Computing vertex-surjective homomorphisms to partially reflexive trees *

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Abstract. A homomorphism from a graph G to a graph H is a vertex mapping $f : V_G \to V_H$ such that $f(u)$ and $f(v)$ form an edge in H whenever u and v form an edge in G. The H-Coloring problem is to test if a graph G allows a homomorphism to a given graph H. A well-known result of Hell and Nešetřil determines the computational complexity of this problem for any fixed graph H. We study a natural variant of this problem, namely the Surjective H-Coloring problem, which is to test whether a graph G allows a homomorphism to a graph H that is (vertex-)surjective. We classify the computational complexity of this problem when H is any fixed partially reflexive tree. Thus we identify the first class of target graphs H for which the computational complexity of Surjective H-Coloring can be determined. For the polynomial-time solvable cases we show a number of parameterized complexity results, especially on graph classes with (locally) bounded expansion.

1 Introduction

A graph is denoted $G = (V_G, E_G)$, where V_G is the set of vertices and E_G is the set of edges. A *homomorphism* from a graph G to a graph H is a mapping $f : V_G \to V_H$ that maps adjacent vertices of G to adjacent vertices of H, i.e., $f(u)f(v) \in E_H$ whenever $uv \in E_G$.

The problem H-Coloring is to test whether a given graph G allows a homomorphism to a graph H called the *target*. Throughout our paper we assume that H denotes a *fixed* graph (i.e., not part of the input) except when we consider a parameterized setting and choose $|V_H|$ as the parameter. If H is the complete graph (graph with edges between all pairs of different vertices) on k vertices, then the H-Coloring problem is equivalent to the k-Coloring problem, which is to test whether a graph G allows a mapping $c : V_G \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E_G$.

For a survey on homomorphisms we refer to Hell and Nešetřil [16]. Here, we only mention the classical result in this area, which is the Hell-Nešetřil dichotomy theorem [15]. This theorem states that H-Coloring is solvable in polynomial

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time if H is bipartite, and NP-complete otherwise. Note that H is assumed to have no self-loop xx, as otherwise we can map every vertex of G to x.

A homomorphism f from a graph G to a graph H is *surjective* if for each $x \in V_H$ there exists at least one vertex $u \in V_G$ with $f(u) = x$. This paper studies the problem of deciding if a given graph allows a surjective homomorphism to a fixed target graph H. This problem is called the SURJECTIVE H-COLORING problem. We observe that, for this variant, the presence of a vertex with a self-loop in the target graph H does not make the problem trivial. So, we do allow such vertices in H and call them *reflexive*, whereas vertices with no self-loop are said to be *irreflexive*. A graph is *reflexive* if all its vertices are reflexive, and a graph is *irreflexive* if all its vertices are irreflexive. Throughout the paper, we assume that the input graph G is irreflexive and that the target graph H may contain one or more self-loops. We also assume that both graphs are undirected, finite and have no multiple edges.

Recall that in this paper we assume that H is a fixed graph. When H is part of the input, the problem is called SURJECTIVE COLORING and known to be NP-complete even for very restricted graphs classes, as shown by Golovach et al. [14]. In particular, they proved that it is NP-complete to test whether there exists a surjective homomorphism from a graph G to a graph H even if G and H are

(i) disjoint unions of paths (linear forests);
(ii) disjoint unions of complete graphs;
(iii) trees;
(iv) connected cographs;
(v) connected proper interval graphs;
(vi) connected split graphs.

Only for some special cases, for instance when H is a path [14], the SURJECTIVE COLORING problem can be solved in polynomial time. Hence, there is not much hope for finding non-trivial tractable cases in this direction, and it is therefore natural to fix the target graph H and study the computational complexity of the SURJECTIVE H-COLORING problem.

The SURJECTIVE H-COLORING problem is NP-complete for general graphs when H is a nonbipartite simple graph. This follows from a simple reduction from the corresponding H-COLORING problem, which is NP-complete due to the Hell-Nešetřil dichotomy theorem [15]; we replace an instance graph G of the latter problem by the disjoint union $G + H$ of G and H, and we observe that G allows an homomorphism to H if and only if $G + H$ allows a surjective homomorphism to H. For other cases, the complexity classification of SURJECTIVE H-COLORING is still open; only some partial results are known. In particular, there exist cases of bipartite simple graphs H for which the problem is NP-complete, e.g., when H is the graph obtained from a 6-vertex cycle with one distinct path of length 3 added to each of its six vertices [1]. Recently, SURJECTIVE H-COLORING has been shown to be NP-complete when H is a 4-vertex cycle with a self-loop in every vertex [19]. In this case, the H-COLORING problem is equivalent to the DISCONNECTED CUT problem that is to test whether a graph $G = (V, E)$ has a vertex cut $U \subseteq V$ that in addition induces a disconnected subgraph of G [18]. This problem has also been studied in the context of H-partitions introduced by Dantas et al. [3, 4]. For a survey on the SURJECTIVE H-COLORING problem from a constraint satisfaction point of view we refer to the paper of Bodirsky, Kara and Martin [1]. Below we discuss a number of other problems that are closely related to SURJECTIVE H-COLORING.

1.1 Related Work

Locally surjective homomorphisms. A homomorphism f from a graph G to a graph H is *locally surjective* if f becomes surjective when restricted to the open neighborhood of every vertex u of G. We also say that such an f is an *H-role assignment*, and the corresponding decision is called the H-ROLE ASSIGNMENT problem. Any locally surjective homomorphism is surjective if the target graph is connected but the reverse implication is not true in general.

The computational complexity of the H-ROLE ASSIGNMENT problem has been completely classified with the problem being solvable in polynomial time if and only if the fixed graph H has no edge, or H has an isolated reflexive vertex, or H is bipartite, irreflexive and has an isolated edge. In all other cases, H-ROLE ASSIGNMENT is NP-complete [13]. For more on locally surjective homomorphisms and the locally injective and bijective variants, we refer to the survey of Fiala and Kratochvíl [12].

List-homomorphisms and retractions. Let G and H be two graphs with a list $L(u) \subseteq V_H$ associated to each vertex $u \in V_G$. Then a homomorphism f from G to H is a *list-homomorphism* with respect to the lists L if $f(u) \in L(u)$ for all $u \in V_G$. List-homomorphisms were introduced by Feder and Hell [8] and generalize list-colorings. Feder, Hell and Huang [9] completely classified the computational complexity of the problem that tests whether a graph G allows a list-homomorphism to a fixed graph H with respect to some given lists L. In our context, a special kind of list homomorphisms are of importance, namely the retractions defined below.

Let H be an induced subgraph of a graph G. A homomorphism f from a graph G to H is a *retraction* from G to H if $f(h) = h$ for all $h \in V_H$. In that case we say that G *retracts to* H. A retraction from G to H can be viewed as a list-homomorphism if we choose $L(x) = \{x\}$ for each $x \in V_H$ and $L(u) = V_H$ for each $u \in V_G \setminus V_H$.

The H-RETRACTION problem is to test whether a graph G retracts to a fixed subgraph H. A *pseudoforest* is a graph in which each (connected) component has at most one cycle different from a self-loop. Feder et al. [10] classified the complexity of the H-RETRACTION problem for all fixed pseudoforests H.

Compactions. We stress that a surjective homomorphism is *vertex-surjective* as opposed to the stronger condition of being *edge-surjective*. The latter condition has been defined in the literature as well. A homomorphism from a graph G to a graph H is called *edge-surjective* or a *compaction* if for any edge $xy \in E_H$ with $x \neq y$ there exists an edge $uv \in E_G$ with $f(u) = x$ and $f(v) = y$. Note that the edge-surjectivity condition only holds for edges $xy \in E_H$; there is no such condition on the self-loops $xx \in E_H$. If f is a compaction from G to H, we also say that G *compacts* to H.

The H-Compaction problem is to test whether a graph G compacts to a fixed graph H. Vikas [27–29] determined the computational complexity of this problem for several classes of fixed target graphs, e.g., when H is a reflexive cycle, an irreflexive cycle, or a graph on at most 4 vertices. Recently, Vikas [30] considered the H-Compaction problems for graphs G that belong to some special graph class.

Finally, we observe that in contrast to the Surjective H-Coloring problem, the injective variant has been well studied in the literature; when both G and H are part of the input, the injective variant is equivalent to the Subgraph Isomorphism problem.

1.2 Our Results

We give a complete classification of the computational complexity of the Surjective H-Coloring problem when H is a tree. Because we consider target graphs that may contain self-loops, H is a *partially reflexive tree*, i.e., a connected graph with no cycles different from a self-loop. Let R_H denote the (possibly empty) set of reflexive vertices of a graph H. We say that H is *loop-connected* if R_H induces a connected subgraph of H. Note that H is loop-connected if H is irreflexive, i.e., if $R_H = \emptyset$. Our main result is the following theorem.

Theorem 1. *For any fixed tree H, the* Surjective H-Coloring *problem is polynomial-time solvable if H is loop-connected, and* NP*-complete otherwise.*

We analyze the running time of the polynomial-time solvable cases in Theorem 1. For connected graphs with n vertices and m edges we find a running time of $O(n^k(n+m))$, where k is the number of leaves of H. We show that there is no function f that only depends on k such that this running time can be improved to $f(k) \cdot n^{O(1)}$, unless FPT = W[1], or to $f(k) \cdot n^{o(k)}$, unless the Exponential Time Hypothesis [17] is false. On the positive side, we prove that for any loop-connected tree H, the Surjective Coloring problem parameterized by $|V_H|$ is FPT on any graph class with locally bounded expansion (defined in Section 2). Examples of such graph classes are graphs of bounded genus (e.g. planar graphs), graphs that exclude a fixed (topological) minor and graphs that locally exclude a fixed minor [7].

2 Preliminaries

Graphs and graph homomorphisms. We refer to the text book of Diestel [5] for all graph notions and notations not defined in this section. We start by shortly recalling the following graph-theoretic notions from Section 1. A graph is denoted $G = (V_G, E_G)$, where V_G is the set of vertices and E_G is the set of edges. A vertex is irreflexive if it has no self-loops and it is reflexive otherwise. A graph G is irreflexive or reflexive, if G contains no reflexive vertices or only reflexive vertices, respectively. We let R_G denote the (possibly empty) set of reflexive vertices of a graph G and say that G is loop-connected if $G[R_G]$ is connected; here we use the notation $G[U]$ to denote the subgraph of G *induced by* a set $U \subseteq V_G$, i.e., the graph with vertex set U such that for all $u, v \in U$, there exists an edge between u and v if and only if there exists an edge between u and v in G. A pseudoforest is a graph in which each component has at most one cycle different from a self-loop; here a *component* is a connected subgraph of G that is not contained in any other connected subgraph of G. A partially reflexive tree is a connected graph with no cycles different from a self-loop. If it is clear from the context we omit the adjective "partially reflexive". A homomorphism from a graph G to a graph H is a mapping $f : V_G \to V_H$ such that $f(u)f(v) \in E_H$ whenever $uv \in E_G$, which is called surjective if for each $x \in V_H$ there exists at least one vertex $u \in V_G$ with $f(u) = x$, and which is called a retraction if H is an induced subgraph of G and $f(h) = h$ for all $h \in V_H$. The problems H-Surjective Coloring and H-Retraction are to test whether there exists a surjective homomorphism or a retraction, respectively, from a given graph G to a graph H called the target graph that is fixed, i.e., that is not part of the input. Here, we assume that G is irreflexive, whereas H may contain self-loops. Note that we can make this assumption for the Retraction problem only by a slight adjustment of the definition, namely that G must contain the graph obtained from H by removing all self-loops as an induced subgraph. This adjustment does not influence the computational complexity of the problem.

Let $G = (V, E)$ be a graph. A subset $E' \subseteq E$ is a *matching* of G if no two edges in E' have an endvertex in common. The graph obtained from G after removing a subset $E' \subseteq E$ is denoted by $G - E'$. A subset $V' \subseteq V$ is a *clique* of G if $G[V']$ is a complete graph, i.e., a graph with edges between all pairs of different vertices. The graph obtained from G by removing a subset $V' \subseteq V$ is denoted by $G - V'$; if $V = \{u\}$ we write $G - u$ instead. The *distance* $\mathrm{dist}_G(u, v)$ between a pair of vertices u and v of G is the number of edges on a shortest path between them. For a set $U \subset V_G$ and a vertex $u \in V_G \setminus U$, we define $\mathrm{dist}_G(u, U) = \min_{v \in U} \mathrm{dist}_G(u, v)$. We denote the *(open) neighborhood* of a vertex u in G by $N_G(u) = \{v \neq u \mid uv \in E_G\}$. We define the neighborhood of a set $U \subseteq V_G$ as $N_G(U) = \{v \mid v \in N_G(u) \setminus U \text{ for some } u \in U\}$. We let $\deg_G(u) = |N_G(u)|$ denote the *degree* of a vertex u in a graph G. A *pendant* vertex in a graph is a vertex of degree one. A set $U \subseteq V_G$ is called *independent* if there is no edge between any two vertices of U, and U is called a *cut set* if $G - U$ has more components than G. The *edge contraction* of an edge $e = uv$ in G removes u and v from G, replaces them by a new vertex adjacent to precisely those vertices to which u or v were adjacent, and (only) adds a self-loop incident with this vertex if u or v is reflexive. We denote the resulting graph by G/e.

Let G be a irreflexive graph. We say that we *identify* two vertices u and v of G if we remove them from G and add a new vertex that we make adjacent to every vertex in $N_G(\{u, v\})$. We say that we *glue* a set $W \subseteq V_G$ into a new vertex w^* if we remove all vertices of W and add w^* to G by making it adjacent to every vertex in $N_G(W)$.

Parameterized complexity. Parameterized complexity is a two dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and the other one is a parameter k. A parameterized problem is called *fixed parameter tractable* (FPT) if it can be solved in time $f(k) \cdot n^c$, where f is a function only depending on k, and c is some constant. The basic complexity class for fixed parameter intractability is W[1]. The principal way of showing that a parameterized problem is unlikely to be fixed-parameter tractable is to prove W[1]-hardness by giving a parameterized reduction from a known W[1]-hard problem. We refer to the text books of Downey and Fellows [6] and Niedermeier [24] for a formal definition of this complexity class. The assumption that there is no algorithm that solves the 3-Satisfiability problem in $2^{o(n)}$ time on n-variable formulas is known as the Exponential Time Hypothesis [17]. The Exponential Time Hypothesis has proven to be an effective tool for establishing tight complexity bounds for parameterized problems.

Graph classes with bounded expansion. Graph classes with bounded expansion were introduced by Nešetřil and Ossona de Mendez [20–23]. Later, graph classes with locally bounded expansion were defined by Dvořák, Král' and Thomas [7]. In particular, graphs of bounded treewidth, graphs of bounded degree, graphs that belong to some proper minor-closed graph class, graphs that contain no subgraph isomorphic to a subdivision of a fixed graph, and graphs that can be drawn in a fixed surface in such a way that each edge crosses at most a constant number of other edges have bounded expansion, whereas classes of graphs with locally bounded treewidth or locally excluding a minor have locally bounded expansion.

In order to define the graph classes with (locally) bounded expansion, we need some extra terminology. Let G be a graph. The *eccentricity* of a vertex $v \in V_G$ is the maximum distance between v and any other vertex of G. The *radius* of G is the minimum eccentricity of a vertex. The *edge density* of G is $\frac{|E_G|}{|V_G|}$. A graph F is a *minor* of G if F can be obtained from G by a series of edge contractions, edge deletions and vertex deletions. For an integer $r \geq 0$, we call F an *r-shallow minor* of a graph G if F can be obtained from a subgraph G' of G by contracting all edges of $|V_F|$ non-empty mutually vertex-disjoint subgraphs of G', each of which has radius at most r. A graph class $\mathcal{G}$ has *bounded expansion* if there exists a function $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that, for every integer $r \geq 0$, every r-shallow minor of every graph of $\mathcal{G}$ has edge-density at most $f(r)$. For a vertex u of a graph G and an integer $d \geq 0$, the *d-neighborhood* of u consists of those vertices in G that are at distance at most d from u; note that $N_G(u)$ is not equal to the 1-neighborhood because $u \notin N_G(u)$. A graph class $\mathcal{G}$ has *locally bounded expansion* if there exists a function $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that for every two integers $d, r \geq 0$, for every graph $G \in \mathcal{G}$ and for every $u \in V_G$, every r-shallow minor of the d-neighborhood of $y = u$ in G has edge-density at most $g(d, r)$. By definition, a graph class with bounded expansion has locally bounded expansion, but the converse may not be true. The syntax of the *first-order logic* of graphs includes logical connectives $\vee$,

$\wedge$, $\neg$, $\Leftrightarrow$, $\Rightarrow$, variables for vertices, and quantifiers $\forall$, $\exists$ that can be applied to these variables. The syntax also includes the following two binary relations for two vertex variables u and v, namely "adj(u, v)", which expresses that u and v are adjacent, and "$u = v$", which expresses that u and v are equal. Dvořák, Král' and Thomas [7] showed that graph properties expressible in first-order logic can be tested in linear time on classes of graphs with bounded expansion.

Theorem 2 ([7]). *Let $\mathcal{G}$ be a class of graphs with bounded expansion, and let Π be a first-order property of graphs. Then there exists a linear-time algorithm that correctly decides whether a given graph from $\mathcal{G}$ satisfies Π.*

The same authors [7] also showed a consequence of this result for graph classes with locally bounded expansion. For some problem P, we say that there exists an *almost linear-time* algorithm that solves P if for every $\varepsilon > 0$ there exists an algorithm that solves P with running time $O(n^{1+\varepsilon})$, where n denotes the size of the input instance.

Corollary 1 ([7]). *Let $\mathcal{G}$ be a class of graphs with locally bounded expansion, and let Π be a first-order property of graphs. Then there exists an almost linear-time algorithm that correctly decides whether a given graph from $\mathcal{G}$ satisfies Π.*

3 The Polynomially Solvable Cases of Theorem 1

We use the classification of Feder et al. [10] on the H-Retraction problem when H is a pseudoforest.

Theorem 3 ([10]). *For a fixed pseudoforest H, the H-*Retraction *problem is* NP*-complete if*

(i) H contains a component that is not loop-connected, or
(ii) H contains a cycle on at least 5 vertices, or
(iii) H contains a reflexive cycle on 4 vertices, or
(iv) H contains an irreflexive cycle on 3 vertices.

*In all other cases, the H-*Retraction *problem can be solved in polynomial time.*

We also need the following result.

Proposition 1. *Let H be a fixed graph. If the H-*Retraction *problem can be solved in $f(n, |V_H|)$ time on n-vertex graphs, then the* Surjective H-Coloring *problem can be solved in time $O(n^{|V_H|} \cdot f(n, |V_H|))$.*

Proof. Let $V_H = \{x_1, \ldots, x_{|V_H|}\}$. Let G be an irreflexive graph on n vertices. We consider all ordered sets $U = \{u_1, \ldots, u_{|V_H|}\}$ of $|V_H|$ vertices of G one by one.

For each ordered set U we do as follows. We map u_i to x_i for $i = 1, \ldots, |V_H|$. We then check if $x_i x_j \in E_H$ whenever $u_i u_j \in E_G$. If not, we discard U. If this condition does hold, then we add an edge $u_i u_j$ whenever $x_i x_j \in E_H$ and $u_i u_j \notin E_G$. This leads to a graph G' such that $G'[U]$ is isomorphic to the graph obtained from H after removing all self-loops from H. We solve H-Retraction on G'. If we find a retraction f, then f is a surjective homomorphism from G to H and we return `Yes`. If we do not find a retraction, then we discard U.

After discarding a set U we consider the next ordered set of $|V_H|$ vertices of G, unless we already considered all such sets. In the latter case we return `No`.

Checking adjacencies between the vertices of an ordered set U of $|V_H|$ vertices and constructing the corresponding graph G' costs $O(|V_H|^2)$ time. By our assumption, we can solve H-Retraction in $f(n, |V_H|)$ time. This means that processing each set costs $O(|V_H|^2 f(n, |V_H|))$ time. Because there are at most $n^{|V_H|}$ different ordered sets of $|V_H|$ vertices of G, we find that the total running time is $O(n^{|V_H|} \cdot |V_H|^2 f(n, |V_H|))$, which is $O(n^{|V_H|} \cdot f(n, |V_H|))$, as H is assumed to be fixed. Hence, the result follows. □

Combining Theorem 3 and Proposition 1 yields the following result, which covers the polynomial part of Theorem 1.

Corollary 2. *For a pseudoforest H,* Surjective H-Coloring *can be solved in polynomial time if every component of H is loop-connected, and H contains no cycle on at least* 5 *vertices, no reflexive cycle on* 4 *vertices, and no irreflexive cycle on* 3 *vertices.*

Note that Corollary 2 does not give any specific bound on the running time; Feder et al. [10] do not state such a bound on the running time of their polynomial-time algorithm in Theorem 3. As a side effect of the proof of our FPT result on graph classes with (locally) bounded expansion in Section 3.1, we obtain the following result, a proof of which will be given in a broader context in Section 3.2.

Theorem 4. *Let H be a loop-connected tree with k leaves. Then* Surjective H-Coloring *can be solved in $O(n^k(n+m))$ time on connected graphs with n vertices and m edges.*

3.1 Parameterized Complexity

We first show that there is no function f that only depends on k such that the running time in Theorem 4 can be improved to $f(k) \cdot n^{O(1)}$, unless FPT = W[1]. Let S_k denote the graph obtained from the star $K_{1,k}$ after adding a self-loop to its center. Because S_k is a loop-connected tree with k leaves, the Surjective S_k-Coloring can be solved in $O(n^k(n+m))$ time by Theorem 4. We observe that for all $k \geq 1$ a connected graph G on at least two vertices allows a surjective homomorphism to S_k if and only if G has an independent set of size at least k. Because the Independent Set problem, which asks whether a graph has an independent set of size at least k, is W[1]-complete when parameterized by k (cf. [6]), we immediately obtain the following.

Proposition 2. Surjective S_k-Coloring *is* W[1]*-complete when parameterized by k.*

Our next result shows that the running time in Theorem 4 cannot be improved to $f(k) \cdot n^{o(k)}$, unless the Exponential Time Hypothesis fails. This follows from combining the aforementioned observation that for all $k \geq 1$ a connected graph G on at least two vertices allows a surjective homomorphism to S_k if and only if G has an independent set of size at least k with the result of Chen et al. [2] who showed that there is no algorithm that solves INDEPENDENT SET on n-vertex graphs in time $f(k) \cdot n^{o(k)}$, unless the Exponential Time Hypothesis fails.

Proposition 3. SURJECTIVE S_k-COLORING *cannot be solved in* $f(k) \cdot n^{o(k)}$ *time on* n*-vertex graphs, unless the Exponential Time Hypothesis fails.*

Due to Propositions 2 and 3 it is natural to consider special graph classes in order to improve the running time. For this purpose we consider graph classes with locally bounded expansion. Our aim is to show that SURJECTIVE COLORING is FPT for ordered pairs (G, H) where G belongs to some graph class with locally bounded expansion, H is a loop-connected tree, and $|V_H|$ is the parameter. Due to Corollary 1, we obtain this result if we can show that the existence of a surjective homomorphism from a graph G to a loop-connected tree H can be reduced to a problem that can be expressed in first-order logic. This is our objective for the rest of this section.

The following observation follows immediately from the definition of a surjective homomorphism.

Observation 1 *Let* G *and* H *be two graphs and let* $h : V_G \to V_H$ *be a mapping. Let* $x \in V_H$ *and let* $W \subseteq h^{-1}(x)$*. Let* G' *be the graph obtained from* G *by gluing* W *into* w^**. Let* $h' : V_{G'} \to V_H$ *be the mapping defined as*

$$h'(v) = \begin{cases} h(v), & v \neq w^*, \\ x, & v = w^*. \end{cases}$$

Then the following two statements hold:

(i) if h *is a surjective homomorphism from* G *to* H*, then* h' *is a surjective homomorphism from* G' *to* H*;*
(ii) if h' *is a surjective homomorphism from* G' *to* H*, and* W *is independent or else* x *is reflexive, then* h *is a surjective homomorphism from* G *to* H*.*

Let v be a vertex of a partially reflexive tree H rooted at r. Observe that r defines the parent-child relation between any two adjacent vertices. Then $C(v)$ denotes the set of all children of v, and $D(v) \supseteq C(v)$ denotes the set of all descendants of v. Note that $v \notin D(v)$, and consequently, $v \notin C(v)$ either.

Let H be a loop-connected tree that has a reflexive root r. Let $L_H = \{z_1, \dots, z_k\}$ denote the set that consists of all leaves of H that are not equal to r (should r be a leaf). Let $U = \{u_1, \dots, u_k\}$ be an ordered subset of vertices of a connected graph G. We define a partition of V_G into sets W_x with $x \in V_H$ inductively:

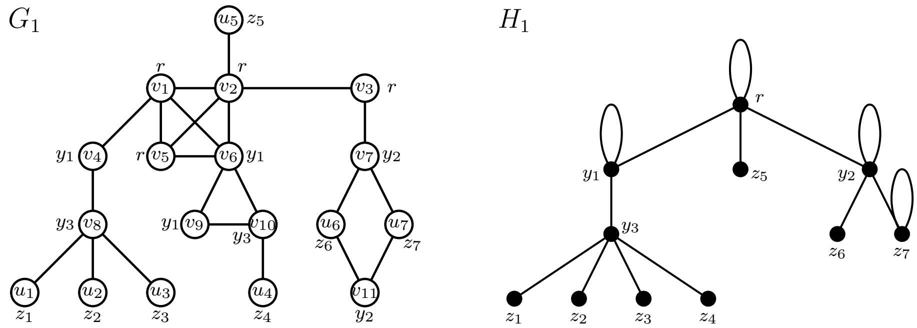

Fig. 1. The U-mapping f_U from a connected graph G_1 with $U = \{u_1, \ldots, u_7\}$ to a loop-connected tree H_1 with reflexive root r that is a surjective homomorphism from G_1 to H_1.

1. Set $W_{z_i} = \{u_i\}$ for $i = 1, \ldots, k$.
2. Let x be in $V_H \setminus (\{r\} \cup L_H)$ such that W_x is not yet defined. Let $Z \subseteq V_H$ be the set of all vertices z of H, for which we already defined corresponding sets W_z. Assuming that $D(x) \subseteq Z$ we set $W_x = \bigcup_{y \in C(x)} N_G(W_y) \setminus \bigcup_{z \in Z} W_z$.
3. Finally, to define W_r, we assume that sets W_z are constructed for all $z \in V_H \setminus \{r\}$, and we set $W_r = V_G \setminus \bigcup_{z \in D(r)} W_z$.

The mapping $f_U \colon V_G \to V_H$ is given by $f_U(v) = x$ if $v \in W_x$. We call this mapping the *U-mapping* from G to H; recall that U is an ordered set, hence G has exactly one U-mapping. See Figure 1 for an example.

Note that an U-mapping from a connected graph G to a loop-connected tree H with a reflexive root does not have to be a surjective homomorphism from G to H; it may not even be a homomorphism if two u-vertices are adjacent. The following lemma is the first of two crucial lemmas. It gives a necessary and sufficient condition for a U-mapping to be a surjective homomorphism, as in the example of Figure 1. Note that $h \neq f_U$ is possible in this lemma. For instance, in the example of Figure 1 we may modify f_U by mapping v_3 to y_2 instead, while still obtaining a surjective homomorphism from G_1 to H_1.

Lemma 1. *Let H be a loop-connected tree that has a reflexive root r. Let $L_H = \{z_1, \ldots, z_k\}$, and let $U = \{u_1, \ldots, u_k\}$ be an ordered vertex subset of a connected graph G. Then there is a surjective homomorphism h from G to H with $h(u_i) = z_i$ for $i = 1, \ldots, k$ if and only if f_U is a surjective homomorphism from G to H.*

Proof. The backward implication holds, because the U-mapping of G maps every u_i to z_i. We prove the forward implication by induction on $|V_H|$.

Let $|V_H| = 1$. Then $L_H = \emptyset$, and consequently, $U = \emptyset$. Moreover, h is equal to the function that maps every vertex of G to r. By definition, h is the $\emptyset$-mapping from G to H.

Let $|V_H| \geq 2$. First suppose that H is a star with r as the central vertex implying that $C(r) = L_H$; note that this case also covers the case when H contains only two vertices. We modify h if necessary by mapping every vertex of $V_G \setminus U$ to r in order to obtain the U-mapping f_U from G to H. Because r is reflexive and h is a homomorphism from G to H, we find that f_U is a homomorphism from G to H. Because h is surjective and $h(u_i) = z_i$ for $i = 1, \ldots, k$, we find that f_U is surjective.

From now on, suppose that H is not a star with central vertex r. Then we can choose a vertex $x \neq r$ with $\emptyset \neq C(x) \subseteq L_H$. We assume without loss of generality that $C(x) = \{z_1, \ldots, z_s\}$ for some $1 \leq s \leq k$. We may also assume without loss of generality that $h^{-1}(z_i) = \{u_i\}$ for $i = 1, \ldots, s$. In order to see this, suppose that $h^{-1}(z_i)$ contains at least one other vertex besides u_i for some $1 \leq i \leq s$. Because h is a homomorphism and the only neighbor of z_i is x, we find that h maps every neighbor of every vertex v in G with $h(v) = z_i$ to either x or to z_i; the latter may only happen if z_i is reflexive. In other words we have that $N_G(v) \subseteq h^{-1}(z_i) \cup h^{-1}(x)$ for all $v \in h^{-1}(z_i)$. Then h can be redefined as follows. If x is a reflexive vertex, then we may map all vertices of $h^{-1}(z_i) \setminus \{u_i\}$ to x. Otherwise, if x is irreflexive, then x has a parent y, because $x \neq r$. Because r is reflexive and x is irreflexive, z_i cannot be reflexive; otherwise $H[R_H]$ is disconnected, and consequently, H would not be loop-connected. Hence, the vertices of $h^{-1}(z_i)$ form an independent set. This means that h maps no neighbor of any vertex v with $h(v) = z_i$ to z_i. Hence, in this case we have that $N_G(v) \subseteq h^{-1}(x)$ for all $v \in h^{-1}(z_i)$. This means that we may map the vertices of $h^{-1}(z_i) \setminus \{u_i\}$ to y; we may even do so if y is irreflexive as the vertices of $h^{-1}(z_i)$ form an independent set.

Let $W = \bigcup_{i=1}^{s} N_G(u_i)$. Note that $W \neq \emptyset$, because G is connected. We find that every neighbor of every u_i is mapped to x, because x is the only neighbor of z_i and h only maps z_i to u_i, as we deduced above. This means that $h(W) = \{x\}$.

Let G' be the connected graph obtained from G by gluing W into w^*. Then, by Observation 1(i), the mapping $h' \colon V_{G'} \to V_H$ such that

$$h'(v) = \begin{cases} h(v), & v \neq w^*, \\ x, & v = w^* \end{cases}$$

is a surjective homomorphism from G' to H.

Let $G'' = G' - \{u_1, \ldots, u_s\}$, and let $H' = H - \{z_1, \ldots, z_s\}$. Then H' is a loop-connected tree, and we choose r to be its (reflexive) root. By construction, every u_i is only adjacent to w^* in G'. This implies that G'' is connected. Recall that $x \neq r$. Hence, $L_{H'} = \{x, z_{s+1}, \ldots, z_k\}$. We let $U' = \{w^*, u_{s+1}, \ldots, u_k\}$. Then $h'' = h'|_{V_{G''}}$ is a surjective homomorphism from G'' to H' that maps w^* to x and u_i to z_i for $i = s+1, \ldots, k$. Then, by the induction hypothesis, we find that the corresponding U'-mapping $f'_{U'}$ from G'' to H' is a surjective homomorphism from G'' to H'. From the definition of the U-mapping f_U from G to H we find that

$$f_U(v) = \begin{cases} f'_{U'}(v), & v \notin \{u_1, \ldots, u_s\} \cup W, \\ f'_{U'}(w^*), & v \in W, \\ z_i, & v \in \{u_1, \ldots, u_s\}. \end{cases}$$

Suppose that x is reflexive. By Observation 1(ii), we obtain that f_U is a surjective homomorphism from G to H. Suppose that x is irreflexive. Recall that h maps every vertex of W to x. Consequently, W is an independent set. Again we use Observation 1(ii) to deduce that f_U is a surjective homomorphism from G to H. This completes the proof of Lemma 1. □

If H is a loop-connected tree and we cannot choose a reflexive vertex to be the root, then H must be irreflexive. In that case we cannot use Lemma 1 and do as follows. Assume that H has at least two vertices. Choose a vertex r to be the root of H, and let r' be a neighbor of r in H. We say that H is *rooted* by the ordered pair (r, r'). Let $L^*_H = \{z_1, \ldots, z_k\}$ consist of all leaves of H that are neither equal to r nor to r' (should r or r' be a leaf). Let $U = \{u_1, \ldots, u_k\}$ be an ordered subset of vertices of a connected bipartite graph G on partition classes V_1 and V_2. Let $(p, q) \in \{(1, 2), (2, 1)\}$. We define a partition of V_G into sets W_x with $x \in V_H$ inductively:

1. Set $W_{z_i} = \{u_i\}$ for $i = 1, \ldots, k$.
2. Let x be in $V_H \setminus (L^*_H \cup \{r, r'\})$ such that W_x is not yet defined. Let $Z \subseteq V_H$ be the set of all vertices z of H, for which we already defined corresponding sets W_z. Assuming that $D(x) \subseteq Z$ we set $W_x = \bigcup_{y \in C(x)} N_G(W_y) \setminus \bigcup_{z \in Z} W_z$.
3. Finally, to define W_r and $W_{r'}$, we assume that sets W_z are constructed for all $x \in V_H \setminus \{r, r'\}$. We set $W_r = V_p \setminus \bigcup_{z \in Z} W_z$ and $W_{r'} = V_q \setminus \bigcup_{z \in Z} W_z$.

The mapping $f^{p,q}_U \colon V_G \to V_H$ is given by $f^{p,q}_U(v) = x$ if $v \in W_x$. We call this mapping the *$U^{p,q}$-mapping* from G to H; recall that U is an ordered set, hence G has exactly one $U^{p,q}$-mapping. See Figure 2 for an example.

Just as in the case of U-mappings, an $U^{p,q}$-mapping from a connected bipartite graph G to an irreflexive tree H does not have to be a surjective homomorphism from G to H. The following lemma is the second crucial lemma. It gives a necessary and sufficient condition for an $U^{p,q}$-mapping $f^{p,q}_U$ to be a surjective homomorphism, as in the example of Figure 2. Note that $h \neq f^{p,q}_U$ is possible in this lemma.

Lemma 2. *Let H be an irreflexive tree rooted by (r, r'). Let $L^*_H = \{z_1, \ldots, z_k\}$, and let $U = \{u_1, \ldots, u_k\}$ be an ordered vertex subset of a connected bipartite graph G on partition classes V_1 and V_2. Then there is a surjective homomorphism h from G to H with $h(u_i) = z_i$ for $i = 1, \ldots, k$, and moreover, with $h^{-1}(r) \subseteq V_p$ and $h^{-1}(r') \subseteq V_q$ if and only if $f^{p,q}_U$ is a surjective homomorphism from G to H.*

Proof. The backward implication holds, because the $U^{p.q}$-mapping of G maps every u_i to z_i. We prove the forward implication by induction on $|V_H|$. Recall that H contains at least two vertices as it is rooted by (r, r').

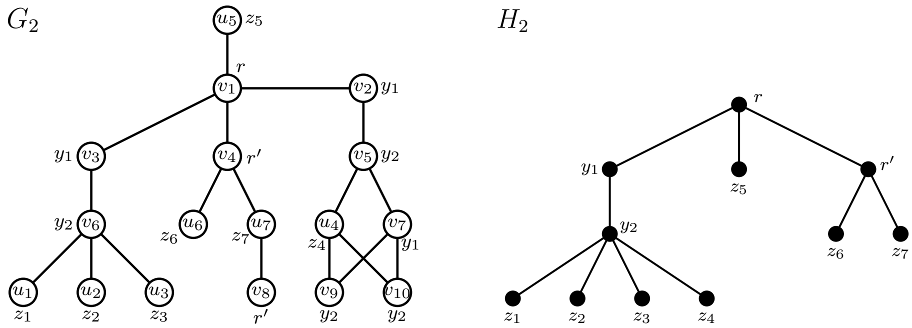

Fig. 2. The $U^{1,2}$-mapping $f_U^{1,2}$ from a connected bipartite graph G_2 with partition classes V_1 and V_2 such that $v_1 \in V_1$, and with $U = \{u_1, \ldots, u_7\}$ to an irreflexive tree H_2 rooted by (r, r') that is a surjective homomorphism from G_2 to H_2.

Let $|V_H| = 2$. Then H only contains r and r'. Then $L_H^* = \emptyset$, and consequently, $U = \emptyset$. Moreover, h is equal to the function that maps every vertex of V_p to r, and every vertex of V_q to r'. By definition, h is the $\emptyset^{p,q}$-mapping from G to H.

Now let $|V_H| \geq 3$. Then we can choose a vertex $x \in V_H$ with $\emptyset \neq C(x)\backslash\{r'\} \subseteq L_H^*$. We assume without loss of generality that $C(x)\backslash\{r'\} = \{z_1, \ldots, z_s\}$ for some $1 \leq s \leq k$. We may also assume without loss of generality that $h^{-1}(z_i) = \{u_i\}$ for $i = 1, \ldots, s$. In order to see this, suppose that $h^{-1}(z_i)$ contains at least two vertices for some $1 \leq i \leq s$. Because h is a homomorphism and H is irreflexive, $h^{-1}(z_i)$ is independent and h maps every neighbor of every vertex v with $h(v) = z_i$ to x, i.e., we have $N_G(v) \subseteq h^{-1}(x)$ for all $v \in h^{-1}(z_i)$. We redefine h as follows by mapping all vertices of $h^{-1}(z_i)\backslash\{u_i\}$ to y, where y is the parent of x unless $x = r$, then we take $y = r'$; note that we take $y = r$ if $x = r'$ as r is the parent of r'. The resulting mapping is also a surjective homomorphism from G to H.

Let $W = \bigcup_{i=1}^{s} N_G(u_i)$. Then $W \neq \emptyset$, because G is connected. Moreover, $h(W) = \{x\}$, because z_i is irreflexive and has x as its only neighbor for $i = 1, \ldots, s$. Let G' be the connected graph obtained from G by gluing W into w^*. Then, by Observation 1(i), the mapping $h' \colon V_{G'} \to V_H$ defined as

$$h'(v) = \begin{cases} h(v), & v \neq w^*, \\ x, & v = w^* \end{cases}$$

is a surjective homomorphism from G' to H.

Let $G'' = G' - \{u_1, \ldots, u_s\}$, and let $H' = H - \{z_1, \ldots, z_s\}$. Then H' is an irreflexive tree containing r and r', and we root it by (r, r'). By construction, every u_i is only adjacent to w^* in G'. This implies that G'' is connected. As we only removed vertices, G'' is bipartite with partition classes $V_1'' \subseteq V_1$ and $V_2'' \subseteq V_2$. Recall that $x \notin \{r, r'\}$. Hence, $L_{H'}^* = \{x, z_{s+1}, \ldots, z_k\}$. We let $U' = \{w^*, u_{s+1}, \ldots, u_k\}$. Then $h'' = h'|_{V_{G''}}$ is a surjective homomorphism from

G'' to H' that maps w^* to x, and u_i to z_i for $i = s+1, \ldots, k$, and moreover, $h''^{-1}(r) \subseteq V''_p$ and $h''^{-1}(r') \subseteq V''_q$. Then, by the induction hypothesis, we find that the corresponding $U'^{p,q}$-mapping $(f^{p,q}_{U'})'$ from G'' to H' is a surjective homomorphism from G'' to H'. From the definition of the $U^{p,q}$-mapping $f^{p,q}_U$ from G to H we find that

$$f^{p,q}_U(v) = \begin{cases} (f^{p,q}_{U'})'(v), & v \notin \{u_1, \ldots, u_s\} \cup W, \\ (f^{p,q}_{U'})'(w^*), & v \in W, \\ z_i, & v \in \{u_1, \ldots, u_s\}. \end{cases}$$

Because $h(W) = \{x\}$ and x is irreflexive, W is independent. We use Observation 1(ii) to deduce that $f^{p,q}_U$ is a surjective homomorphism from G to H. This completes the proof of Lemma 2. $\square$

We are now ready to prove the main result of this section, which shows that Surjective Coloring is FPT for ordered pairs (G, H) where G belongs to some graph class with locally bounded expansion, H is a loop-connected tree, and $|V_H|$ is the parameter.

Theorem 5. *Let $\mathcal{G}$ be a graph class of locally bounded expansion, and let H be a loop-connected tree. Then the problem* Surjective H-Coloring *can be solved in almost linear time on $\mathcal{G}$.*

Proof. By Corollary 1, we have proven Theorem 5 after showing that the existence of a surjective homomorphism from G to H can be reduced in constant time to a problem that can be expressed in first-order logic.

Let H be a loop-connected tree. Let G be a graph with components $G_1, \ldots, G_p$ for some $p \geq 1$. Then G allows a surjective homomorphism to H if and only if every G_i allows a surjective homomorphism to some H_i for connected induced subgraphs $H_1, \ldots, H_p$ of H such that $V_H = \bigcup_{i=1}^{p} V_{H_i}$. We can construct all possible ordered tuples $(H_1, \ldots, H_p)$ in constant time by brute force, as H is fixed. Hence, we may assume that $p = 1$, i.e., that G is connected.

We distinguish between the cases $R_H \neq \emptyset$ and $R_H = \emptyset$. First suppose that $R_H \neq \emptyset$. If H has one vertex, then G has a trivial surjective homomorphism, namely the homomorphism that maps every vertex of G to the single reflexive vertex of H. We now assume that H has at least two vertices. We choose a root vertex r in H, which defines the parent-child relation between every pair of adjacent vertices. Because $R_H \neq \emptyset$, we may assume that r is reflexive. We let $\{z_1, \ldots, z_k\}$ be the set of all non-root leaves of H. By Lemma 1, there is a surjective homomorphism from G to H if and only if there is an ordered subset $U = \{u_1, \ldots, u_k\}$ of vertices of G such that f_U is a surjective homomorphism from G to H.

We first show how to construct a first-order logic formula ϕ_x for every $x \in V_H$ such that for every $v \in V_G$, $\phi_x(v)$ expresses the property $v \in W_x$, or equivalently, the property $f_U(v) = x$. For this purpose we use the inductive definition of W_x. For $i = 1, \ldots, k$, we define $\phi_{z_i}(v)$ as $v = u_i$. Let $x \in V_H \setminus \{r, z_1, \ldots, z_k\}$. Let $Z \subseteq V_H$ be the set of all vertices z of H for which the formulas ϕ_z have already been constructed. Assuming that $D(x) \subseteq Z$, we let $\phi_x(v)$ express the following properties that together describe the property $v \in W_x$:

1. there are $y \in C(x)$ and $u \in N_G(v)$, such that $\phi_y(u)$ holds;
2. for all $z \in Z$ and all $u \in V_G$, if $\phi_z(u)$ then $u \neq v$.

Finally, to define $\phi_r(v)$, we assume that formulas ϕ_z have been constructed for all $z \in V_H \setminus \{r\}$. Then $\phi_r(v)$ expresses the following property: for all $z \in D(r)$ and all $u \in V_G$, if $\phi_z(u)$ then $u \neq v$.

We can now express the property that there is an ordered set of vertices $U = \{u_1, \ldots, u_k\}$ of G such that f_U is a surjective homomorphism from G to H: there are $u_1, \ldots, u_k$ such that $u_i \neq u_j$ if $i \neq j$, and for all $x \in V_H$, there is $v \in V_G$ such that $x = f_{\{u_1,\ldots,u_k\}}(v)$ (expressing the surjectivity property), and for all $v, w \in V_G$, $v \neq w$, there are $x, y \in V_H$ such that the following three conditions (expressing the homomorphism property) hold:

(i) $f_U(v) = x$ and $f_U(w) = y$;
(ii) if $x = y$, then $\mathrm{adj}(v, w)$ if and only if $x \in R_H$;
(iii) if $x \neq y$, then $\mathrm{adj}(v, w)$ if and only if x, y are adjacent in H.

We observe that the formulas ϕ_u are constructed in constant time, as H is fixed.

Now suppose that $R_H = \emptyset$. We answer `No` if G is not bipartite, because only bipartite graphs allow a homomorphism to a bipartite graph. Hence, assume that G is bipartite with partition classes V_1 and V_2. If H has one vertex, then G has a surjective homomorphism to H if and only if G also has one vertex. Let H have at least two vertices. Choose a vertex r to be the root of H, and let r' be a neighbor of r in H. We let $\{z_1, \ldots, z_k\}$ be the set of all leaves of H distinct from r, r'. By Lemma 2, there is a surjective homomorphism from G to H if and only if there is an ordered subset $U = \{u_1, \ldots, u_k\}$ of vertices of G and a pair $(p, q) \in \{(1, 2), (2, 1)\}$ such that $f_U^{(p,q)}$ is a surjective homomorphism from G to H. By an analysis similar to the case when $R_H \neq \emptyset$, we can express in first-order logic the property that there is an ordered set of vertices $U = \{u_1, \ldots, u_k\}$ of G such that $f_U^{1,2}$ or $f_U^{2,1}$ is a surjective homomorphism from G to H. Just as in the case when $R_H \neq \emptyset$ this takes constant time. $\square$

3.2 A Remark on the Running Time Analysis

Lemmas 1 and 2 immediately yield an $O(n + m)$ time algorithm that solves H-Retraction on a connected graph G with n vertices and m edges when H is a loop-connected tree. This can be seen as follows. Let H' denote the induced subgraph of G that is isomorphic to H. Then H' fixes the set U. Suppose that $R_H \neq \emptyset$. We observe that the construction of f_U respects H'. Hence, by Lemma 1, we only have to construct f_U and check if the obtained mapping is a surjective homomorphism from G to H. This takes $O(n + m)$ time. If $R_H = \emptyset$, we first check whether G is bipartite, say with partition classes V_1 and V_2, as otherwise the answer is `No`. We also recall that for every homomorphism h from G to H

either $h^{-1}(x) \subseteq V_1$ or $h^{-1}(x) \subseteq V_2$ for each $x \in V_H$. Hence we can use Lemma 2 to derive the same running time.

Note that we can also obtain an $O(n+m)$ running time for H-Retraction if G is not connected and H is a loop-connected tree. The reason is that H will be an induced subgraph of a component of G, because H is a connected graph. If H contains a reflexive vertex, then we map the vertices of the other components of G to this vertex. If H is irreflexive, then every component of G must be bipartite, and we map the vertices of the other components of G to an edge of H should H contain at least one edge (if H consists of a single vertex, then the problem is trivial).

The $O(n+m)$ running time can also be obtained by analyzing the algorithm of Feder et al. [10]. However, they do not define the mappings f_U and $f_U^{p,q}$ explicitly. We had to do this in order to prove Theorem 5.

By Proposition 1, we obtain an $O(n^{|V_H|}(n+m))$ time algorithm that solves Surjective H-Coloring on a graph G when H is a loop-connected tree. If G is connected, then we may obtain a considerable improvement, because the number of leaves of H can be considerably less than the total number of vertices of H. In that case, we consecutively check all ordered k-vertex sets U and apply Lemma 1 or 2, respectively. Because the number of different sets U is $O(n^k)$, we find a total running time of $O(n^k(n+m))$. Note that in the case that $R_H = \emptyset$, we must also consider the pairs $(p,q) = (1,2)$ and $(p,q) = (2,1)$. However, this only influences the constant hidden in the big-O notation. Note that this is a proof of Theorem 4.

4 The NP-Complete Cases of Theorem 1

In this section we show that the Surjective H-Coloring problem is NP-complete for any fixed tree H that is not loop-connected. In order to do this, we need some additional technical lemmas and observations.

Observation 2 *Let h be a homomorphism from a graph G to a graph H. Let u and v be in V_G with $h(u) = x$ and $h(v) = y$. Then $\mathrm{dist}_G(u,v) \geq \mathrm{dist}_H(x,y)$.*

Observation 3 *Let h be a homomorphism from a graph G to a partially reflexive tree H. Let u, v, w form a triangle in G. Then h maps at least two vertices of $\{u, v, w\}$ to the same reflexive vertex in H.*

Recall that H/e denotes the graph obtained from a graph H after contracting an edge e.

Observation 4 *Let $e = xy$ be an edge of a graph H with $x, y \in R_H$. Let z be the (reflexive) vertex obtained by contracting xy. If h is a surjective homomorphism from a graph G to H, then*

$$h'(v) = \begin{cases} h(v), & v \in V_G \setminus h^{-1}(\{x,y\}) \\ z, & v \in h^{-1}(\{x,y\}) \end{cases}$$

is a surjective homomorphism from G to H/e.

Lemma 3. *Let H be a connected graph with $R_H \neq \emptyset$. Let x be a pendant irreflexive vertex of H. Let $H' = H - x$. If h is a surjective homomorphism from a graph G to H, then there is a surjective homomorphism h' from G to H', such that $h'(v) = h(v)$ for all vertices $v \in V_G \setminus h^{-1}(x)$.*

Proof. Let h' be a function that maps every $v \in V_G \setminus h^{-1}(x)$ to $h(v)$. We show how to extend h' to V_G. Let y be the (unique) neighbor of x in H. If $y \in R_H$, then we set $h'(v) = y$ for all $v \in h^{-1}(x)$. Otherwise, the assumption that $R_H \neq \emptyset$ implies that y is adjacent to a vertex $z \neq x$, and we set $h'(v) = z$ for all $v \in h^{-1}(x)$. Because x is irreflexive, $h^{-1}(x)$ is an independent set. Hence, h' is a surjective homomorphism from G to H' (even if $h'(v) = z$ for all $v \in h^{-1}(x)$ and z is irreflexive). $\square$

Lemma 4. *Let $\ell \geq 2$ be an integer, and H be a tree with $R_H \neq \emptyset$ such that*

1. *for every two different vertices $x, y \in R_H$, $\mathrm{dist}_H(x, y) \geq \ell$;*
2. *for every irreflexive leaf $x \in V_H$ and every $y \in R_H$, $\mathrm{dist}_H(x, y) \geq \ell$.*

Let G be a connected graph with a set $U \subset V_G$ such that $h(U) \subseteq R_H$ for some surjective homomorphism h from G to H. Let $u \in V_G \setminus U$ be a vertex that has $\mathrm{dist}_G(u, U) < \ell$ and whose neighborhood is a clique. Let $G' = G - u$. Then $h' = h|_{V_{G'}}$ is a surjective homomorphism from G' to H.

Proof. Because h is a homomorphism from G to H, we find that h' is a homomorphism from G' to H. Hence we are left to prove that h' is surjective, i.e., that $h'(V_{G'}) = V_H$. This will be true, if there is a vertex v in G' that h maps to $z = h(u)$.

First suppose that either $z \in R_H$ or else that z is a leaf not in R_H. Because $\mathrm{dist}_G(u, U) < \ell$, there is a vertex $v \in U$ such that $\mathrm{dist}_G(u, v) < \ell$; note that v belongs to G'. Observation 2 combined with conditions 1 and 2, respectively, tells us that u and v cannot be mapped to two different vertices of R_H. Hence, $h(v) = h(u) = z$.

Now suppose that z is not in R_H and that z is not a leaf. Then z is an inner vertex of an x, y-path P for two distinct leaves x, y in H. Let r, s be two vertices of G such that $h(r) = x$ and $h(s) = y$, and let Q be a shortest r, s-path in G; observe that $u \notin \{r, s\}$. Because H is a tree, P is the only path between x and y. Then, $V_P \subseteq h(V_Q)$. Moreover, u is not an inner vertex of Q, because the neighborhood of u is a clique and Q is a shortest path, and consequently, an induced path in G. Therefore, Q is a path in G'. Consequently, G' contains a vertex v (namely a vertex that lies on Q) with $h(v) = z$. $\square$

In our hardness proof, we reduce from a variant of the Matching-Cut problem. This problem is to test whether a connected graph G has a *matching-cut* M, i.e., a matching $M \subseteq E_G$ such that $G - M$ is disconnected. Patrignani and Pizzonia [25] prove that this problem is NP-complete. We call two vertices s and t of a graph G the *(matching) roots* of G if s and t belong to two different components of $G - M$ for every matching-cut of G (should G have at least one matching-cut). This leads to the following variant that is useful for our purposes.

MATCHING-CUT WITH ROOTS
Instance: a connected graph G of minimum degree at least two with roots s, t.
Question: does G have a matching-cut?

We emphasize that by definition, the roots s and t are part of the input of every instance of MATCHING-CUT WITH ROOTS, i.e., we do not have to check whether the specified vertices s and t are roots as they are given to us. It is stated in Lemma 5 that MATCHING-CUT WITH ROOTS is NP-complete. This lemma is essentially due to Patrignani and Pizzonia [25] as it immediately follows from the following small observation in their hardness reduction from the NOT-ALL-EQUAL-3-SATISFIABILITY problem, which is an NP-complete problem [26]. For a given instance of NOT-ALL-EQUAL-3-SATISFIABILITY, Patrignani and Pizzonia [25] construct a connected graph G of minimum degree at least two with the following property: G contains two disjoint sets F and T of vertices (that compose a so-called *false chain* and *true chain*, respectively) such that for every matching-cut M, the sets F and T are in distinct components of $G-M$. We use their construction and choose $s \in F$ and $t \in T$ respectively.

Lemma 5. *The* MATCHING-CUT WITH ROOTS *problem is* NP*-complete.*

We are now ready to prove the main result of this section.

Theorem 6. *For any fixed tree H that is not loop-connected, the* SURJECTIVE H-COLORING *problem is* NP*-complete.*

Proof. Because checking if a given mapping is a surjective homomorphism can be done in polynomial time, the problem belongs to NP. In order to prove NP-hardness we reduce from the problem MATCHING-CUT WITH ROOTS, which is NP-complete by Lemma 5. We start with some auxiliary constructions. Let H be a tree that is not loop-connected. We choose two vertices $p, q \in V_H$ that belong to two different components of $H[R_H]$ in such a way that $\text{dist}_H(p,q) \leq \text{dist}_H(x,y)$ for any pair x, y that are in two different components of $H[R_H]$. Let $\ell = \text{dist}_H(p,q)$. By definition, $\ell \geq 2$. Let H_1 and H_2 be two different components of the forest obtained from H after removing the edge incident with q in the unique p,q-path in H. Assume that $p \in V_{H_1}$ and $q \in V_{H_2}$. We construct graphs F_i for $i = 1, 2$ (see Figure 3) as follows:

1. For each vertex $x \in V_{H_i} \setminus R_H$, we introduce a vertex $t_x^{(1)}$;
2. For each vertex $x \in V_{H_i} \cap R_H$, we introduce two adjacent vertices $t_x^{(1)}, t_x^{(2)}$;
3. For each edge $xy \in E_{H_i}$, we add an edge between any $t_x^{(h)}$ and any $t_y^{(j)}$.

We say that $t_p^{(1)}, t_p^{(2)}$ are the *roots* of F_1, and $t_q^{(1)}, t_q^{(2)}$ are the *roots* of F_2.

We now describe our polynomial-time reduction from MATCHING-CUT WITH ROOTS to SURJECTIVE H-COLORING. Let G be a connected graph that has minimum degree at least two and that has matching roots s and t. Note that we may assume without loss of generality that G is irreflexive. Recall that by definition s and t are separated by every matching-cut in G (if a matching-cut exists). From F_1, F_2, and G we construct a graph G' (see Figure 4) as follows:

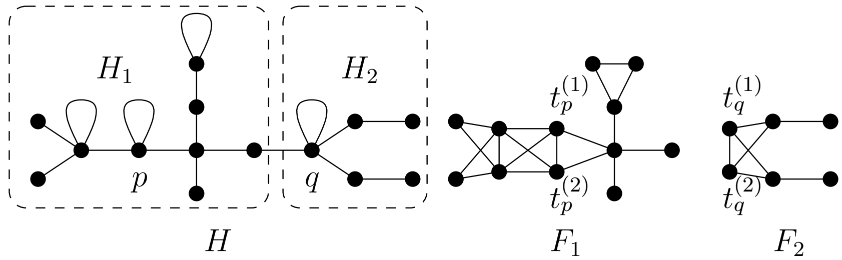

Fig. 3. The construction of the graphs F_1 and F_2 from a tree H that is not loop-connected; note that $\ell = 3$ in this example and that the pair (p, q) is not unique.

1. For each $u \in V_G$ we construct a clique C_u on $\max\{\deg_G(u), 3\}$ vertices if $u \notin \{s, t\}$ and on $\deg_G(u) + 2$ vertices if $u \in \{s, t\}$. We denote $d = \deg_G(u)$ vertices of C_u by $g_{u,e_1}, \ldots, g_{u,e_d}$ to indicate that they correspond to the edges $e_1, \ldots, e_d$ that are incident with u in G. Because G has minimum degree at least two, C_u has at most one other vertex if $u \notin \{s, t\}$; otherwise C_u has two other vertices. If C_u has one other vertex then we denote this vertex by $g_u^{(1)}$, and if C_u has two other vertices then we denote these vertices by $g_u^{(1)}$ and $g_u^{(2)}$, respectively.
2. For each edge $e = uv \in E_G$, the vertices $g_{u,e}, g_{v,e}$ are identified if $\ell = 2$, and the vertices $g_{u,e}, g_{v,e}$ are joined by a path P_e of length $\ell - 2$ if $\ell > 2$. For $\ell = 2$, we let P_e be the single vertex $g_{u,e} = g_{v,e}$.
3. We add F_1 by identifying $t_p^{(1)}, g_s^{(1)}$ and by identifying $t_p^{(2)}, g_s^{(2)}$.
4. We add F_2 by identifying $t_q^{(1)}, g_t^{(1)}$ and by identifying $t_q^{(2)}, g_t^{(2)}$.

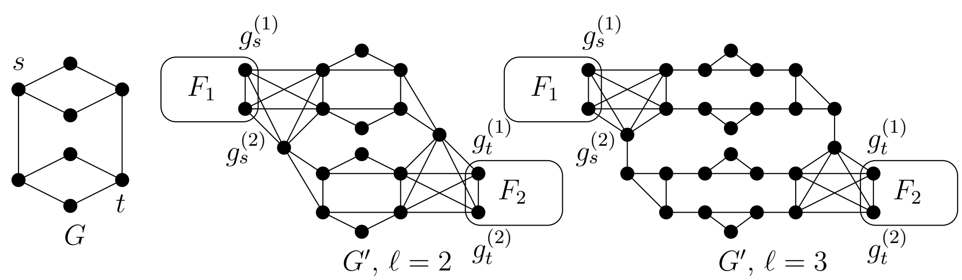

Fig. 4. The construction of G'.

We claim that G has a matching-cut if and only if there exists a surjective homomorphism from G' to H.

First suppose that G has a matching-cut M. Note that in G' this matching-cut is represented by a set $\mathcal{P}$ of $|M|$ mutually vertex-disjoint paths P_e, such that no two vertices of any two different paths P_e and $P_{e'}$ are adjacent. Moreover, if we remove all vertices of all paths in $\mathcal{P}$ then we disconnect G'. In particular, if $\ell = 1$ then the paths in $\mathcal{P}$ are single vertices, which form an independent set that disconnects G'. Let V_1 be the vertex set of the component of $G - M$ that contains s, and let $V_2 = V_G \setminus V_1$. Note that $t \in V_2$, because $s \in V_1$ and s, t are the two given roots of G. We define a mapping $h : V_{G'} \to V_H$ as follows.

Consider an edge $e = uv \in E_G$. If u and v are both in V_1 or both in V_2, then we let h map every vertex from P_e to p or q, respectively. Suppose that one of u, v, say u, belongs to V_1, whereas the other one, v, belongs to V_2. Let $P_e = a_1 \cdots a_{\ell-1}$ (note that $a_1 = g_{u,e}$ and that $a_{\ell-1} = g_{v,e}$). Let $px_1 \cdots x_{\ell-1}q$ denote the p, q-path in H. We let h map a_i to x_i for $i = 1, \ldots, \ell - 1$. Finally, we let h map every vertex $t_x^{(i)} \in V_{F_1} \cup V_{F_2}$ to x. We refer to Figure 5 for an example.

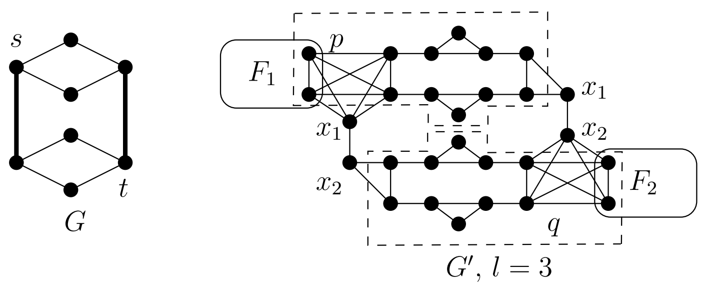

Fig. 5. An example of the surjective homomorphism h from G' to H that is obtained from a matching-cut in G. As the matching-cut in G we took the two vertical edges in G that are displayed in bold. In G' we did not denote any vertex labels but they are all the same as in Figure 4. Instead, we show that all displayed vertices in the top dotted area including $g_s^{(1)}$ and $g_s^{(2)}$ are mapped to p by h, whereas all vertices in the bottom dotted area including $g_t^{(1)}$ and $g_t^{(2)}$ are mapped to q. Also note that the two matching edges in G are in 1-to-1 correspondence with the two paths (of length 1) in G', the ends of which are mapped to x_1 and x_2.

We claim that h is a surjective homomorphism from G' to H. This can be seen as follows. Recall that the paths in $\mathcal{P}$ are in 1-to-1 correspondence to the edges in M. Hence, V_1 corresponds to one component in the graph obtained from G' after removing the vertices of the paths in $\mathcal{P}$. This means that h maps all vertices of every clique C_u either to one of p, x_1 or else to one of $q, x_{\ell-1}$. Because M is a matching-cut of G, we find that h maps at most one vertex of any clique C_u not to p or q. In that case, h maps such a vertex to x_1 or to $x_{\ell-1}$ depending whether u is an end-vertex of an edge in M that belongs to V_1 or V_2. Finally, h maps every vertex $t_x^{(i)} \in V_{F_1} \cup V_{F_2}$ to x. This does not violate the definition of a homomorphism either, because the only vertices of the subgraphs F_1 and F_2 of G' that have neighbors outside F_1 and F_2 are $g_s^{(1)}$, $g_s^{(2)}$, and $g_t^{(1)}, g_t^{(2)}$, respectively, and these vertices are mapped to p or q, respectively. We conclude that h is a homomorphism from G' to H. Because M contains at least one edge, there is at least one edge $uv \in V_G$ with $u \in V_1$ and $v \in V_2$. Hence, the vertices $x_1, \ldots, x_{\ell-1}$ are in $h(V_{G'})$. Then, because h maps every vertex $t_x^{(i)} \in V_{F_1} \cup V_{F_2}$ to x, we find that $h(V_{G'}) = V_H$. Hence, h is surjective.

Now suppose that there exists a surjective homomorphism h from G to H. Throughout the proof we make heavily use of Observation 3 as we have given the cliques C_u size at least three. By repeatedly applying this observation, we find that h maps all but at most one vertex of every clique C_u in G' to one or more reflexive vertices of H. By the definition of a homomorphism these reflexive vertices belong to the same component of $H[R_H]$. Claim 1 states that G' must contain two cliques C_u and C_v that contain vertices that are mapped to reflexive vertices from two different components of $H[R_H]$. We first show that Claim 1 is all we need to finish the proof. Then afterward we will prove Claim 1.

Claim 1. There are two vertices $u, v \in V_G$ such that $h(C_u) \cap R_H$ and $h(C_v) \cap R_H$ belong to the vertex sets of two different components of $H[R_H]$.

Assuming that Claim 1 holds we can do as follows. We choose a component D of $H[R_H]$ such that the set $V_1 = \{v \in V_G \mid h(C_v) \cap V_D \neq \emptyset\}$ is nonempty. Let $V_2 = V_G \setminus V_1$. Claim 1 tells us that $V_2 \neq \emptyset$. Consider the edge-cut $M = \{uv \in E_G \mid u \in V_1, v \in V_2\}$. Let $e = uv$ be an arbitrary edge in M. By Observation 3 we find that h maps at least $|C_u| - 1$ vertices of C_u to V_D, and at least $|C_v| - 1$ vertices of C_v to the vertices of some other component D' of $H[R_H]$. Let P be the shortest path in H with endpoints in D and D'. By definition, P has length at least ℓ. Therefore, all vertices of P_e must be mapped to inner vertices of P. Hence P has length ℓ, all internal vertices of P are irreflexive, and the vertices $g_{u,e} \in C_u$ and $g_{v,e} \in C_v$ are mapped to irreflexive vertices of H. Because at most one vertex of C_u or C_v can be mapped to a vertex outside D or D', respectively, M contains no other edges incident with u or v. This means that M is a matching-cut in G meaning that we are done subject to proving Claim 1. This is what we do below.

We prove Claim 1 as follows. In order to obtain a contradiction, suppose that there is a component D of $H[R_H]$ with $h(C_u) \cap R_H \subseteq V_D$ for every $u \in V_G$. Let H' be the tree obtained from H by contracting all edges between different reflexive vertices, and recursively removing all irreflexive pendant vertices from H that are at distance at most $\ell - 1$ from R_H; see Figure 6 for an example. By Observation 4 and Lemma 3, we obtain a surjective homomorphism h' from G' to H'. Note that the components of $H'[R_{H'}]$ are isolated vertices. Let z be the vertex in H' that is obtained by contracting the edges in $H[D]$. Then for any $u \in V_G$, we have $h'(C_u) \cap R_{H'} = \{z\}$. Let X be the set of irreflexive vertices that we have removed from H when we constructed H'. Note that H' has no pendant irreflexive vertices that are adjacent to reflexive vertices, because we would have put such vertices in X as $\ell \geq 2$.

We consider the graphs F_1 and F_2. Recall that the vertices of each F_i correspond to the vertices of H_i, and that for each reflexive vertex $x \in V_{H_i}$ we introduced two adjacent vertices $t_x^{(1)}, t_x^{(2)}$ of F_i that are adjacent to exactly the same neighbors in G'. Hence, by Observation 3, any surjective homomorphism

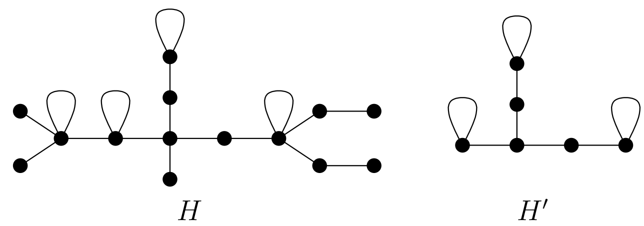

Fig. 6. An example of a graph H and the corresponding graph H'.

from G' to H' maps at least one of the vertices $t_x^{(1)}, t_x^{(2)}$ to a vertex of $R_{H'}$. By symmetry, we may assume without loss of generality that h' maps every $t_x^{(1)}$ to a vertex in $R_{H'}$. Let $U = \{t_x^{(1)} \mid x \in R_H\} \subseteq V_{F_1} \cup V_{F_2}$ be the set that will correspond to the set U in Lemma 4. In order to apply this lemma we do as follows. We consider the vertices of X in the order in which they were removed from H. For each vertex $x \in X$, we remove the corresponding vertex $t_x^{(1)}$ from G'. After we have finished, we have obtained a graph G''. Let F_1' and F_2' be the subgraphs of G'' induced by the remaining vertices of F_1 and F_2, respectively. Note that we never destroy the connectivity of G' while removing a vertex $t_x^{(1)}$. Moreover, at the moment we remove a vertex $t_x^{(1)}$, it is of distance at most $\ell - 1$ from the set U due to the definition of X. Hence we may apply Lemma 4 every time we remove a vertex $t_x^{(1)}$ with $x \in X$. Then in the end we find that $h'' = h'|_{V_{G''}}$ is a surjective homomorphism from G'' to H'. Note that for each $u \in V_G$ we have $h''(C_u) \cap R_{H'} = \{z\}$, because $h'(C_u) \cap R_{H'} = \{z\}$. We modify h'' into a mapping $f\colon V_{G''} \to V_{H'}$ that is defined as follows:

1. for each edge $e \in E_G$, $f(a) = z$ if a is an inner vertex of P_e;
2. for each $u \in V_G$, $f(g) = z$ if $g \in C_u$;
3. for each $u \in F_1' \cup F_2'$, $f(u) = h''(u)$.

We claim that f is a surjective homomorphism from G'' to H'. If $f = h''$ then this is the case, because h'' is a surjective homomorphism from G'' to H'. Assume that $f \neq h''$.

First suppose that $e = uv$ is an edge of G such that h'' does not map all inner vertices of P_e to z. Let a be an inner vertex of P_e with $y = h''(a)$ and $\text{dist}_{H'}(y, z) = \max_{b \in V_{P_e}} \text{dist}_{H'}(h''(b), z)$. Note that $\text{dist}_{H'}(y, z) < \ell$, because the length of P_e is $\ell - 2$ and because h'' maps at most one vertex of C_u and at most one vertex of C_v to an irreflexive vertex, whereas h'' maps all other vertices of $C_u \cup C_v$ to z, due to Observation 3 combined with the fact that $h''(C_u) \cap R_{H'} = \{z\}$ for all $u \in V_G$. Hence, y is an inner vertex of a path P in H' with irreflexive inner vertices that either joins z with another vertex in $R_{H'}$ or that joins z with an irreflexive leaf of H'. Let y' be the neighbor of y in H' that lies between z and y in P; note that $y' = z$ is possible. Because h'' maps at most one vertex of C_u and at most one vertex of C_v to an irreflexive vertex and all other vertices of $C_u \cup C_v$ to z, we obtain $h''(N_{G''}(a)) = \{y'\}$. This means that we can do as follows. If $y' = z$, then we remap a to z. Otherwise, there is a neighbor y'' of y' that lies on P between z and y', and we remap a to y''. This new mapping is a homomorphism from G'' to H'. In order to prove that it is vertex-surjective, we observe that G'' contains a path, the vertices of which are mapped to the vertices of P by h''. Therefore, there is a vertex $b \neq a$ in this path that is mapped to y. By repeatedly applying this procedure, we get a surjective homomorphism that maps all inner vertices of each path P_e to z. From now on we assume that $h''(a) = z$ for every inner vertex a of every path P_e.

Now suppose that there is a vertex $u \in V_G$ such that there is a vertex $g \in C_u$ that h does not map to z. Because C_u is a clique with at least three vertices, g is the unique vertex of C_u with this property due to Observation 3. Moreover, $h''(N_{G'}(g)) = \{z\}$. Hence we can remap g to z. By the same arguments as before, the modified mapping is a surjective homomorphism from G' to H'. We repeat this procedure as long as necessary. In this way, we obtain f and find that f is a surjective homomorphism from G'' to H'.

We now define a mapping $f_e\colon E_{G''} \to E_{H'}$ that maps the edges of G'' to the edges of H' such that for each $ab \in E_{G''}$, we have $f_e(ab) = f(a)f(b)$. Let $E^*_{H'}$ be the set of all edges of H' that are not self-loops. Because f is a surjective homomorphism from G'' to H' and G'' is connected, we find that $E^*_{H'} \subseteq f_e(E_{G''})$. Let $xy \in E^*_{H'}$. Because $E^*_{H'} \subseteq f_e(E_{G''})$, there exists an edge $ab \in E_{G''}$ with $f_e(ab) = xy$ (so, $f(a) = x$ and $f(b) = y$). Because all components of $H'[R_{H'}]$ are isolated vertices, no two reflexive vertices of H' are adjacent. This means that at least one of the vertices x, y, say y, is irreflexive. Then, by the definition of f, we find that $ab \in E_{F'_1} \cup E_{F'_2}$. There are four types of edges in F'_1 and F'_2:

A. an edge $t^{(1)}_{x'}t^{(2)}_{x'}$ for each reflexive vertex $x' \in V_H$;

B. an edge $t^{(i)}_{x'}t^{(j)}_{y'}$ for each pair of different reflexive vertices $x', y' \in V_H$ and each pair $i, j \in \{1, 2\}$;

C. an edge $t^{(1)}_{x'}t^{(j)}_{y'}$ for each irreflexive vertex $x' \in V_H$, each reflexive vertex $y' \in V_H$ and each $j \in \{1, 2\}$;

D. an edge $t^{(1)}_{x'}t^{(1)}_{y'}$ for each pair of different irreflexive vertices $x', y' \in V_H$.

Suppose that ab is an edge of type A or type B. Then, as y is irreflexive, we apply Observation 3 to deduce that x is reflexive and that $f(N_{G''}(b)) = \{x\}$. Because H' has no irreflexive leaves adjacent to reflexive vertices by construction, y has another neighbor in H' not equal to x. Hence, y is an inner vertex of a path P in H' with irreflexive inner vertices that either joins x with another vertex in $R_{H'}$ or that joins x with an irreflexive leaf of H'. Because G'' is connected and f is vertex-surjective, G'' contains a path, the vertices of which are mapped to the vertices of P by f. Because $f(N_{G''}(b)) = \{x\}$, we find that b is not on this path. Hence, there must be an edge in G'' of type C or D that is mapped to xy by f_e. So, we may assume that ab is of type C or D.

If ab is of type C, then we may assume without loss of generality that $ab = t^{(1)}_{x'}t^{(1)}_{y'}$. Then Observation 3 tells us that either $f_e(t^{(1)}_{x'}t^{(2)}_{y'}) = xy$ or $f_e(t^{(1)}_{x'}t^{(2)}_{y'})$

is a self-loop in H'. Let m_C and m_D denote the number of edges of type C and the number of edges of type D, respectively. Then the above observation for the edges of type C combined with the fact that $E^*_{H'} \subseteq f_e(E_{G''})$ implies that $|E^*_{H'}| \leq m_C/2 + m_D$. However, we also have $m_C/2 + m_D \leq |E^*_{H'}| - 1$, as in the construction of F_1, F_2, and hence also in the construction of F'_1, F'_2, we removed the edge on the path from p to q that was incident with q from H, and this particular edge is in $E^*_{H'}$ as well. By this contradiction we have proven Claim 1. This completes the proof of Theorem 6. □

5 Future Research

We have shown that for any partially reflexive tree H, the Surjective H-Coloring problem is polynomial-time solvable if H is loop-connected and NP-complete otherwise. Determining a complete complexity classification of the Surjective H-Coloring seems a very challenging open problem, and even conjecturing a possible dichotomy (between P and NP-complete) is difficult.

A natural question that also gives an indication on why this problem is so challenging is whether the three problems H-Compaction, H-Retraction and Surjective H-Coloring are polynomially equivalent to each other for each target graph H. Also, the computational complexity classifications of the H-Compaction problem and H-Retraction problem, respectively, are still far from being completed. The well-known Feder-Vardi conjecture [11] states that the $\mathcal{H}$-Constraint Satisfaction problem, where $\mathcal{H}$ is some fixed finite target structure, has a dichotomy. Feder and Vardi [11] showed that this conjecture is equivalent to the conjecture that H-Retraction has a dichotomy.

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