HARMONIC FUNCTIONS ON RANK ONE ASYMPTOTICALLY HARMONIC MANIFOLDS

GERHARD KNIEPER AND NORBERT PEYERIMHOFF

Abstract. Asymptotically harmonic manifolds are simply connected complete Riemannian manifolds without conjugate points such that all horospheres have the same constant mean curvature h. In this article we present results for harmonic functions on rank one asymptotically harmonic manifolds X with mild curvature boundedness conditions. Our main results are (a) the explicit calculation of the Radon-Nykodym derivative of the visibility measures, (b) an explicit integral representation for the solution of the Dirichlet problem at infinity in terms of these visibility measures, and (c) a result on horospherical means of bounded eigenfunctions implying that these eigenfunctions do not admit non-trivial continuous extensions to the geometric compactification $\overline{X}$.

Contents

1. Introduction

Manifolds with asymptotically harmonic metrics were first introduced by Ledrappier ([Led, Thm. 1]) in the special case of negative curvature in connection with rigidity of measures related to the Dirichlet problem (harmonic measure) and the dynamics of the geodesic flow (Bowen-Margulis measure). One of the equivalent characterisations of asymptotically harmonic metrics there was that all horospheres have constant mean curvature $h \geq 0$. We express this geometric property in terms of Jacobi tensors (see Definion

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1.1 below). Let (X,g) be a complete Riemannian manifold without conjugate points and let $\pi : SX \to X$ be the canonical footpoint projection from the unit tangent bundle. For $v \in SX$ let $c_v : \mathbb{R} \to X$ be the unique geodesic given by $c_v'(0) = v$. Let $S_{v,r}$ and $U_{v,r}$ be the orthogonal Jacobi tensors along c_v, defined by $S_{v,r}(0) = U_{v,r}(0) = \mathrm{id}$ and $S_{v,r}(r) = 0$ and $U_{v,r}(-r) = 0$. Note that we have $U_{v,r}(t) = S_{-v,r}(-t)$. The stable and unstable Jacobi tensors S_v and U_v are then defined as the Jacobi tensors along c_v with initial conditions $S_v(0) = U_v(0) = \mathrm{id}$ and $S_v'(0) = \lim_{r\to\infty} S_{v,r}'(0)$ and $U_v'(0) = \lim_{r\to\infty} U_{v,r}'(0)$. They are related by $U_v(t) = S_{-v}(-t)$. For simplicity of notation, we introduce $U(v) = U_v'(0)$ and $S(v) = S_v'(0)$. (For more detailed information on Jacobi tensors see, e.g., [Kn1].)

Definition 1.1. *An* asymptotically harmonic manifold (X,g) *is a complete, simply connected Riemannian manifold without conjugate points such that for all $v \in SX$ we have $\operatorname{tr} U(v) = h$ for a constant $h \geq 0$.*

The manifolds considered in this article are *rank one* asymptotically harmonic manifolds. The notion of rank has been introduced by Ballmann, Brin and Eberlein in [BBE] for nonpositively curved manifolds as the dimension of the parallel Jacobi fields along geodesics. Since we do not assume nonpositive curvature, the notion of rank has to be understood in the following generalized sense given in [Kn2, Def. 3.1]:

Definition 1.2. *Let (X,g) be a complete simply connected Riemannian manifold without conjugate points. For $v \in SX$ let $D(v) = U(v) - S(v)$ and we define*
$$\mathrm{rank}(v) = \dim(\ker D(v)) + 1$$
and
$$\mathrm{rank}(X) = \min\{\mathrm{rank}(v) \mid v \in SX\}.$$

In [KnPe2], we proved equivalence of the following four properties for asymptotically harmonic manifolds X under the mild curvature boundedness condition

$$\|R\| \leq R_0 \quad \text{and} \quad \|\nabla R\| \leq R_0' \tag{1.1}$$

for some constants $R_0, R_0' > 0$: (a) X has rank one, (b) X has Anosov geodesic flow, (c) X is Gromov hyperbolic, and (d) X has purely exponential volume growth with growth rate $h_{vol} = h$. These equivalences were first proved for noncompact harmonic manifolds in [Kn2] and for asymptotically harmonic manifolds admitting compact quotients in [Zi1]. Besides negatively curved symmetric spaces, Damek-Ricci spaces provide examples of rank one harmonic and therefore also asymptotically harmonic manifolds, since they all have purely exponential volume growth. As a consequence, all Damek-Ricci spaces are Gromov hyperbolic. (Note that all non-symmetric Damek-Ricci spaces admit zero-curvature.) In this article, we use the above equivalences to study harmonic functions on rank one asymptotically harmonic manifolds (X,g) satisfying (1.1). Let us discuss the results of this paper in more detail.

In Sections 2 and 3, we introduce the geometric boundary $X(\infty)$ via equivalence classes of geodesic rays and the canonical maps $\varphi_p : S_pX \to X(\infty)$, $\varphi_p(v) = c_v(\infty)$. These maps have natural extensions $\bar\varphi_p$ to the geometric compactification $\overline{X} = X \cup X(\infty)$, and we show that these extensions are homeomorphisms. The visibility measures $\{\mu_p\}$ on $X(\infty)$ are then defined as follows:

Definition 1.3. *Let $\mathcal{M}_1(X(\infty))$ denote the space of Borel probability measures on $X(\infty)$. For every $p \in X$, let $\mu_p \in \mathcal{M}_1(X(\infty))$ be defined by*

$$\int_{X(\infty)} f(\xi)\, d\mu_p(\xi) = \frac{1}{\omega_n} \int_{S_pX} f(\varphi_p(v))\, d\theta_p(v) \quad \forall f \in C(X(\infty)),$$

where $n = \dim(X)$ and ω_n is the volume of the $(n-1)$-dimensional standard unit sphere and $d\theta_p$ is the volume element of S_pX induced by the Riemannian metric. μ_p is called the visibility measure *of (X,g) at the point p.*

Sections 4 and 5 are concerned with the explicit calculation of the Radon-Nykodym derivative of the visibility measures. To state the result (Theorem 1.4 below), we need Busemann functions. Let $v \in S_qX$ and $\xi = c_v(\infty) \in X(\infty)$. Then the Busemann function (associated to $v \in S_qX$ or to $(q,\xi) \in X \times X(\infty)$) is defined as

$$b_v(p) = b_{q,\xi}(p) = \lim_{t\to\infty} d(c_v(t),p) - t. \tag{1.2}$$

Theorem 1.4. *Let (X,g) be a rank one asymptotically harmonic manifold satisfying* (1.1)*. Let $(\mu_p)_{p\in X}$ be the associated family of visibility measures. Then these measures are pairwise absolutely continuous and we have*

$$\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb_{q,\xi}(p)}.$$

An analogous result on the Radon-Nykodym derivative for asymptotically harmonic manifolds in the case of pinched negative curvature was given in [CaSam, Prop. 6.1].

Since our rank one asymptotically harmonic manifolds (X,g) are Gromov hyperbolic and have positive Cheeger constants (see [KnPe2, Prop. 5.3]), the general theory of Ancona [Anc1, Anc2] implies that the geometric boundary and the Martin boundary agree and that the Dirichlet problem at infinity can be solved. In Section 6 we give an alternative direct proof of this latter fact and give an explicit integral representation for the solution of the Dirichlet problem at infinity in terms of the visibility measures:

Theorem 1.5. *Let (X,g) be a rank one asymptotically harmonic manifold satisfying* (1.1)*. Let $f : X(\infty) \to \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function $H_f : X \to \mathbb{R}$ such that*

$$\lim_{x\to\xi} H_f(x) = f(\xi). \tag{1.3}$$

Moreover, H_f has the following integral presentation:

$$H_f(x) = \int\limits_{X(\infty)} f(\xi) d\mu_x(\xi),$$

where $\{\mu_x\}_{x\in X} \subset \mathcal{M}_1(X(\infty))$ are the visibility probability measures.

A related result in the setting of harmonic manifolds can be found in Zimmer [Zi2, Thm. 1]. Moreover, the solution of the Dirichlet problem at infinity for general nonpositively curved rank one manifolds admitting compact quotients was shown by Ballmann [Ba].

In Section 8 we consider eigenfunctions $\Delta f + \lambda f = 0$, $\lambda \in \mathbb{R}\backslash\{0\}$ on rank one asymptotically harmonic manifolds X satisfying (1.1). We show that if such an eigenfunction $f \in C^\infty(X, \mathbb{C})$ has a continuous extension to the boundary $X(\infty)$, then the extension must be necessarily trivial, in contrast to Theorem 1.5 for harmonic functions. The proof is based on taking horospherical means. Since horospheres $\mathcal{H}$ are noncompact, the averages have to be taken via *compact exhaustions* $\{K_j\}$ with smooth boundaries ∂K_j. We first observe in Section 8 (see Theorem 8.1) that, for continuous functions $f : \overline{X} = X \cup X(\infty) \to \mathbb{R}$ and horospheres $\mathcal{H}$ centered at $\xi \in X(\infty)$ with compact exhaustion $\{K_j\}$, we have

$$\lim_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\mathrm{vol}_{n-1}(K_j)} = f(\xi). \tag{1.4}$$

The expression (8.1) is called *the horospherical mean of f* with respect to the exhaustion $\{K_j\}$. In Section 7, we prove that all horospheres in these spaces have polynomial volume growth, which implies that they admit *(compact) isoperimetric exhaustions* $\{K_j\}$, that is,

$$\frac{\mathrm{vol}_{n-2}(\partial K_j)}{\mathrm{vol}_{n-1}(K_j)} \to 0 \quad \text{as } j \to \infty. \tag{1.5}$$

The main result of Section 8 is that, for all $\lambda \in \mathbb{R}\backslash\{0\}$, the horospherical means (with respect to isoperimetric exhaustions) of bounded eigenfunctions are zero.

Theorem 1.6. *Let (X, g) be a rank one asymptotically harmonic manifold of dimension n satisfying (1.1) and $h > 0$ be the mean curvature of all horospheres. Let $\lambda \neq 0$ be a real number and $f \in C^\infty(X)$ be a bounded function satisfying $\Delta f + \lambda f = 0$ and $\mathcal{H} \subset X$ be a horosphere with* isoperimetric exhaustion *$\{K_j\}$. Then we have*

$$\lim_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\mathrm{vol}_{n-1}(K_j)} = 0. \tag{1.6}$$

This result leads to the following above mentioned fact, complementing Theorem 1.5.

Theorem 1.7. *Let (X, g) be a rank one asymptotically harmonic manifold satisfying (1.1). Let $\lambda \in \mathbb{R}\backslash\{0\}$ and $f \in C^\infty(X)$ be an eigenfunction $\Delta f + \lambda f = 0$. If f has a continuous extension $F \in C(\overline{X})$ then we have necessarily $F|_{X(\infty)} \equiv 0$.*

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2. Uniform divergence of geodesics

In this section, we prove that for every distance $d > 0$ and any angle $\alpha > 0$ there exists a $t_0 > 0$, such that any two unit speed geodesics c_1, c_2 starting at the same point and differing by an angle $\geq \alpha$ will diverge uniformly in the sense that $d(c_1(t), c_2(t)) \geq d$ for all $t \geq t_0$. For the proof, we start with the following lemma.

Lemma 2.1. *Let (X, g) be a manifold without conjugate points and, for $v \in SX$, let A_v be the orthogonal Jacobi tensor along c_v satisying $A_v(0) = 0$ and $A_v'(0) = \mathrm{id}$. Then we have*

(i) $A_v(t) = U_v(t) \int_0^t (U_v^* U_v)^{-1}(u) du,$

(ii) $(U_v'(0) - S_{v,t}'(0))^{-1} = \int_0^t (U_v^* U_v)^{-1}(u) du.$

Proof. Since the endomorphism $U_v(u)$ is non-singular and Lagrangian for all $u \in \mathbb{R}$, we conclude from [Kn2, Prop. 2.1] that

$$A_v(t) = U_v(t) \left(\int_0^t (U_v^* U_v)^{-1}(u) du \, C_1 + C_2 \right)$$

with suitable constant tensors C_1 and C_2. Evaluating and differentiating this identity at $t = 0$ yields $C_2 = 0$ and $C_1 = \mathrm{id}$, finishing the proof of (i). The statement (ii) can be found in [Kn2, Lemma 2.3]. □

Proposition 2.2. *Let (X, g) be a rank one asymptotically harmonic manifold satisfying* (1.1). *Then there exist constants $a, \rho > 0$ such that*

$$\|A_v(t)\, x\| \geq a e^{\frac{\rho}{2} t} \|x\|$$

for all $v \in SX$, $x \in v^\perp \subset TX$ and $t \geq 1$.

Proof. We conclude from [KnPe2, Thm. 1.3] that there exists $\rho > 0$ such that $D(v) = U(v) - S(v) \geq \rho \cdot \mathrm{id}$. Using [KnPe2, Prop. 2.5] and the fact that $S_v(t)$ is non-singular for $t \geq 0$, we conclude that there exists $a_2 > 0$ such that $\|S_v^{-1}(t) y\| \geq \frac{1}{a_2} e^{\frac{\rho}{2} t} \|y\|$ for all $y \in (\Phi^t v)^\perp$, where $\Phi^t : SX \to SX$ denotes the geodesic flow. Using $S_{\Phi^t w}(u) y_u = S_w(u + t)(S_w^{-1}(t) y)_u$ (where y_u is the parallel translation of $y \in (\Phi^t w)^\perp$ along c_v) with $u = -t$ and $-v = \Phi^t w$ yields

$$\|U_v(t) y\| = \|S_{-v}(-t) y\| = \|S^{-1}_{-\Phi^t v}(t) y_t\| \geq \frac{1}{a_2} e^{\frac{\rho}{2} t} \|y\|.$$

Lemma 2.1 yields

$$\|A_v(t)(U_v'(0) - S_{v,t}'(0)) y\| = \|U_v(t) y\| \geq \frac{1}{a_2} e^{\frac{\rho}{2} t} \|y\|,$$

i.e.,

$$\|A_v(t) x\| \geq \frac{1}{a_2 \|U_v'(0) - S_{v,t}'(0)\|} e^{\frac{\rho}{2} t} \|x\| \geq \frac{1}{a_2 (\|U_v'(0)\| + \|S_{v,t}'(0)\|)} e^{\frac{\rho}{2} t} \|x\|.$$

The proposition follows now from $\|U_v'(0)\| \leq \sqrt{R_0}$ and

$$\|S_{v,t}'(0)\| = \|A_v'(t) A_v^{-1}(t)\| \leq \sqrt{R_0} \coth(\sqrt{R_0}),$$

which can be found in [KnPe2, Lem. 2.2] □

Using this we derive the uniform divergence of geodesics described above.

Corollary 2.3. *Let $c_v : [0,\infty) \to X$ and $c_w : [0,\infty) \to X$ be two geodesics with $v, w \in S_pX$. Then*

$$d(c_v(t), c_w(t)) \geq a(t)\angle(v,w)$$

where $a : [0,\infty) \to [0,\infty)$ is a function (not depending on $p \in X$) with $\lim_{t\to\infty} a(t) = \infty$.

Proof. Let $c : [0,1] \to X$ be a geodesic connecting $c_v(t)$ with $c_w(t)$. Then c is given by

$$c(s) = \exp_p r(s)v(s),$$

where $v(s) \in S_pX$ and $r(s) > 0$ for all $0 \leq s \leq 1$, and $v(0) = v$, $v(1) = w$ and $r(0) = r(1) = t$. Then

$$\begin{aligned} c'(s_0) &= D\exp_p(r(s_0)v(s_0))(r'(s_0)v(s_0) + r(s_0)v'(s_0)) \\ &= r'(s_0)c'_{v(s_0)}(r(s_0)) + A_{v(s_0)}(r(s_0))(v'(s_0)). \end{aligned}$$

Since $c'_{v(s_0)}(r(s_0)) \perp A_{v(s_0)}(r(s_0))(v'(s_0))$, we obtain

$$\begin{aligned} \left\|c'(s_0)\right\|^2 &= (r'(s_0))^2 + ||A_{v(s_0)}(r(s_0))v'(s_0)||^2 \\ &\geq ||A_{v(s_0)}(r(s_0))v'(s_0)||^2. \end{aligned}$$

If there exists $s_0 \in [0,1]$ such that $r(s_0) \leq \frac{t}{2}$ then using the triangle inequality we have $d(c_v(t), c_w(t)) \geq t \geq (t/\pi)\angle(v,w)$. If this is not the case, we obtain for all $t > 0$

$$\begin{aligned} d(c_v(t), c_w(t)) = \text{length}(c) &\geq \int_0^1 ||A_{v(s)}(r(s))v'(s)||ds \\ &\geq ae^{\frac{\varrho}{4}t} \int_0^1 ||v'(s)||ds \\ &\geq ae^{\frac{\varrho}{4}t}\angle(v,w). \end{aligned}$$

The corollary follows now with the choice

$$a(t) = \min\left\{\frac{t}{\pi}, ae^{\frac{\varrho}{4}t}\right\}.$$

□

Remark. *Note that the proof shows that the function $a(t)$ describing the divergence of geodesics has at least linear growth.*

3. The geometric compactification

Let (X,g) be a rank one asymptotically harmonic manifold satisfying (1.1). The *geometric boundary* $X(\infty)$ is the set of equivalence classes of asymptotic geodesic rays. Two geodesic rays $c_1, c_2 : [0,\infty) \to X$ are called *asymptotic*, if there exists $C > 0$ with $d(c_1(t), c_2(t)) \leq C$ for all $t \geq 0$. The equivalence class of a geodesic ray c is denoted by $c(\infty)$.

Let $p \in X$ and consider the map $\varphi_p : S_pX \to X(\infty)$ with $\varphi_p(v) = c_v(\infty)$. Uniform divergence of geodesics implies that φ_p is injective. Our next aim is to prove surjectivity of φ_p. For this, we first prove general results which will also be useful later on. The first result requires besides no conjugate points only a lower bound on the sectional curvature of X.

Proposition 3.1. *Let $p, q_0, p_0 \in X$ be three different points such that $1 < r = d(p,p_0) = d(q_0,p_0)$ and $d(p,q_0) < r-1$. Let β be the radial projection (from p_0) of the geodesic connecting p and q_0 into $S_r(p_0)$. Then there is a function $b : \mathbb{R} \to (0,\infty)$ such that*

$$d_{S_r(p_0)}(p,q_0) \leq \text{length}(\beta) \leq \left(\max_{|s|\leq r} b(s)\right) d(p,q_0),$$

where $d_{S_r(p_0)}$ is the intrinsic distance of the sphere $S_r(p_0)$.

Proof. Let $\gamma : [0, d(p,q_0)] \to X$ be the geodesic connecting p and q_0. We first write γ and β in polar coordinates, i.e.,

$$\gamma(t) = \exp_{p_0}(d(t)v(t)), \quad \beta(t) = \exp_{p_0}(rv(t))$$

with $d(t) = d(p_0,\gamma(t)) > 1$ and $v : [0, d(p,q_0)] \to S_{p_0}X$. Then we have

$$\gamma'(t) = d'(t)c'_{v(t)}(d(t)) + A_{v(t)}(d(t))(v'(t))$$

and $\beta'(t) = A_{v(t)}(r)(v'(t))$. Note that the lower bound on sectional curvature yields the existence of a function $b : \mathbb{R} \to [0,\infty)$ such that for all $v \in SX$ and $r \geq 1$ we have $\|S_{v,r}(t)\| \leq b(t)$ (see proof of Lemma 2.16 in [Kn1]). Using $A_v(r)A_v^{-1}(x) = S_{v,x}(x-r)$ we therefore obtain

$$\begin{aligned}
\|\beta'(t)\| &= \|A_{v(t)}(r)v'(t)\| \leq \|A_{v(t)}(r)A^{-1}_{v(t)}(d(t))\| \cdot \|A_{v(t)}(d(t))v'(t)\| \\
&= \|S_{v(t),d(t)}(d(t)-r)\| \cdot \|A_{v(t)}(d(t))v'(t)\| \\
&\leq b(d(t)-r)\sqrt{\|A_{v(t)}(d(t))v'(t)\|^2 + \|d'(t)c'_{v(t)}(d(t)\|^2} \\
&\leq \left(\max_{|s|\leq r} b(s)\right) \|\gamma'(t)\|.
\end{aligned}$$

The last inequality above follows from $r - d(p,q_0) \leq d(t) = d(p_0,\gamma(t)) \leq r + d(p,q_0)$ and $d(p.q_0) \leq r$. ☐

For the next result, we need to introduce for every $v \in SX$ and $r > 0$ the function $b_{v,r}(p) = d(c_v(r),p) - r$ and the Busemann function $b_v(p) = \lim_{r\to\infty} b_{v,r}(p)$. Since we also use a uniform bound on on the norm of Jacobi tensor $S_{v,r}(t)$ for all $t \geq 0$ and $r \geq 1$ as has been derived in [KnPe2, Cor. 2.6] we need the assumption on X made at the beginning of this section.

Corollary 3.2. *Let $p, q \in X$, $r > 2d(p,q)+1$, $v \in S_pX$ and $w = -\operatorname{grad} b_{v,r}(q) \in S_qX$. Then there exists a constant $C = C(r) > 0$ such that*

$$d(c_v(t), c_w(t)) \leq (1 + 2Ce^{-\frac{\rho}{2}t})d(p,q) \quad \text{for all } 0 \leq t \leq r.$$

Proof. Let $p_0 = c_v(r)$, $q_0 = c_w(d(p_0,q) - r) \in S_r(p_0)$ and $w_0 = c'_w(d(p_0,q) - r)$. Then we have

$$d(p,q_0) \leq d(p,q) + d(q,q_0) \leq 2d(p,q) < r-1.$$

Let $\beta : [0,1] \to S_r(p_0)$ be the intrinsic geodesic in $S_r(p_0)$ connecting p and q_0. Let $d_{p_0}(x) = d(p_0,x)$ and $N(x) = -\operatorname{grad} d_{p_0}(x)$ for $x \neq p_0$. Let

$\beta_t : [0,1] \to S_{r-t}(p_0)$ defined by $\beta_t(s) = c_{N(\beta(s))}(t)$ for $t \in [0,r)$. Then $\beta_t'(s) = S_{N(\beta(s)),r}(t)(\beta'(s))_t$, which implies, using [KnPe2, Cor. 2.6],

$$\|\beta_t'(s)\| \le \|S_{N(\beta(s)),r}(t)\| \cdot \|\beta'(s)\| \le a_2 e^{-\frac{\varrho}{2}t}\|\beta'(s)\|.$$

Consequently,

$$d(c_v(t), c_{w_0}(t)) \le \mathrm{length}(\beta_t) \le a_2 e^{-\frac{\varrho}{2}t} d_{S_r(p_0)}(p,q_0) \le C e^{-\frac{\varrho}{2}t} d(p,q_0)$$

with $C = a_2 \max_{|s|\le r} b(s)$, using Proposition 3.1. This implies

$$\begin{aligned} d(c_v(t), c_w(t)) &\le d(c_v(t), c_{w_0}(t)) + d(c_{w_0}(t), c_w(t)) \\ &\le C e^{-\frac{\varrho}{2}t} d(p,q_0) + d(q,q_0) \\ &\le (1 + 2Ce^{-\frac{\varrho}{2}t}) d(p,q). \end{aligned}$$

□

Now we prove surjectivity of φ_p: Let $c : [0,\infty) \to X$ be a geodesic ray with $w = c'(0) \in S_qX$. Let $v = -\operatorname{grad} b_w(p) \in S_pX$. Then c_v is asymptotic to c_w by Corollary 3.2 with $r = \infty$. Therefore $\varphi_p(v) = c(\infty)$ and φ_p is surjective.

We define $\overline{X} = X \cup X(\infty)$ and introduce for every $p \in X$ the following bijective map $\bar\varphi_p : \overline{B_1(p)} \to \overline{X}$, where $\overline{B_1(p)} \subset T_pX$ is the closed ball of radius 1:

$$\bar\varphi_p(v) = \begin{cases} \varphi_p(v) & \text{if } \|v\| = 1, \\ \exp_p\left(\frac{1}{1-\|v\|}v\right) & \text{if } \|v\| < 1. \end{cases}$$

We define a topology on $\overline{X}$ such that the bijective map $\bar\varphi_p : \overline{B_1(p)} \to \overline{X}$ is a homeomorphism. Next we show that this topology on $\overline{X}$ does not depend on the reference point p. For that we need to show that $\bar\varphi_{p,q} = \bar\varphi_q^{-1} \circ \bar\varphi_p : \overline{B_1(p)} \to \overline{B_1(q)}$ is a homeomorphism.

For the continuity of $\bar\varphi_{p,q}$ note first that

$$\bar\varphi_{p,q}(v) = \begin{cases} -\operatorname{grad} b_v(q) & \text{if } \|v\| = 1, \\ \exp_q^{-1}\left(\exp_p\left(\frac{1}{1-\|v\|}v\right)\right) & \text{if } \|v\| < 1. \end{cases}$$

Let $v_n \in \overline{B_1(p)}$ such that $v_n \to v \in \overline{B_1(p)}$. If $\|v\| < 1$, the continuity of $\bar\varphi_{p,q}$ at v follows from the continuity of the exponential maps. If $\|v\| = 1$, it suffices to consider two cases: in the first case we have $0 \ne \|v_n\| < 1$ for all n and $\|v_n\| \to 1$, and in the second case we have $\|v_n\| = 1$ for all n. We present the prove of the first case, the second case goes analogously: Note that we have

$$\exp_q^{-1}\left(\exp_p\left(\frac{1}{1-\|v_n\|}v_n\right)\right) = \\ -\frac{d(q, c_{v_n}(f(\|v_n\|))}{1 + d(q, c_{v_n}(f(\|v_n\|))} \operatorname{grad} b_{\frac{v_n}{\|v_n\|}, f(\|v_n\|)}(q) = w_n,$$

where $f(x) = \frac{x}{1-x}$. We need to show that $w_n \to -\operatorname{grad} b_v(q)$. Choose a convergent subsequence $w_{n_j} \in S_qX$ with limit $w \in S_qX$. Then there exists a constant $a > 0$ such that for all sufficiently large $n \in \mathbb{N}$

$$d(c_{v_n}(t), c_{w_n}(t)) \le a \quad \text{for all } 0 \le t \le f(\|v_n\|) = r_n,$$

by Corollary 3.2. Note that $r_n \to \infty$. This implies that

$$d(c_v(t), c_w(t)) \leq a \quad \text{for all } t \geq 0,$$

i.e., c_v and c_w are asymptotic geodesic rays. By Corollary 3.2, c_v and $c_{-\operatorname{grad}_v(q)}$ are also asymptotic. Therefore, by the injectivity of φ_q, we have $w = -\operatorname{grad}_v(q)$. This finishes the proof that $\bar{\varphi}_{p,q}$ is a homeomorphism.

This topology on $\overline{X}$ was first introduced for Hadamard manifolds by Eberlein-O'Neill [EON] and is called *cone topology.* The points in $X(\infty) \subset \overline{X}$ are called *points at infinity.* Note that a sequence $x_n \in X$ converges in the cone topology to a point at infinity if and only if for every $p \in X$ we have $d(x_n, p) \to \infty$ and for every $\epsilon > 0$ there exists $n(\epsilon)$ such that $\angle_p(x_n, x_m) < \epsilon$ for all $n, m \geq n(\epsilon)$. We write "$\angle_p(x_n, x_m) \to 0$ as $n, m \to \infty$" for the latter.

4. Gromov hyperbolicity

We start this section by introducing the Gromov product.

Definition 4.1. *Let (X, d) be a metric space and $x_0 \in X$ a reference point. The* Gromov product $(x|y)_{x_0}$ *of $x, y \in X$ is defined as*

$$(x|y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$$

Note that the Gromov product $(x|y)_{x_0}$ is non-negative, by the triangle inequality. A metric space (X, d) is called a *geodesic space*, if any two points $x, y \in X$ can be connected by a geodesic, i.e., if there exists a curve $\sigma_{xy} : [0, d(x, y)] \to X$ connecting x and y, such that $d(\sigma_{xy}(s), \sigma_{xy}(t)) = |t - s|$ for all $s, t \in [0, d(x, y)]$.

Definition 4.2. *A geodesic space (X, d) is called* δ-hyperbolic *if every geodesic triangle Δ is δ-thin, i.e., every side of Δ is contained in the union of the δ-neighborhoods of the other two sides. If a geodesic space (X, d) is δ-hyperbolic for some $\delta \geq 0$, we call (X, d) a Gromov hyperbolic space.*

Let us recall the following two general results for Gromov hyperbolic spaces.

Proposition 4.3. *(see* [CDP, Chapter 1, Prop. 3.6]*) Let (X, d) be a δ-hyperbolic space. Then we have for all $x_0, x, y, z \in X$:*

$$(x|y)_{x_0} \geq \min\{(x|z)_{x_0}, (y|z)_{x_0}\} - 8\delta.$$

Proposition 4.4. *(see* [CDP, Chapter 3, Lem. 2.7]*) Let (X, d) be a δ-hyperbolic space. Then we have for all $x_0, x, y \in X$:*

$$(x|y)_{x_0} \leq d(x_0, \sigma_{xy}) \leq (x|y)_{x_0} + 32\delta.$$

Now assume that X is a rank one asymptotically harmonic manifold satisfying (1.1) and, therefore, a Gromov hyperbolic space, by [KnPe2, Thm. 1.5]. We show now that two sequences $\{x_n\}, \{y_n\} \subset X$ have the same limiting behavior at infinity in the cone topology if and only if

$$\lim_{n\to\infty} (x_n|y_n)_p = \infty.$$

We note that condition $\lim_{n,m\to\infty}(x_n|x_m)_p = \infty$ is used for general Gromov hyperbolic space as a definition for convergence to infinity (see [BS, Section 2.2]).

Theorem 4.5. *Let X be a rank one asymptotically harmonic manifold satisfying* (1.1)*. Let $p \in X$ and $\{x_n\}, \{y_n\}$ be two sequences in X. The following are equivalent.*

(a) *We have $d(x_n, p), d(y_n, p) \to \infty$ and $\angle_p(x_n, y_n) \to 0$ for $n \to \infty$.*
(b) *$(x_n|y_n)_p \to \infty$ for $n \to \infty$.*

Proof. *(b)* $\Rightarrow$ *(a)*: X is δ-hyperbolic for some $\delta \geq 0$. Let $(x_n|y_n)_p \to \infty$. We know from Proposition 4.4 that $d(p, x_n), d(p, y_n) \geq (x_n|y_n)_p$, which shows that $d(p, x_n), d(p, y_n) \to \infty$ as $n \to \infty$. It remains to show that $\angle_p(x_n, y_n) \to 0$. Let U_{px_n}, U_{py_n} be δ-tubes around the geodesic arcs σ_{px_n} and σ_{py_n}. Then the geodesic $\sigma_{x_n y_n}$ must contain a point $p_1 \in U_{px_n} \cap U_{py_n}$. We conclude from Proposition 4.4 that

$$d(p_1, p) \geq d(\sigma_{x_n y_n}, p) \geq (x_n|y_n)_p.$$

Let γ_1 and γ_2 be the shortest curves connecting p_1 with σ_{px_n} and σ_{py_n} at the points $\hat{x}_n$ and y_n', see Figure 1. Then $d(p_1, \hat{x}_n), d(p_1, y_n') \leq \delta$, which implies $d(\hat{x}_n, y_n') \leq 2\delta$ and

$$d(\hat{x}_n, p), d(y_n', p) \geq (x_n|y_n)_p - \delta.$$

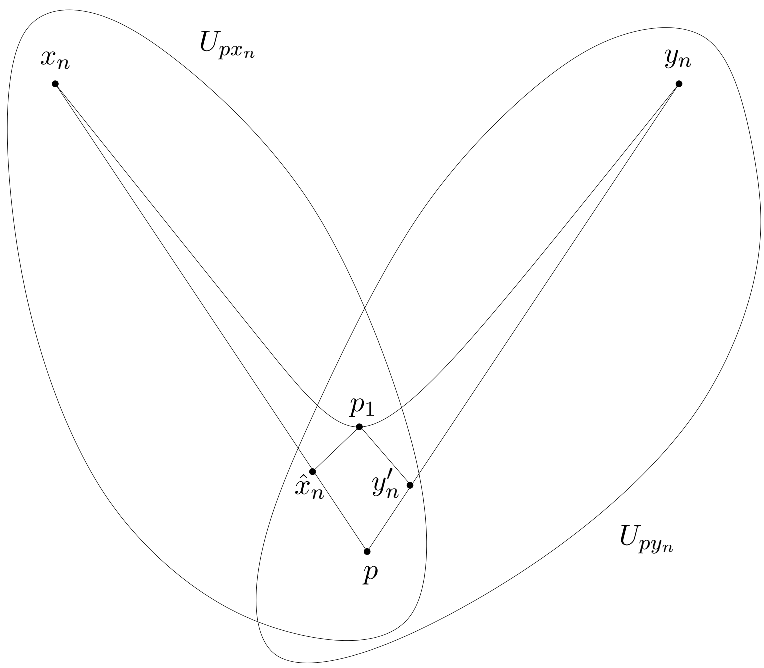

FIGURE 1. Illustration of the proof of $(b) \Rightarrow (a)$ in Theorem 4.5

We assume, without loss of generality, that $d(\hat{x}_n, p) \geq d(y_n', p)$. Let $\hat{y}_n \in \sigma_{py_n}$ be such that $d(p, \hat{y}_n) = d(p, \hat{x}_n)$. This implies that

$$d(\hat{x}_n, p) = d(\hat{y}_n, p) \geq (x_n|y_n)_p - \delta.$$

Since

$$d(y_n', p) \leq d(\hat{y}_n, p) = d(\hat{x}_n, p) \leq d(y_n', p) + d(\hat{x}_n, y_n') \leq d(y_n', p) + 2\delta,$$

and since $y_n', \hat{y}_n$ lie on the same geodesic arc σ_{py_n}, we have $d(y_n', \hat{y}_n) \le 2\delta$. This implies that

$$d(\hat{y}_n, \hat{x}_n) \le d(y_n', \hat{x}_n) + d(\hat{y}_n, y_n') \le 2\delta + 2\delta = 4\delta.$$

Using Corollary 2.3, we conclude that

$$4\delta \ge \text{length}(\sigma_{\hat{x}_n\hat{y}_n}) \ge a(d(\hat{x}_n, p))\angle_p(x_n, y_n).$$

Since $d(\hat{x}_n, p) \to \infty$, we also have $a(d(\hat{x}_n, p)) \to \infty$, which implies that $\angle_p(x_n, y_n) \to 0$.

$(a) \Rightarrow (b)$: Assume $\angle_p(x_n, y_n) \to 0$ and $d(x_n, p), d(y_n, p) \to \infty$ for $n \to \infty$. For all $R > 0$, there exists $n_0(R) \ge 0$, such that for all $n \ge n_0(R)$:

$$d(p, x_n), d(p, y_n) \ge R \quad \text{and} \quad d(c_{px_n}(R), c_{py_n}(R)) \le 1, \tag{4.1}$$

since $\angle_p(x_n, y_n) \to 0$ for $n \to \infty$. Note that the constant $n_0(R)$ does not depend on p, but only on the values $d(p, x_n), d(p, y_n)$ and $\angle_p(x_n, y_n)$, since X has a uniform lower curvature bound.

We show now the following: *The geodesic arc $\sigma_{x_ny_n}$ has empty intersection with the open ball $B_{R-\frac{1}{2}}(p)$ for all $n \ge n_0(R)$.*

If $\sigma_{x_ny_n} \cap B_R(p) = \emptyset$, there is nothing to prove. If $\sigma_{x_ny_n} \cap B_R(p) \neq \emptyset$, there exists a first $t_0 > 0$ and a last $t_1 > 0$ such that

$$q_1 = \sigma_{x_ny_n}(t_0),\ q_2 = \sigma_{x_ny_n}(t_1) \in S_R(p),$$

where $S_R(p)$ denotes the sphere of radius $R > 0$ around p (see Figure 2). Then we have

$$d(q_1, q_2) = l(\sigma_{x_ny_n}) - d(x_n, q_1) - d(y_n, q_2).$$

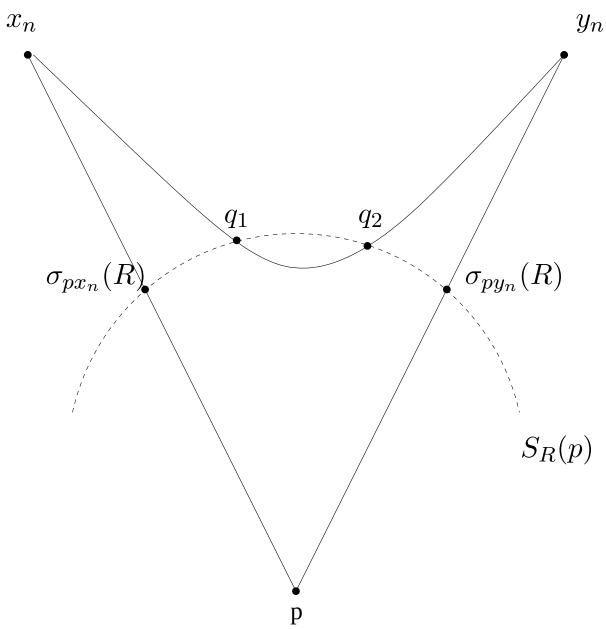

FIGURE 2. Illustration of the proof of $(a) \Rightarrow (b)$ in Theorem 4.5

Using (4.1), we have

$$\begin{aligned} l(\sigma_{x_n y_n}) &\le d(x_n, \sigma_{px_n}(R)) + d(\sigma_{px_n}(R), \sigma_{py_n}(R)) + d(\sigma_{py_n}(R), y_n) \\ &\le d(x_n, \sigma_{px_n}(R)) + d(y_n, \sigma_{py_n}(R)) + 1, \end{aligned}$$

which implies that

$$\begin{aligned} d(q_1, q_2) \le (d(x_n, \sigma_{px_n}(R)) - d(x_n, q_1)) \\ + (d(y_n, \sigma_{py_n}(R)) - d(y_n, q_2)) + 1. \end{aligned} \tag{4.2}$$

Since $d(p, x_n) = R + d(\sigma_{px_n}(R), x_n) \le d(q_1, x_n) + R$ (by the triangle inequality), we obtain $d(x_n, q_1) - d(x_n, \sigma_{px_n}(R)) \ge 0$, and similarly $d(y_n, q_2) - d(y_n, \sigma_{py_n}(R)) \ge 0$. This, together with (4.2) shows $d(q_1, q_2) \le 1$. But then the geodesic segment of $\sigma_{x_n y_n}$ between q_1 and q_2 cannot enter the ball $B_{R-\frac{1}{2}}(p)$.

Therefore, we have for all $n \ge n_0(R)$,

$$R - \frac{1}{2} \le d(p, \sigma_{x_n y_n}) \le (x_n | y_n)_p + 32\delta,$$

using Proposition 4.4. This shows that

$$(x_n | y_n)_p \to \infty \quad \text{as } n \to \infty.$$

□

5. Visibility measures and their Radon-Nykodym derivative

Let (X, g) be a rank one asymptotically harmonic manifold of dimension n. The boundary $X(\infty) \subset \overline{X}$ is homeomorphic to the sphere S^{n-1} and equipped with the relative topology of the cone topology. Moreover, we have a family of visibility measures $\{\mu_p \in \mathcal{M}_1(X(\infty))\}_{p \in X}$, which were introduced in Definition 1.3. We will see that any two visibility measures $\mu_p, \mu_q \in \mathcal{M}_1(X(\infty))$ are absolutely continuous, by calculating their Radon-Nykodym derivative via a limiting process. Similar calculations were carried out in [CaSam, Section 6.1] for asymptotically harmonic manifolds with pinched negative curvature.

Lemma 5.1. *For all $p, q \in X$ there exists $t(p, q) > 0$ such that for all $t \ge t(p, q)$ and all $v \in S_q X$ the geodesic ray $c_v : [0, \infty) \to X$ intersects $S_t(p)$ in a unique point $F_t(v)$ (see Figure 3). In particular, the map $F_t : S_q X \to S_t(p)$ is bijective.*

Proof. Let $a(t)$ be as in Corollary 2.3. Choose t_0 such that for all $t \ge t_0$ we have $2d(p, q) \le a(t)$. Define

$$t(p, q) = \max\{d(p, q) + 1, t_0\}.$$

In particular, q lies in the ball of radius t around p, for all $t \ge t(p, q)$, and hence for all $v \in S_q X$ the geodesic ray $c_v : [0, \infty) \to X$ intersects $S_t(p)$. Let $t \ge t(p, q)$, and assume that $q' = c_v(t_1)$ is an intersection point of $c_v([0, \infty))$ and $S_t(p)$ such that $c_v'(t_1)$ is either pointing into $B_t(p)$ or is tangent to $S_t(p)$, i.e.,

$$\angle(c_v'(t_1), c_w'(t)) \ge \pi/2,$$

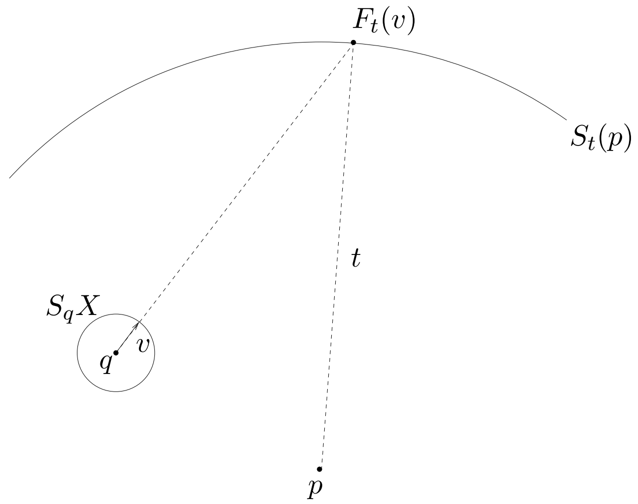

FIGURE 3. Illustration of the map $F_t : S_qX \to S_t(p)$

where $w \in S_pX$ is the unique vector such that $c_w(t) = q'$. Using the triangle inequality we obtain

$$t - d(p, q) \leq t_1 \leq t + d(p, q)$$

Using Corollary 2.3, we obtain for all $s \geq 0$

$$d(c_v(t_1 - s), c_w(t - s)) \geq a(s)\pi/2.$$

In particular for $s = t$ this yields

$$a(t)\pi/2 \leq d(c_v(t_1 - t), p) \leq d(c_v(t_1 - t), q) + d(q, p) \leq 2d(p, q) \leq a(t),$$

which is a contradiction. Hence, a second intersection point between the geodesic ray $c_v([0, \infty))$ and $S_t(p)$ cannot occur. □

Proposition 5.2. *Let (X, g) be a complete, simply connected noncompact manifold without conjugate points and $p, q \in X$. Consider the map $F_t : S_qX \to S_t(p)$, where $F_t(v)$ is the first intersection point of the geodesic ray $c_v : [0, \infty) \to X$ with $S_t(p)$. If q is contained in the ball of radius t about p, this map is well defined. Then the Jacobian of F_t is given by*

$$\text{Jac}\, F_t(v) = \frac{\det A_v(d(q, F_t(v)))}{\langle N_p(F_t(v)), N_q(F_t(v))\rangle}, \tag{5.1}$$

where $N_x(y) = (\text{grad}\, d_x)(y)$ and $d_x(y) = d(x, y)$.

Note that (5.1) agrees with [CaSam, (6.3)]. For convenience of the readers, we provide our own proof of this formula.

Proof. Choose a curve $\gamma : (-\epsilon, \epsilon) \to S_qX$ with $\gamma(0) = v \in S_qX$. Then

$$F_t(\gamma(s)) = \exp_q(d(q, F_t(\gamma(s))) \cdot \gamma(s)),$$

and, using the chain rule and the product rule,

$$DF_t(v)(\gamma'(0)) =$$
$$D\exp_q(d(q, F_t(v)) \cdot v)(\langle N_q(F_t(v)), DF_t(v)\gamma'(0)\rangle v + d(q, F_\gamma(v)) \cdot \gamma'(0)).$$

Note that $\gamma'(0) \perp v$. We have

$$D\exp_q(tv)(tw) = Y(t)(w) = J(t),$$

where Y is the Jacobi tensor along c_v with $Y(0) = 0$ and $Y'(0) = \mathrm{id}$, and therefore J is a Jacobi field along c satisfying $J(0) = 0$ and $J'(0) = w$. Note that Y and A_v are related by $A_v = Y\Big|_{(c_v')^\perp}$. In particular, we have $D\exp_q(tv)(tv) = tc_v'(t)$. This yields

$$\begin{aligned} &DF_t(v)(\gamma'(0)) \\ &\quad = \langle N_q(F_t(v)), DF_t(v)\gamma'(0)\rangle \; c_v'(d(q, F_t(v))) + A_v(d(q, F_t(v)))(\gamma'(0)). \end{aligned}$$

Consequently,

$$\begin{aligned} DF_t(v)(\gamma'(0)) &= \\ &\langle N_q(F_t(v)), DF_t(v)\gamma'(0)\rangle N_q(F_t(v)) + A_v(d(q, F_t(v)))(\gamma'(0)). \end{aligned} \tag{5.2}$$

Next, we introduce the map

$$\begin{aligned} L_x : N_p(x)^\perp &\to N_q(x)^\perp, \\ L_x(w) &= w - \langle w, N_q(x)\rangle N_q(x). \end{aligned}$$

Then (5.2) can be rewritten as

$$L_{F_t(v)} \circ DF_t(v) = A_v(d(q, F_t(v))). \tag{5.3}$$

To finish the proof of the above Proposition, we need the following lemma.

Lemma 5.3. $\mathrm{Jac}\, L_x = |\langle N_p(x), N_q(x)\rangle|$.

Proof. Consider

$$N_p(x)^\perp \cap N_q(x)^\perp = \{w \in T_xX \mid \langle w, N_p(x)\rangle = 0 \text{ and } \langle w, N_q(x)\rangle = 0\}.$$

Then $N_p(x)^\perp \cap N_q(x)^\perp$ has co-dimension one in $N_p(x)^\perp$ and L_x is the identity on $N_p(x)^\perp \cap N_q(x)^\perp$. Let

$$w_0 = N_q(x) - \langle N_q(x), N_p(x)\rangle N_p(x) \in N_p(x)^\perp.$$

The vector w_0 is orthogonal to $N_p(x)^\perp \cap N_q(x)^\perp$ since for all $w \in N_p(x)^\perp \cap N_q(x)^\perp$ we have $\langle w, N_p(x)\rangle = 0$ and $\langle w, N_q(x)\rangle = 0$, and therefore

$$\langle w, w_0\rangle = \underbrace{\langle w, N_q(x)\rangle}_{=0} - \langle N_q(x), N_p(x)\rangle \underbrace{\langle w, N_p(x)\rangle}_{=0} = 0.$$

Moreover, $L_x w_0$ is also orthogonal to $N_p(x)^\perp \cap N_q(x)^\perp$:

$$\begin{aligned} L_x w_0 &= w_0 - \langle w_0, N_q(x)\rangle N_q(x) \\ &= \langle N_p(x), N_q(x)\rangle (\langle N_p(x), N_q(x)\rangle N_q(x) - N_p(x)), \end{aligned}$$

and consequently $\langle w, L_x w_0\rangle = 0$ for all w satisfying $\langle w, N_p(x)\rangle = \langle w, N_q(x)\rangle = 0$. This shows that

$$\mathrm{Jac}\, L_x = \frac{||L_x w_0||}{||w_0||}.$$

Since

$$\|L_x w_0\|^2 = \langle N_p(x), N_q(x)\rangle^2 (1 - \langle N_p(x), N_q(x)\rangle^2)$$

and

$$\|w_0\|^2 = 1 - \langle N_p(x), N_q(x)\rangle^2,$$

we obtain

$$\begin{aligned}\operatorname{Jac} L_x &= \left(\frac{\langle N_p(x), N_q(x)\rangle^2 (1 - \langle N_p(x), N_q(x)\rangle^2)}{1 - \langle N_p(x), N_q(x)\rangle^2}\right)^{1/2} \\ &= |\langle N_p(x), N_q(x)\rangle|,\end{aligned}$$

which yields the lemma. □

Finally, (5.3) implies that

$$\operatorname{Jac} F_t(v) = \frac{\det A_v(d(q, F_t(v)))}{\operatorname{Jac}\ L_{F_t}(v)} = \frac{\det A_v(d(q, F_t(v)))}{\langle N_p(F_t(v)), N_q(F_t(v))\rangle},$$

finishing the proof of the proposition. □

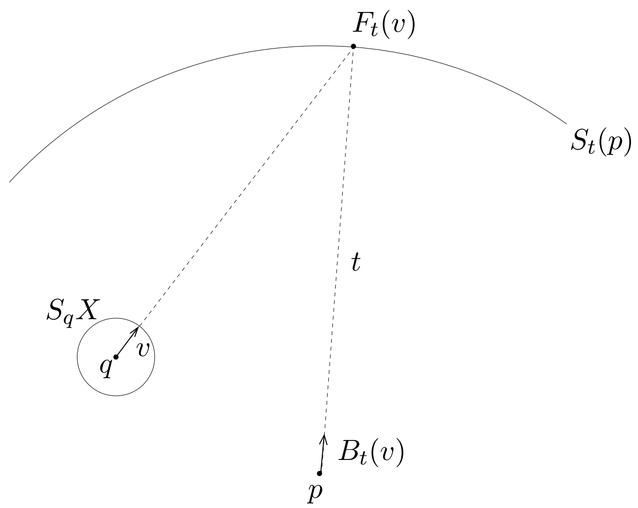

FIGURE 4. Illustration of the map $B_t : S_qX \to S_pX$

Corollary 5.4. *Let (X, g) be a complete, simply connected noncompact manifold without conjugate points and $p, q \in X$. Let $B_t : S_qX \to S_pX, v \mapsto \frac{1}{t}\exp_p^{-1} \circ F_t(v)$ (see Figure 4). Then we have*

$$\operatorname{Jac} B_t(v) = \frac{\det A_v(d(q, F_t(v)))}{\det A_u(t)} \cdot \frac{1}{\langle N_p(F_t(v)), N_q(F_t(v))\rangle},$$

where $u = B_t(v)$.

Proof. Let $u \in S_pX$. Then $D\exp_p(tu) : u^\perp \to T_{\exp_p(tu)} S_t(p)$ is given by $D\exp_p(tu)(w) = \frac{1}{t} A_u(t)(w)$, and therefore with $u = B_t(v)$,

$$\begin{aligned}\operatorname{Jac} B_t(v) &= \frac{1}{\det A_u(t)} \cdot \operatorname{Jac} F_t(v) \\ &= \frac{\det A_v(d(q, F_t(v)))}{\det A_u(t)} \cdot \frac{1}{\langle N_p(F_t(v)), N_q(F_t(v))\rangle}.\end{aligned}$$

□

From now on, (X, g) denotes a rank one asymptotically harmonic manifold satisfying (1.1) with $n = \dim(X)$. Let $f \in C(X(\infty))$. We know from Lemma 5.1 that $B_t : S_qX \to S_pX$ is a bijection, for $t > 0$ large enough. Then we have with $f_1 = f \circ \varphi_p$:

$$\begin{aligned}\int_{X(\infty)} f(\xi)\, d\mu_p(\xi) &= \frac{1}{\omega_n}\int_{S_pX} f_1(w)\, d\theta_p(w)\\ &= \frac{1}{\omega_n}\int_{S_qX} (f_1 \circ B_t)(v)(\mathrm{Jac}\, B_t)(v)\, d\theta_q(v).\end{aligned}$$

We will show that

(i) $\lim_{t\to\infty} B_t = (\varphi_p)^{-1} \circ \varphi_q$,

(ii) There exist constants $t_0 > 0$ and $C > 0$ such that

$$|\,\mathrm{Jac}\, B_t(v)| \le C \quad \forall\, v \in S_qX,\ t \ge t_0.$$

(iii) We have, for all $v \in S_qX$,

$$\lim_{t\to\infty} \mathrm{Jac}\, B_t(v) = e^{-hb_v(p)},$$

where b_v is the Busemann function introduced in (1.2). Having these facts, we conclude with Lebesgue's dominated convergence that

$$\begin{aligned}\int_{X(\infty)} f(\xi)\, d\mu_p(\xi) &= \lim_{t\to\infty}\frac{1}{\omega_n}\int_{S_qX} (f_1 \circ B_t)(v)(\mathrm{Jac}\, B_t)(v)\, d\theta_q(v)\\ &= \frac{1}{\omega_n}\int_{S_qX} (f\circ\varphi_q)(v) e^{-hb_v(p)}\, d\theta_q(v)\\ &= \int_{X(\infty)} f(\xi) e^{-hb_{q,\xi}(p)}\, d\mu_q(\xi),\end{aligned}$$

with $b_{q,\xi} = b_v$ with $\xi = c_v(\infty)$ and $v \in S_qX$. This proves Theorem 1.4 from the Introduction:

Theorem 1.4. *Let (X, g) be a rank one asymptotically harmonic manifold satisfying* (1.1)*. Let $(\mu_p)_{p\in X}$ be the associated family of visibility measures. Then these measures are pairwise absolutely continuous and we have*

$$\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb_{q,\xi}(p)}.$$

It remains to prove properties (i), (ii) and (iii) listed above.

Proof of (i): Let $t_n \to \infty$ and $s_n \ge 0$, $w_n = B_{t_n}(v) \in S_pX$ such that $y_n = \exp_q(s_n v) = \exp_p(t_n w_n)$. We obviously have $s_n \to \infty$ and $y_n \to \varphi_q(v)$. Let w_{n_j} be a convergent subsequence of $w_n = B_{t_n}(v)$ with limit $w \in S_pX$. Then we have $y_{n_j} \to \varphi_p(w)$ and

$$\varphi_q(v) = \varphi_p(w).$$

This shows that $\lim_{n\to\infty} B_{t_n}(v) = (\varphi_p)^{-1} \circ \varphi_q(v)$. □

For the proof of (ii), we need the following lemma:

Lemma 5.5. *For every $\epsilon > 0$, there exists $t_0 > 0$ such that we have for all $v \in S_qX$*

$$|\langle N_p(F_t(v)), N_q(F_t(v))\rangle - 1| < \epsilon \quad \forall\, t \ge t_0.$$

Proof. This is an easy consequence of Corollary 2.3. □

Proof of (ii): We start with the formula (see [Kn2, p. 676])

$$\det A_v(t) = \frac{\det U_v(t)}{\det(U_v'(0) - S_{v,t}'(0))} = \frac{e^{ht}}{\det(U_v'(0) - S_{v,t}'(0))}.$$

Then

$$\frac{\det A_v(d(q, F_t(v))}{\det A_{u_t}(t)} = \frac{e^{hd(q,F_t(v))} \det(U_{u_t'}(0) - S_{u_t,t}'(0))}{\det(U_v'(0) - S_{v,d(q,F_t(v))}'(0))e^{ht}},$$

where $u_t = B_t(v) \in S_pX$.

Let $\epsilon > 0$ be chosen. Since $\det(U_v'(0) - S_{v,t}'(0))$ converges monotonically to a universal constant $A > 0$ (see [KnPe2, Theorem 1.3] and use the fact that X is rank one), we conclude with Dini that the convergence is uniformly on compact sets. Therefore, there exists $t_0 \geq 0$ such that $A \leq \det(U_w'(0) - S_{w,t}'(0)) \leq A + \epsilon$ for all $w \in S_pX \cup S_qX$ and $t \geq t_0$. Using Lemma 5.5 and increasing $t_0 > 0$, if necesary, we can also assume that

$$\langle N_p(F_t(v)), N_q(F_t(v)) \rangle \geq \frac{1}{2}$$

for all $t \geq t_0$. Since $d(q, F_t(v)) \leq t + d(p, q)$, we conclude from Corollary 5.4 for all $t \geq t_0$ and all $v \in S_qX$,

$$|\operatorname{Jac} B_t(v)| \leq 2\frac{A+\epsilon}{A} e^{hd(p,q)}.$$ □

Proof of (iii): This is a immediate consequence of Lemma 5.5 and the following Lemma:

Lemma 5.6. *Using the notation above we have that*

$$\lim_{t\to\infty} \frac{\det A_v(d(q, F_t(v)))}{\det A_{u_t}(t)} = e^{-hb_v(p)},$$

where $u_t = B_t(v)$.

Proof. We recall that

$$\frac{\det A_v(d(q, F_t(v)))}{\det A_{u_t}(t)} = e^{h(d(q,F_t(v))-t)} \frac{\det(U_{u_t'}(0) - S_{u_t,t}'(0))}{\det(U_v'(0) - S_{v,d(q,F_t(v))}'(0))}$$

and

$$\frac{\det(U_{u_t'}(0) - S_{u_t,t}'(0))}{\det(U_v'(0) - S_{v,d(q,F_t(v))}'(0))} \to \frac{A}{A} = 1.$$

Now the lemma follows from

$$\begin{aligned} \lim_{t\to\infty} d(q, F_t(v)) - t &= \lim_{t\to\infty} d(q, F_t(v)) - d(p, F_t(v)) \\ &= \lim_{s\to\infty} d(q, c_v(s)) - d(p, c_v(s)) \\ &= \lim_{s\to\infty} s - d(p, c_v(s)) = -b_v(p). \end{aligned}$$

□

Remark Theorem 1.4 has an analogue for simply connected, noncompact harmonic manifolds (X, g) without the rank one condition and replacing the geometric boundary $X(\infty)$ by the Busemann boundary (see [KnPe1, Theorem 12.6]). There, we have $\det A_v(t) = f(t)$ for all $v \in SX$, where $f(t)$ is the volume density function, and $f(t)$ is an exponential polynomial. Moreover, the uniform divergence of geodesics (Corollary 2.3) holds there without the rank one condition. These results are not known for general asymptotically harmonic manifolds.

6. Solution of the Dirichlet problem at infinity

Since rank one asymptotically harmonic manifolds (X, g) satisfying (1.1) are Gromov hyperbolic with positive Cheeger constant (see [KnPe2]), general results of Ancona yield that the Martin boundary and the geometric boundary coincide ([Anc2, Théorème 6.2]) and that the Dirichlet problem at infinity has a solution ([Anc2, Théorème 6.7]). In this section we give an alternative direct proof that the Dirichlet problem at infinity has a solution for these manifolds by providing a concrete integral formula of the solution using the visibility measures. Moreover, this shows that the visibility measures coincide with the harmonic measures on $X(\infty)$.

A crucial step for our result of this section is to show that $\lim_{x\to\xi} \mu_x = \delta_\xi$, where δ_ξ is the δ-distribution at ξ. This abstract condition will follow from the next proposition. To state it, we introduce for $v_0 \in S_pX$ and $\delta > 0$ the cone

$$C(v_0, \delta) = \{c_v(t) \mid \ t \in [0, \infty], \angle(v_0, v) \leq \delta\}.$$

Note that the set of all truncated cones $C(v_0, \delta) \cap B_R(p)^c$ together with all open balls $B_r(q)$ define a basis of the cone topology of the geometric compactification $\bar{X}$.

Proposition 6.1. *Let (X, g) be a rank one asymptotically harmonic manifold satisfying (1.1). Let $p \in X$ and $\delta > 0$. Then there exists a constant $C_1 = C_1(\delta) > 0$ such that for all $v \in S_pX$*

$$b_v(q) \geq d(p, q) - C_1 \quad \textit{for all } q \in X \backslash C(v, \delta).$$

Proof. Let $p \in X$ and $\delta > 0$ be given. Then there exists a constant $C_1 > 0$ such that

$$0 \leq 2(c_v(t)|q)_p \leq C_1 \quad \forall\, t \geq 0 \quad \forall\, v \in S_pX \quad \forall\, q \in X \backslash C(v, \delta), \tag{6.1}$$

where $(\cdot|\cdot)_p$ is the Gromov product introduced in Definition 4.1. If this were false, then we could find sequences $t_n \geq 0$, $v_n \in S_pX$ and $q_n \in X \backslash C(v_n, \delta)$ such that

$$(c_{v_n}(t_n)|q_n)_p \to \infty.$$

Let $q_n = c_{w_n}(r_n)$ with $w_n \in S_pX$ and $r_n = d(q_n, p)$. This would mean, by Theorem 4.5, that $d(p, q_n) \to \infty$ and $\angle_p(v_n, w_n) \to 0$, which is a contradiction to $q_n \in X \backslash C(v_n, \delta)$.

(6.1) means that

$$d(p, q) - (d(c_v(t), q) - t) \leq C_1 \quad \forall\, t \geq 0.$$

Taking the limit $t \to \infty$, we obtain

$$d(p, q) - b_v(q) = d(p, q) - \lim_{t\to\infty} (d(c_v(t), q) - t) \leq C_1,$$

finishing the proof. □

REMARK The statement of the proposition includes the fact that any horoball $\mathcal{H}$, centered at $\xi = c_v(\infty) \in X(\infty)$, ends up inside any given cone $C(v,\delta)$, when being translated to a horoball $\widetilde{\mathcal{H}}$ along the stable direction (see the illustration in Figure 5). (Note that the horoballs centered at ξ can be described by $\{q \in X \mid b_v(q) \le -C\}$, and that these horoballs become smaller and shrink towards the limit point ξ, as $C \in \mathbb{R}$ increases to infinity.)

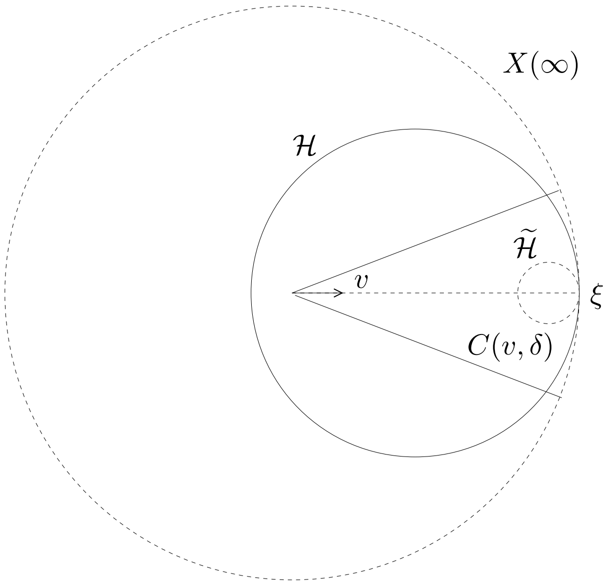

FIGURE 5. Geometric property necessary for the solution of the Dirichlet problem at infinity

REMARK Proposition 6.1 does not hold if (X, g) is the Euclidean space. In this case, every horoball is a halfspace, which lies never inside a given cone.

Now we state our main result of this section, namely, the solution of the Dirichlet problem at infinity for rank one asymptotically harmonic manifolds satisfying (1.1) via an explicit integral formula involving the visibility measures (see Theorem 1.5 from the Introduction).

Theorem 1.5. *Let (X, g) be a rank one asymptotically harmonic manifold satisfying* (1.1). *Let $f : X(\infty) \to \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function $H_f : X \to \mathbb{R}$ such that*

$$\lim_{x\to\xi} H_f(x) = f(\xi). \tag{6.2}$$

Moreover, H_f has the following integral presentation:

$$H_f(x) = \int_{X(\infty)} f(\xi) d\mu_x(\xi),$$

where $\{\mu_x\}_{x\in X} \subset \mathcal{M}_1(X(\infty))$ are the visibility probability measures.

Proof. (a) We show first that $\int\limits_{X(\infty)} f(\xi)d\mu_x(\xi)$ is a harmonic function. Let $p \in X$. Then

$$\Delta_x \int\limits_{X(\infty)} f(\xi)d\mu_x(\xi) = \Delta_x \int\limits_{X(\infty)} f(\xi)e^{-hb_{p,\xi}(x)}d\mu_p(\xi).$$

Let $K \subset X$ be a compact set. Then $x \mapsto f(\xi)e^{-hb_{p,\xi}(x)}$ is bounded for all $x \in K$ and all $\xi \in X(\infty)$, because of $|b_{p,\xi}(x)| \leq d(p,x)$. Moreover $\Delta_x f(\xi)e^{-hb_{p,\xi}(x)} = 0$ and $b_{p,\xi}(\cdot)$ is smooth, because of $\Delta_x b_{p,\xi} = h$. Therefore,

$$\Delta_x \int\limits_{X(\infty)} f(\xi)d\mu_x(\xi) = \int\limits_{X(\infty)} f(\xi) \underbrace{\Delta_x e^{-hb_{p,\xi}(x)}}_{=0} d\mu_p(\xi) = 0.$$

(b) Now we prove

$$\lim_{x\to\xi_0} \int\limits_{X(\infty)} f(\xi)d\mu_x(\xi) = f(\xi_0).$$

Let $\xi_0 = c_{v_0}(\infty)$ with $v_0 \in S_pX$. Without loss of generality, we can assume that $f(\xi_0) = 0$ (by subtracting a constant if necessary). Let $\epsilon > 0$ be given. Then there exists $\delta > 0$, such that

$$|\ f(c_v(\infty))\ | \leq \epsilon \quad \forall\, v \in S_pX \text{ with } \angle_p(v_0, v) \leq \delta.$$

We split the integral representing $H_f(x)$ in the following way:

$$\omega_n|H_f(x)| \leq \left| \int_{S_pX\ \setminus\ \{v\ |\ \angle(v_0,v)\leq\delta\}} f(c_v(\infty))\ e^{-hb_v(x)}\ d\theta_p(v) \right| + \left| \int_{\{v\ |\ \angle(v_0,v)\leq\delta\}} f(c_v(\infty))\ e^{-hb_v(x)}\ d\theta_p(v) \right|.$$

Now, using Proposition 6.1, we obtain for all $x \in C(v_0, \delta/2)$ and $C_1 = C_1(\delta/2)$

$$\begin{aligned} \omega_n|H_f(x)| \leq \|f\|_\infty \int_{S_pX\ \setminus\ \{v\ |\ \angle(v_0,v)\leq\delta\}} & e^{-h(d(p,x)-C_1)}\ d\theta_p(v) + \\ \epsilon \int_{\{v\ |\ \angle(v_o,v)\leq\delta\}} & e^{-hb_v(x)}\ d\theta_p(v) \leq \\ \|f\|_\infty\ \omega_n\ e^{hC_1}\ e^{-hd(p,x)} + \epsilon & \underbrace{\int_{S_pX} e^{-hb_v(x)} d\theta_p(v)}_{=\int_{S_xX} d\theta_x(v)=\omega_n} \leq \\ & \omega_n \left(\epsilon + \|f\|_\infty\ e^{hC_1}\ e^{-hd(p,x)}\right). \end{aligned}$$

Let $x_n = c_{v_n}(r_n)$ with $v_n \in S_pX$ and $r_n \geq 0$ be a sequence converging to $\xi_0 \in X(\infty)$. Then we have $r_n = d(p,x_n) \to \infty$ and $\angle_p(v_0, v_n) \to 0$. Since $\epsilon > 0$ was arbitrary, the above estimate shows that

$$H_f(x) \to 0 \quad \text{for}\, x \to \xi_0.$$

(c) Uniqueness of the solution follows from the maximum principle. □

Remark The above considerations show that rank one asymptotically harmonic manifolds (X, g) with reference point $x_0 \in X$ satisfying (1.1) admit Poisson kernels of the form $P(x, \xi) = e^{-hb_{x_0,\xi}(x)}$.

These Poisson kernels can be used to define a map $\varphi : X \ni x \to P(x, \xi) d\mu_{x_0}(\xi) \in \mathcal{P}(X(\infty))$, where $\mathcal{P}(X(\infty))$ is the space of all probability measures on ∂X which are absolutely continuous to μ_{x_0}. $\mathcal{P}(X(\infty))$ carries a natural Riemannian metric G, called the *Fisher-Information metric* (see [Fr] or [ItSa1] for more details). The following was proved in [ItSa1, Prop. 1] for homogeneous Hadamard manifolds of dimension n: if (X, g) admits Poisson kernels of the form $P(x, \xi) = e^{-cb_{x_0,\xi}(x)}$ with $c > 0$, then the Poisson kernel map $\varphi : X \to \mathcal{P}(X(\infty))$ satisfies $\varphi^* G = \frac{c^2}{n} g$, i.e., that φ is a homothety. Examples of such spaces are rank one symmetric spaces of non-compact type and Damek-Ricci spaces. Conversely, the following was shown in [ItSa2, Thm 1.3]: If (X, g) is an n-dimensional Hadamard manifold admitting a Poisson kernel map $\varphi : X \to \mathcal{P}(X(\infty))$, which is both a *homothety* with constant $\frac{c^2}{n}$, $c > 0$ and *minimal*, then (X, g) is necessarily asymptotic harmonic with horospheres of mean curvature c. These results provide an interesting characterization of asymptotic harmonic manifolds via the Poisson kernel map.

7. Polynomial volume growth of horospheres

Let (X, g) be a rank one asymptotically harmonic manifold satisfying (1.1). Let $W^s(v) \subset SX$ be a strong stable manifold through $v \in SX$. Its projection $\mathcal{H}_v = \pi W^s(v) \subset X$ is a horosphere orthogonal to v. Let $p = \pi(v)$. Consider a curve

$$\beta : [0, 1] \to \mathcal{H}_v$$

with length$(\beta) \le r$. Let $\gamma : [0, 1] \to W^s(v)$ be the lift of β in the strong stable manifold and $\beta_t = \pi \Phi^t \gamma$, where Φ^t is the geodesic flow on SX. We conclude from [KnPe2, Corollary 2.6] that

$$\mathrm{length}(\beta_t) \le a_2 r e^{-\frac{\rho}{2} t}$$

for all $t \ge 0$. Hence length$(\beta_t) \le 1$ for all

$$t \ge t_0 := \frac{2 \log(a_2 r)}{\rho}.$$

Since the curvature of X and the second fundamental form of horospheres are bounded, the Gauss equation implies that the sectional curvatures of horospheres are bounded, as well. Therefore, by the volume comparison theorem, any ball of radius 1 in any horosphere has an intrinsic volume bounded by some constant $A > 0$:

$$\mathrm{vol}_{\mathcal{H}}(B_1(q)) \le A \quad \forall\, \mathcal{H} \text{ horospheres } \forall\, q \in \mathcal{H}.$$

This implies that

$$\begin{aligned} \mathrm{vol}_{\mathcal{H}_v}(B_r(p)) &\le \mathrm{vol}_{\mathcal{H}_v}(\Phi^{-t_0}(B_1(\pi \circ \Phi^{t_0}(v)))) \\ &\le e^{ht_0}\, \mathrm{vol}_{\mathcal{H}_{\Phi^{t_0}(v)}}(B_1(\pi \circ \Phi_{t_0}(v))) \le A e^{ht_0} = A' r^{\frac{2h}{\rho}}, \end{aligned}$$

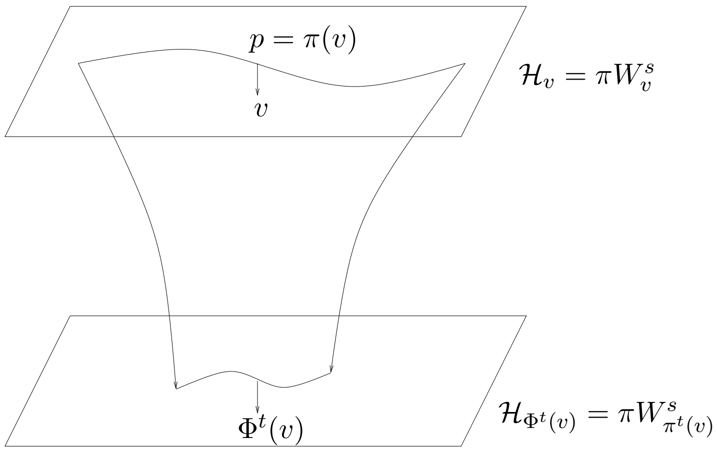

FIGURE 6. Contraction of the geodesic flow on stable horospheres

with $A' = Aa_2^{\frac{2h}{\rho}}$. This proves that all horospheres have polynomial volume growth in (X, g). □

8. Horospherical means and bounded eigenfunctions

In this section, we are mainly concerned with horospherical means of bounded eigenfunctions on rank one asymptotically harmonic manifolds X satisfying (1.1). Before we consider the special class of eigenfunctions, we first state a general result for *all* continuous functions on the geometric compactification $\overline{X}$. The underlying space is also more general than just rank one asymptotically harmonic manifolds.

Theorem 8.1. *Let (X, g) be a complete, simply connected Riemannian manifold without conjugate points of dimension n. Assume that the geometric compactification $\overline{X} = X \cup X(\infty)$ carries a topology such that the maps $\bar{\varphi}_p : \overline{B_1(p)} \to \overline{X}$ are homeomorphisms for all $p \in X$ (see Section 3 for details). Moreover, we assume that the following holds for every horosphere $\mathcal{H} \subset X$:*

(a) *We have* $\mathrm{vol}_{n-1}(\mathcal{H}) = \infty$.
(b) *For every ball $B_r(p)$ of radius $r > 0$ around $p \in X$, we have*

$$\mathrm{vol}_{n-1}(\mathcal{H} \cap B_r(p)) < \infty.$$

(c) *The closure of $\mathcal{H}$ in the geometric compactification $\overline{X}$ satisfies*

$$\overline{\mathcal{H}} = \mathcal{H} \cup \{\xi\},$$

where $\xi \in X(\infty)$ is the center of $\mathcal{H}$.

Then we have for every horosphere $\mathcal{H} \subset X$ centered at $\xi \in X(\infty)$, every compact exhaustion $\{K_j\}$, and every continuous function $f : \overline{X} \to \mathbb{R}$:

$$\lim_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\mathrm{vol}_{n-1}(K_j)} = f(\xi). \tag{8.1}$$

Proof. Let $\mathcal{H}$ be centered at $\xi \in X(\infty)$ and $p_0 \in X$. We show first indirectly that for every open neighbourhood $U \subset \overline{X}$ of ξ there exists $R > 0$ such that

$$\mathcal{H} \subset B_R(p_0) \cup U. \tag{8.2}$$

Assume that there exists $x_n \in \mathcal{H}$ with $x_n \notin U$ and $d(p_0, x_n) \to \infty$. Then, after choosing a subsequence if necessary, we have $x_n \to \xi' \in X(\infty)$ with $\xi' \neq \xi$. But this is ruled out by (c).

Let $\{K_j\}$ be a compact exhaustion and $\epsilon > 0$ be given. Then there exists an open neighbourhood $U \subset \overline{X}$ of ξ such that

$$|f(q) - f(\xi)| < \epsilon \quad \text{for all } q \in U.$$

Let $R > 0$ such that (8.2) is satisfied. Let $K_{j,0} = K_j \cap B_r(p_0)$ and $K_{j,1} = K_j \backslash K_{j,0} \subset U$. Then (a) and (b) yield $1/\operatorname{vol}(K_j) \int_{K_{j,0}} f \to 0$ and $\operatorname{vol}(K_{j,1})/\operatorname{vol}(K_j) \to 1$, which imply

$$f(\xi) - \epsilon \leq \liminf_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\operatorname{vol}_{n-1}(K_j)} \leq \limsup_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\operatorname{vol}_{n-1}(K_j)} \leq f(\xi) + \epsilon.$$

This shows (8.1), since $\epsilon > 0$ was arbitrary. □

The following proposition states that Theorem 8.1 is applicable in our setting of rank one asymptotically harmonic manifolds.

Proposition 8.2. *Let (X, g) be a rank one asymptotically harmonic manifold of dimension n satisfying* (1.1). *Then every horosphere $\mathcal{H} \subset X$ satisfies properties (a), (b), and (c) in Theorem 8.1.*

Before we present the proof of the proposition, we first introduce some useful notation. Let $\mathcal{H} \subset X$ be a horosphere. Then there exists $p_0 \in \mathcal{H}$ and $v \in S_{p_0}X$ such that $\mathcal{H} = b_v^{-1}(0)$. Let $\mathcal{H}_t = b_v^{-1}(t)$ and $\eta_t : X \to X$ be the flow associated to $\operatorname{grad} b_v$. Then $\mathcal{H} = \mathcal{H}_0$, $\eta_t : \mathcal{H}_0 \to \mathcal{H}_t$ and, for every $A \subset \mathcal{H}_0$ and $A(t) = \eta_t(A) \subset \mathcal{H}_t$ we have (see [PeSa, Prop. 3.1])

$$\operatorname{vol}_{n-1}(A(t)) = e^{ht} \operatorname{vol}_{n-1}(A). \tag{8.3}$$

Proof. (a) Assume there is a horosphere $\mathcal{H} \subset X$ with $\operatorname{vol}_{n-1}(\mathcal{H}) < \infty$. Using the above notation associated to $\mathcal{H}$, we see that the horoball

$$\mathcal{B} = \bigcup_{t \leq 0} \mathcal{H}_t = b_v^{-1}((-\infty, 0])$$

must also be of finite volume, since

$$\operatorname{vol}_n(\mathcal{B}) = \int_{-\infty}^{0} e^{ht}\, dt \operatorname{vol}_{n-1}(\mathcal{H}) = \frac{1}{h} \operatorname{vol}_{n-1}(\mathcal{H}).$$

But $\mathcal{B}$ contains the balls $B_r(c_v(r)) \subset X$ with arbitrarily large radii $r > 0$, whose volumes become arbitrarily large because of Proposition 2.2. This is a contradiction.

(b) Let $\mathcal{H}$ be a horosphere and $B_r(p) \subset X$ a ball. Let $A = \mathcal{H} \cap B_r(p)$ and assume that $\operatorname{vol}_{n-1}(A) = \infty$. Let $A_1 = \bigcup_{0 \leq t \leq 1} A(t)$. Then we also have $\operatorname{vol}_n(A_1) = \infty$, by (8.3). But $A_1 \subset B_{r+1}(p)$, and $B_{r+1}(p)$ has finite volume. This is, again, a contradiction.

(c) Since $\mathcal{H} = b_v^{-1}(0)$ is closed in X, we only need to show that $\mathcal{H}$ has no other accumuluation points in $X(\infty)$ other than ξ. We proceed indirectly.

Assume there exist $x_n \in \mathcal{H}$ with $d(p, x_n) \to \infty$ and $\lim x_n = \xi' \in X(\infty)$ and $\xi' \neq \xi$. Then we can find $\delta > 0$ such that $\xi' = c_w(\infty)$ for some $w \in S_{p_0}X$ with $\angle(w, v) > \delta$. Using the remark after Proposition 6.1, we know that there exists $s < 0$ such that $\mathcal{H}_s = \eta_s(\mathcal{H}) \subset C(v, \delta)$. Let $x_n(s) = \eta_s(x_n) \in \mathcal{H}_s$. Since $d(x_n, x_n(s)) = s$, we still have $x_n(s) \to \xi'$ and $x_n(s) \in \mathcal{H}_s \subset C(v, \delta)$ and, therefore, $\angle(w, v) \leq \delta$, which is a contradiction. □

Next we prove the main result of this section for bounded eigenfunctions (see Theorem 1.6 in the Introduction). The proof is similar to the proof of Theorem 1 in [KP].

Theorem 1.6. *Let (X, g) be a rank one asymptotically harmonic manifold of dimension n satisfying* (1.1) *and $h > 0$ be the mean curvature of all horospheres. Let $\lambda \neq 0$ be a real number and $f \in C^\infty(X)$ be a bounded function satisfying $\Delta f + \lambda f = 0$ and $\mathcal{H} \subset X$ be a horosphere with* isoperimetric exhaustion $\{K_j\}$. *Then we have*

$$\lim_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\mathrm{vol}_{n-1}(K_j)} = 0. \tag{8.4}$$

Remark Since horospheres have polynomial volume growth, the intrinsic balls of suitably chosen increasing radii r_j satisfy

$$\frac{\mathrm{vol}_{n-2}(\partial B_{\mathcal{H}}(r_j))}{\mathrm{vol}_{n-1}(B_{\mathcal{H}}(r_j))} \to 0.$$

A suitable choice of sets K_j are regularized spheres, as explained in [KP, p. 665]. But there might be many more increasing sets satisfying this asymptotic isoperimetric property.

Proof. We give an indirect proof. Assume that (8.4) is not satisfied. Then we can assume – by replacing $\{K_j\}$ by a subsequence, if needed – that there exists $c \neq 0$ such that

$$\lim_{j\to\infty} \frac{\int_{K_j} f(x)dx}{\mathrm{vol}_{n-1}(K_j)} = c.$$

Let $\eta_t : \mathcal{H}_0 \to \mathcal{H}_t$ be again the flow defined above after Proposition 8.2. Let $K_j(t) = \eta_t(K_j) \subset \mathcal{H}_t$. Recall that we have

$$\mathrm{vol}_{n-1}(K_j(t)) = e^{ht}\,\mathrm{vol}_{n-1}(K_j).$$

Since X has a lower sectional curvature bound, there exists $C > 0$ such that

$$\mathrm{vol}_{n-2}(\partial K_j(t)) \leq e^{C|t|}\,\mathrm{vol}_{n-2}(\partial K_j).$$

This implies that, *on every compact interval $I \subset [0, \infty)$*, we have

$$\left\| \frac{\mathrm{vol}_{n-2}(\partial K_j(\cdot))}{\mathrm{vol}_{n-1}(K_j(\cdot))} \right\|_{\infty, I} \to 0, \quad \text{as } j \to \infty.$$

Define

$$g_j(t) = \frac{\int_{K_j(t)} f(x)dx}{\mathrm{vol}_{n-1}(K_j(t))} \quad \forall\, t \in \mathbb{R}.$$

Since $\|g_j\|_\infty \leq \|f\|_\infty$, using diagonal arguments, we find a subsequence g_{j_k} such that $g_{j_k}(t) \to g(t)$, for all rational $t \in \mathbb{Q}$. Since $|\nabla f|$ is uniformly bounded by Yau's gradient estimate [Yau, Theorem 3], f is uniformly continuous and therefore, the sequence g_{j_k} is equicontinuous. This implies that we have $g_{j_k} \to g$ pointwise to a continuous limit and $g(0) = c \neq 0$.

Next we show that g satisfies

$$g'' + hg' + \lambda g = 0, \tag{8.5}$$

in the distributional sense. Let $\psi \in C_0^\infty(\mathbb{R})$ be a test function. Then we have

$$\int_{-\infty}^{\infty} g_j(t)(\psi''(t) - h\psi'(t) + \lambda h)dt = \int_{-\infty}^{\infty} \frac{\int_{K_j(t)} f(x)dx}{\mathrm{vol}_{n-1}(K_j(t))}(\psi''(t) - h\psi'(t) + \lambda\psi)dt.$$

Let $\tilde{f} : \mathcal{H} \times (-\infty, \infty) \to \mathbb{R}$ be defined as $\tilde{f}(x,t) := f(\eta_t(x))$. The tranformation formula yields:

$$\int_{K_j(t)} f(x)dx = \int_{K_j} f \circ \eta_t(x) \overbrace{\mathrm{Jac}\ \eta_t(x)}^{e^{ht}} dx = e^{ht} \int_{K_j} \tilde{f}(x,t)dx.$$

Therefore, we have $g_j(t) = 1/\,\mathrm{vol}_{n-1}(K_j) \int_{K_j} f(\eta_t x)dx$, and

$$\begin{aligned}
g_j''(t) + hg_j'(t) + \lambda g_j &= \frac{1}{\mathrm{vol}_{n-1}(K_j)} \int_{K_j} \frac{d^2}{dt^2} f(\eta_t x) + h\frac{d}{dt} f(\eta_t x) + \lambda f(\eta_t x)dx \\
&= \frac{1}{\mathrm{vol}_{n-1}(K_j(t))} \int_{K_j(t)} \underbrace{\Delta_x f(x) + \lambda f(x)}_{=0} - \Delta_{\mathcal{H}_t} f(x)dx \\
&= -\frac{1}{\mathrm{vol}_{n-1}(K_j(t))} \int_{K_j(t)} \Delta_{\mathcal{H}_t} f(x)dx \\
&= \frac{1}{\mathrm{vol}_{n-1}(K_j(t))} \int_{\partial K_j(t)} \langle \mathrm{grad}_{\mathcal{H}_t} f(x), \nu_x \rangle dx,
\end{aligned}$$

where ν_x denotes the outward unit vector of $\partial K_j(t) \subset \mathcal{H}_t$. Since $\mathrm{supp}\,\psi \subset \mathbb{R}$ is compact, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} g_j(t)(\psi''(t) - h\psi'(t) + \lambda\psi(t))dt &= \int_{-\infty}^{\infty} (g_j''(t) + hg_j'(t) + \lambda g_j(t))\psi(t)dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\mathrm{vol}_{n-1}(K_j(t))} \int_{\partial K_j(t)} \langle \mathrm{grad}_{\mathcal{H}(t)} f(x), \nu_x \rangle dx\, \psi(t)dt.
\end{aligned}$$

Taking absolute value and using, again, Yau's gradient estimate [Yau, Theorem 3], we obtain

$$\left| \int_{-\infty}^{\infty} g_j(t)(\psi''(t) - h\psi'(t) + \lambda\psi(t))dt \right|$$
$$\leq \int_{\operatorname{supp}\psi} \frac{\mathrm{vol}_{n-2}(\partial K_j(t))}{\mathrm{vol}_{n-1}(K_j(t))} \, \| \operatorname{grad}_X f \|_\infty \, \|\psi\|_\infty \, dt \to 0,$$

as $j \to \infty$. By Lebesgue's dominated convergence, and since $\|g\|_\infty, \|g_j\|_\infty \leq \|f\|_\infty$, we conclude that

$$\int_{-\infty}^{\infty} g(t)(\psi''(t) - h\psi'(t) + \lambda\psi(t))dt = 0,$$

i.e., the continuous function g satisfies (8.5) in the distributional sense. Therefore, g is smooth and satisfies $g'' + hg' + \lambda g = 0$ in the classical sense. This implies that g is of the general form

$$g(t) = c_1 e^{\left(-\frac{h}{2} + \sqrt{\left(\frac{h}{2}\right)^2 - \lambda}\right)t} + c_2 e^{\left(-\frac{h}{2} - \sqrt{\left(\frac{h}{2}\right)^2 - \lambda}\right)t} \tag{8.6}$$

if $\lambda \neq (h/2)^2$ and

$$g(t) = c_1 e^{-\frac{h}{2}t} + c_2 t e^{-\frac{h}{2}t}$$

if $\lambda = (h/2)^2$. It is straightforward to check for $\lambda \neq 0$ that every choice of $(c_1, c_2) \neq (0,0)$ leads to an unbounded function $g(t)$. But g must be bounded because of $\|g\|_\infty \leq \|f\|_\infty$. Therefore we conclude that $(c_1, c_2) = (0,0)$ in contradiction to $g(0) = c \neq 0$, finishing the indirect proof. □

Examples of bounded eigenfunctions. *(a) Let X be a rank one symmetric space of non-compact type and $M = X/\Gamma$ be a compact quotient. Then every non-constant Δ_M-eigenfunction $f \in C^\infty(M)$ gives rise to a bounded lift $\widetilde{f} \in C^\infty(X)$ which is also a Δ_X-eigenfunction to the same eigenvalue. Since $\widetilde{f}$ is non-constant and Γ-periodic, it does not admit a continuous extension to the compacitification $\overline{X}$.*

(b) Let $X^{(p,q)}$ be a Damek-Ricci space with p, q defined as in [Rou]. *Then $X^{(p,q)}$ is an asymptotically harmonic manifold with $h = p/2 + q$ and there exist radial eigenfunctions $\varphi_\mu \in C^\infty(X^{(p,q)})$ satisfying*

$$\Delta\varphi_\mu + \left(\mu^2 + \left(\frac{h}{2}\right)^2\right)\varphi_\mu = 0 \quad \textit{and } \varphi_\mu(e) = 1,$$

where $\mu \in \mathbb{C}$ and $e \in X^{(p,q)}$ denotes the neutral element in the Damek-Ricci space considered as a solvable group. If $0 < i\mu < h/2$, we have (see [Rou, p. 78]*)*

$$\varphi_\mu(r) \sim c(\mu) e^{(i\mu - h/2)r} \qquad \textit{as } r \to \infty$$

with suitable constants $c(\mu) \in \mathbb{R}\backslash\{0\}$. This means that φ_μ is a bounded eigenfunction with trivial *continuous extension to the compactification $\overline{X^{(p,q)}}$.*

Now we are in a position to prove our final result (see Theorem 1.7 in the Introduction) which states that the above examples are the only two possible cases with regards to continuous extensions of bounded eigenfunctions f: either f cannot be extended to $\overline{X}$ or the extension is trivial.

Theorem 1.7. *Let (X, g) be a rank one asymptotically harmonic manifold satisfying* (1.1)*. Let $\lambda \in \mathbb{R}\backslash\{0\}$ and $f \in C^\infty(X)$ be an eigenfunction $\Delta f + \lambda f = 0$. If f has a continuous extension $F \in C(\overline{X})$ then we have necessarily $F|_{\partial X} \equiv 0$.*

Proof. Assume that $\lambda \neq 0$ and that an eigenfunction $\Delta f + \lambda f = 0$ has a continuous extension F on the compactification $\overline{X}$. Then we know from Theorem 8.1 that all horospherical means of f over horospheres centered at $\xi \in X(\infty)$ agree with $F(\xi)$. On the other hand, we conclude from Theorem 1.6 that all horospherical means with isoperimetric exhaustions have to vanish. Moreover, every horosphere in X has polynomial volume growth and, therefore, admits isoperimetric exhaustions. This implies that $F|_{\partial X} \equiv 0$. □

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