COUNTING ONE SIDED SIMPLE CLOSED GEODESICS ON FUCHSIAN THRICE PUNCTURED PROJECTIVE PLANES

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ABSTRACT. We prove that there is a true asymptotic formula for the number of one sided simple closed curves of length $\leq L$ on any Fuchsian real projective plane with three points removed. The exponent of growth is independent of the hyperbolic metric, and it is noninteger, in contrast to counting results of Mirzakhani for simple closed curves on orientable Fuchsian surfaces.

1. INTRODUCTION

Let $\Sigma := P^2(\mathbf{R}) - \{3 \text{ points}\}$, the three times punctured real projective plane. It is the fixed topological surface of interest in this paper. A hyperbolic metric[1] J of finite area on Σ gives a way to measure the length of curves. For fixed J, any isotopy class of nonperipheral simple closed curve $[\gamma]$ on Σ has a unique geodesic representative, and we call the length of this geodesic with respect to J simply the length of $[\gamma]$.

It is known by work of Mirzakhani [11] that for a fixed finite area hyperbolic metric J on an *orientable* surface S, the number $n_J(L)$ of isotopy classes of simple closed curves of length $\leq L$ has an asymptotic formula:

Theorem 1 (Mirzakhani)**.**

$$n_J(L) = cL^d + o(L^d)$$

where $c = c(J) > 0$ and $d = d(S) > 0$ is the **integer** *dimension of the space of compactly supported measured laminations on S.*

In the case of the once punctured torus, a stronger form of Theorem 1 was obtained previously by McShane and Rivin [12].

An isotopy class of simple closed curve in Σ is said to be *one sided* if cutting along this curve creates only one boundary component, or in other words, a thickening of this curve is homeomorphic to a Möbius band. The point of the current paper is

M. Magee was supported in part by NSF award DMS-1701357.

[1] A hyperbolic metric is a complete Riemannian metric of constant curvature -1.

to establish an asymptotic formula for $n_J^{(1)}(L)$, the number of isotopy classes of *one sided* simple closed curves of length $\leq L$ with respect to a given hyperbolic metric J on Σ.

Theorem 2. *There is a **noninteger** parameter $\beta > 0$ such that for any finite area hyperbolic metric J on $P^2(\mathbf{R}) - \{3 \text{ points}\}$,*

$$n_J^{(1)}(L) = cL^\beta + o(L^\beta)$$

for some $c = c(J) > 0$.

The parameter β appeared for the first time in the work of Baragar [1, 2, 3] in connection with the affine varieties

$$V_{n,a} : \quad x_1^2 + x_2^2 + \ldots + x_n^2 = a x_1 x_2 \ldots x_n.$$

These varieties have a rich automorphism group that contains an embedded copy of $\mathcal{G} := C_2^{*n}$, the free product of a cyclic group of size 2 with itself n times. Baragar proved that for $o \in V(\mathbf{Z})$ the following limit exists and is independent of o:

$$\lim_{R \to \infty} \frac{\log |\mathcal{G}.o \cap B_{\ell^\infty}(R)|}{\log \log R} = \beta(n) > 0.$$

The variety $V_{4,1}$ was connected to the Teichmüller space of Σ by Huang and Norbury in [8]. The value β of Theorem 2 is therefore $\beta := \beta(4)$ that Baragar estimated to be in the range

$$2.430 < \beta(4) < 2.477.$$

Using Baragar's result, Huang and Norbury proved in [8] for an arbitrary finite area hyperbolic metric J on Σ that[2]

$$\lim_{L \to \infty} \frac{\log n_J^{(1)}(L)}{\log L} = \beta.$$

A true asymptotic count for the integer points $V_{n,a}(\mathbf{Z})$ was obtained[3] by Gamburd, Magee and Ronan in [5, Theorem 3].

Theorem 3 (Gamburd-Magee-Ronan)**.** *Let $o \in V_{n,a}(\mathbf{Z})$ and $\beta = \beta(n)$ as for Baragar* [1]*. There is $c(o) > 0$ such that*

$$|\mathcal{G}.o \cap B_{\ell^\infty}(R)| = c(\log R)^\beta + o\left((\log R)^\beta\right).$$

[2] This statement corrects the statement in [8, Theorem 3].

[3] In fact the paper [5] treats slightly more general varieties than $V_{n,a}$.

This is a strengthening of Baragar's result analogous to the main Theorem 2. In the sequel we show how to combine and refine the arguments of [5] and [8] to prove Theorem 2.

We also point out the recent preprint of Gendulphe [6] who has begun a systematic investigation into the issues of growth rates of simple geodesics on general non-orientable surfaces.

Acknowledgements. We would like to thank Yi Huang for explaining his work to us, and to the anonymous referees who made helpful suggestions.

2. Orbits on Teichmüller space

The *curve complex* of Σ is the simplicial complex whose vertices are isotopy classes of one sided simple closed curves, and a collection of $k+1$ curves span a k-simplex if they pairwise intersect once[4]. We write Z for this complex that was introduced by Huang and Norbury in [8], and its 1-skeleton was studied earlier by Scharlemann in [15]. It is a pure complex of dimension 3, that is, all maximal simplices are 3 dimensional. Throughout the paper we use the notation Z^k for the k-simplices of Z.

The collection of all finite area hyperbolic metrics on Σ, up to isotopy, is called the *Teichmüller space of* Σ and denoted by $\mathcal{T}(\Sigma)$. It has a natural real analytic structure.

Let V be the affine subvariety of $\mathbf{C}^4$ cut out by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1 x_2 x_3 x_4. \tag{2.1}$$

It was proven by Hu, Tan and Zhang in [7, Theorem 1.1] that the automorphism group of the complex variety V is given by

$$\Lambda \rtimes (N \rtimes S_4)$$

where

(1) N is the group of transformations that change the sign of an even number of variables.
(2) S_4 is the symmetric group on 4 letters that acts by permuting the coordinates of $\mathbf{C}^4$.

[4]This is not the standard definition of the curve complex. The curve complex of an orientable surface was introduced by Harvey in [Har81] (see also Hatcher and Thurston [HT80]). The simplices of these curve complexes are given by isotopy classes of **disjoint** simple closed curves.

(3) Λ is a nonlinear group generated by *Markoff moves*, e.g.

$$m_1(x_1, x_2, x_3, x_4) = (x_2x_3x_4 - x_1, x_2, x_3, x_4)$$

replaces x_1 by the other root of the quadratic obtained by fixing x_2, x_3, x_4 in (2.1). Similarly there are moves m_2, m_3, m_4 that flip the roots in the other coordinates, and m_1, m_2, m_3, m_4 generate a subgroup

$$\Lambda \cong C_2 * C_2 * C_2 * C_2 \tag{2.2}$$

of $\mathrm{Aut}(V)$ where the m_i correspond to the generators of the C_2 factors.

Since the abstract group C_2^{*4} acts in different ways in the sequel, we let

$$\mathcal{G} := C_2^{*4}.$$

We obtain an action of $\mathcal{G}$ on $V(\mathbf{R}_+)$ by the identification (2.2).

Huang and Norbury in [8] prove that $V(\mathbf{R}_+)$ can be identified with $\mathcal{T}(\Sigma)$ by the following map. Let $\Delta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be an ordering of a 3-simplex of Z. Let $\ell_{\alpha_j}(J)$ be the length of the geodesic representative of α_j in the metric of J. Define a map

$$\Theta_\Delta(J) := (x_{\alpha_1}(J), x_{\alpha_2}(J), x_{\alpha_3}(J), x_{\alpha_4}(J))$$

where

$$x_{\alpha_i}(J) := \sqrt{2\sinh\left(\frac{1}{2}\ell_{\alpha_i}(J)\right)}. \tag{2.3}$$

Building on work of Penner [14], Huang and Norbury show

Theorem 4 ([8, Proposition 8 and Section 2.4])**.** *For any ordering of the curves $\Delta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in a 3-simplex of Z,*

$$\Theta_\Delta : \mathcal{T}(\Sigma) \to V(\mathbf{R}_+)$$

is a real analytic diffeomorphism.

Let Z^3_{ord} denote tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a 3-simplex of Z. It is more symmetric to consider instead of Theorem 4, the pairing

$$\langle \bullet, \bullet \rangle : \mathcal{T}(\Sigma) \times Z^3_{\mathrm{ord}} \to V(\mathbf{R}_+),$$
$$\langle J, \Delta \rangle := \Theta_\Delta(J).$$

Huang and Norbury note for fixed $\Delta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in Z^3_{\mathrm{ord}}$ there is a unique way to 'flip' each α_i, $1 \le i \le 4$, to another one sided simple closed curve, say α_1' in the case of α_1 being flipped, so that e.g. $\Delta' = (\alpha_1', \alpha_2, \alpha_3, \alpha_4)$ is in Z^3_{ord}, i.e. α_1'

intersects each of $\alpha_2, \alpha_3, \alpha_4$ once. Similarly flipping the other α_i yields new elements of Z^3_{ord}.

This yields an action of $\mathcal{G}$ on Z^3_{ord} where the generator of the first C_2 factor always acts by flipping the first curve and so on. Recall also the action of $\mathcal{G}$ on $V(\mathbf{R}_+)$. The pairing $\langle \bullet, \bullet \rangle$ is equivariant for the action of $\mathcal{G}$ on the second factor:

$$\langle J, g.(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\rangle = g.\langle J, (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\rangle, \quad g \in \mathcal{G}.$$

3. Dynamics of the markoff moves

Our approach to counting relies on establishing the following properties for points $x \in V(\mathbf{R}_+)$ in various contexts.

- **A:** The largest entry of x appears in exactly one coordinate.
- **B:** If x_j is the largest coordinate of x then the largest entry of $m_j(x)$ is smaller than x_j, that is, $(m_j(x))_i < x_j$ for all i.
- **C:** If x_j is not the unique largest coordinate of x then it becomes the largest after the move m_j, that is, $(m_j(x))_j > (m_j(x))_i$ for all $i \neq j$.

We will have use for the following theorem due to Hurwitz [9], building on work of Markoff [10].

Theorem 5 (Markoff, Hurwitz: Infinite descent)**.** *If $x \in V(\mathbf{Z}_+) - (2,2,2,2)$ then Properties **A**, **B** and **C** hold for x.*

Proof. Hurwitz showed the corresponding result for the point $(1,1,1,1) \in V'(\mathbf{Z}_+)$ where V' is defined by

$$V': \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4x_1x_2x_3x_4$$

and $\mathcal{G}$ now acts on $V'(\mathbf{Z})$ via renormalized Markoff moves m'_i, e.g.

$$m'_1(x_1, x_2, x_3, x_4) = (4x_2x_3x_4 - x_1, x_2, x_3, x_4).$$

It is easy to check that the map $V'(\mathbf{Z}_+) \to V(\mathbf{Z}_+)$, $x \mapsto 2x$ is a $\mathcal{G}$-equivariant bijection, and in particular respects properties $\mathbf{A}, \mathbf{B}, \mathbf{C}$ for the two types of Markoff moves. $\square$

Corollary 6. *Every x in $V(\mathbf{Z}_+)$ has every entry $x_j \geq 2$ and is obtained by a unique series of nonrepeating m_j from $(2,2,2,2)$.*

The following observation will be used several times in the remainder of the section.

Lemma 7. *For any $o \in V(\mathbf{R}_+)$, the coordinates of $\mathcal{G}.o$ form a discrete set.*

Proof. For fixed $\Delta \in Z^3_{\text{ord}}$ let J be such that $\langle J, \Delta \rangle = o$. Then $\mathcal{G}.o = \langle J, \mathcal{G}.\Delta \rangle$ and the coordinates are all obtained as $\sqrt{2 \sinh\left(\frac{1}{2}\ell\right)}$ where ℓ is the length of some one sided simple closed curve in Σ w.r.t. J. Since these values of ℓ are discrete in $\mathbf{R}_+$ and $\sinh^{1/2}$ has bounded below derivative in $\mathbf{R}_+$ we are done. □

Lemma 7 has the following fundamental consequence that makes our counting arguments work.

Lemma 8. *For every point $o \in V(\mathbf{R}_+)$ there is some $\epsilon = \epsilon(o) > 0$ such that for all $x \in \mathcal{G}.o$ we have*

$$x_i x_j \geq 2 + \epsilon, \quad 1 \leq i < j \leq 4.$$

Proof. Let $x \in \mathcal{G}.o$ and without loss of generality suppose $x_1 \leq x_2 \leq x_3 \leq x_4$. Since from (2.1)

$$x_1 x_2 x_3 x_4 - x_3^2 - x_4^2 = x_1^2 + x_2^2 > 0$$

we obtain

$$x_1 x_2 x_3 x_4 > x_3^2 + x_4^2 \geq 2 x_3 x_4$$

implying that $x_1 x_2 > 2$. Since x_1 and x_2 are related by (2.3) to lengths of simple closed curves with respect to a hyperbolic metric $J = J(o)$, they take on discrete values in $\mathcal{G}.o$ that are bounded away from 0 and so the possible values of $x_1 x_2$ with $2 < x_1 x_2 < 3$ are discrete. This completes the proof[5]. □

We also need the following theorem that establishes Theorem 5 for an arbitrary orbit of $\mathcal{G}$, outside a compact set depending on the orbit. This gives an adaptation of [5, Proposition 14] to the current setting.

Theorem 9. *For given $o \in V(\mathbf{R}_+)$, there is a compact S_4-invariant set $K = K(\mathcal{G}.o) \subset V(\mathbf{R}_+)$ such that Properties **A**, **B** and **C** hold for $x \in \mathcal{G}.o - K$. Call a move that takes place at a non-largest entry of x **outgoing.** The set $\mathcal{G}.o - K$ is preserved under outgoing moves.*

Proof. Fix o throughout the proof. Lemma 8 tells us that for some $\epsilon > 0$, $x_i x_j \geq 2 + \epsilon$ for all $x \in \mathcal{G}.o$ and $1 \leq i < j \leq 4$. Let $K_0 := \{(x_1, x_2, x_3, x_4) \in V(\mathbf{R}_+) : \|x\|_\infty \leq 10\}$. We will choose K such that $K_0 \subset K$.

[5]An alternative interesting 'geometric' proof of Lemma 8 was pointed out to us by a referee. Fixing a hyperbolic metric J, then as depicted in [8, Figure 4], for any two one sided simple closed geodesics α_i, α_j on Σ w.r.t. J that intersect in a point, there is a unique two sided simple closed geodesic β w.r.t. J that is disjoint from α_1 and α_2. A formula of Norbury [13, Equation (6)] gives with $x_i = \sqrt{2 \sinh\left(\frac{\ell_{\alpha_i}(J)}{2}\right)}$ (according to (2.3)), that $x_i x_j = \sqrt{2\left(1 + \cosh\left(\frac{\ell_\beta(J)}{2}\right)\right)}$. Lemma 8 follows since the lengths of closed geodesics w.r.t. J are bounded below by a positive constant.

A. Take $x \in \mathcal{G}.o$. We'll prove something stronger than property **A** for suitable choice of K, and use this later in the proof. Suppose for simplicity $x_1 \le x_2 \le x_3 \le x_4$. Write $x_4 = x_3 + \delta$ and assume $\delta < \delta_0$ where $\delta_0 < 1$ is small enough to ensure

$$(1 + \delta x_3^{-1})x_1 x_2 - 1 - (1 + \delta x_3^{-1})^2 > \frac{\epsilon}{2} \tag{3.1}$$

given $x_3 \ge 9$ (which we know to be the case since $x \notin K_0$). We will enlarge K_0 so that this is a contradiction. From (2.1)

$$x_1^2 + x_2^2 + x_3^2(1 + (1 + \delta x_3^{-1})^2 - (1 + \delta x_3^{-1})x_1 x_2) = 0, \tag{3.2}$$

so $x_3^2(3 + (1 + \delta x_3^{-1})^2 - (1 + \delta x_3^{-1})x_1 x_2) > 0$ and hence

$$x_1 x_2 \le \frac{3 + (1 + \delta x_3^{-1})^2}{(1 + \delta x_3^{-1})} < 5 \tag{3.3}$$

given $x \notin K_0$ (so $x_3 \ge 9$) and the assumption $\delta < 1$. On the other hand (3.1) and (3.2) now imply that if $\eta > 0$ is a lower bound for all coordinates of $\mathcal{G}.o$ then

$$x_3^2 \le \frac{2}{\epsilon}\left(x_1^2 + x_2^2\right) \le \frac{4}{\epsilon}x_2^2 \le \frac{4}{\eta^2 \epsilon}x_1^2 x_2^2 \le \frac{100}{\eta^2 \epsilon},$$

where the last inequality is from (3.3).

Now let

$$K_1 = \{x \in V(\mathbf{R}_+) \,:\, \|x\|_\infty \le \frac{10}{\eta\sqrt{\epsilon}} + 1\} \cup K_0.$$

We proved there is $\delta = \delta(o)$ such that for $x \in \mathcal{G}.o - K_1$, there is an entry of x that is $\ge \delta$ more than all the other entries.

B. Take $x \in \mathcal{G}.o - K_1$ with $x_1 \le x_2 \le x_3 < x_4$. We follow the method of Cassels [4, pg. 27]. Consider the quadratic polynomial

$$f(T) = T^2 - x_1 x_2 x_3 T + x_1^2 + x_2^2 + x_3^2.$$

Then f has roots at x_4 and x_4' where x_4' is the last entry of $m_4(x)$. Property **B** holds at x unless $x_3 < x_4 \le x_4'$, in which case $f(x_3) > 0$ giving

$$x_3^2(4 - x_1 x_2) \ge x_3^2(2 - x_1 x_2) + x_1^2 + x_2^2 > 0.$$

Therefore $x_1 x_2 < 4$. By discreteness of the coordinates of $\mathcal{G}.o$ this means there are finitely many possibilities for x_1 and x_2. Now $x_4' \ge x_4$ directly implies

$$x_1 x_2 x_3 \ge 2x_4$$

so $2x_4^2 \le x_1 x_2 x_3 x_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and so

$$(x_4 + x_3)(x_4 - x_3) \le x_1^2 + x_2^2 \le M$$

for some M depending on the finitely many possible values for x_1, x_2. Since we know $x_4 - x_3 \geq \delta$ we obtain

$$x_4 + x_3 \leq \frac{M}{\delta}$$

so $x_3 \leq x_4 \leq M\delta^{-1}$. Let $K_2 := \{x \in V(\mathbf{R}_+) \ : \ \|x\|_\infty \leq M\delta^{-1}\} \cup K_1$. This establishes **B** for $x \in \mathcal{G}.o - K_2$.

C. Take $x \in \mathcal{G}.o$ with $x_1 \leq x_2 \leq x_3 \leq x_4$. Then for $1 \leq j \leq 3$,

$$(m_j(x))_j = \frac{x_1x_2x_3x_4}{x_j} - x_j \geq x_1x_2x_4 - x_3 = x_4\left(x_1x_2 - \frac{x_3}{x_4}\right) \geq x_4(1+\epsilon) > x_4.$$

by Lemma 8. This establishes **C**.

We established **A, B** for $x \in \mathcal{G}.o - K$ with $K = K_2$, and **C** for any $x \in \mathcal{G}.o$. It is clear from the previous that $\mathcal{G}.o - K$ is stable under outgoing moves. □

4. The topology of the curve complex

Our first goal in this section is to prove the following topological theorem. Let G be the graph whose vertices are 3-simplices $\{\alpha, \beta, \gamma, \delta\}$ of Z with an edge between two vertices if they share a dimension 2 face.

Theorem 10. *G is a 4-regular tree.*

Theorem 10 is stated by Huang and Norbury in [8, pg. 9] as a consequence of the following theorem of Scharlemann:

Theorem 11 ([15, Theorem 3.1])**.** *The* 1*-skeleton of Z is the* 1*-skeleton of the complex obtained by repeated stellar subdivision of the dimension* 2 *faces of a tetrahedron.*

It is not spelled out in detail in [8] how Theorem 10 follows from Theorem 11, we do so here for the reader's convenience[6].

Proof of Theorem 10. Let Y be the 1-skeleton of Z. Then Z is the clique complex[7] of Y. If Y_n is the subcomplex of Y obtained by n generations of stellar subdivisions of the faces of a regular tetrahedron, and Z_n is the clique complex of Y_n, then we get a filtration of Z by the Z_n.

The key observation is the following. Any 3-simplex Δ in Z_n that is not in Z_{n-1} has to contain a vertex v that is in Z_n and not in Z_{n-1}. Since v has been created as a

[6]We thank both Yi Huang and an anonymous referee for explaining how to simplify the proof of Theorem 10 we originally gave.

[7]Recall that the clique complex of a graph H has the same vertex set as H and a k-simplex for each clique (complete subgraph) of H of size $k+1$.

subdivision point of a triangle, it has valence 3 in Y_n and its neighbors w_1, w_2, w_3 are in Y_{n-1}. The triangle formed by w_1, w_2, w_3 on some face of the regular tetrahedron was part of the $(n-1)$st subdivision. Hence the 2-simplex given by w_1, w_2, w_3 is a face of a 3-simplex in Z_{n-1}. Moreover, this is the only 3-simplex in Z_n that shares a face with Δ.

Now let G_n be the subgraph of G induced by vertices of G contained in Z_n. Then G_0 is a single vertex, and we argue by induction that G_n is connected and acyclic. Let Δ be a vertex of G_n that is not in G_{n-1}. The discussion of the previous paragraph implies each vertex of G_n that is not contained in G_{n-1} has valence 1 and its neighbor is in G_{n-1}. This easily implies by induction that G_n is connected and acyclic for all n. Hence G is connected and acyclic, and since G is 4-regular the result follows. □

In the rest of this section we prove that smaller pieces of G are connected and acyclic. Specifically, for any simplex $\Delta \in Z$ we may form G_Δ, the subgraph of G induced by vertices containing Δ. For example, if Δ is a 2-simplex then G_Δ has two vertices and an edge representing a flip between them. If Δ is a 3-simplex then G_Δ has only one vertex, Δ. More generally,

Proposition 12. *For all $\Delta \subset Z$, G_Δ is a tree.*

Proof. Since G is acyclic it suffices to prove G_Δ is connected. We give the proof that G_δ is connected in the case δ is a vertex of Z, the case Δ is an edge is similar and we have already discussed the other cases.

Suppose δ is a vertex of $\Delta, \Delta' \in Z^3$. We aim to connect Δ to Δ' by flips that don't touch δ. Order Δ and Δ' so that δ is the final element of each. Let J be the hyperbolic metric provided by Theorem 4 such that $\langle J, \Delta\rangle = (2,2,2,2)$. Since $\Delta' = (\beta_1, \beta_2, \beta_3, \delta)$, $\langle J, \Delta'\rangle = (x_1, x_2, x_3, 2)$ for some $x_1, x_2, x_3 \in \mathbf{Z}$. By adapting the method used to prove Corollary 6, the infinite descent (Theorem 5) for $V(\mathbf{Z}_+)$ now yields a series of flips that never modifies δ, starts at Δ' and ends at some $\Delta'' = (\gamma_1, \gamma_2, \gamma_3, \delta) \in Z^3_{\text{ord}}$ with $\langle J, \Delta''\rangle = (2,2,2,2)$. Also note that by combining Theorem 5 and Theorem 10, there is a unique $\Delta_0 \in Z^3$ such that $\langle J, \Delta_0\rangle = (2,2,2,2)$ for any ordering of Δ_0. Therefore up to reordering, $\Delta'' = \Delta_0 = \Delta$ as required. □

5. Proof of theorem 2

Let Γ denote the mapping class group of Σ. Mapping classes in Γ may permute the punctures of Σ. The group Γ acts simplicially on Z in the obvious way.

Recall that for each 3-dimensional simplex $\Delta = \{\alpha, \beta, \gamma, \delta\} \in Z$, there is a unique flip of α that produces a new simplex $\{\alpha', \beta, \gamma, \delta\}$. Further to this, Huang and Norbury [8] construct a corresponding unique mapping class $\gamma^1_\Delta \in \Gamma$ that maps $\{\alpha, \beta, \gamma, \delta\}$ to $\{\alpha', \beta, \gamma, \delta\}$, similarly γ^2_Δ performs a flip at β and so on.

The mapping class elements γ^i_Δ can be extended to a cocycle for the group action of $\mathcal{G}$ on Z^3_{ord}. In other words, for every $\Delta \in Z^3_{\text{ord}}$ and $g \in \mathcal{G}$ there is a mapping class group element $\gamma(g, \Delta)$ such that $\gamma(g, \Delta)\Delta = g\Delta$. For example, if g_1 is the generator of the first factor of $\mathcal{G}$ then $\gamma(g, \Delta) = \gamma^1_\Delta$.

Proposition 13. *For any given Δ, if $\tilde{\Delta} \in Z^3_{\text{ord}}$ is an ordering of Δ then the map*

$$\begin{aligned} \mathcal{G} &\to Z^3 \\ g &\mapsto \gamma(g, \tilde{\Delta}).\Delta \end{aligned} \tag{5.1}$$

is a bijection.

Proof. The map $g \mapsto \gamma(g, \tilde{\Delta})\Delta$ yields a graph homomorphism from the Cayley graph of $\mathcal{G}$, a 4-regular tree, to G. Recall that G is also a 4-regular tree by Theorem 10. The homomorphism is locally injective. Therefore $g \mapsto \gamma(g, \tilde{\Delta})\Delta$ is a bijection. □

The next proposition allows us to pass from counting over G to counting over simple closed curves (our goal), up to finite subsets at either side of the passage.

Proposition 14. *Let $J \in \mathcal{T}(\Sigma)$ and for arbitrary fixed $\Delta_0 \in Z^3_{\text{ord}}$ let $o := \langle J, \Delta_0 \rangle$. Let K be a compact S_4-invariant subset of $V(\mathbf{R}_+)$ containing the set $K(\mathcal{G}.o)$ from Theorem 9. Since K is S_4-invariant, the condition $\langle J, \Delta \rangle \notin K$ is independent of the ordering of $\Delta \in Z^3$, and so well defined. The map*

$$\Phi : \{ \ \Delta \in Z^3 \ : \ \langle J, \Delta \rangle \notin K \ \} \to Z^0,$$

$$\Phi : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \mapsto \{\alpha_i\} \ : \ \ell_{\alpha_i}(J) = \max_{1 \le j \le 4} \ell_{\alpha_j}(J) \tag{5.2}$$

is a well defined injection whose image is all but finitely many elements of Z^0.

Proof. That Φ is well defined is immediate from Theorem 9, Property **A**.

Suppose $\delta \in Z^0$ is the longest curve in each of Δ, Δ' with respect to J, with $\langle J, \Delta \rangle, \langle J, \Delta' \rangle \notin K$. By Proposition 12 there is a series of flips taking Δ to Δ' and never modifying δ. By Property **C** of Theorem 9, the first flip creates a curve longer than δ w.r.t. J. This continues, since $\mathcal{G}.o - K$ is stable under outgoing moves, and it is therefore impossible to reach $\Delta' \neq \Delta$ since δ is the largest curve of Δ', but not of any intermediate simplex of the sequence that was generated. This establishes injectivity of Φ.

As for the final statement that the image of (5.2) misses only finitely many curves, let $\delta \in Z^0$. We aim to find $(\alpha', \beta', \gamma, \delta)$ for which δ is the longest curve with respect to J. Say that δ is *bad* if $\langle J, \Delta\rangle \in K$ for some Δ containing δ. Otherwise say δ is *good*. Since K is compact, and the set of lengths of one sided simple closed curves in J is discrete, there are only finitely many bad δ. We will prove all good δ are in the image of Φ. For good δ, begin with any $\Delta \in Z^3_{\text{ord}}$ such that $\langle J, \Delta\rangle \notin K$ and δ is last in Δ. If δ is the longest curve of Δ with respect to J then we are done. Otherwise let $(x_1, x_2, x_3, x_4) = \langle J, \Delta\rangle$. Using Property **B** of Theorem 9, apply moves at the largest entries of (x_1, x_2, x_3, x_4) (which do not correspond to δ) until δ becomes the longest curve. The resulting $(y_1, y_2, y_3, y_4) = \langle J, \Delta'\rangle$ cannot be in K, so we are done since $\delta = \Phi(\Delta')$. □

We have put all the pieces in place to use the methods of Gamburd, Magee and Ronan [5] to prove Theorem 2. We now give an overview of the method of [5] and explain how what we have already proved extends the method to the current setting.

Step 1. *(loc. cit.)* begins with a compact set K such that for $x \in \mathcal{G}.o - K$, properties **A**, **B**, and **C** hold. Here, we take K to be the set provided by Theorem 9. It is then deduced from **A**, **B**, and **C** that the number of distinct entries of $x \in \mathcal{G}.o - K$ cannot decrease during an outgoing move. There is a further regularization of K in [5, Section 2.4], by adding to K a large ball $B_{\ell^\infty}(R)$ if necessary, in order to assume that if for example $x_1 \le x_2 \le x_3 \le x_4$ with $(x_1, x_2, x_3, x_4) \in \mathcal{G}.o - K$ then

$$x_3 \ge \frac{1}{2} x_4^{\frac{1}{3}},$$

$$\frac{3\log(1 - 2x_4^{-\frac{1}{3}}) - 3\log 2}{\log x_4} \ge -\frac{1}{2},$$

and $x_4 \ge 10$. These inequalities play a role in technical estimates throughout the proof, in particular, the proof of [5, Lemma 21]. It is possible to increase K to ensure these hold (and the corresponding inequalities for other ordering of the coordinates of x) for the same reasons as in *(loc. cit.)*. Also, without loss of generality, $o \in K$.

Step 2. Recall the quantity $n_J^{(1)}(L)$ from our main Theorem 2. Fix $\Delta_0 \in Z^3_{\text{ord}}$ and let $o := \langle J, \Delta_0\rangle$. Let K be the enlarged compact set from Step 1.

Putting Propositions 13 and 14 (for the the current K) together gives us

$$
\begin{aligned}
n_J^{(1)}(L) &:= \sum_{\alpha \in Z^0} \mathbf{1}\{\ell_\alpha(J) \le L\} \\
\text{(Proposition 14)} &= \sum_{\Delta\colon \langle J,\Delta\rangle \notin K} \mathbf{1}\left\{\max\langle J,\tilde{\Delta}\rangle \le \sqrt{2\sinh\left(\frac{1}{2}L\right)}\right\} + O_J(1) \\
\text{(Proposition 13)} &= \sum_{g\in\mathcal{G}\colon \mathcal{G}.o\notin K} \mathbf{1}\left\{\max g.o \le \sqrt{2\sinh\left(\frac{1}{2}L\right)}\right\} + O_J(1), \qquad (5.3)
\end{aligned}
$$

where for $\Delta \in Z^3$ we wrote $\tilde{\Delta}$ for an arbitrary lift of Δ to Z^3_{ord}. Since

$$\sqrt{2\sinh\left(\frac{1}{2}L\right)} = \sqrt{e^{L/2}-e^{-L/2}} = e^{L/4}(1+O(e^{-L}))$$

the required asymptotic formula for (5.3) as $L\to\infty$ will follow from an estimate of the form

$$\sum_{g\in\mathcal{G}\colon \mathcal{G}.o\notin K} \mathbf{1}\{\max g.o \le e^L\} = c(o)L^\beta + o(L^\beta). \qquad (5.4)$$

Note as in [5] that the set $\mathcal{G}.o - K$ breaks up into a finite union

$$\mathcal{G}.o - K = \cup_{i=1}^N \mathcal{O}_i$$

where each $\mathcal{O}_i$ is the orbit of a point $o_i \in \mathcal{G}.o - K$ under outgoing moves. The points o_i are each one move outside of K. The fact there are finitely many o_i requires the discreteness of $\mathcal{G}.o$ and the compactness of K. Each $\mathcal{O}_i$ has the form

$$\mathcal{O}_i = \{m_{j_M}\dots m_{j_3}m_{j_2}m_{j_1}o_i \;:\; M\ge 0,\; j_i \ne j_{i+1}, j_1 \ne j_0(i)\}$$

where $j_0(i)$ is such that $m_{j_0(i)}o_i \in K$, or in other words, $m_{j_0(i)}$ is not outgoing on o_i. It can be deduced from **A, B, C** and preceding remarks that each orbit $\mathcal{O}_i$ can be identified with a subset $\mathcal{G}_i \subset \mathcal{G}$ via a bijection

$$g \in \mathcal{G}_i \mapsto g.o \in \mathcal{O}_i.$$

Moreover the $\mathcal{G}_i$ are disjoint. Therefore

$$
\begin{aligned}
\sum_{g\in\mathcal{G}\colon \mathcal{G}.o\notin K} \mathbf{1}\{\max g.o \le e^L\} &= \sum_{i=1}^N \sum_{g\in\mathcal{G}_i} \mathbf{1}\{\max g.o \le e^L\} \\
&= \sum_{i=1}^N \sum_{x\in\mathcal{O}_i} \mathbf{1}\{\max x \le e^L\}. \qquad (5.5)
\end{aligned}
$$

This reduces the count for $n_J^{(1)}(L)$ to a count for each of a finite number of orbits under outgoing moves in a region where **A**, **B** and **C** hold.

Step 3. The methods of [5] now take over, with one important thing to point out. A version of Lemma 8 is crucially used during the proof of [5, Lemma 20]. In that instance [5] can make a better bound than we have[8], but what is really important is the existence of the uniform $\epsilon > 0$ in Lemma 8. This establishes a weaker, but qualitatively the same, version of [5, Lemma 20] that plays the same role in the proof. The rest of the arguments of [5] go through without change to establish

Theorem 15 (Gamburd-Magee-Ronan, adapted)**.** *For each $1 \le i \le N$ there is a constant $c(\mathcal{O}_i) > 0$ such that*

$$\sum_{x \in \mathcal{O}_i} \mathbf{1}\{\max x \le e^L\} = c(\mathcal{O}_i) L^\beta + o(L^\beta).$$

Using Theorem 15 in (5.5) completes the proof of Theorem 2.

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