SIMILARITY AND COMMUTATORS OF MATRICES OVER PRINCIPAL IDEAL RINGS

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Abstract. We prove that if R is a principal ideal ring and $A \in \mathrm{M}_n(R)$ is a matrix with trace zero, then A is a commutator, that is, $A = XY - YX$ for some $X, Y \in \mathrm{M}_n(R)$. This generalises the corresponding result over fields due to Albert and Muckenhoupt, as well as that over $\mathbb{Z}$ due to Laffey and Reams, and as a by-product we obtain new simplified proofs of these results. We also establish a normal form for similarity classes of matrices over PIDs, generalising a result of Laffey and Reams. This normal form is a main ingredient in the proof of the result on commutators.

1. Introduction

Let R denote an arbitrary ring. If a matrix $A \in \mathrm{M}_n(R)$ is a commutator, that is, if $A = [X, Y] = XY - YX$ for some $X, Y \in \mathrm{M}_n(R)$, then A must have trace zero. The problem of when the converse holds goes back at least to Shoda [15] who showed in 1937 that if K is a field of characteristic zero, then every $A \in \mathrm{M}_n(K)$ with trace zero is a commutator. Shoda's argument fails in positive characteristic, but Albert and Muckenhoupt [1] found another argument valid for all fields. The first result for rings which are not fields was obtained by Lissner [9] who proved that if R is a principal ideal domain (PID) then every $A \in \mathrm{M}_2(R)$ with trace zero is a commutator. A motivation for Lissner's work was the relation with a special case of Serre's problem on projective modules over polynomial rings, nowadays known as the Quillen-Suslin theorem (see [9, Sections 1-2]). Lissner's result on commutators in $\mathrm{M}_2(R)$ for R a PID was rediscovered by Vaserstein [21] and Rosset and Rosset [14], respectively. Vaserstein also formulated the problem of whether every $A \in \mathrm{M}_n(\mathbb{Z})$ with trace zero is a commutator for $n \geq 3$ (see [21, Section 5]). A significant breakthrough was made by Laffey and Reams [7] who settled Vaserstein's problem in the affirmative. However, their proofs involve steps which are special to the ring of integers $\mathbb{Z}$ and do not generalise to other rings in any straightforward way. The most crucial step of this kind is an appeal to Dirichlet's theorem on primes in arithmetic progressions. The analogue of Dirichlet's theorem, although true in the ring $\mathbb{F}_q[x]$, fails for other Euclidean domains such as $\mathbb{C}[x]$ or discrete valuation rings. Nevertheless, in [6] Laffey asked whether any matrix with trace zero over a Euclidean domain is a commutator. Until now this appears to have remained an open problem even for $n = 3$, except for the cases where R is a field or $\mathbb{Z}$.

In the present paper we answer Laffey's question by proving that if R is any PID and $A \in \mathrm{M}_n(R)$ is a matrix with trace zero, then A is a commutator. This

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is achieved by extending the methods of Laffey and Reams and in particular removing the need for Dirichlet's theorem. Another of our main results is a certain (non-unique) normal form for similarity classes of matrices over PIDs, itself a generalisation of a result proved in [7] over $\mathbb{Z}$. The normal form, while interesting in its own right and potentially for other applications, is also a key ingredient in the proof of the main result on commutators.

We now describe the contents of the paper in more detail. In Section 2 we define regular elements in $\mathrm{M}_n(R)$ where R is a commutative ring with identity, and state some of their basic properties. Regular elements play a central role in the problem of writing matrices as commutators because of the criterion of Laffey and Reams, treated in Section 3. The criterion says that if R is a PID and $A, X \in \mathrm{M}_n(R)$ with X regular mod every maximal ideal of R, then a necessary and sufficient condition for A to be a commutator is that $\mathrm{tr}(X^r A) = 0$ for $r = 0, 1, \ldots, n-1$. This was proved in [7] for $R = \mathbb{Z}$, but the proof goes through for any PID with only a minor modification.

In Section 4 we apply the Laffey-Reams criterion for fields to give a short proof of the theorem of Albert and Muckenhoupt mentioned above. We actually prove a stronger and apparently new result, namely that in the commutator one of the matrices may be taken to be regular (see Proposition 4).

Section 5 is concerned with similarity of matrices over PIDs, that is, matrices up to conjugation by invertible elements. Our first main result is Theorem 5.6 stating that every non-scalar element in $\mathrm{M}_n(R)$ is similar to one in a special form. This result was established by Laffey and Reams over $\mathbb{Z}$. However, a crucial step in their proof uses the fact that 2 is a prime element in $\mathbb{Z}$, and the analogue of this does not hold in an arbitrary PID. To overcome this, our proof involves an argument based on the surjectivity of the map $\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/I)$ for an ideal I, which in a certain sense lets us avoid any finite set of primes, in particular those of index 2 in R (see Lemma 5.1). This argument is evident especially in the proof of Proposition 5.3. Apart from this, our proof uses the methods of [7], although we give a different argument, avoiding case by case considerations, and have made Lemma 5.2 explicit.

Our second main result is Theorem 6.3 whose proof occupies Section 6, and follows the lines of [7, Section 4]. There are two new key ideas in our proof. First, there is again an argument which at a certain step allows us to avoid finitely many primes, including those of index 2 in R. This step in the proof is the choice of q and uses Lemma 6.1 *i)*. Secondly, we apply Lemma 6.2 to obtain a set of generators of the centraliser of a certain matrix modulo a product of distinct primes; see (6.8). It is this set of generators together with our choice of q and an appropriate choice of t in (6.10) which allows us to avoid Dirichlet's theorem. It is interesting to note that the proofs of our main results, Theorems 5.6 and 6.3, despite being rather different, both involve the technique of avoiding finitely many primes, in particular those of index 2 in R. Our proof of Theorem 6.3 also simplifies parts of the proof of Laffey and Reams over $\mathbb{Z}$ since we avoid some of the case by case considerations present in the latter. By a theorem of Hungerford, Theorem 6.3, once established, easily extends to any principal ideal ring (not necessarily an integral domain); see Corollary 6.4.

The final Section 7 discusses the possibility of generalising Theorem 6.3 to other classes of rings such as Dedekind domains, and mentions some known counterexamples.

We end this introduction by mentioning some recent work on matrix commutators. In [11] Mesyan proves that if R is a ring (not necessarily commutative) and $A \in \mathrm{M}_n(R)$ has trace zero, then A is a sum of two commutators. This result was proved for commutative rings in earlier unpublished work of Rosset. In [5] Khurana and Lam study "generalised commutators", that is, elements of the form $XYZ - ZYX$, where $X, Y, Z \in \mathrm{M}_n(R)$. They establish in particular that if R is a PID, then every element in $\mathrm{M}_n(R)$, $n \geq 2$, is a generalised commutator. Although these results may seem closely related to the commutator problem studied in the present paper, the proofs are in fact very different.

Notation and terminology. We use $\mathbb{N}$ to denote the natural numbers $\{1, 2, \dots\}$.

Let R be a commutative ring with identity. We denote the set of maximal ideals of R by $\operatorname{Specm} R$ and the ring of $n \times n$ matrices over R by $\mathrm{M}_n(R)$. For $A, B \in \mathrm{M}_n(R)$ we call $[A, B] = AB - BA$ the *commutator* of A and B. Let $A \in \mathrm{M}_n(R)$. A matrix $B \in \mathrm{M}_n(R)$ is said to be *similar* to A if there exists a $g \in \mathrm{GL}_n(R)$ such that $gAg^{-1} = B$. The transpose of A is denoted by A^T and the trace of A by $\operatorname{tr}(A)$. We write $C_{\mathrm{M}_n(R)}(A)$ for the centraliser of A in $\mathrm{M}_n(R)$, that is,

$$C_{\mathrm{M}_n(R)}(A) = \{B \in \mathrm{M}_n(R) \mid [A, B] = 0\}.$$

Let $f(x) = a_0 + a_1 x + \dots + x^n \in R[x]$ be the characteristic polynomial of A. We will refer to the *companion matrix* associated to A (or to f) as the matrix $C \in \mathrm{M}_n(R)$ such that

$$C = (c_{ij}) = \begin{cases} c_{i,i+1} = 1 & \text{for } 1 \leq i \leq n-1, \\ c_{ni} = -a_{i-1} & \text{for } 1 \leq i \leq n, \\ c_{ij} = 0 & \text{otherwise.} \end{cases}$$

The identity matrix in $\mathrm{M}_n(R)$ is denoted by 1 or sometimes 1_n. For $u, v \in \mathbb{N}$ we write E_{uv} for the matrix units, that is, $E_{uv} = (e_{ij})$ with $e_{uv} = 1$ and $e_{ij} = 0$ otherwise. The size of the matrices E_{uv} is suppressed in the notation and will be determined by the context.

2. Regular elements

Throughout this section R is assumed to be a commutative ring with identity. Let $\mathbf{G}$ be a reductive algebraic group over a field K with algebraic closure $\overline{K}$. An element $x \in G = \mathbf{G}(\overline{K})$ is called *regular* if $\dim C_G(x)$ is minimal, and it is known that this minimal dimension equals the rank $\operatorname{rk} G$ (see [18] and [2, Section 14]). Similarly, if $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$ an element $X \in \mathfrak{g}(\overline{K})$ is called *regular* if $\dim C_G(X) = \operatorname{rk} G$, where G acts on $\mathfrak{g}$ via the adjoint action. In the case $\mathbf{G} = \mathrm{GL}_n$ there are several equivalent characterisations of regular elements in $\mathfrak{g}(K) = \mathrm{M}_n(K)$. More precisely, the following is well-known:

Proposition 2.1. *Let K be a field and $X \in \mathrm{M}_n(K)$. Then the following are equivalent*

i) X is regular,
ii) There exists a vector $v \in K^n$ such that $\{v, Xv, \dots, X^{n-1}v\}$ is a basis for K^n over K,
iii) The set $\{1, X, \dots, X^{n-1}\}$ is linearly independent over K,

iv) X *is similar to its companion matrix* C *as well as to* C^T*,*
v) $C_{\mathrm{M}_n(K)}(X) = K[X]$.

In the following we will use the properties of regular elements expressed in Proposition 2.1 without explicit reference. Regular elements of $\mathrm{M}_n(K)$ are sometimes called *non-derogatory* or *cyclic.* For matrices over R we make the following definition.

Definition 2.2. A matrix $X \in \mathrm{M}_n(R)$ is called *regular* if there exists a vector $v \in R^n$ such that $\{v, Xv, \dots, X^{n-1}v\}$ is a basis for R^n over R.

Proposition 2.3. *Let* $X \in \mathrm{M}_n(R)$*. Then the following are equivalent*

i) X *is regular,*
ii) X *is similar to its companion matrix* C*,*
iii) X *is similar to* C^T*.*

Proof. The proof that *i)* and *ii)* are equivalent follows easily from Definition 2.2 and is the same as in the classical case of matrices over fields. The equivalence between *ii)* and *iii)* follows from work of Estes and Guralnick [3]; see e.g. [16]. □

Let S be a commutative ring with identity. If $\phi : R \to S$ is a homomorphism we also use ϕ to denote the induced homomorphism $\mathrm{M}_n(R) \to \mathrm{M}_n(S)$.

Lemma 2.4. *Let* $\phi : R \to S$ *be a homomorphism. If* $X \in \mathrm{M}_n(R)$ *is regular, then* $\phi(X)$ *is regular.*

Proof. Suppose that X is regular. By definition there exists a vector $v \in R^n$ such that $\{v, Xv, \dots, X^{n-1}v\}$ is an R-basis for R^n. Then $\{v\otimes 1, Xv\otimes 1, \dots, X^{n-1}v\otimes 1\}$ is an S-basis for $R^n \otimes_R S$ (cf. [8, XVI, Proposition 2.3]). Let $\phi(v) \in S^n$ be the image of v under component-wise application of ϕ. Under the isomorphism $R^n \otimes_R S \to S^n$, the elements $X^i v \otimes 1$ are sent to $\phi(X)^i \phi(v)$, so $\{\phi(v), \phi(X)\phi(v), \dots, \phi(X)^{n-1}\phi(v)\}$ is a basis for S^n. Thus $\phi(X)$ is regular. □

If $\mathfrak{p}$ is an ideal of R we use $X_{\mathfrak{p}}$ to denote the image of $X \in \mathrm{M}_n(R)$ under the canonical map $\pi : \mathrm{M}_n(R) \to \mathrm{M}_n(R/\mathfrak{p})$, that is, $X_{\mathfrak{p}} = \pi(X)$. In general, an element in $\mathrm{M}_n(R)$ which is regular modulo every maximal ideal may not be regular. However, if R is a local ring, the situation is favourable:

Lemma 2.5. *Assume that* R *is a local ring with maximal ideal* $\mathfrak{m}$*. Then* $X \in \mathrm{M}_n(R)$ *is regular if and only if* $X_{\mathfrak{m}} \in \mathrm{M}_n(R/\mathfrak{m})$ *is regular.*

Proof. If X is regular, then $X_{\mathfrak{m}}$ is regular by Lemma 2.4. Conversely, suppose that $X_{\mathfrak{m}}$ is regular and choose $v \in (R/\mathfrak{m})^n$ such that $(R/\mathfrak{m})^n = (R/\mathfrak{m})[X_{\mathfrak{m}}]v$. Let $\hat{v} \in R^n$ be a lift of v. Then $R^n = R[X]\hat{v} + \mathfrak{m}M$ for some submodule M of R^n, and Nakayama's lemma yields $R^n = R[X]\hat{v}$, so X is regular. □

Proposition 2.6. *Let* R *be an integral domain with field of fractions* F*, and let* $X \in \mathrm{M}_n(R)$*. If* X *is regular at a closed point of* $\operatorname{Spec} R$*, then* X *is regular at the generic point. In other words, if* $X_{\mathfrak{m}}$ *is regular for some maximal ideal* $\mathfrak{m}$ *of* R*, then* X *is regular as an element of* $\mathrm{M}_n(F)$*.*

Proof. Suppose that $X_{\mathfrak{m}}$ is regular for some maximal ideal $\mathfrak{m}$ of R. Let $R_{\mathfrak{m}}$ be the localisation of R at $\mathfrak{m}$, and let $j : R \to R_{\mathfrak{m}}$ be the canonical homomorphism. Since the diagram

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d] & R_{\mathfrak{m}} \arrow[d] \\ R/\mathfrak{m} \arrow[r, "\cong"] & R_{\mathfrak{m}}/\mathfrak{m} \end{tikzcd}$$

commutes, Lemma 2.5 implies that $j(X)$ is regular. If $\sum_{i=0}^{n-1} r_i X^i = 0$ for some $r_i \in R$, then $\sum_{i=0}^{n-1} j(r_i) j(X)^i = 0$. But since $j(X)$ is regular, we must have $j(r_i) = 0$ for all $i = 0, \dots, n-1$. Since R is an integral domain j is injective, so $r_i = 0$ for $i = 0, \dots, n-1$. Now, if $\sum_{i=0}^{n-1} s_i X^i = 0$ for some $s_i \in F$, then clearing denominators shows that $s_i = 0$ for all $i = 0, \dots, n-1$. Hence, by Proposition 2.1 *iii)* the matrix X is regular as an element of $\mathrm{M}_n(F)$. $\square$

The following result has appeared in [22, Proposition 6].

Lemma 2.7. *Let $A = (a_{ij}) \in \mathrm{M}_n(R)$ be a matrix such that $a_{i+1,i} = 1$ for all $1 \leq i \leq n-1$ and $a_{ij} = 0$ for all $i \geq j+2$. Then A is regular.*

Proof. Let $v = (1, 0, \dots, 0)^T \in R^n$. Then the matrix

$$B = (v, Av, \dots, A^{n-1}v)$$

is upper triangular with 1s on the diagonal, so $B \in \mathrm{SL}_n(R)$. In particular, the columns of B form a basis for R^n, so A is regular. $\square$

Note that if $A = (a_{ij}) \in \mathrm{M}_n(R)$ is a matrix such that $a_{i,i+1} = 1$ for all $1 \leq i \leq n-1$ and $a_{ij} = 0$ for all $j \geq i+2$, then the above lemma together with Proposition 2.3 implies that A is regular.

3. The criterion of Laffey and Reams

Throughout this section R is a PID and F its field of fractions. In Theorem 3.3 we give a criterion, discovered by Laffey and Reams [7, Section 3], for a matrix in $\mathrm{M}_n(R)$ to be a commutator. This criterion plays an important role in our proof of the main theorem. Laffey and Reams proved the criterion for matrices over fields and over $\mathbb{Z}$, and we only need minor modifications of their proofs, together with Proposition 2.6, to prove it over arbitrary PIDs.

The following result is from [7, Section 3]. We reproduce the proof here for completeness.

Proposition 3.1. *Let K be a field and $X \in \mathrm{M}_n(K)$ be regular. Let $A \in \mathrm{M}_n(K)$. Then $A = [X, Y]$ for some $Y \in \mathrm{M}_n(K)$ if and only if $\mathrm{tr}(X^r A) = 0$ for all $r = 0, \dots, n-1$.*

Proof. Since $\{1, X, \dots, X^{n-1}\}$ is linearly independent over K the subspace

$$V = \{B \in \mathrm{M}_n(K) \mid \mathrm{tr}(X^r B) = 0 \text{ for } 0, 1, \dots, n-1\}$$

has dimension $n^2 - n$. The kernel of the linear map $\mathrm{M}_n(K) \to \mathrm{M}_n(K)$, $Y \mapsto [X, Y]$ is equal to the centraliser $C_{\mathrm{M}_n(K)}(X)$, which has dimension n since X is regular. Thus the image $[X, \mathrm{M}_n(K)]$ of the map $Y \mapsto [X, Y]$ has dimension $n^2 - n$. But if $A \in [X, \mathrm{M}_n(K)]$ there exists a $Y \in \mathrm{M}_n(K)$ such that for every $r = 0, 1, \dots, n-1$ we have

$$\mathrm{tr}(X^r A) = \mathrm{tr}(X^r(XY - YX)) = \mathrm{tr}(X^{r+1}Y) - \mathrm{tr}(X^r Y X) = 0.$$

Thus $A \in V$ and so $[X, \mathrm{M}_n(K)] \subseteq V$. Since $\dim V = \dim[X, \mathrm{M}_n(K)]$ we conclude that $V = [X, \mathrm{M}_n(K)]$. □

Proposition 3.2. *Let $X \in \mathrm{M}_n(R)$ be such that $X_{\mathfrak{p}}$ is regular for every maximal ideal $\mathfrak{p}$ in R. Suppose that $M \in \mathrm{M}_n(F)$ is such that $[X, M] \in \mathrm{M}_n(R)$. Then there exists an $Y \in \mathrm{M}_n(R)$ such that $[X, M] = [X, Y]$.*

Proof. There exists an element $m \in R$ such that $mM \in \mathrm{M}_n(R)$, and we have $[X, mM] = m[X, M]$. Assume that $d \in R$ is chosen so that it has the minimal number of irreducible factors with respect to the property that $[X, C] = d[X, M]$ for some $C \in \mathrm{M}_n(R)$. If d is a unit we are done, so assume that p is an irreducible factor of d. Then $[X, C] \in p\,\mathrm{M}_n(R)$, so $X_{(p)}$ commutes with $C_{(p)}$. But since $X_{(p)}$ is regular, we have $C_{(p)} = f(X_{(p)})$, for some polynomial $f(T) \in R[T]$. Hence $C - f(X) = pD$ for some $D \in \mathrm{M}_n(R)$. But this implies that $[X, C] = [X, pD] = p[X, D]$ and thus $(dp^{-1})[X, M] = [X, D]$, giving a contradiction to our choice of d. Hence d is a unit and so $[X, M] = [X, Y]$ with $Y = d^{-1}C \in \mathrm{M}_n(R)$. □

Proposition 3.3. *Let $A \in \mathrm{M}_n(R)$ and let $X \in \mathrm{M}_n(R)$ be such that $X_{\mathfrak{p}}$ is regular for every maximal ideal $\mathfrak{p}$ in R. Then $A = [X, Y]$ for some $Y \in \mathrm{M}_n(R)$ if and only if $\mathrm{tr}(X^r A) = 0$ for $r = 0, \dots, n-1$.*

Proof. Clearly the condition $\mathrm{tr}(X^r A) = 0$ for all $r \geq 0$ is necessary for A to be of the form $[X, Y]$ with $Y \in \mathrm{M}_n(R)$. Conversely, suppose that $\mathrm{tr}(X^r A) = 0$ for $r = 0, 1, \dots, n-1$. By Proposition 2.6 X is regular as an element in $\mathrm{M}_n(F)$ so Proposition 3.1 implies that $A = [X, M]$ for some $M \in \mathrm{M}_n(F)$. But now the result follows from Proposition 3.2. □

4. Commutators over fields

Let K be a field. Using the criterion of Laffey and Reams over fields (Proposition 3.1) we give a swift proof of the theorem of Albert and Muckenhoupt [1] that every matrix with trace zero in $\mathrm{M}_n(K)$ is a commutator.

Let R be a commutative ring with identity. Note that if $A, X, Y \in \mathrm{M}_n(R)$ are such that $A = [X, Y]$, then for every $g \in \mathrm{GL}_n(R)$ we have $gAg^{-1} = [gXg^{-1}, gYg^{-1}]$. Thus A is a commutator if and only if any matrix similar to A is.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $k = \lfloor n/2 \rfloor$. The following matrices were considered by Laffey and Reams [7, Section 4] who also established the properties stated below.

$$P_n = (p_{ij}) = \begin{cases} p_{ii} = 1 & \text{for } i = 2, 4, \dots, 2k, \\ p_{i,i-2} = 1 & \text{for } i = 3, 4, \dots n, \\ p_{ij} = 0 & \text{otherwise.} \end{cases}$$

Depending on the context we will consider P_n as an element of $\mathrm{M}_n(R)$ for various rings R. For any $m \in \mathbb{N}$ and $a \in R$ we will use $J_m(a)$ to denote the $m \times m$ Jordan block with eigenvalue a and 1s on the subdiagonal. The matrix P_n is similar to $J_k(1) \oplus J_{n-k}(0)$ (cf. [7, p. 681]), and thus it is regular by Lemma 2.7.

For any $A = (a_{ij}) \in \mathrm{M}_n(R)$ let

$$c(A) := \sum_{i=1}^{k} a_{2i,2i}.$$

Suppose now that $a_{ij} = 0$ for $j \geq i+2$. Observe that for any $r \in \mathbb{N}$, P_n^r has the same diagonal as P_n and the (i,j) entry of P_n^r is 0 if $i \neq j$ and $i < j+2$. Thus

$$\operatorname{tr}(P_n^r A) = c(A), \text{ for } r \in \mathbb{N}. \tag{4.1}$$

Proposition 4.1. *Let K be a field and let $A \in \mathrm{M}_n(K)$ be a matrix with trace zero. Then $A = [X, Y]$ for some $X, Y \in \mathrm{M}_n(K)$, where X is regular. More precisely, if A is non-scalar X can be chosen to be conjugate to P_n, while if A is scalar we can take $X = J_n(0)$.*

Proof. Assume first that A is non-scalar. It then follows from the rational normal form that A is similar to a matrix $B = (b_{ij})$ with $b_{12} = 1$ and $b_{ij} = 0$ for $j \geq i+2$, so we have $A = gBg^{-1}$ for some $g \in \mathrm{GL}_n(K)$. Define $z \in \mathrm{SL}_n(K)$ as

$$z = 1 + c(B)E_{21}.$$

Then the (i,j) entry of $z^{-1}Bz$ is 0 for $j \geq i+2$ and $c(z^{-1}Bz) = 0$, so by (4.1) we have $\operatorname{tr}(P_n^r z^{-1}Bz) = 0$ for $r = 0, \ldots, n-1$. By Proposition 3.1 it follows that $B = [zP_nz^{-1}, Y]$ for some $Y \in \mathrm{M}_n(K)$, and thus $A = [gzP_n(gz)^{-1}, gYg^{-1}]$.

Assume on the other hand that A is a scalar. Then $\operatorname{tr}(J_n(0)^r A) = 0$ for $r = 0, \ldots, n-1$, and Proposition 3.1 implies that $A = [J_n(0), Y]$, for some $Y \in \mathrm{M}_n(K)$. □

5. Matrix similarity over a PID

In this section we extend the results of [7, Section 2] on similarity of matrices over $\mathbb{Z}$ to matrices over an arbitrary PID R.

Lemma 5.1. *Let $A \in \mathrm{M}_n(R)$ be non-scalar, and let S be a finite set of maximal ideals of R such that $A_{\mathfrak{p}} \in \mathrm{M}_n(R/\mathfrak{p})$ is non-scalar for every $\mathfrak{p} \in S$. Then A is similar to a matrix $B = (b_{ij}) \in \mathrm{M}_n(R)$ such that $b_{12} \notin \mathfrak{p}$ for all $\mathfrak{p} \in S$.*

Proof. It is well known that for any PID R and any ideal $\mathfrak{a}$ of R the natural map

$$\mathrm{SL}_n(R) \longrightarrow \mathrm{SL}_n(R/\mathfrak{a}) \tag{5.1}$$

is surjective. This follows for example from the fact that $R/\mathfrak{a}$ is the product of local rings and that over local rings SL_n is generated by elementary matrices (see [13, 2.2.2 and 2.2.6]). Moreover, if we take $\mathfrak{a} = \prod_{\mathfrak{p} \in S} \mathfrak{p}$ the Chinese remainder theorem implies that we have an isomorphism

$$\mathrm{SL}_n(R/\mathfrak{a}) \xrightarrow{\sim} \prod_{\mathfrak{p} \in S} \mathrm{SL}_n(R/\mathfrak{p}). \tag{5.2}$$

Let $\mathfrak{p} \in S$. Since $A_{\mathfrak{p}}$ is non-scalar and $R/\mathfrak{p}$ is a field the rational canonical form for matrices in $\mathrm{M}_n(R/\mathfrak{p})$ implies that there exists a $g_{\mathfrak{p}} \in \mathrm{GL}_n(R/\mathfrak{p})$ such that $g_{\mathfrak{p}} A_{\mathfrak{p}} g_{\mathfrak{p}}^{-1}$ is a matrix whose $(1,2)$ entry is non-zero. Any element $x \in \mathrm{GL}_n(R/\mathfrak{p})$, can be written as $x = ty$, where $t \in \mathrm{GL}_n(R/\mathfrak{p})$ is diagonal and $y \in \mathrm{SL}_n(R/\mathfrak{p})$. Any $M \in \mathrm{M}_n(R/\mathfrak{p})$ has non-zero $(1,2)$ entry if and only if tMt^{-1} does. We may therefore take $g_{\mathfrak{p}}$ to be in $\mathrm{SL}_n(R/\mathfrak{p})$. Suppose that $g_{\mathfrak{p}}$ is chosen in this way for every $\mathfrak{p} \in S$. By the surjectivity of the maps (5.1) and (5.2), there exists a $g \in \mathrm{SL}_n(R)$ such that the image of g in $\mathrm{SL}_n(R/\mathfrak{p})$ is $g_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$. Let $B = (b_{ij}) = gAg^{-1}$. Then B is a matrix such that b_{12} is non-zero modulo every $\mathfrak{p} \in S$. □

The following lemma will be used repeatedly in the proof of Proposition 5.3 and Theorem 5.6. It can informally be described as saying that if the off-diagonal entries in a row (column) of a matrix $A \in \mathrm{M}_n(R)$ with $n \geq 3$ have a greatest common divisor d, then A is similar to a matrix in which the corresponding row (column) has off-diagonal entries $d, 0, \ldots, 0$.

Lemma 5.2. *Let $A = (a_{ij}) \in \mathrm{M}_n(R)$, $n \geq 3$. Let $1 \leq u \leq n$ and $1 \leq v \leq n$ be fixed. Let $b \in R$ be a generator of the ideal $(a_{uj} \mid 1 \leq j \leq n,\ j \neq u)$, and let $c \in R$ be a generator of the ideal $(a_{iv} \mid 1 \leq i \leq n,\ i \neq v)$. Then A is similar to a matrix $B = (b_{ij})$ such that if $u = 1$ we have $b_{u2} = b$ and $b_{uj} = 0$ for all $3 \leq j \leq n$, and if $u \geq 2$ we have $b_{u1} = b$ and $b_{uj} = 0$ for all $2 \leq j \leq n$ such that $j \neq u$. Moreover, A is similar to a matrix $C = (c_{ij})$ such that if $v = 1$ we have $c_{2v} = c$ and $c_{iv} = 0$ for all $3 \leq i \leq n$, and if $v \geq 2$ we have $c_{1v} = c$ and $c_{iv} = 0$ for all $2 \leq i \leq n$ such that $i \neq v$.*

Proof. The proof follows the lines of [12, Ch. III, Section 2]. For $1 \leq i < j \leq n$ and $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{SL}_2(R)$, let

$$\begin{aligned} M_{ij} &= M_{ij}(x, y, z, w) \\ &= 1_n + (x-1)E_{ii} + yE_{ij} + zE_{ji} + (w-1)E_{jj} \in \mathrm{SL}_n(R). \end{aligned}$$

Note that $M_{ij}^{-1} = M_{ij}(w, -y, -z, x)$. Let $3 \leq j \leq n$. Direct computation shows that the first row in $B_1 := M_{2j}^{-1} A M_{2j}$ is

$$\begin{aligned} (a_{11}, a_{12}x + a_{13}z, a_{12}y + a_{13}w, a_{14}, \ldots, a_{1n}) &\quad \text{if } j = 3, \\ (a_{11}, a_{12}x + a_{1j}z, a_{13}, \ldots, a_{1,j-1}, a_{12}y + a_{1j}w, a_{1,j+1}, \ldots, a_{1n}) &\quad \text{if } j > 3. \end{aligned}$$

Now let $3 \leq j \leq n$ be the smallest integer such that $a_{1j} \neq 0$ (if no such j exists the assertion of the lemma holds trivially for A and $u = 1$). Let $d \in R$ be a generator of (a_{12}, a_{1j}) and let $y, w \in R$ be such that

$$yd = a_{1j}, \quad wd = -a_{12}.$$

Then $(y, w) = (1)$ and hence $x, z \in R$ may be determined so that $xw - yz = 1$. Thus $a_{12}x + a_{1j}z = -d$. With these values of x, y, z, w all the entries of B_1 in positions $(1,3), \ldots, (1,j)$ are zero, and the $(1,2)$ entry generates the ideal (a_{12}, a_{1j}). Repeating the process, let $j < k \leq n$ be the smallest integer such that $a_{1k} \neq 0$. Then $B_2 := M_{2k}^{-1} B_1 M_{2k}$ has all its entries $(1,3), \ldots, (1,k)$ zero and its $(1,2)$ entry generates the ideal (a_{12}, a_{1j}, a_{1k}). Proceeding in this way, we obtain a matrix $B = (b_{ij})$ similar to A such that b_{12} is a generator of $(a_{1j} \mid 2 \leq j \leq n)$ and $b_{1j} = 0$ for $3 \leq j \leq n$ (the generator b_{12} can be replaced by any other generator of $(a_{1j} \mid 2 \leq j \leq n)$ by a diagonal similarity transformation of B). This shows the existence of B for $u = 1$. For $u \geq 2$, observe that if we let $W_u = (w_{ij}^{(u)}) \in \mathrm{GL}_n(R)$ be any permutation matrix such that $w_{1u}^{(u)} = w_{u1}^{(u)} = 1$, then

$$A' = (a'_{ij}) = W_u A W_u^{-1}$$

is a matrix such that $a'_{11} = a_{uu}$ and $\{a'_{1j} \mid 2 \leq j \leq n\} = \{a_{uj} \mid 1 \leq j \leq n, u \neq j\}$. Informally, the off-diagonal entries in the u-th row of A are the same as the off-diagonal entries in the first row of A', up to a permutation. Thus the existence of B for $u \geq 2$ follows from the argument for $u = 1$ above.

For the existence of C, apply the above result on the rows of A to the transpose A^T. We obtain that A^T is similar to a matrix C^T, where C is of the desired form. Thus A is similar to C. □

Proposition 5.3. *Let $A \in \mathrm{M}_3(R)$ be non-scalar. Then A is similar to a matrix $B = (b_{ij}) \in \mathrm{M}_3(R)$ such that $b_{12} \mid b_{ij}$ for all $i \neq j$ and $b_{12} \mid (b_{ii} - b_{jj})$ for all $1 \leq i, j \leq 3$.*

Proof. Write $A = aI + bA'$, where $a, b \in R$, $b \neq 0$ and where, if $A' = (a'_{ij})$, we have $(a'_{ii} - a'_{jj}, a'_{ij} \mid i \neq j, 1 \leq i, j \leq 3) = (1)$. Note that $A'_{\mathfrak{p}}$ is non-scalar for every maximal ideal $\mathfrak{p}$ of R and that the proposition will follow for A if we can show it for A', that is, if we can show that A' is similar to a matrix whose $(1,2)$ entry is a unit. Without loss of generality we may therefore assume that $A = A'$ so that A satisfies

$$(a_{ii} - a_{jj}, a_{ij} \mid i \neq j, 1 \leq i, j \leq 3) = (1).$$

Note that any matrix similar to A will also satisfy this. Let

$$S := \{\mathfrak{p} \in \operatorname{Specm} R \mid |R/\mathfrak{p}| = 2\}.$$

Note that S is a finite set since in any PID (or any Dedekind domain) there are only finitely many maximal ideals of any given finite index. Since $A_{\mathfrak{p}}$ is not scalar for any maximal ideal $\mathfrak{p}$ of R, Lemma 5.1 implies that A is similar to a matrix $B = (b_{ij})$ such that $b_{12} \notin \mathfrak{p}$ for all $\mathfrak{p} \in S$. Among all such matrices choose one for which the number of distinct primes which divide b_{12} is least possible, and subject to this, for which the number of not necessarily distinct prime factors is minimal. By Lemma 5.2 applied to the first row in B, we see that there exists a matrix B' similar to B whose $(1,3)$ entry is zero and whose $(1,2)$ entry, being equal to a generator of (b_{12}, b_{13}), has no more distinct prime factors than b_{12}. Hence we may assume that B has been replaced by B' so that $b_{13} = 0$. We thus have the following condition on B:

(∗) The matrix $B = (b_{ij})$ is similar to A, $b_{12} \notin \mathfrak{p}$ for all $\mathfrak{p} \in S$ and among all $(1,2)$ entries of matrices with these properties b_{12} has the least number of distinct prime factors. Among all matrices with these properties B is such that b_{12} has the least number of not necessarily distinct prime factors. Moreover, $b_{13} = 0$.

Note first that by Lemma 5.2 applied to the second column in B, there exists a matrix similar to B whose $(1,2)$ entry is a generator of (b_{12}, b_{32}). Thus, by $(*)$ we must have $b_{12} \mid b_{32}$, so $b_{32} = b_{12}a$ for some $a \in R$. Let

$$B_1 = (b^{(1)}_{ij}) = (1 - E_{31}a)B(1 - E_{31}a)^{-1}.$$

Then $b^{(1)}_{12} = b_{12}$ and $b^{(1)}_{13} = b^{(1)}_{32} = 0$ so that

$$B_1 = \begin{pmatrix} b^{(1)}_{11} & b_{12} & 0 \\ b^{(1)}_{21} & b^{(1)}_{22} & b^{(1)}_{23} \\ b^{(1)}_{31} & 0 & b^{(1)}_{33} \end{pmatrix}.$$

In particular, B_1 satisfies $(*)$.

Claim 5.4. The entry b_{12} divides both $b^{(1)}_{33} - b^{(1)}_{11}$ and $b^{(1)}_{31}$.

Let $y \in R$. The first row of the matrix $(1 + E_{13}y)B_1(1 + E_{13}y)^{-1}$ is

$$(b^{(1)}_{11} + yb^{(1)}_{31},\, b_{12},\, y(b^{(1)}_{33} - b^{(1)}_{11} - yb^{(1)}_{31})).$$

Thus, by $(*)$ and Lemma 5.2 applied to the first row in $(1+E_{13}y)B_1(1+E_{13}y)^{-1}$ we conclude that b_{12} divides $y(b_{33}^{(1)} - b_{11}^{(1)} - yb_{31}^{(1)})$ for any $y \in R$. Let

$$(b_{12}) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_\nu^{e_\nu}$$

be the factorisation of (b_{12}), where $\nu \in \mathbb{N}$, $e_i \in \mathbb{N}$ and the ideals $\mathfrak{p}_i \in \operatorname{Specm} R$ are distinct for $1 \le i \le \nu$. By $(*)$ and the definition of S we know that $|R/\mathfrak{p}_i| \ge 3$ for any $1 \le i \le \nu$. Hence there exist elements $y_i, y_i' \in R/\mathfrak{p}_i$ such that

$$y_i \neq 0, \quad y_i' \neq 0, \quad y_i \neq y_i', \quad \text{for } i = 1, \ldots, \nu. \tag{5.3}$$

By the Chinese remainder theorem we have

$$R/(b_{12}) \cong \prod_{i=1}^{\nu} R/\mathfrak{p}_i^{e_i}.$$

Let λ and λ' be the elements in $R/(b_{12})$ which correspond to $(y_1, \ldots, y_\nu), (y_1', \ldots, y_\nu') \in \prod_{i=1}^{\nu} R/\mathfrak{p}_i^{e_i}$ under the above isomorphism. By (5.3) each of λ, λ' and $\lambda - \lambda'$ is a unit in $R/(b_{12})$. In other words, each of λ, λ' and $\lambda - \lambda'$ is coprime to b_{12}. Let $\hat{\lambda}, \hat{\lambda}' \in R$ be representatives of λ, λ', respectively. We know from the above that b_{12} divides $y(b_{33}^{(1)} - b_{11}^{(1)} - yb_{31}^{(1)})$ for any $y \in R$. In particular, choosing $y = \hat{\lambda}, \hat{\lambda}'$, respectively, we obtain $b_{31}^{(1)}(\hat{\lambda} - \hat{\lambda}') \in (b_{12})$, hence $b_{31}^{(1)} \in (b_{12})$ and $b_{33}^{(1)} - b_{11}^{(1)} \in (b_{12})$. This proves the claim.

By Claim 5.4 there exist elements $\alpha, \beta \in R$ such that

$$b_{33}^{(1)} - b_{11}^{(1)} = \alpha b_{12} \quad \text{and} \quad b_{31}^{(1)} = \beta b_{12}.$$

Let

$$B_2 = (b_{ij}^{(2)}) = (1 + E_{21}(-\alpha+\beta))(1+E_{31})B_1(1+E_{31})^{-1}(1+E_{21}(-\alpha+\beta))^{-1}.$$

Then $b_{12}^{(2)} = b_{32}^{(2)} = b_{12}$ and $b_{13}^{(2)} = b_{31}^{(2)} = 0$ so that

$$B_2 = \begin{pmatrix} b_{11}^{(2)} & b_{12} & 0 \\ b_{21}^{(2)} & b_{22}^{(2)} & b_{23}^{(2)} \\ 0 & b_{12} & b_{33}^{(2)} \end{pmatrix}.$$

Moreover, let

$$B_2' = (1 - E_{31})B_2(1-E_{31})^{-1} = \begin{pmatrix} b_{11}^{(2)} & b_{12} & 0 \\ b_{23}^{(2)} + b_{21}^{(2)} & b_{22}^{(2)} & b_{23}^{(2)} \\ b_{33}^{(2)} - b_{11}^{(2)} & 0 & b_{33}^{(2)} \end{pmatrix}$$

and

$$B_2'' = (w - E_{33})B_2(w-E_{33})^{-1} = \begin{pmatrix} b_{33}^{(2)} & b_{12} & 0 \\ b_{23}^{(2)} + b_{21}^{(2)} & b_{22}^{(2)} & b_{21}^{(2)} \\ b_{11}^{(2)} - b_{33}^{(2)} & 0 & b_{11}^{(2)} \end{pmatrix},$$

where $w = (w_{ij})$ is the anti-diagonal permutation matrix, that is, $w_{i,n-i+1} = 1$ and $w_{ij} = 0$ otherwise. We will now show that B_2 has the property that $b_{12} \mid b_{ij}^{(2)}$ for all $i \neq j$ and $b_{12} \mid (b_{ii}^{(2)} - b_{jj}^{(2)})$ for all $1 \le i, j \le 3$. This follows from the following fact applied to the matrices B_2' and B_2''.

Claim 5.5. Suppose that $C = (c_{ij}) \in \mathrm{M}_n(R)$ satisfies $(*)$ and that $c_{32} = 0$. Then $c_{12} \mid c_{ij}$ for all i, j such that $(i,j) \neq (2,1)$ and $i \neq j$, and $c_{12} \mid (c_{ii} - c_{jj})$ for all $1 \le i, j \le 3$.

To prove the claim, let $x \in R$ and

$$X = (x_{ij}) = (1 + E_{32}x)C(1 + E_{32}x)^{-1}.$$

Then

$$x_{12} = c_{12},$$
$$x_{32} = x(c_{22} - c_{33} - xc_{23}),$$

and by Lemma 5.2 applied to the second column in X and $(*)$ we conclude that c_{12} divides $x(c_{22} - c_{33} - xc_{23})$ for any $x \in R$. By the same argument as in the proof of Claim 5.4 we then obtain

$$c_{12} \mid (c_{22} - c_{33}) \quad \text{and} \quad c_{12} \mid c_{23}.$$

Next, for $y \in R$ let

$$Y = (y_{ij}) = (1 + E_{13}y)C(1 + E_{13}y)^{-1}.$$

Recall that $c_{13} = 0$ since C satisfies $(*)$. We have

$$y_{12} = c_{12},$$
$$y_{13} = y(c_{33} - c_{11} - yc_{31}),$$

and by Lemma 5.2 applied to the first row in Y and the same argument as for the matrix X (that is, using $(*)$ and the same argument as in the proof of Claim 5.4) we obtain

$$c_{12} \mid (c_{33} - c_{11}) \quad \text{and} \quad c_{12} \mid c_{31},$$

whence also $c_{12} \mid (c_{22} - c_{11})$. This proves Claim 5.5 for C.

Applying Claim 5.5 to the matrices B_2' and B_2'', respectively, we conclude that B_2 has the property that $b_{12} \mid b_{ij}^{(2)}$ for all $i \neq j$ and $b_{12} \mid (b_{ii}^{(2)} - b_{jj}^{(2)})$ for all $1 \leq i, j \leq 3$. Since B_2 is similar to B (and B is similar to A), we have

$$(b_{ii} - b_{jj}, b_{ij} \mid i \neq j, 1 \leq i, j \leq 3) = (1),$$

so b_{12} must be a unit. This proves the proposition. □

We now use Proposition 5.3 to prove the corresponding result for matrices in $\mathrm{M}_n(R)$ for all $n \geq 3$. More precisely, we have

Theorem 5.6. *Let $A \in \mathrm{M}_n(R)$ with $n \geq 3$, be non-scalar. Then A is similar to a matrix $B = (b_{ij}) \in \mathrm{M}_n(R)$ such that $b_{12} \mid b_{ij}$ for all $i \neq j$ and $b_{12} \mid (b_{ii} - b_{jj})$ for all $1 \leq i, j \leq n$. Moreover, B may be chosen with $b_{ij} = 0$ for all i, j such that $j \geq i + 2$ and $1 \leq i \leq n - 2$.*

Proof. As in the proof of Proposition 5.3, we may assume that

$$(a_{ii} - a_{jj}, a_{ij} \mid i \neq j, 1 \leq i, j \leq n) = (1),$$

and choose a matrix B satisfying the following condition

$(*)$ The matrix $B = (b_{ij})$ is similar to A, $(b_{12}, 2) = (1)$, $b_{1j} = 0$ for $j \geq 3$ and among all $(1, 2)$ entries of matrices with these properties b_{12} has the least number of distinct prime factors. Among all matrices with these properties B is such that b_{12} has the least number of not necessarily distinct prime factors.

If for some i, j the entry b_{12} does not divide $b_{ii} - b_{jj}$, then b_{12} does not divide $b_{11} - b_{vv}$ for some v. If $v \geq 4$ let $W_v = (w_{ij}^{(v)}) \in \mathrm{GL}_n(R)$ be any permutation matrix such that $w_{11}^{(v)} = w_{22}^{(v)} = 1$, $w_{v3}^{(v)} = 1$ and $w_{3v}^{(v)} = 1$. Then $W_v B W_v^{-1}$ has $(1,2)$ entry equal to b_{12} and $(3,3)$ entry equal to b_{vv}, so we may assume that b_{12} does not divide $b_{11} - b_{22}$ or $b_{11} - b_{33}$. Consider the submatrix

$$B_0 = (b_{ij})_{1 \le i,j \le 3}$$

of B and note that any similarity $B_0 \mapsto g^{-1} B_0 g$ for $g \in \mathrm{GL}_3(R)$ may be achieved by $B \mapsto (g \oplus I_{n-3}) B (g \oplus I_{n-3})^{-1}$. By the minimality property of b_{12} expressed in $(*)$ and the argument in the proof of Proposition 5.3 applied to B_0 we conclude that b_{12} divides both $b_{11} - b_{22}$ and $b_{11} - b_{33}$, which is a contradiction. Thus

$$b_{12} \mid (b_{ii} - b_{jj}) \text{ for all } 1 \le i,j \le n \quad \text{and} \quad b_{12} \mid b_{ij} \text{ for all } i \ne j,\ 1 \le i,j \le 3.$$

Similarly, for any $4 \le v \le n$ the matrix $W_v B W_v^{-1}$ has $(3,1)$ entry equal to b_{v1}, so by $(*)$ and the argument in the proof of Proposition 5.3 applied to B_0 we conclude that $b_{12} \mid b_{v1}$. Hence

$$b_{12} \mid b_{v1} \text{ for all } 4 \le v \le n.$$

Furthermore, by $(*)$ and Lemma 5.2 applied to the second column in B, we see that

$$b_{12} \mid b_{i2} \text{ for all } i \ne 2.$$

Let $1 \le u, v \le n$ be such that $u \ge 3$ and $v \ne u$. For $x \in R$ let

$$X_u = (x_{ij}^{(u)}) = (1 + E_{u2}) B (1 + E_{u2})^{-1},$$

so that $x_{v2}^{(u)} = b_{v2} - b_{vu}$ and in particular $x_{12}^{(u)} = b_{12}$. By $(*)$ and Lemma 5.2 applied to the second column in X_u we see that $b_{12} \mid x_{v2}^{(u)}$ and since $b_{12} \mid b_{v2}$ we conclude that $b_{12} \mid b_{vu}$. Hence

$$b_{12} \mid b_{vu} \text{ for all } u \ge 3,\ v \ne u.$$

We have thus shown that B has the property that $b_{12} \mid b_{ij}$ for all $i \ne j$ and $b_{12} \mid (b_{ii} - b_{jj})$ for all $1 \le i, j \le n$.

For the second statement we follow [12, III, 2]. Conjugating B by $1_2 \oplus M_{3j} \in \mathrm{GL}_n(R)$ for a suitable $M_{3j} \in \mathrm{GL}_{n-2}(R)$ (cf. the proof of Lemma 5.2), we can replace B by a matrix B_1 in which the first row equals that of B and whose $(2,j)$ entries are zero whenever $j \ge 4$. Conjugating B_1 by $1_3 \oplus M_{4j} \in \mathrm{GL}_n(R)$ for a suitable $M_{4j} \in \mathrm{GL}_{n-3}(R)$, we can replace B_1 by a matrix B_2 in which the first two rows equal those of B_1 and whose $(3,j)$ entries are zero whenever $j \ge 5$. Proceeding inductively in this way, we obtain a matrix $C = (c_{ij})$ similar to B such that $c_{12} = b_{12}$ and $c_{ij} = 0$ for i,j such that $j \ge i+2$ and $1 \le i \le n-2$. But since $B \equiv b_{11} 1_n \bmod (b_{12})$ we also have $C \equiv b_{11} 1_n \bmod (b_{12})$, so C has the desired form. □

Using Theorem 5.6 it is now easy to prove the following result. The following proof is entirely analogous to that of Laffey and Reams for $R = \mathbb{Z}$.

Proposition 5.7. *Let $A \in \mathrm{M}_n(R)$, $n \ge 3$ have trace zero, and suppose that for every $\mathfrak{p} \in \operatorname{Specm} R$ and every $a \in R/\mathfrak{p}$, $a \ne 0$ we have $A_{\mathfrak{p}} \ne a 1_n$. Then A is similar to a matrix $B = (b_{ij}) \in \mathrm{M}_n(R)$ where $b_{ii} = 0$ for all $1 \le i \le n$.*

Proof. If $A_{\mathfrak{p}} = 0$ for some $\mathfrak{p}$, we can write $A = mA'$, where $m \in R$ and A' is such that for every $\mathfrak{p} \in \operatorname{Specm} R$ and every $a \in R/\mathfrak{p}$ we have $A'_{\mathfrak{p}} \ne a 1_n$. Since A' must be non-scalar Theorem 5.6 implies that A' is similar to a matrix $A'' = (a''_{ij})$ such that $a''_{12} \mid a''_{ij}$ for all $i \ne j$ and $a''_{12} \mid (a''_{ii} - a''_{jj})$ for all $1 \le i, j \le n$. Since A'' satisfies $A''_{\mathfrak{p}} \ne a 1_n$ for any $\mathfrak{p} \in \operatorname{Specm} R$ and $a \in R/\mathfrak{p}$, the entry a''_{12} must be a unit. We may therefore assume without loss of generality that $A = A''$, so that in particular a_{12} is a unit.

We now prove that A is similar to a matrix with zero diagonal by induction on n. If $n = 2$, the matrix

$$(1 + E_{21}a_{11}a_{12}^{-1})A(1 + E_{21}a_{11}a_{12}^{-1})^{-1}$$

has zero diagonal. If $n > 2$, conjugating A by a matrix of the form $1 + \alpha E_{n1}$, $\alpha \in R$, we may assume that $a_{n2} = 1$, and then conjugating A by a matrix of the form $1 + \beta E_{21}$, $\beta \in R$, we may further assume that $a_{11} = 0$. Thus we may assume that A is of the form

$$\begin{pmatrix} 0 & x \\ y^T & A_1 \end{pmatrix},$$

where $x, y \in R^{n-1}$, $A_1 = (a_{ij}^{(1)}) \in \mathrm{M}_{n-1}(R)$ with $a_{n-1,1}^{(1)} = 1$ and $\mathrm{tr}(A_1) = 0$. By Theorem 5.6 A_1 is similar to a matrix $A_2 = (a_{ij}^{(2)})$ such that $a_{12}^{(2)} \mid a_{ij}^{(2)}$ for all $i \neq j$ and $a_{12}^{(2)} \mid (a_{ii}^{(2)} - a_{jj}^{(2)})$ for all $1 \leq i, j \leq n$. Since $(A_2)_{\mathfrak{p}} \neq a1_n$ for all $\mathfrak{p} \in \operatorname{Specm} R$ and $a \in R/\mathfrak{p}$, the entry $a_{12}^{(2)}$ must be a unit. So by induction there exists a $Q \in \mathrm{GL}_{n-1}(R)$ such that $QA_1Q^{-1} = B_1$ is a matrix with zeros on the diagonal. But thenan

$$B = (1_1 \oplus Q)A(1_1 \oplus Q)^{-1}$$

has the desired form. □

A matrix in $\mathrm{M}_n(R)$ satisfying the conditions on the matrix B in Theorem 5.6 will be said to be in *Laffey-Reams form.*

6. Proof of the main result

In this section we give a proof of our main theorem on commutators, Theorem 6.3. We first prove a couple of lemmas used in the proof.

Lemma 6.1. *Let R be a PID. Then the following holds:*

i) Let $a, b \in R$ be such that $(a, b) = (1)$, and let S be a finite set of maximal ideals of R. Then there exists an $x \in R$ such that for all $\mathfrak{p} \in S$ we have $a + bx \notin \mathfrak{p}$.

ii) Let $\alpha, \beta \in R$ and let S be a finite set of maximal ideals of R such that for all $\mathfrak{p} \in S$ we have $|R/\mathfrak{p}| \geq 3$ and $\alpha \notin \mathfrak{p}$. Then there exists a $t \in R$ such that for all $\mathfrak{p} \in S$ we have $t \notin \mathfrak{p}$ and $\alpha t + \beta \notin \mathfrak{p}$.

Proof. To prove *i)*, take x to be a generator of the product

$$\prod_{\substack{\mathfrak{p} \in S \\ a \notin \mathfrak{p}}} \mathfrak{p}$$

and let $x = 1$ if there is no $\mathfrak{p} \in S$ such that $a \notin \mathfrak{p}$. Suppose that $\mathfrak{p} \in S$ is such that $a \in \mathfrak{p}$. If $a + bx \in \mathfrak{p}$, then $bx \in \mathfrak{p}$ and since $(a, b) = (1)$ we have $x \in \mathfrak{p}$, which contradicts the definition of x. On the other hand, suppose that $\mathfrak{p} \in S$ is such that $a \notin \mathfrak{p}$. If $a + bx \in \mathfrak{p}$, then by the definition of x we have $bx \in \mathfrak{p}$, so $a \in \mathfrak{p}$, which is a contradiction. Thus in either case, $a + bx \notin \mathfrak{p}$.

Next, we prove *ii)*. For any $\mathfrak{p} \in S$, the condition $|R/\mathfrak{p}| \geq 3$ implies that there exist two distinct non-zero elements $t_{\mathfrak{p}}^{(1)}, t_{\mathfrak{p}}^{(2)} \in R/\mathfrak{p}$. If $\alpha_{\mathfrak{p}} t_{\mathfrak{p}}^{(i)} + \beta_{\mathfrak{p}} = 0$ for $i = 1, 2$, then in particular $\alpha_{\mathfrak{p}} = 0$, contradicting the hypothesis $\alpha \notin \mathfrak{p}$. Thus there exists a non-zero element $t_{\mathfrak{p}} \in R/\mathfrak{p}$ such that that $\alpha_{\mathfrak{p}} t_{\mathfrak{p}} + \beta_{\mathfrak{p}} \neq 0$. The surjectivity of the map $R \to \prod_{\mathfrak{p}\in S} R/\mathfrak{p}$ now yields the desired element $t \in R$. $\square$

The following result is the Chinese remainder theorem for centralisers of matrices over quotients of R. It will be used at a crucial step in our proof of Theorem 6.3.

Lemma 6.2. *Let R be a PID, $X \in \mathrm{M}_n(R)$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_\nu$, $\nu \in \mathbb{N}$ be distinct maximal ideals in R. Then the map*

$$\begin{aligned} C_{\mathrm{M}_n(R/(\mathfrak{p}_1\cdots\mathfrak{p}_\nu))}(X_{(\mathfrak{p}_1\cdots\mathfrak{p}_\nu)}) &\longrightarrow \prod_{i=1}^{\nu} C_{\mathrm{M}_n(R/\mathfrak{p}_i)}(X_{\mathfrak{p}_i}) \\ g &\longmapsto (g_{\mathfrak{p}_1}, \dots, g_{\mathfrak{p}_\nu}), \end{aligned}$$

is an isomorphism.

Proof. Let $\mathcal{C} = C_{\mathrm{M}_n(R/(\mathfrak{p}_1\cdots\mathfrak{p}_\nu))}(X_{(\mathfrak{p}_1\cdots\mathfrak{p}_\nu)})$. Then $\mathcal{C}$ is a module over R. By the Chinese remainder theorem we have an isomorphism $R/(\mathfrak{p}_1 \cdots \mathfrak{p}_\nu) \to \prod_{i=1}^{\nu} R/\mathfrak{p}_i$ given by $a \mapsto (a_{\mathfrak{p}_1}, \dots, a_{\mathfrak{p}_\nu})$, and tensoring this by $\mathcal{C}$ yields

$$\begin{aligned} \mathcal{C} &\cong R/(\mathfrak{p}_1 \cdots \mathfrak{p}_\nu) \otimes_R \mathcal{C} \cong \Big(\prod_{i=1}^{\nu} R/\mathfrak{p}_i\Big) \otimes_R \mathcal{C} \cong \prod_{i=1}^{\nu} (R/\mathfrak{p}_i \otimes_R \mathcal{C}) \\ &\cong \prod_{i=1}^{\nu} C_{\mathrm{M}_n(R/\mathfrak{p}_i)}(X_{\mathfrak{p}_i}). \end{aligned}$$

Tracking the maps shows that the effect of the above isomorphisms on elements is given by

$$g \longmapsto 1 \otimes g \longmapsto (1_{\mathfrak{p}_1}, \dots, 1_{\mathfrak{p}_\nu}) \otimes g \longmapsto (1_{\mathfrak{p}_1} \otimes g, \dots, 1_{\mathfrak{p}_\nu} \otimes g) \longmapsto (g_{\mathfrak{p}_1}, \dots, g_{\mathfrak{p}_\nu}).$$

$\square$

We now give the proof of our main theorem. Note that our proof in the case $n = 2$ is different from the case $n \geq 3$, and that for $n = 2$ our argument yields the stronger result that any $A \in \mathrm{M}_2(R)$ with trace zero can be written as $A = [X, Y]$ for some $X, Y \in \mathrm{M}_2(R)$ with $X_{\mathfrak{p}}$ regular for every $\mathfrak{p} \in \operatorname{Specm} R$.

Theorem 6.3. *Let R be a PID and let $A \in \mathrm{M}_n(R)$ be a matrix with trace zero. Then $A = [X, Y]$ for some $X, Y \in \mathrm{M}_n(R)$.*

Proof. For $n = 1$ the result is trivial. First assume that $n = 2$ and let $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$. Choose $x, y \in R$ such that $cx + by = 0$. By dividing by a suitable common factor if necessary, we can ensure that $(x, y) = (1)$. Then the matrix

$$X = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$$

is regular modulo any maximal ideal $\mathfrak{p}$ of R because $(x, y) = (1)$ implies that for any $\mathfrak{p}$ either $x_{\mathfrak{p}}$ or $y_{\mathfrak{p}}$ is non-zero. Since $\mathrm{tr}(XA) = 0$, Proposition 3.3 yields $A = [X, Y]$ for some $Y \in \mathrm{M}_2(R)$.

Assume now that $n \geq 3$. If A is a scalar matrix we obviously have $\mathrm{tr}(J_n(0)^r A) = 0$ for all $r \geq 0$, so Proposition 3.3 yields the desired conclusion. We may therefore henceforth assume that A is non-scalar. Write $A = (a_{ij})$ for $1 \leq i, j \leq n$. By Theorem 5.6 we may assume that A is in Laffey-Reams form. If $d \in R$ is such that $(a_{11}, a_{ij}, a_{ii} - a_{jj} \mid i \neq j, 1 \leq i, j \leq n) = (d)$, then $(a_{ij} \mid 1 \leq i, j \leq n) = (d)$ so we can write $A = dA'$ where $A' = (a'_{ij}) \in \mathrm{M}_n(R)$ is in Laffey-Reams form and

$(a'_{11}, a'_{12}) = (1)$. It thus suffices to assume that $A = A'$ so that $(a_{11}, a_{12}) = (1)$, $a_{12} \mid a_{ij}$ for $i \neq j$, $a_{12} \mid (a_{ii} - a_{jj})$ for $1 \leq i, j \leq n$, and $a_{ij} = 0$ for $j \geq i + 2$. Let $k = \lfloor n/2 \rfloor$. For $x, y, q \in R$ define the matrix $X = (x_{ij}) \in \mathrm{M}_n(R)$ by

$$(x_{ij}) = \begin{cases} x_{ii} = -y & \text{for } i = 2, 4, \dots, 2k, \\ x_{21} = x, & \\ x_{31} = q, & \\ x_{j,j-1} = 1 & \text{for } j = 3, 4, \dots, n, \\ x_{ij} = 0 & \text{otherwise.} \end{cases}$$

Recall that for any $B = (b_{ij}) \in \mathrm{M}_n(R)$ we write $c(B) = \sum_{i=1}^{k} b_{2i,2i}$.We have

$$\mathrm{tr}(XA) = xa_{12} + a_{23} + \cdots + a_{n-1,n} - yc(A).$$

We claim that $\mathrm{tr}(XA) = 0$ implies that $\mathrm{tr}(X^r A) = 0$ for all $r \geq 0$. To see this, observe that the matrix $X^2 + yX$ is lower triangular and its (i, j) entry is 0 if $j \geq i - 1$. Since $\mathrm{tr}(E_{ij}A) = 0$ if $j < i - 1$ (since $a_{ij} = 0$ for $j \geq i + 2$), it follows that $\mathrm{tr}((X^2 + yX)A) = 0$, so if $\mathrm{tr}(XA) = 0$ we get $\mathrm{tr}(X^2 A) = 0$. More generally, using the fact that X is lower triangular, we have $\mathrm{tr}((X^r + yX^{r-1})A) = 0$, and working inductively we get $\mathrm{tr}(X^r A) = 0$ for all $r \geq 0$.

Assume for the moment that $a_{12} \mid c(A)$ and let $M = 1 - c(A)a_{12}^{-1}E_{21} \in \mathrm{M}_n(R)$. Then

$$c(MAM^{-1}) = 0,$$

so Proposition 3.3 together with (4.1) and the fact that P_n is regular imply $MAM^{-1} = [P_n, Y]$, for some $Y \in \mathrm{M}_n(R)$. Thus in this case $A = [M^{-1}P_nM, M^{-1}YM]$, so we may henceforth assume that

$$a_{12} \nmid c(A). \tag{6.1}$$

We now show that there exist elements $x, y \in R$ with $(x, y) = (1)$ and such that $\mathrm{tr}(XA) = 0$. To this end, consider the equation

$$xa_{12} + a_{23} + \cdots + a_{n-1,n} = yc(A), \qquad x, y \in R.$$

Since a_{12} divides $a_{23}, \dots, a_{n-1,n}$, this may be written

$$a_{12}(x + l) = yc(A), \tag{6.2}$$

for some $l \in R$. Let $d \in R$ be a generator of $(a_{12}, c(A))$. Let $\tilde{a}_{12}, \widetilde{c(A)} \in R$ be such that $a_{12} = \tilde{a}_{12}d$ and $c(A) = \widetilde{c(A)}d$, respectively. Then (6.2) is equivalent to

$$\begin{aligned} x &= h\widetilde{c(A)} - l \\ y &= h\tilde{a}_{12}, \end{aligned}$$

for an arbitrary $h \in R$. Choose h to be a generator of the product of all maximal ideals $\mathfrak{p}$ of R such that $\tilde{a}_{12} \in \mathfrak{p}$ and $l \notin \mathfrak{p}$ (and let $h = 1$ if no such $\mathfrak{p}$ exist). Suppose that $x, y \in (p)$ for some prime element $p \in R$. Then $y \in (p)$ and so $\tilde{a}_{12} \in (p)$ or $h \in (p)$. If $\tilde{a}_{12} \in (p)$ and $l \notin (p)$, then $h \in (p)$, so $x \notin (p)$. If $\tilde{a}_{12} \in (p)$ and $l \in (p)$, then $h \notin (p)$ and since $(\tilde{a}_{12}, \widetilde{c(A)}) = (1)$ we have $x \notin (p)$. Furthermore, if $h \in (p)$ then $l \notin (p)$ so $x \notin (p)$. Thus $(x, y) = (1)$. If y is a unit then $\tilde{a}_{12}$ must be a unit, and so $a_{12} \mid c(A)$, contradicting (6.1). Thus y is not a unit, and so $x^2a_{12} \notin (ya_{12})$. Since a_{12} divides each of $a_{11}-a_{22}$, a_{21}, a_{31} and a_{32}, we have $xy(a_{11}-a_{22})-y^2(a_{21}+ya_{31}+xa_{32}) \in (ya_{12})$. Thus, we must have

$$x^2a_{12}+xy(a_{11}-a_{22})-y^2(a_{21}+ya_{31}+xa_{32}) \neq 0. \tag{6.3}$$

From now on let x and y be as above, so that $(x,y)=(1)$ and $\operatorname{tr}(XA)=0$. Next, we specify the entry q in X.

Let S_0 be the set of maximal ideals $\mathfrak{p}$ of R such that $x^2a_{12}+xy(a_{11}-a_{22})-y^2(a_{21}+ya_{31}+xa_{32}) \in \mathfrak{p}$, and let

$$S = S_0 \cup \{\mathfrak{p} \in \operatorname{Specm} R \mid |R/\mathfrak{p}| = 2\}.$$

Note that S is a finite set because of (6.3) together with the fact that for any PID R' (or any Dedekind domain), there are only finitely many $\mathfrak{p} \in \operatorname{Specm} R'$ such that $|R'/\mathfrak{p}|=2$. By Lemma 6.1 *i)* we can thus choose $q \in R$ such that

$$x+qy \notin \mathfrak{p}, \quad \text{for all } \mathfrak{p} \in S.$$

Assume from now on that q has been chosen in this way. Let V be the set of maximal ideals of R such that $x+qy \in \mathfrak{p}$, that is,

$$V = \{\mathfrak{p} \in \operatorname{Specm} R \mid x+qy \in \mathfrak{p}\}.$$

Note that $x+qy \neq 0$ since $(x,y)=(1)$ and y is not a unit (as we saw above, using (6.1)), so V is a finite set. By the choice of q we thus have in particular that

$$\mathfrak{p} \in V \Longrightarrow x^2a_{12}+xy(a_{11}-a_{22})-y^2(a_{21}+ya_{31}+xa_{32}) \notin \mathfrak{p}. \tag{6.4}$$

Note that for every $\mathfrak{p} \in V$ we have $y \notin \mathfrak{p}$ since $(x,y)=(1)$. Note also that $S \cap V = \varnothing$.

We claim that $X_{\mathfrak{p}} \in \mathrm{M}_n(R/\mathfrak{p})$ is regular for every maximal ideal $\mathfrak{p}$ not in V. To show this, let $\mathfrak{p} \in (\operatorname{Specm} R) \setminus V$ and let

$$M = \begin{pmatrix} x+qy & 0 \\ q & 1 \end{pmatrix} \oplus 1_{n-2} \in \mathrm{M}_n(R).$$

Since $x+qy \notin \mathfrak{p}$ the image $M_{\mathfrak{p}} \in \mathrm{M}_n(R/\mathfrak{p})$ of M is invertible and, letting $y_{\mathfrak{p}}$ denote the image of y in $R/\mathfrak{p}$, we have

$$M_{\mathfrak{p}}X_{\mathfrak{p}}M_{\mathfrak{p}}^{-1} = (m_{ij}) = \begin{cases} m_{ii} = -y_{\mathfrak{p}} & \text{for } i = 2,4,\ldots,2k, \\ m_{j,j-1} = 1 & \text{for } j = 2,3,\ldots,n, \\ m_{ij} = 0 & \text{otherwise.} \end{cases}$$

It follows from Lemma 2.7 that $M_{\mathfrak{p}}X_{\mathfrak{p}}M_{\mathfrak{p}}^{-1}$ is regular, and thus $X_{\mathfrak{p}}$ is regular. In particular, if V is empty then $X_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Specm} R$.

Furthermore, since $x+qy \neq 0$ we can consider the matrix M as an element in $\mathrm{GL}_n(F)$, where F is the field of fractions of R. Then MXM^{-1}, and thus X, is regular as an element in $\mathrm{M}_n(F)$, by Lemma 2.7. By our choice of x and y we have $\operatorname{tr}(X^rA)=0$ for $r=0,1,\ldots,n-1$, so Proposition 3.1 implies that we can write $A=[X,Q]$, for some $Q \in \mathrm{M}_n(F)$. Clearing denominators in Q we find that there exists a non-zero element $m_0 \in R$ such that $m_0A \in [X, \mathrm{M}_n(R)]$. We now highlight a step which we will refer to in the following:

(∗) Let $m \in R$ be such that it has the minimal number of (not necessarily distinct) prime factors among all $m' \in R$ such that $m'A \in [X, \mathrm{M}_n(R)]$, and let $Q \in \mathrm{M}_n(R)$ be such that $mA = [X,Q]$.

We show that if a maximal ideal contains m then it lies in V. Suppose that $\mathfrak{p}=(p) \in (\operatorname{Specm} R) \setminus V$ and that $m \in \mathfrak{p}$. Then $0=[X_{\mathfrak{p}}, Q_{\mathfrak{p}}]$, and since $X_{\mathfrak{p}}$ is regular there exists a polynomial $f \in R[T]$ such that $Q = f(X) + pQ'$ for some $Q' \in \mathrm{M}_n(R)$, so $mA = [X, f(X) + pQ'] = [X, pQ']$ and thus $mp^{-1}A = [X, Q']$, which contradicts $(*)$. Thus, if $m \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Specm} R$, then we must have $\mathfrak{p} \in V$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_\nu$, $\nu \in \mathbb{N}$ be the elements of V such that $m \in \mathfrak{p}_i$. For each $\mathfrak{p}_i$, choose a generator $p_i \in R$, so that $\mathfrak{p}_i = (p_i)$, for $i = 1, \dots, \nu$. We then have

$$(m) = (p_1^{e_1} p_2^{e_2} \cdots p_\nu^{e_\nu}),$$

for some $e_i \in \mathbb{N}$, $1 \le i \le \nu$.

The strategy is now to show that X can be replaced by a matrix X_1 which is regular mod $\mathfrak{p}$ for every $\mathfrak{p} \in V$. Let

$$N = 1 + qE_{21} \in \mathrm{M}_n(R).$$

For ease of calculation we will consider the matrices

$$A_0 = NAN^{-1}, \quad X_0 = NXN^{-1}, \quad Q_0 = NQN^{-1}.$$

Let $\mathfrak{p} \in V$ be any of the ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_\nu$. We have

$$(X_0)_\mathfrak{p} = \begin{pmatrix} 0 & 0 \\ 0 & W_\mathfrak{p} \end{pmatrix} = (0) \oplus W_\mathfrak{p}, \tag{6.5}$$

where $W_\mathfrak{p} \in \mathrm{M}_{n-1}(R/\mathfrak{p})$ is regular. We wish to determine the dimension of the centraliser

$$C(\mathfrak{p}) := C_{\mathrm{M}_n(R/\mathfrak{p})}((X_0)_\mathfrak{p}).$$

Since $(x, y) = (1)$, we have $y_\mathfrak{p} \ne 0$, so the Jordan form of $(X_0)_\mathfrak{p}$ is

$$J_k(-y_\mathfrak{p}) \oplus J_{n-k-1}(0) \oplus J_1(0),$$

where $k = \lfloor n/2 \rfloor$, as before. We have an isomorphism of $R/\mathfrak{p}$-vector spaces

$$C(\mathfrak{p}) \cong C_{\mathrm{M}_k(R/\mathfrak{p})}(J_k(-y_\mathfrak{p})) \oplus C_{\mathrm{M}_{n-k}(R/\mathfrak{p})}(J_{n-k-1}(0) \oplus J_1(0)).$$

Since $\dim C_{\mathrm{M}_k(R/\mathfrak{p})}(J_k(-y_\mathfrak{p})) = k$ it remains to determine the dimension of

$$C_{\mathrm{M}_{n-k}(R/\mathfrak{p})}(J_{n-k-1}(0) \oplus J_1(0)).$$

A matrix

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \in \mathrm{M}_{n-k}(R/\mathfrak{p}),$$

where H_{11} is a $(n-k-1) \times (n-k-1)$ block, H_{22} is a 1×1 block, and the other blocks are of compatible sizes, commutes with $J_{n-k-1}(0) \oplus J_1(0)$ if and only if

$$H_{11} J_{n-k-1}(0) = J_{n-k-1}(0) H_{11}, \quad H_{12} \in \begin{pmatrix} R/\mathfrak{p} \\ 0 \end{pmatrix}, \quad H_{21} \in (0, R/\mathfrak{p}).$$

Hence $\dim C_{\mathrm{M}_{n-k}(R/\mathfrak{p})}(J_{n-k-1}(0) \oplus J_1(0)) = n - k - 1 + 1 + 1 + 1$, and so

$$\dim C(\mathfrak{p}) = n + 2,$$

that is, $(X_0)_\mathfrak{p}$ is subregular (cf. [17]). Next, we need the dimension of $(R/\mathfrak{p})[(X_0)_\mathfrak{p}]$ (the algebra of polynomials in $(X_0)_\mathfrak{p}$ over the field $R/\mathfrak{p}$). Since $(R/\mathfrak{p})[(X_0)_\mathfrak{p}] \cong (0) \oplus (R/\mathfrak{p})[W_\mathfrak{p}]$ and $W_\mathfrak{p}$ is regular, we have $\dim(R/\mathfrak{p})[(X_0)_\mathfrak{p}] = n - 1$.

We now find a basis for $C(\mathfrak{p})$. We know that $(R/\mathfrak{p})[(X_0)_\mathfrak{p}]$ is an $(n-1)$-dimensional subspace of $C(\mathfrak{p})$. Moreover, direct verification shows that E_{11} and $E_{12} + y_\mathfrak{p} E_{13}$ are in $C(\mathfrak{p})$. Let $\kappa = n + 1 - 2\lfloor (n+1)/2 \rfloor$, that is, κ is 0 if n is odd and 1 if n is even. Then we also have

$$E_{n1} + \kappa y_\mathfrak{p} E_{n-1,1} \in C(\mathfrak{p}).$$

Since $(X_0)_\mathfrak{p}$ is lower triangular and the first column of $(X_0)_\mathfrak{p}^i$ is 0 for all $i \in \mathbb{N}$, the intersection of $(R/\mathfrak{p})[(X_0)_\mathfrak{p}]$ with the $R/\mathfrak{p}$-span $\langle E_{11}, E_{12} + y_\mathfrak{p} E_{13}, E_{n1} + \kappa y_\mathfrak{p} E_{n-1,1}\rangle$ is 0. Since $\{E_{11}, E_{12}+y_\mathfrak{p} E_{13}, E_{n1}+\kappa y_\mathfrak{p} E_{n-1,1}\}$ is linearly independent, $\dim(R/\mathfrak{p})[(X_0)_\mathfrak{p}] = n-1$ and $\dim C(\mathfrak{p}) = n+2$, we must have

$$C(\mathfrak{p}) = \langle (R/\mathfrak{p})[(X_0)_\mathfrak{p}], E_{11}, E_{12} + y_\mathfrak{p} E_{13}, E_{n1} + \kappa y_\mathfrak{p} E_{n-1,1}\rangle. \tag{6.6}$$

We observe that the matrix $E_{n1} + \kappa y E_{n-1,1} \in \mathrm{M}_n(R)$, whose image in $\mathrm{M}_n(R/\mathfrak{p})$ is $E_{n1} + \kappa y_\mathfrak{p} E_{n-1,1}$, satisfies

$$E_{n1} + \kappa y E_{n-1,1} \in C_{\mathrm{M}_n(R)}(X_0). \tag{6.7}$$

Let $\mathfrak{a} = \prod_{i=1}^{\nu} \mathfrak{p}_i$, so that $\mathfrak{a} = (p_1 \cdots p_\nu)$. By (6.6) and Lemma 6.2 we have

$$C_{\mathrm{M}_n(R/\mathfrak{a})}(X_\mathfrak{a}) = \langle (R/\mathfrak{a})[(X_0)_\mathfrak{a}], E_{11}, E_{12} + y_\mathfrak{a} E_{13}, E_{n1} + \kappa y_\mathfrak{a} E_{n-1,1}\rangle. \tag{6.8}$$

Since $[X_0, Q_0] = mA_0$ we have $([X_0, Q_0])_\mathfrak{a} = 0$, that is, $(Q_0)_\mathfrak{a} \in C_{\mathrm{M}_n(R/\mathfrak{a})}((X_0)_\mathfrak{a})$. Hence, by (6.8)

$$Q_0 = f(X_0) + \alpha E_{11} + \beta(E_{12} + yE_{13}) + \gamma(E_{n1} + \kappa y E_{n-1,1}) + p_1 \cdots p_\nu D,$$

for some $\alpha, \beta, \gamma \in R$, $f(T) \in R[T]$ and $D \in \mathrm{M}_n(R)$. Using (6.7) we get

$$\begin{aligned} [X_0, Q_0] &= [X_0, f(X_0) + \alpha E_{11} + \beta(E_{12} + yE_{13}) \\ &\qquad + \gamma(E_{n1} + \kappa y E_{n-1,1}) + p_1 \cdots p_\nu D] \\ &= [X_0, \alpha E_{11} + \beta(E_{12} + yE_{13}) + p_1 \cdots p_\nu D] \\ &= [X_0, Q_1], \end{aligned} \tag{6.9}$$

where

$$Q_1 := \alpha E_{11} + \beta(E_{12} + yE_{13}) + p_1 \cdots p_\nu D.$$

Let $i \in \mathbb{N}$ be such that $1 \leq i \leq \nu$. If $(\alpha, \beta) \subseteq \mathfrak{p}_i$ then $[X_0, Q_1] \in p_i[X_0, \mathrm{M}_n(R)]$ and so $mp_i^{-1}A \in [X, \mathrm{M}_n(R)]$, contradicting $(*)$. Thus either $\alpha \notin \mathfrak{p}_i$ or $\beta \notin \mathfrak{p}_i$. We show that the case where $\alpha \in \mathfrak{p}_i$ and $\beta \notin \mathfrak{p}_i$ cannot arise. Since $mA_0 = [X_0, Q_0] = [X_0, Q_1]$, we have $m \cdot \mathrm{tr}(Q_1 A_0) = 0$, whence $\mathrm{tr}(Q_1 A_0) = 0$. Together with $\alpha \in \mathfrak{p}_i$ and $\beta \notin \mathfrak{p}_i$ this implies that

$$\mathrm{tr}((E_{12} + yE_{13})A_0) \in \mathfrak{p}_i.$$

Recalling that $A_0 = NAN^{-1}$ we thus get

$$-q^2 a_{12} + q(a_{11} - a_{22}) + a_{21} + ya_{31} - qya_{32} \in \mathfrak{p}_i$$

and, after multiplying by y^2,

$$-q^2y^2 a_{12} + qy^2(a_{11} - a_{22}) + y^2(a_{21} + ya_{31} - qya_{32}) \in \mathfrak{p}_i.$$

Since $\mathfrak{p}_i \in V$ we have $qy \in -x + \mathfrak{p}_i$ and so

$$x^2 a_{12} + xy(a_{11} - a_{22}) - y^2(a_{21} + ya_{31} + xa_{32}) \in \mathfrak{p}_i.$$

But by our choice of q we have

$$x^2 a_{12} + xy(a_{11} - a_{22}) - y^2(a_{21} + ya_{31} + xa_{32}) \notin \mathfrak{p},$$

for all $\mathfrak{p} \in V$, which together with (6.4) yields a contradiction. Therefore we cannot have $\alpha \in \mathfrak{p}_i$ and $\beta \notin \mathfrak{p}_i$, so we must have $\alpha \notin \mathfrak{p}_i$. We have thus shown that

$$\alpha \notin \mathfrak{p}_i, \quad \text{for all } i = 1, \ldots, \nu.$$

By Lemma 6.1 *ii)* and our choice of S there exists a $t \in R$ such that

$$t \notin \mathfrak{p}_i \quad \text{and} \quad \alpha t + y \notin \mathfrak{p}_i, \quad \text{for all } i = 1, \ldots, \nu. \tag{6.10}$$

Define the matrix

$$X_1 = X_0 + tQ_1.$$

Let $\mathfrak{p}$ be any of the ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_\nu$. Let $\alpha_\mathfrak{p}, \beta_\mathfrak{p}, t_\mathfrak{p}$ denote the images of α, β and t in $R/\mathfrak{p}$, respectively. As before, let $y_\mathfrak{p}$ denote the image of y in $R/\mathfrak{p}$. If we let

$$L_\mathfrak{p} = \begin{pmatrix} 1 & \beta_\mathfrak{p}\alpha_\mathfrak{p}^{-1} & y_\mathfrak{p}\beta_\mathfrak{p}\alpha_\mathfrak{p}^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus 1_{n-3} \in \mathrm{M}_n(R/\mathfrak{p}),$$

then direct verification shows that $L_\mathfrak{p}(X_1)_\mathfrak{p}L_\mathfrak{p}^{-1} = \alpha_\mathfrak{p}t_\mathfrak{p}E_{11} \oplus W_\mathfrak{p}$, where $W_\mathfrak{p}$ is the matrix in (6.5). Since $W_\mathfrak{p}$ is regular and neither of its eigenvalues 0 or $-y_\mathfrak{p}$ equals $\alpha_\mathfrak{p}t_\mathfrak{p}$ by (6.10), the matrix $\alpha_\mathfrak{p}t_\mathfrak{p}E_{11} \oplus W_\mathfrak{p}$, and hence $(X_1)_\mathfrak{p} \in \mathrm{M}_n(R/\mathfrak{p})$, is regular. We thus see that $(X_1)_{\mathfrak{p}_i}$ is regular for all $i = 1, \ldots, \nu$.

By (6.9) we have

$$mA_0 = [X_0, Q_0] = [X_0, Q_1] = [X_1, Q_1],$$

and since $(X_1)_{\mathfrak{p}_i}$ is regular and $m \in \mathfrak{p}_i$ for all $i = 1, \ldots, \nu$, we get $Q_1 = g_i(X_1) + p_iQ_1^{(i)}$, for some $g_i(T) \in R[T]$ and $Q_1^{(i)} \in \mathrm{M}_n(R)$. Thus

$$mA_0 = [X_1, g_i(X_1) + p_iQ_1^{(i)}] = p_i[X_1, Q_1^{(i)}],$$

and so $mp_i^{-1}A_0 = [X_1, Q_1^{(i)}]$. Repeating the argument if necessary, we obtain $mp_i^{-e_i}A_0 \in [X_1, \mathrm{M}_n(R)]$. Running through each $i = 1, \ldots, \nu$ we obtain $A_0 = [X_1, Y]$ for some $Y \in \mathrm{M}_n(R)$, and hence $A = [N^{-1}X_1N, NYN^{-1}]$. □

By a theorem of Hungerford [4] every principal ideal ring (PIR) is a finite product of rings, each of which is a homomorphic image of a PID. Together with Theorem 6.3 this immediately implies the following:

Corollary 6.4. *Let R be a PIR (not necessarily an integral domain) and let $A \in \mathrm{M}_n(R)$, $n \geq 2$, be a matrix with trace zero. Then $A = [X, Y]$ for some $X, Y \in \mathrm{M}_n(R)$.*

We end this section by proving a strengthened version of Theorem 6.3 for $n = 3$.

Proposition 6.5. *Let R be a PID and let $A \in \mathrm{M}_3(R)$ be a matrix with trace zero. Then $A = [X, Y]$ for some $X, Y \in \mathrm{M}_3(R)$ such that $X_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Specm} R$.*

Proof. As in the proof of Theorem 6.3 we may assume that A is in Laffey-Reams form. Define the matrix

$$X = \begin{pmatrix} 0 & 0 & 0 \\ x & -y & 0 \\ q & z & 0 \end{pmatrix} \in \mathrm{M}_3(R).$$

The same argument as in the proof of Theorem 6.3 shows that $\operatorname{tr}(XA) = 0$ implies that $\operatorname{tr}(X^rA) = 0$ for all $r \geq 0$. Let $a'_{23} \in R$ be such that $a_{23} = a_{12}a'_{23}$, and let $d \in R$ be a generator of $(a_{12}, c(A))$. Let $\tilde{a}_{12}, \widetilde{c(A)} \in R$ be such that $a_{12} = \tilde{a}_{12}d$ and $c(A) = \widetilde{c(A)}d$, respectively. The condition $\operatorname{tr}(XA) = 0$ is then equivalent to

$$\begin{aligned} x &= h\widetilde{c(A)} - a'_{23}z \\ y &= h\tilde{a}_{12}, \end{aligned}$$

for an arbitrary $h \in R$. We claim that the system of equations

$$\begin{cases} x = h\widetilde{c(A)} - a'_{23}z \\ y = h\tilde{a}_{12} \\ xz + qy = 1 \end{cases} \tag{6.11}$$

has a solution in $x, y, q, z, h \in R$. Indeed, substituting the first two equations in the last, we get

$$-a'_{23}z^2 + h(\widetilde{c(A)}z + q\tilde{a}_{12}) = 1,$$

and since $(\widetilde{c(A)}, \tilde{a}_{12}) = (1)$ we can choose z and q in R such that $\widetilde{c(A)}z + q\tilde{a}_{12} = 1$, and it then remains to take

$$h = 1 + a'_{23}z^2.$$

Suppose now that $x, y, q, z, h \in R$ is a solution of (6.11), and let $\mathfrak{p} \in \operatorname{Specm} R$. We show that $X_{\mathfrak{p}}$ is regular. The characteristic polynomial of X is

$$\lambda^2(\lambda + y) \in R[\lambda].$$

We have

$$X^2 = \begin{pmatrix} 0 & 0 & 0 \\ -xy & y^2 & 0 \\ xz & -yz & 0 \end{pmatrix}.$$

Thus, if $y \notin \mathfrak{p}$ then $(X_{\mathfrak{p}})^2 \neq 0$, and if $y \in \mathfrak{p}$, then we must have $xz \notin \mathfrak{p}$, so $(X_{\mathfrak{p}})^2 \neq 0$ also in this case. Furthermore, since $xz + qy = 1$ we have

$$X(X + y) = E_{31} \neq 0.$$

Thus the minimal polynomial of $X_{\mathfrak{p}}$ must equal the characteristic polynomial, so $X_{\mathfrak{p}}$ is regular. Since we have $\operatorname{tr}(X^r A) = 0$ for all $r \geq 0$, Proposition 3.3 implies that $A = [X, Y]$, for some $Y \in \mathrm{M}_3(R)$. □

We remark that while the matrix X in the proof of the above proposition is regular modulo every $\mathfrak{p} \in \operatorname{Specm} R$, it is not necessarily regular. Moreover, while for $n = 4$ we can find an analogous matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & -y & 0 & 0 \\ q & z & 0 & 0 \\ 0 & 0 & 1 & -y \end{pmatrix}$$

such that $\operatorname{tr}(AX) = 0$ and $xz + yq = 1$, in this case the matrix $X_{\mathfrak{p}}$ fails to be regular for any $\mathfrak{p} \in \operatorname{Spec} R$ such that $z \in \mathfrak{p}$.

7. Further directions

If R is a field we have shown that every $A \in \mathrm{M}_n(R)$ with trace zero can be written $A = [X, Y]$ where $X, Y \in \mathrm{M}_n(R)$ and X is regular. Our proof of Theorem 6.3 shows that for any PID R, $n \geq 2$ and every $A \in \mathrm{M}_n(R)$ with trace zero we have $A = [X, Y]$ for some $X, Y \in \mathrm{M}_n(R)$ where $X_{\mathfrak{p}}$ is regular for all but finitely many maximal ideals $\mathfrak{p}$ of R. Moreover, our proof of Theorem 6.3 for $n = 2$ together with Proposition 6.5 say that when $n \leq 3$ the matrix X can be chosen such that $X_{\mathfrak{p}}$ is regular for all maximal ideals $\mathfrak{p}$.

Problem. For $n \geq 4$ and $A = [X, Y]$, is it always possible to choose X such that $X_{\mathfrak{p}}$ is regular for all maximal ideals $\mathfrak{p}$?

This problem is interesting insofar as a proof, if possible, would be likely to yield a substantially simplified proof of Theorem 6.3.

It is natural to ask for generalisations of Theorem 6.3 to rings other than PIRs. We first mention some counter-examples. It was shown by Lissner [9] that the analogue of Theorem 6.3 fails when $n = 2$ and $R = k[x, y, z]$, where k is a field, and more generally that for $R = k[x_1, \ldots, x_{2n-1}]$ there exist matrices in $\mathrm{M}_n(R)$ with trace zero which are not commutators (see [9, Theorem 5.4]). Rosset and Rosset [14, Lemma 1.1] gave a sufficient criterion for a 2×2 trace zero matrix over any commutative ring not to be a commutator. They showed however, that a Noetherian integral domain cannot satisfy their criterion unless it has dimension at least 3. This means that their criterion is not an obstruction to a 2×2 trace zero matrix over a one or two-dimensional Noetherian domain being a commutator. Still, if R is the two-dimensional domain $\mathbb{R}[x, y, z]/(x^2+y^2+z^2-1)$ it can be shown that there exists a matrix in $\mathrm{M}_2(R)$ with trace zero which is not a commutator (this example goes back to Kaplansky; see [19, Section 4, Example 1], [10, p. 532] or [14, Section 3]).

A ring R is called an *OP-ring* if for every $n \geq 1$ every vector in $\bigwedge^{n-1} R^n$ is decomposable, that is, of the form $v_1 \wedge \cdots \wedge v_{n-1}$ for some $v_i \in R^n$. This is equivalent to saying that every vector in R^n is an outer product (hence the acronym OP). The notion of OP-ring was introduced in [10]. In particular, for $n = 3$ the condition on R of being an OP-ring is equivalent to the condition that every trace zero matrix in $\mathrm{M}_2(R)$ is a commutator (see [9, Section 3]). It is known that every Dedekind domain is an OP-ring [10, p. 534] and that every polynomial ring in one variable over a Dedekind domain is an OP-ring [20, Theorem 1.2]. This prompts the following problem:

Problem. Let R be a Dedekind domain and assume that $A \in \mathrm{M}_n(R)$, $n \geq 2$, has trace zero. Is it true that $A = [X, Y]$ for some $X, Y \in \mathrm{M}_n(R)$?

Since Dedekind domains are OP-rings the question has an affirmative answer for $n = 2$, and one could ask the same question for any OP-ring. In the setting of matrices over a Dedekind domain the methods we have used to prove Theorem 6.3 are of little use because they rely crucially on the underlying ring being both atomic and Bézout, which implies that it is a PID.

References

1. A. A. Albert and Benjamin Muckenhoupt, *On matrices of trace zero*, Michigan Math. J. **4** (1957), 1–3.
2. F. Digne and J. Michel, *Representations of finite groups of Lie type*, Cambridge University Press, Cambridge, 1991.
3. Dennis R. Estes and Robert M. Guralnick, *Representations under ring extensions: Latimer-MacDuffee and Taussky correspondences*, Adv. in Math. **54** (1984), no. 3, 302–313.
4. Thomas W. Hungerford, *On the structure of principal ideal rings*, Pacific J. Math. **25** (1968), 543–547.
5. Dinesh Khurana and T.Y. Lam, *Generalized commutators in matrix rings*, Linear and Multilinear Algebra **60** (2012), no. 7, 797–827.
6. T. J. Laffey, *Lectures on integer matrices*, unpublished lecture notes (1997).
7. Thomas J. Laffey and Robert Reams, *Integral similarity and commutators of integral matrices*, Linear Algebra Appl. **197/198** (1994), 671–689.
8. S. Lang, *Algebra*, GTM 211, Springer-Verlag, New York, 2002.
9. David Lissner, *Matrices over polynomial rings*, Trans. Amer. Math. Soc. **98** (1961), 285–305.
10. ———, *Outer product rings*, Trans. Amer. Math. Soc. **116** (1965), 526–535.

11. Zachary Mesyan, *Commutator rings*, Bull. Austral. Math. Soc. **74** (2006), no. 2, 279–288.
12. Morris Newman, *Integral matrices*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 45.
13. Jonathan Rosenberg, *Algebraic K-theory and its applications*, GTM, vol. 147, Springer-Verlag, New York, 1994.
14. Myriam Rosset and Shmuel Rosset, *Elements of trace zero that are not commutators*, Comm. Algebra **28** (2000), no. 6, 3059–3072.
15. Kenjiro Shoda, *Einige Sätze über Matrizen*, Jap. J. Math. **13** (1937), 361–365.
16. Louis Solomon, *Similarity of the companion matrix and its transpose*, Linear Algebra Appl. **302/303** (1999), 555–561, With an appendix by R. Guralnick.
17. T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups, LNM, Vol. 131, Springer, Berlin, 1970, pp. 167–266.
18. Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 49–80.
19. Richard G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277.
20. Jacob Towber, *Complete reducibility in exterior algebras over free modules*, J. Algebra **10** (1968), 299–309.
21. L. N. Vaserstein, *Noncommutative number theory*, Algebraic K-theory and Algebraic Number Theory, Contemp. Math., vol. 83, AMS, Providence, RI, 1989, pp. 445–449.
22. L. N. Vaserstein and E. Wheland, *Commutators and companion matrices over rings of stable rank* 1, Linear Algebra Appl. **142** (1990), 263–277.

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