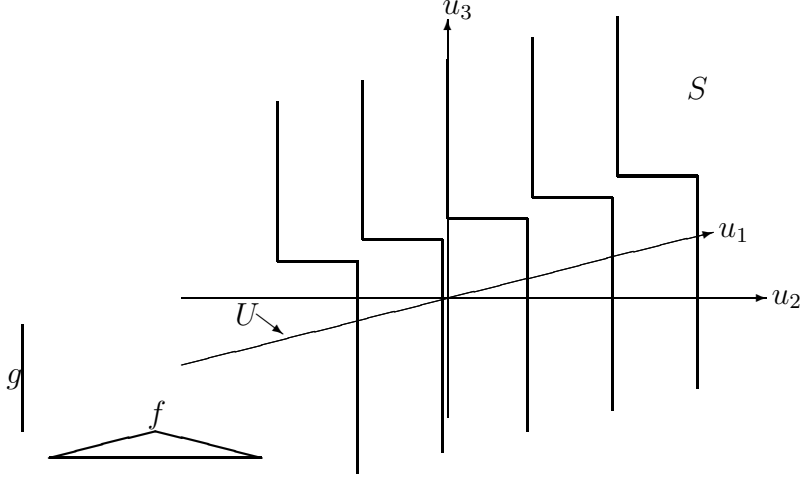


FIGURE 1. The regions  $S$  and  $U$  and the shape of the support of the annihilating relations  $f$  and  $g$

at which the expanding hypothesis is essential rather than convenient. Write  $\mathcal{P}$  for the  $\sigma$ -algebra defined by the common refinement  $\bigvee_{\mathbf{n} \in U} \alpha^{\mathbf{n}}(P)$ .

Let  $[0] = \{x \in X : x_{\mathbf{n}} = 0 \text{ for } \mathbf{n} \in S\}$ . There is a natural projection map  $\pi$  from  $X$  onto  $(\mathbb{T}^k)^{S \cup U}$ . Let  $G = \pi([0])$ , and notice that the properties of  $g$  and  $f$  ensure that  $G$  is a finite group. Let  $\mathcal{A}$  be the  $\sigma$ -algebra determined by the partition  $\bigvee_{\mathbf{n} \in S} \alpha^{\mathbf{n}}(P)$ . Notice that our assumptions ensure that  $\mathcal{A}$  is the same  $\sigma$ -algebra as the pre-image of the whole Borel  $\sigma$ -algebra on  $(\mathbb{T}^k)^S$ .

Given any point  $z \in [0]$ , let

$$F_z(x) = \mu_x^{\mathcal{A}}([x+z]_{\mathcal{A} \vee \mathcal{P}}),$$

where  $\mu_x^{\mathcal{A}}$  is the conditional measure, and  $[x]_{\mathcal{C}}$  denotes the  $\mathcal{C}$ -atom containing  $x$ . Notice that  $F_z = F_{z'}$  whenever  $\pi(z) = \pi(z')$ . If  $\zeta \in G$  and  $z \in [0]$  with  $\pi(z) = \zeta$  we set  $F_{\zeta} = F_z$ . Furthermore,

$$-\log F_0(x) = I_{\mu}(\mathcal{P}|\mathcal{A})(x)$$

is the information function. More is true: results from [3] may be used to show that

$$(3) \quad \int_X -\log F_0(x) d\mu = H_{\mu}(\mathcal{P}|\mathcal{A}) = h_{\mu}(\alpha|_{\Lambda})$$

where  $h_\mu(\alpha|_\Lambda)$  denotes the entropy with respect to  $\mu$  of the  $\mathbb{Z}^2$ -action generated by  $\alpha^{e_2}$  and  $\alpha^{e_3}$ . Notice that the second equality in (3) is a Rokhlin formula, expressing the entropy of this  $\mathbb{Z}^2$ -action as the information contained in the present given the information contained in the future with respect to a generator; one subtlety that needs to be dealt with in proving this is that  $\mathcal{P}$  is not a finite partition.

We sketch the proof of (3). First, since the coefficients of the triangular polynomial  $f$  are  $\pm 1$  at the corners, it follows that there exists some non-negative integer  $\ell$  such that

$$\mathcal{P} \vee \mathcal{A} = \bigvee_{n=0}^{\ell} \alpha^{ne_1} P \vee \mathcal{A}.$$

This shows that the integral of the information function is finite, and that  $F_0(x) > 0$  a.e. Furthermore, again by the properties of the triangular polynomial,

$$(4) \quad \bigvee_{\mathbf{n} \in U, |\mathbf{n}_1| < N} \alpha^{\mathbf{n}} P \vee \bigvee_{\mathbf{n} \in S, |\mathbf{n}_1| < N} \alpha^{\mathbf{n}} P = \bigvee_{\mathbf{n} \in U, 0 \leq \mathbf{n}_1 < \ell} \alpha^{\mathbf{n}} P \vee \bigvee_{\mathbf{n} \in S, |\mathbf{n}_1| < N} \alpha^{\mathbf{n}} P$$

holds for every  $N > \ell$ . Here we use the fact that the support of  $f$  is triangular. The increasing Martingale theorem for entropy and (4) then shows (3).

The next step is to understand how  $F_\xi$  varies as  $\xi$  runs through the finite group  $G$ . First,  $F_0(x) > 0$  and  $\sum_\xi F_\xi(x) = 1$  for a.e.  $x$ . Since  $G$  is finite, some power  $m$  of  $\alpha^{e_1}$  maps  $\xi$  to itself, so

$$F_\xi(x) = F_\xi(\alpha^{me_1}x).$$

This implies — since  $\alpha^{e_1}$  is totally ergodic — that  $F_\xi(x) = p_\xi$  is a.e. constant in  $x$ . However, for a different  $\xi$  the constant may be different. We claim that

$$H = \{\zeta \in G: p_\zeta > 0\}$$

forms a subgroup of  $G$ . This is proved in [3] in detail; to see why it should be true one may argue as follows. For  $\xi_1, \xi_2 \in H$ , positivity of  $F_{\xi_1}(x)$  means that  $x + \xi_1$  is a configuration allowed with positive  $\mu$ -measure; positivity of  $F_{\xi_2}(x + \xi_1)$  then means that  $\xi_1 + \xi_2$  is also allowed with the same  $\mu$ -measure so  $\xi_1 + \xi_2 \in H$ . The full proof requires some care in the removal of a null set of exceptional behaviour.

It follows that

$$F_\xi(x) = \frac{1}{|H|} \text{ for all } \xi \in H.$$

Notice that the group  $H$  is trivial if and only if  $h_\mu(\alpha|_\Lambda)$  vanishes. Since the proposition is trivial in this case, we assume without loss of generality that  $|H| = \exp h_\mu(\alpha|_\Lambda) > 1$ .

At this point we have shown that the measure  $\mu$  must be invariant under translation by a finite, non-trivial, group (the subgroup  $H$  of  $G$ ) when restricted to a small  $\sigma$ -algebra of sets (the sets measurable with respect to  $\mathcal{A} \vee \mathcal{P}$ ). In order to extend this to invariance on a larger  $\sigma$ -algebra, we replicate a version of the argument above on a larger scale.

Given an integer  $N$ , define

$$\begin{aligned} S_N &= \{\mathbf{m} \in \mathbb{Z}^3: m_2 \geq 0, m_3 \geq N\} \cup \{\mathbf{m} \in \mathbb{Z}^3: m_2 \geq N\} \\ U_N &= \{\mathbf{m} \in \mathbb{Z}^3: 0 \leq m_2 < N, 0 \leq m_3 < N\} \end{aligned}$$

and the projection map  $\pi_N : X \rightarrow (\mathbb{T}^k)^{S_N \cup U_N}$  accordingly. If  $M < N$ , let  $B(M, N)$  denote the set of points  $x \in X$  with  $x_{\mathbf{n}} = 0$  for all  $\mathbf{n} \in S_N$  except for those coordinates in the shaded part of Figure 2. Let  $H_N$  be the group defined by the construction above applied to the scaled action generated by  $\alpha^{Ne_1}, \alpha^{Ne_2}, \alpha^{Ne_3}$ . Notice that

$$h_\mu(\alpha|_{N\Lambda}) = N^2 h_\mu(\alpha|_\Lambda),$$

so  $|H_N| = N^2 |H|$ . The corresponding  $\sigma$ -algebras  $\mathcal{A}_N = \bigvee_{\mathbf{n} \in S_N} \alpha^{\mathbf{n}}(\mathcal{P})$  are nested since  $S_N$  moves away from 0, and

$$\mathcal{A}_N \searrow \mathcal{N}_X = \{\emptyset, X\}.$$

Because of the relations  $g$  and  $f$ , there is some fixed number  $K$  such that every little square in Figure 2 can at most contribute a factor of  $K$  to  $|\pi_N(B(M, N))|$ . Therefore

$$\log |\pi_N(B(M, N)) \cap H_N| \leq 2MN \log K.$$

It follows that for any  $M$  there is an  $N$  such that  $H_N \not\subset B(M, N)$ . In particular, if  $Q = Q_{M,n}$  denotes the ‘inner’ square of side  $N - M$  in Figure 2 then there exists an  $x \in H_N$  such that  $x|_Q \neq 0$ . So we may choose for each  $N$  an element  $x^{(N)} \in H_N$  with  $x_{\mathbf{n}}^{(N)} \neq 0$  for some  $\mathbf{n} = \mathbf{n}(N) \in Q_{M,N}$ . After shifting (via the action) the picture by  $\mathbf{n}$ , this gives a point  $y^{(N)}$  with  $y_0^{(N)} \neq 0$ . Notice that

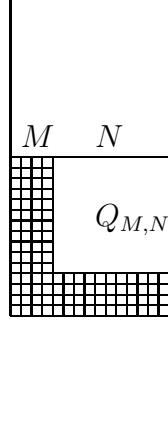


FIGURE 2. The relative size of  $B(M, N)$  and  $H_N$  as  $N \rightarrow \infty$ , viewed in the plane generated by the last two standard basis vectors.

after shifting the point  $y^{(N)}$  satisfies that  $\mu(B + y^{(N)}) = \mu(B)$  for all  $B \in \bigvee_{\mathbf{m} \in [-M, M]^3} \alpha^{\mathbf{m}} P$ . By compactness there is a non-trivial  $y \in X$  with the property that  $\mu$  is invariant under translation by  $y$  (and therefore by the smallest closed subgroup of  $X$  containing  $y$ ) on the whole  $\sigma$ -algebra  $\mathcal{B}_X$ .

We are now ready to prove Proposition 6. Let

$$Y = \{x \in X: \mu \text{ is } x\text{-invariant}\},$$

and put  $Z = X/Y$ . Then  $(Z, \mu, \alpha)$  has property *ET* and still satisfies all of the assumptions of Proposition 6. So if  $h(\alpha_Z|_\Lambda) > 0$  we would have to find that  $\mu$  is translation-invariant with respect to a non-trivial subgroup, contradicting the choice of  $Y$ . It follows that  $h(\alpha_Z|_\Lambda) = 0$ .

## 5. REMARKS

A central question in this type of algebraic rigidity is the following. If  $X$  and  $Y$  are zero-entropy mixing algebraic  $\mathbb{Z}^d$ -actions,  $d > 1$ , do they exhibit isomorphism rigidity? Bhattacharya has shown that

the answer is ‘no’ in general (though he has gone on to show that an extension of the notion of affine map allows rigidity to be recovered — measurable isomorphisms still arise in rigid families and in particular are continuous). Nothing has yet been shown to preclude a positive answer to the following question (cf. [14, Conj. 9.1]): do zero-entropy mixing algebraic  $\mathbb{Z}^d$ -actions,  $d > 1$  on connected groups exhibit isomorphism rigidity?

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