On nonparametric predictive inference for asset and European option trading in the binomial tree model

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ABSTRACT

This paper introduces a novel method for asset and option trading in a binomial scenario. This method uses nonparametric predictive inference (NPI), a statistical methodology within imprecise probability theory. Instead of inducing a single probability distribution from the existing observations, the imprecise method used here induces a set of probability distributions. Based on the induced imprecise probability, one could form a set of conservative trading strategies for assets and options. By integrating NPI imprecise probability and expectation with the classical financial binomial tree model, two rational decision routes for asset trading and for European option trading are suggested. The performances of these trading routes are investigated by computer simulations. The simulation results indicate that the NPI based trading routes presented in this paper have good predictive properties.

Keywords: imprecise probability, nonparametric predictive inference, asset trading, European option trading, asset risk pricing, option risk pricing.

1 Introduction

In recent years, imprecise probability has gained increasing popularity in the study of uncertainty of various phenomena [10, 15]. In the imprecise probability framework, there is a statistical methodology named nonparametric predictive inference (NPI) [6, 7, 8] which is of low structure and make no assumption of underlying distribution. NPI currently has applications in the fields of engineering and medical treatments [1, 9]. Researches from those fields have shown that NPI has good statistical properties and gives reliable predictive results. However, only little effort has been dedicated to applications of NPI in finance [4].

In this paper, NPI for Bernoulli data is applied to a basic financial binomial model in finance with formulation of different trading strategies. Demonstration focuses on symmetric binomial model for ease of calculation, the method can easily be adapted to asymmetric binomial model. Relevant trading strategies for assets and for European options are suggested. One should note that the content of European option trading presented in this paper is different from He et al.'s work [12], as they consider NPI imprecise expectation as a alternative pricing model to the classical Cox-Ross-Rubinstein (CRR) pricing model for financial options and investigate the trading outcome of these two pricing models under different market conditions. In this paper, we use the CRR price as a market benchmark price and NPI imprecise probability quantities as a guideline for different trading decisions for related financial products. Computer simulations are conducted to assess the performance and win-loss profile of each individual route. The viability of trading routes based on this method is confirmed by simulation results.

In section 2, the definitions of classical probability and imprecise probability are recalled and briefly compared which gives the motivation for the use of imprecise probability to quantify uncertainty. Based on $A_{(n)}$ assumption, mass function of NPI for bernoulli data is formally constructed. And NPI formulae for events of interest in this paper are given.

Section 3 briefly restates the concept of the binomial tree model in finance, followed by the details of application of NPI methods for asset, European call option and European put option trading based on the binomial tree model.

In Section 4, simulations of different trading routes are conducted in order to investigate their performance. Performance indexes for different trading routes are evaluated and discussed. It is confirmed by the simulation results that the NPI trading routes for assets, European call options and European put options have positive average payoff.

In section 5, some novel aspects of this new method are briefly discussed and related potential future research directions are suggested.

2 Nonparametric predictive inference

In this section, basic concepts of imprecise probability and NPI are reviewed. As the concept of imprecise probability is likely to be novel for most readers, the definitions for both classical probability and imprecise probability [18, 19, 20] are firstly stated and compared. Sequentially, the foundamental $A_{(n)}$ assumption for NPI is introduced. Based on $A_{(n)}$ assumption, NPI mass function for Bernoulli data in single and multiple future stages are constructed respectively.

Classical Probability space

Given a sample-space Ω, a σ algebra $\mathscr{A}$ of a collection of events in Ω and a single probability function $p : \mathscr{A} \longrightarrow [0, 1]$

The triple $[\Omega; \mathscr{A}; p]$ is called a probability space if p satisfies the Kolmogorov axioms (I-III):

I: $p(\theta) \geqslant 0 \quad \forall \theta \in \mathscr{A}$

II: $p(\Omega) = 1$

III: If $\theta_i \in \mathscr{A}$ for $i \in \mathbb{N}$ and $\theta_i \cap \theta_j = \emptyset$ for $i \neq j$,

then $p(\cup_{i\in\mathbb{N}}\theta_i) = \sum_{i\in\mathbb{N}} p(\theta_i)$

a p satisfied above is called a probability for the measurable space $(\Omega; \mathscr{A})$ given a measurable space $(\Omega; \mathscr{A})$, we denote the set of all the probabilities p on this space as P.

$P = \{p | p \text{ satisfies Kolmogorov axiom (I-III)}\}$

Imprecise probability space

Given a sample-space Ω, a σ algebra $\mathscr{A}$ of a collection of events in Ω and a mass functions m mapping from elements in $\mathscr{A}$ to $[0, 1]$, $m : \mathscr{A} \longrightarrow [0, 1]$.

The triple $[\Omega, \mathscr{A}, m]$ is an imprecise probability space $\mathscr{I}$ if it satisfies the following conditions:

I: $m(\emptyset) = 0$

II: $\sum\limits_{\epsilon\in\mathscr{A}} m(\epsilon) = 1$

Given one imprecise probability space $\mathscr{I} = [\Omega, \mathscr{A}, m]$ as defined above, the corresponding upper probability $\overline{p}$ and lower probability $\underline{p}$ based on the mass function m for an event $\mu \in \mathscr{A}$ are defined as:

$$\overline{p}(\mu) = \sum_{\substack{\epsilon\in\mathscr{A} \\ \epsilon\cap\mu\neq\emptyset}} m(\epsilon) \quad \text{and} \quad \underline{p}(\mu) = \sum_{\substack{\epsilon\in\mathscr{A} \\ \epsilon\subset\mu}} m(\epsilon)$$

From the definition, one could know the lower and upper probability satisfy:

inequality $\quad 0 \leq \underline{p}(\mu) \leq \overline{p}(\mu) \leq 1$ and

conjugacy relation $\quad \overline{p}(\mu^c) + \underline{p}(\mu) = 1 \qquad \forall \mu \in \mathscr{A}$

A subset of all probabilities P_m induced by a m

Given a measurable space $(\Omega; \mathscr{A})$, a set of all probabilities P on this space and a mass function

$m(\cdot)$ on this space, one can induce a subset P_m of all probabilities by the mass function m

$$\begin{aligned} P_m &= \{p \mid p \in P,\ \underline{p}_m(\mu) \leq p(\mu) \leq \overline{p}_m(\mu) \quad \forall \mu \in \mathscr{A}\} \\ &= \{p \mid p \in P,\ \sum_{\substack{\epsilon \in \mathscr{A} \\ \epsilon \subset \mu}} m(\epsilon) \leq p(\mu) \leq \sum_{\substack{\epsilon \in \mathscr{A} \\ \epsilon \cap \mu \neq \emptyset}} m(\epsilon) \quad \forall \mu \in \mathscr{A}\} \end{aligned}$$

Given an imprecise probability space $[\Omega,\ \mathscr{A},\ m]$, and a measurable function $f : \Omega \longrightarrow \mathbb{R}$, we denote the upper expectation probability of f by p_f^u and lower expectation probability of f by p_f^l, these are:

$$p_f^u = \underset{p \in P_m}{argsup} \sum_{\omega \in \Omega} f(\omega) p(\omega)$$
$$p_f^l = \underset{p \in P_m}{arginf} \sum_{\omega \in \Omega} f(\omega) p(\omega)$$

Given an imprecise probability space $[\Omega,\ \mathscr{A},\ m]$, and a measurable function $f : \Omega \longrightarrow \mathbb{R}$, we denote the upper expectation of f by $\overline{E}(f)$ and the lower expectation probability of f by $\underline{E}(f)$, these are:

$$\overline{E}(f) = \sum_{\omega \in \Omega} f(\omega) p_f^u(\omega)$$
$$\underline{E}(f) = \sum_{\omega \in \Omega} f(\omega) p_f^l(\omega)$$

From the above definitions, one can see that if one chooses to model uncertainties with imprecise probabilities, one will induce a set of probabilities to quantify the uncertainties of interest instead of a single probability. By doing this, the model can be more robust than its precise probability counterpart as a set of probabilities is more likely to cover the true underlying probability of the uncertainties. Also, imprecise model also would be a more appropriate model to reflect the lack of perfect information when gathering perfect information is unachievable in practices.

$A_{(n)}$ Assumption

NPI [2, 7, 8] is based on Hill's assumption $A_{(n)}$ [13]. This assumption is suitable for situations where no probability distribution regarding a future random quantity is assumed. The assumption $A_{(n)}$ is stated as follows:

Given n exchangeable real-valued observations $y_1, y_2, ..., y_n$ with order statistics of data $y_{(1)} < y_{(2)} < .... < y_{(n)}$. We define $y_{(0)} = -\infty$, $y_{(n+1)} = \infty$ and assume $p(y_i = y_j) = 0$ for $i \neq j$. The $y_1, y_2, ..., y_n$ divide the real-line into n + 1 intervals $I_g = (y_{(g-1)}, y_{(g)})$ for $g = 1, 2, ..., n+1$. The assumption $A_{(n)}$ states that future random quantity y_{n+i}, for $i \geq 1$ will fall equally likely into each interval.

$$p(Y_{n+i} \in I_g) = \frac{1}{n+1} \quad \text{for} \quad g = 1, 2...n+1; \;\; i \in \mathbb{N}$$

NPI mass function for Bernoulli data in any single future stage.
Coolen (1998) developed NPI for Bernoulli data by combining $A_{(n)}$ with an assumed underling latent variable representation [6]. Suppose one has n Bernoulli observations $x_1, x_2, ..., x_n \in \{0, 1\}$. Assume a latent thresold variable l and a sequence of latent real variables y_i corresponding to each observation x_i with order statistics $y_{(1)} < y_{(2)} < .... < y_{(n)}$ such that for all the data $x_i = 1$ if and only if $y_i < l$, $x_i = 0$ if and only if $y_i > l$. The number of success in the data is $j = |\{i : x_i = 1\}| = |\{i : y_i < l\}|$. Given a future random quantity X_{n+i} in any future single stage i. $\forall i \in \mathbb{N}$. The NPI for Bernoulli data mass function is defined by:

$$\begin{aligned} m(X_{n+i} = 1) = m(Y_{n+i} < l) &= \tfrac{j}{n+1} \\ m(X_{n+i} = 0) = m(Y_{n+i} > l) &= \tfrac{n-j}{n+1} \\ m(X_{n+i} = \{0, 1\}) = m(Y_{n+i} < l \;\text{ or }\; Y_{n+i} > l) &= \tfrac{1}{n+1} \end{aligned}$$

NPI mass function for Bernoulli data in any multiple future stages.

NPI for Bernoulli data has been developed for inference on multiple future stages [5, 6].

Let $\mathbb{N}_a^b$ denote the set of integers from a to b inclusively, so $\mathbb{N}_a^b = \{x | x \in \mathbb{N} \text{ and } a \leq x \leq b\}$. Let $D_{(n)}$ be the historical data of previous n time units with j upward movement or “successes”. $D_{(n)} = \{x_i\}_{i=-n+1}^{0}$ with $x_i \in \{0, 1\}$, $|\{i : x_i = 1 \wedge x_i \in D_{(n)}\}| = j$.
Let $S_T = \sum\limits_{i=1}^{T} X_{n+i}$ denote the number of upward movement or “successes” in the future T stages. Assume A_n up to A_{n+T-1}, for future T stage, there is in total $\binom{n+T}{n}$ possible orderings of which number of elements of $\{Y_{n+i}\}_{i=1}^{i=T}$ in between each interval I_g is different and all the possible ordering of on the real line are equally likely.

Let $\mathcal{P}(\cdot)$ denote the power set operator and $\mathscr{C}(\cdot)$ denote the consecutive integer set generation operation on consecutive integer set of the form $\mathbb{N}_0^T$, $\mathscr{C}(\mathbb{N}_0^T) = \{\mathbb{N}_{j_1}^{j_2} | j_1, j_2 \in \mathbb{N}_0^T, 0 \leq j_1 \leq j_2 \leq T\}$

NPI mass function for future T stages $m_T^{D_{(n)}} : \mathcal{P}(\mathbb{N}_0^T) \longrightarrow [0, 1]$ is then constructed as

$$m_T^{D(n)}(S_T \in \epsilon) = \begin{cases} \binom{j-1+j_1}{j_1}\binom{n-j-1+T-j_2}{T-j_2} \times \binom{n+T}{n}^{-1} & \epsilon \in \mathscr{C}(\mathbb{N}_0^T) \;\text{ and }\; \epsilon = \mathbb{N}_{j_1}^{j_2} \\ 0 & \epsilon \in \mathcal{P}(\mathbb{N}_0^T) \setminus \mathscr{C}(\mathbb{N}_0^T) \end{cases}$$

The idea behind this construction is the following. The event that has the number of successes between j_1 to j_2 in future T stages is equivalent to the event that has **at least** j_1 successes and **at least** $T - j_2$ failures. Thus one counts all the possible orderings that satisfied the equivalent event.

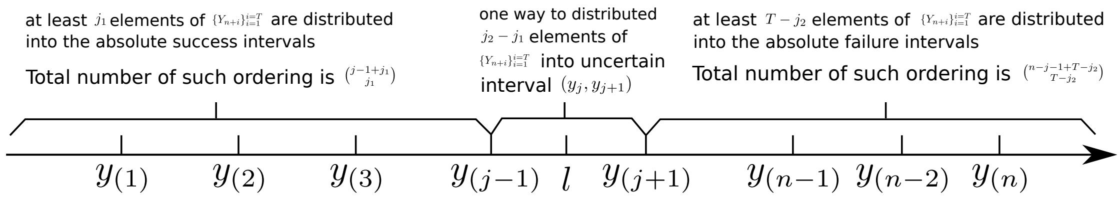


Figure 1: Illustration of mass function construction in multiple future stages

The sum of the mass function values over all $\epsilon \in \mathscr{C}(\mathbb{N}_0^T)$ is one.

$$\sum_{\epsilon \in \mathscr{C}(\mathbb{N}_0^T)} m_T^{D_{(n)}}(S_T \in \epsilon) = 1$$

Intuitively, one can interpret the mass function value for a atomic event $S_T \in \mathbb{N}_{j_1}^{j_1}$ or simply $S_T \in \{j_1\}$ as the probability mass that **has to** be assigned to the event $S_T \in \{r\}$. but for event which is the union atomic events, for example $S_T \in \mathbb{N}_{j_1}^{j_2}$, the mass function value is the **shared probability mass** of $\mathbb{N}_{j_1}^{j_2}$ and this mass value could assign to any event $S_T \in \mathbb{N}_{i_1}^{i_2}$ which is subset of $\mathbb{N}_{j_1}^{j_2}$ when one is taking the upper probability of $S_T \in \mathbb{N}_{i_1}^{i_2}$.

By the above construction of NPI mass function for Bernoulli data, one can find the imprecise probability of any T future stage event $\epsilon \in \mathcal{P}(\mathbb{N}_0^T)$

Let $S_T \in \underset{i \in A}{\cup} \mathbb{N}_{j_1^i}^{j_2^i}$ where A is some index set, then:

$$\overline{\underline{p}}_T^{D_{(n)}}(S_T \in \underset{i \in A}{\cup} \mathbb{N}_{j_1^i}^{j_2^i}) = \left[\underline{p}_T^{D_{(n)}}(S_T \in \underset{i \in A}{\cup} \mathbb{N}_{j_1^i}^{j_2^i}), \overline{p}_T^{D_{(n)}}(S_T \in \underset{i \in A}{\cup} \mathbb{N}_{j_1^i}^{j_2^i})\right] = \left[\sum_{\substack{\mathbb{N}_{i_1}^{i_2} \in \mathscr{C}(\mathbb{N}_0^T) \\ \mathbb{N}_{i_1}^{i_2} \subset (\underset{i \in A}{\cup} \mathbb{N}_{j_1^i}^{j_2^i})}} m_T^{D_{(n)}}(\mathbb{N}_{i_1}^{i_2}), \sum_{\substack{\mathbb{N}_{i_1}^{i_2} \in \mathscr{C}(\mathbb{N}_0^T) \\ \mathbb{N}_{i_1}^{i_2} \cap (\underset{i \in A}{\cup} \mathbb{N}_{j_1^i}^{j_2^i}) \neq \emptyset}} m_T^{D_{(n)}}(\mathbb{N}_{i_1}^{i_2})\right]$$

For example, using the above mass function, we have:

$$\overline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_m^T) = \sum_{\substack{\epsilon \in \mathscr{C}(\mathbb{N}_0^T) \\ \epsilon \cap \mathbb{N}_m^T \neq \emptyset}} m_T^{D(n)}(S_T \in \epsilon) = \binom{n+T}{n}^{-1} \times \sum_{i=m}^{T} \left[\binom{j+i}{j}\binom{n-j-1+T-i}{T-i}\right]$$

$$\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_m^T) = \sum_{\substack{\epsilon \in \mathscr{C}(\mathbb{N}_0^T) \\ \epsilon \subset \mathbb{N}_m^T}} m_T^{D(n)}(S_T \in \epsilon) = \binom{n+T}{n}^{-1} \times \sum_{i=m}^{T} \left[\binom{j-1+i}{i}\binom{n-j+T-i}{T-i}\right]$$

$$\overline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_0^m) = \sum_{\substack{\epsilon \in \mathscr{C}(\mathbb{N}_0^T) \\ \epsilon \cap \mathbb{N}_0^m \neq \emptyset}} m_T^{D(n)}(S_T \in \epsilon) = \binom{n+T}{n}^{-1} \times \sum_{i=0}^{m} \left[\binom{j-1+i}{i}\binom{n-j+T-i}{T-i}\right]$$

$$\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_0^m) = \sum_{\substack{\epsilon \in \mathscr{C}(\mathbb{N}_0^T) \\ \epsilon \subset \mathbb{N}_0^m}} m_T^{D(n)}(S_T \in \epsilon) = \binom{n+T}{n}^{-1} \times \sum_{i=0}^{m} \left[\binom{j+i}{i}\binom{n-j-1+T-i}{T-i}\right]$$

Moreover, given a function $f(S_T)$ of a the random variable $S_T \in \mathbb{N}_0^T$. One can use the mass function $m_T^{D_{(n)}}(\cdot)$ as tool to find the upper expectation probability $p^u_{f(S_T)}$ for $f(S_T)$, the probability which maximizes the expectation of $f(S_T)$ and the lower expectation probability $p^l_{f(S_T)}$ for $f(S_T)$, the probability which minimizes the expectation of $f(S_T)$. So in essence, a upper expectation probability or a lower expectation probability is a element in the subset P_m of all probabilities induced by the mass function (introduced in the early definition), such that this probability maximize or minimize the expectation of the function of the random variable.

The objects of interest in this paper are asset and European option in binomial tree model. Since asset and call option are monotonic increasing functions of S_T and put option is monotonic decreasing function S_T, only explicit formulas of the upper expectation probability for monotonic function of S_T are presented. Using the above formula, one can deduce the lower and the upper expectation probability for monotonic increasing function $f_\uparrow(S_T)$ of S_T, $\forall m \in \mathbb{N}_0^T$:

$$\begin{aligned}
p^l_{f_\uparrow(S_T)}(f_\uparrow(S_T = m)) =& \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_m^T) - \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_{m+1}^T) \\
=& \binom{j-1+m}{m}\binom{n-j+T-m}{T-m}\binom{n+T}{n}^{-1} \\
p^u_{f_\uparrow(S_T)}(f_\uparrow(S_T = m)) =& \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_0^m) - \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_0^{m-1}) \\
=& \binom{j+m}{m}\binom{n-j-1+T-m}{T-m}\binom{n+T}{n}^{-1}
\end{aligned}$$

The notion behind these formulas is the following. The lower probability of a event is the sum of all the probability masses that **has to** be assigned to the event. As the function is monotonic, to find the lower expectation probability, one assigns the largest possible probability mass to the smaller values and least possible probability mass to the larger values.

Thus for T future stages, one starts with the probability assignment at value 0, $p^l_{f_\uparrow(S_T)}(f_\uparrow(S_T = 0)) = \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_0^T) - \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_1^T)$, and $\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_0^T) = 1$ as $S_T \in \mathbb{N}_0^T$ is the whole set, it has all the probability mass, $\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_1^T)$ is then the probability mass that has to assign to $\mathbb{N}_1^T$, the difference between these quantity is the maximum probability mass that could be assigned to value 0, which in essence include probability mass that **has to** assign to 0 and probability mass that could be but not necessary need to assigned to 0. One then move to the next maximum probability mass assignment for the next smallest value (in this case, it is 1) with residual probability mass $\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_1^T)$. $p^l_{f_\uparrow(S_T)}(f_\uparrow(S_T = 1)) = \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_1^T) - \underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_2^T)$. $\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_2^T)$ is the probability mass that has to assign to $\mathbb{N}_2^T$ and residual probability mass we left is $\underline{p}_T^{D_{(n)}}(S_T \in \mathbb{N}_1^T)$, so the difference is the maximum probability mass that could be assigned to 1, which in essence include probability mass that **has to** assign to 1 and probability mass that could be but not necessary need to assigned to 1 but not 0. One repeats this process until all the value has desire probability mass assignment and thus find the lower expectation probability for $f_\uparrow(S_T)$. To find the upper expectation probability $p^u_{f(S_T)}$ for $f(S_T)$, one only need to reverse the logic above.

By similar reasoning, for monotonic decreasing function $f_\downarrow(S_T)$ of S_T follows:

$$p^l_{f_\downarrow(S_T)}(f_\downarrow(S_T=m)) = p^u_{f_\uparrow(S_T)}(f_\uparrow(S_T=m)) \quad p^u_{f_\downarrow(S_T)}(f_\downarrow(S_T=m)) = p^l_{f_\uparrow(S_T)}(f_\uparrow(S_T=m))$$

The lower and the upper expectation probabilities subsequently allow one to find the lower expectation $\underline{E}$ and upper expectation $\overline{E}$ of $f_\uparrow(S_T)$ and $f_\downarrow(S_T)$

$$\begin{aligned}
\underline{E}f_\uparrow(S_T) &= \sum_m f_\uparrow(S_T=m)p^l_{f_\uparrow(S_T)}(f_\uparrow(S_T=m)) \\
\overline{E}f_\uparrow(S_T) &= \sum_m f_\uparrow(S_T=m)p^u_{f_\uparrow(S_T)}(f_\uparrow(S_T=m)) \\
\underline{E}f_\downarrow(S_T) &= \sum_m f_\downarrow(S_T=m)p^l_{f_\downarrow(S_T)}(f_\downarrow(S_T=m)) \\
\overline{E}f_\downarrow(S_T) &= \sum_m f_\downarrow(S_T=m)p^u_{f_\downarrow(S_T)}(f_\downarrow(S_T=m))
\end{aligned}$$

3 Methodology for NPI in the binomial model

In this section, binomial tree model is stated and relevant notation is introduced. NPI based trading strategies for both asset and European option are formulated, followed by their motivations.

3.1 Financial symmetric binomial tree model.

For the model we consider in this paper we make the following assumptions. In short period of time interval, the participants in a market of an asset tends to have homogeneous behaviour, the upward or downward movement of the asset price driven by the participants in each time unit (in this short period of time) could be considered approximately as an i.i.d. Bernoulli random variable, so the number of upwards moment in T time units is suitably modelled by a random variable with a set of binomial-like probability distributions induced by NPI; The quantity upward or downward movement in a short time interval is close to each other and it is thus modelled by a factor u and factor d respectively; There exists a identical risk-free interest rate r in each time step for investment and short selling in the market.

Binomial tree can be categorised into three types, the upward binomial tree ($ud > 1$), the symmetric binomial tree ($ud = 1$) and the downward binomial tree ($ud = 1$). In this paper, only symmetric binomial tree is presented for the ease of calculation, asymmetric tree model can be easily adapted by changing parameters u and d.

To guarantee the market is arbitrage free, one also assume that $d < e^r < u$.

Let a_0 denote the asset price at time 0, the objects of interest are asset value $A_T(S_T) = a_0u^{S_T}d^{T-S_T} = a_0u^{2S_T-T}$ at time T. Call option value $\Lambda_c(A_T(S_T), K) = (A_T(S_T) - K)^+$ at time T and put option $\Lambda_p(A_T(S_T), K) = (K - A_T(S_T))^+$ value at time T. Then, $A_T(S_T)$ and

$\Lambda_c(A_T(S_T), K)$ are monotonic increasing functions of S_T while $\Lambda_p(A_T(S_T), K)$ is a monotonic decreasing function of S_T.

In the NPI probability framework, one can learn from past n time steps historical data and induce a imprecise probability space on the asset price A_T of future time T. From the imprecise probability space, one can further deduce the lower and the upper probability of some event of interest. Also one could find the upper expectation $\overline{E}$ and the lower expectation $\underline{E}$ of any finance derivative (the asset itself, option, future, etc.) upon a specific asset at future time T.

3.2 NPI Asset trading strategies in binomial model

Using the induced imprecise probability space, two decision routes for asset trading are suggested and performances are investigated by simulation. Let $B(t) = e^{rt}$ denote the discount rate for time of length t

Route 1.1
Set threshold value $0.5 < w < 1$. From NPI setting, one could know $0 < \underline{p}(A_T(S_T) > a_0B(T)) < \overline{p}(A_T(S_T) > a_0B(T)) < 1$ if $S_T \subsetneq \mathbb{N}_0^T$ and $S_T \neq \emptyset$.

$$\begin{cases} \text{Buy the asset at current time, sell it at time } T \text{ if } \underline{p}(A_T(S_T) > a_0B(T)) > w \\ \\ \text{Short the asset at for } a_0 \text{ at current time, invest the money at risk free rate and} \\ \text{close the short position at time } T \text{ if } \underline{p}(A_T(S_T) < a_0) > w \\ \\ \text{Invest at risk free rate and receive } a_0B(T) \text{ at time } T \text{ if none of above satisfied} \end{cases}$$

Route 1.2
Trading route based on the lower expectation of asset price $\underline{E}(A_T)$ and the upper expectation of asset price $\overline{E}(A_T)$ at time T

$$\begin{cases} \text{Buy the asset at current time, sell it at time } T \text{ if } \underline{E}(A_T) > a_0B(T) \\ \\ \text{Short the asset for } a_0 \text{ at current time, invest the money at risk free rate} \\ \text{and close the short position at time } T \text{ if } \overline{E}(A_T) < a_0 \\ \\ \text{Invest at risk free rate and receive } a_0B(T) \text{ at time } T \text{ if none of above satisfied} \end{cases}$$

Motivation behind Route 1.1
One will be in favour of buying the asset if the asset price at T future time steps $A_T(S_T)$ is greater than the risk free investment payoff $a_0B(T)$ at future time T. So a prudent individual

using imprecise probability will choose to invest if the lower probability of this favourable event $A_T(S_T) > a_0B(T)$ is greater than a chosen threshold value $w > 0.5$. One should note $w > 0.5$ is a suggested threshold value interval, threshold value $w < 0.5$ can also be used if one tends to be more risk prone.

If the lower probability of event $A_T(S_T) < a_0$ is greater than w ($w > 0.5$), then one would expect the price of asset is more likely less than current price a_0 and thus one would short sell the asset. If none of the above conditions satisfied, one would choose the risk free investment to have guaranteed value $a_0B(T)$ at time T.

One can show that only one of actions can be taken in Route 1.1: Using inequality and conjugate property of imprecise probability, one can have the following:

$$\begin{aligned}
&\text{if } \underline{p}(A_T(S_T) > a_0B(T)) > w \\
&\text{then, by congugacy property, one has } 1 - \overline{p}(A_T(S_T) < a_0B(T)) > w \\
&\text{then, by imprecise enquality, one has} \\
&\underline{p}(A_T(S_T) < a_0) < \overline{p}(A_T(S_T) < a_0) < \overline{p}(A_T(S_T) < a_0B(T)) < 1 - w < w
\end{aligned}$$

Therefore, only one action could be taken in Route 1.1

Motivation behind Route 1.2

If the lower expectation of asset price A_T at future time T is greater than the value $a_0B(T)$ received when investing at risk free rate, then one would prefer to buy the asset. If the upper expectation of asset price A_T at future time T is less than the current price a_0, then one would prefer to short sell and invest the money at risk free rate expecting receive at least $a_0B(T)-\overline{E}(A_T)$ at future time T.

It is easy to show that only one of actions could be taken in Route 1.2: In the imprecise probability framework, one has $\underline{E}(A) \leq \overline{E}(A)$ and in the context, one has $a_0 < a_0B(T)$. So $\underline{E}(A_T) > a_0B(T)$ and $\overline{E}(A_T) < a_0$ could not be satisfied at the same time. Similar reasoning could be used to show all the actions in NPI trading routes for European options are mutually exclusive in the below sections.

3.3 NPI European option trading strategies in binomial model

Cox, Ross, & Rubinstein (CRR) developed the binomial options pricing model in 1979 [11]. In the CRR model, it is assumed that the asset price binomial tree is symmetric ($ud = 1$) and there is no transaction cost when trading. By replicating the performance of the European option with a self-financing portfolio in each time step, one could find a risk neutral measure $q = \frac{e^r-d}{u-d}$. Hence there exists a unique arbitrage-free price $\Lambda^{\mathbb{Q}}(A_t, K) = B(T-t)^{-1}E_{\mathbb{Q}}(\Lambda(A_T, K)|\mathscr{F}_t)$ for European option with strike price K mature date T at time t with underlying asset price A_t.

In the CRR model, there is no real probability or "risk involved" in the derivation of arbitrage-free price. In other words, anyone in the market who is willing to bid a price $y_t > \Lambda^{\mathbb{Q}}(A_t, K)$ or ask a price $y_t < \Lambda^{\mathbb{Q}}(A_t, K)$ for European option $\Lambda^{\mathbb{Q}}(A_T, K)$ will become a free money source for an arbitrager. This behaviour is still rational if at time t the expected present value of option product under one's risk measure is greater than the arbitrage-free price of the option, $\Lambda^{\mathbb{P}} = B_{T-t}^{-1} E_{\mathbb{P}}(\Lambda(A_T, K)|\mathscr{F}_t) > \Lambda^{\mathbb{Q}}(A_t, K)$, when one considers to buy, or one's expected present value under one's risk measure is less than the arbitrage-free price of the option, $\Lambda^{\mathbb{P}} = B_{T-t}^{-1} E_{\mathbb{P}}(\Lambda(A_T, K)|\mathscr{F}_t) < \Lambda(A_t, K)$, when one considers to sell.

One should note that the main focus of study in this paper is to study performance of different trading routes using NPI imprecise risk measure. And we admits the arbitrage-free price derived by the CRR model and also using it as the current market price when the simulations are conducted. The formulation of all trading routes integrate concepts of arbitrage-free price and NPI risk imprecise measure and expectation (minimum selling price or maximum buying price). This is a crucial difference from He et al.'s work [12] where they use NPI expectation as a alternative option pricing model and investigate the trading result between CCR believer and NPI believer under different market scenarios.

With the NPI imprecise probability and expectation, one could rationally have following decision routes when trading with European call option and put option.

Route 2.1

For European call option, set threshold value $0.5 < w < 1$. From NPI setting, $0 < \underline{p}((A_T(S_T) - K)^+ > B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)) < \overline{p}((A_T(S_T) - K)^+ > B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)) < 1$ if $S_T \subsetneq \mathbb{N}_0^T$ and $S_T \neq \emptyset$. One then could rationally have the following trading route based on the lower and the upper probability of desirable events for call option.

$$\begin{cases} \text{Buy the call option at current time, exercise the call option and sell the corresponding underlying} \\ \text{asset at market price } A_T(S_T) \text{ at time } T \text{ if } \underline{p}[(A_T(S_T) - K)^+ > B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)] > w \\ \\ \text{Short the call option at current time for } \Lambda_c^{\mathbb{Q}}(a_0, K)\text{, invest the money for risk free rate and close} \\ \text{the short position at time } T \text{ if } \underline{p}[(A_T(S_T) - K)^+ < \Lambda_c(a_0, K)] > w \\ \\ \text{Invest } \Lambda_c^{\mathbb{Q}}(a_0, K) \text{ at risk free rate and receive } \Lambda_c^{\mathbb{Q}}(a_0, K)B(T) \text{ at time } T \text{ if none of above satisfied} \end{cases}$$

Route 2.2

Trading route based on the upper expectation of call option payoff $\underline{E}((A_T - K)^+)$ and the lower expectation of call option payoff $\overline{E}((A_T - K)^+)$ at time T

$$\begin{cases} \text{Buy the call option at current time, exercise the call option and sell the corresponding underlying} \\ \text{asset at market price } A_T(S_T) \text{ at time } T \text{ if } \underline{E}[(A_T - K)^+] > B(T)\Lambda_c^{\mathbb{Q}}(a_0, K) \\ \\ \text{Short the call option at current time, invest the money received at risk free rate and close the short} \\ \text{position at time } T \text{ if } \overline{E}[(A_T - K)^+] < \Lambda_c^{\mathbb{Q}}(a_0, K) \\ \\ \text{Invest } \Lambda_c^{\mathbb{Q}}(a_0, K) \text{ at risk free rate and receive } \Lambda_c^{\mathbb{Q}}(a_0, K)B(T) \text{ at time } T \text{ if none of above satisfied} \end{cases}$$

Motivation behind Route 2.1

Consider the event $(A_T(S_T) - K)^+ > B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)$ that the payoff of the call option at time T is greater than the payoff $B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)$ received at time T by investing non-arbitrage price $\Lambda_c^{\mathbb{Q}}(a_0, K)$ derived using CRR model at risk free rate, if the lower probability of this event is greater than threshold value w ($w > 0.5$), then one would prefer to buy this call option and expect to earn more than $B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)$ in future time T.

On the contrary, consider the lower probability of event $(A_T(S_T) - K)^+ < \Lambda_c^{\mathbb{Q}}(a_0, K)$ that the payoff the call option at future time T is less than the non-arbitrage price, if the lower probability of this event is greater than threshold value w ($w > 0.5$), then one would expect the payoff of call option more likely to be less than the non-arbitrage price $\Lambda_c^{\mathbb{Q}}(a_0, K)$ and thus one would short sell the call option. If none of above conditions are satisfied, one would choose the risk free investment to have guaranteed value $B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)$ at time T.

Motivation behind Route 2.2

When the lower expectation of call option payoff $\underline{E}[(A_T - K)^+]$ at future time T is greater than the payoff $B(T)\Lambda_c^{\mathbb{Q}}(a_0, K)$ received at time T by investing the same amount of monetary resource of non-arbitrage call option price $\Lambda_c^{\mathbb{Q}}(a_0, K)$ at time 0 into a risk free rate, one would prefer to buy the call option and expect to receive at least $\underline{E}[(A_T - K)^+]$ at time T.

If the upper expectation of call option payoff $\overline{E}[(A_T - K)^+]$ at future time T is less than the current call option non-arbitrage price $\Lambda_c^{\mathbb{Q}}(a_0, K)$, one would rationally choose to short the call option and invest the money received into risk free rate, expecting monetary value in hand at time T is at least $B(T)\Lambda_c^{\mathbb{Q}}(a_0, K) - \overline{E}[(A_T - K)^+]$.

Route 2.3

For European put option, set threshold value $0.5 < w < 1$. From NPI setting, $0 < \underline{p}((K - A_T(S_T))^+ > B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)) < \overline{p}((K - A_T(S_T))^+ > B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)) < 1$ if $V \subsetneq \mathbb{N}_0^T$ and $V \neq \emptyset$. One then could rationally have the following trading route based on the lower and the upper probability of desirable events for put option.

$$\begin{cases} \text{Buy the put option at current time, buy the underlying asset with market price } A_T(S_T) \\ \text{and exericise the put option at time } T \text{ if } \underline{p}[(K - A_T(S_T))^+ > B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)] > w \\ \\ \text{Short the put option at current time,invest the money received at risk free rate and close} \\ \text{the short position at time } T \text{ if } \underline{p}[(K - A_T(S_T))^+ < \Lambda_p^{\mathbb{Q}}(a_0, K)] > w \\ \\ \text{Invest at risk free rate and receive } B(T)\Lambda_p^{\mathbb{Q}}(a_0, K) \text{ at time } T \text{ if none of above satisfied} \end{cases}$$

Route 2.4

Trading route based on the lower expectation of European put option payoff $\underline{E}((K - A_T)^+)$ and the upper expectation of put option payoff $\overline{E}((K - A_T)^+)$ at future time T

$$\begin{cases} \text{Buy the put option at current time, buy the underlying asset with market price } A_T(S_T) \\ \text{and exericise the put option at time } T \text{ if } \underline{E}[(K - A_T)^+] > B(T)\Lambda_p^{\mathbb{Q}}(a_0, K) \\ \\ \text{Short the put option at current time,invest the money received at risk free rate and close} \\ \text{the short position at time } T \text{ if } \overline{E}[(K - A_T)^+] < \Lambda_p^{\mathbb{Q}}(a_0, K) \\ \\ \text{Invest at risk free rate and receive } B(T)\Lambda_p^{\mathbb{Q}}(a_0, K) \text{ at time } T \text{ if none of above satisfied} \end{cases}$$

Motivation behind Route 2.3

Consider the event $(K - A_T(S_T))^+ > B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)$ that the payoff of the put option at time T is greater than the payoff $B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)$ received at time T by investing non-arbitrage price $\Lambda_p^{\mathbb{Q}}(a_0, K)$ derived using CRR model at risk free rate, if the lower probability of this event is greater than threshold value w ($w > 0.5$), then one would prefer to buy this put option and expect to earn more than $B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)$ in future time T.

Also consider the lower probability of event $(K - A_T)^+ < \Lambda_p^{\mathbb{Q}}(a_0, K)$ that the payoff the put option at future time T is less than the non-arbitrage price, if the lower probability of this event is greater than threshold value w ($w > 0.5$), then one would expect the payoff of put option more likely to be less than the non-arbitrage price $\Lambda_p^{\mathbb{Q}}(a_0, K)$ and thus one would short sell the put option. If none of above conditions are satisfied, one would choose the risk free investment to have guaranteed value $B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)$ at time T.

Motivation behind Route 2.4

When the lower expectation of put option payoff $\underline{E}[(K - A_T)^+]$ at future time T is greater than the payoff $B(T)\Lambda_p^{\mathbb{Q}}(a_0, K)$ received at time T by investing the same amount of monetary resource of non-arbitrage put option price $\Lambda_p^{\mathbb{Q}}(a_0, K)$ at time 0 into a risk free rate, one would prefer to buy the put option and expect to receive at least $\underline{E}[(K - A_T)^+]$ at time T.

If the upper expectation of put option payoff $\overline{E}[(K - A_T)^+]$ at future time T is less than the current put option non-arbitrage price $\Lambda_p^{\mathbb{Q}}(a_0, K)$, one would rationally choose to short the put option and invest the money received into risk free rate, expecting monetary value in hand at time T is at least $B(T)\Lambda_p^{\mathbb{Q}}(a_0, K) - \overline{E}[(K - A_T)^+]$.

4 Simulations

In this section we present the results of simulation studies with several goals. First, to verify the predictive property of NPI trading routes in asset and European option. Second, to evaluate and compare the performance of different NPI trading routes in asset and European option. Third, to identify the effectiveness and efficiency of data learning in NPI imprecise probability.

Data are generated from whole family of Bernoulli distribution. Firstly, one draw a random number p from Uniform(0,1) and then generate $n + T$ data points from Bernoulli(p).

As stated in the assumptions, each time steps is considered to be small. So in this paper, each time step is assumed to be one trading day and the discounting rate r is set at 0.0007 in the simulation. Other parameters includes upward movement $u = 1.03$, downward movement $d = 1/u$, initial asset price $a_0 = 100$. Strike price for call option $K_c = 103$ for put option $K_p = 98$.

All the decisions routes are simulated 100,000 times using the statistical software R.

4.1 Performance of Route 1.1 and 1.2

Let $N \in (1, 100000)$ be the index of simulation trial. The performances of asset trading decision routes are measured by four statistics of the present value pay-off function $f_i^A(n, T, w, route_i; N)$ in 100000 simulations. $f_i^A(n, T, w, route_i; N)$ is defined as follow:

$$
\begin{aligned}
&f_i^A(n, T, w, route_i; N) \\
&= \begin{cases} A_T(s_T^N)B(T)^{-1} - a_0 & \text{if choose to buy} \\ a_0 - A_T(s_T^N)B(T)^{-1} & \text{if choose to short sell and invest in risk free rate} \\ 0 \quad \text{if invest in risk free rate} \end{cases}
\end{aligned}
$$

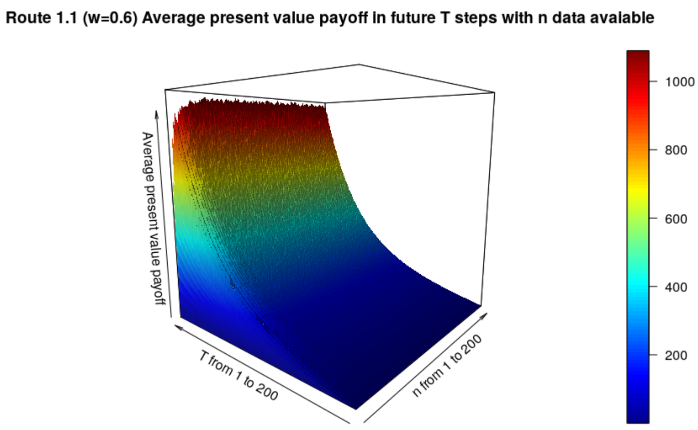
where the input n is the length of data from the asset price history one could learn; T is the future time that the this function is evaluate; w is the thresold value of trading route if existed; $route_i$ is the trading route this function take; N is the index of simulation trial.

Four statistics of this function measure from 100000 simulations are:

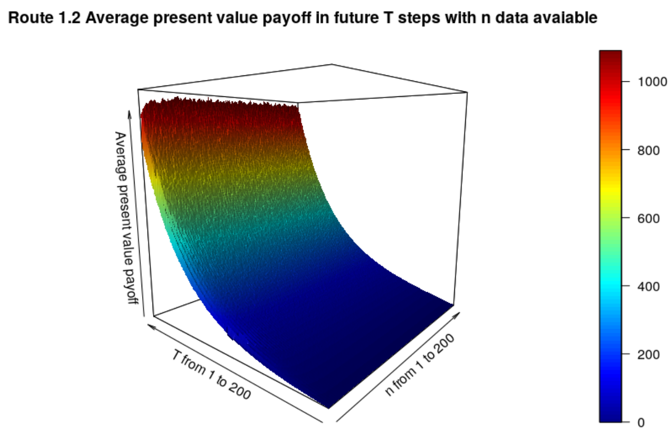
Average present value payoff $\overline{f_i^A} = \frac{\sum_N f_i^A}{100000}$ win-loss ratio $R_{wlA}^i = \frac{|\{N:f_i^A>0\}|}{|\{N:f_i^A<0\}|}$

win rate $R_{wrA}^i = \frac{|\{N:f_i^A>0\}|}{100{,}000}$ loss rate $R_{lrA}^i = \frac{|\{N:f_i^A)<0\}|}{100{,}000}$

One should know the sum of win rate and loss rate is not equal to 1, as the NPI trading decision route allows "inaction" when the quantification of desirable events' uncertainty is quite imprecise.

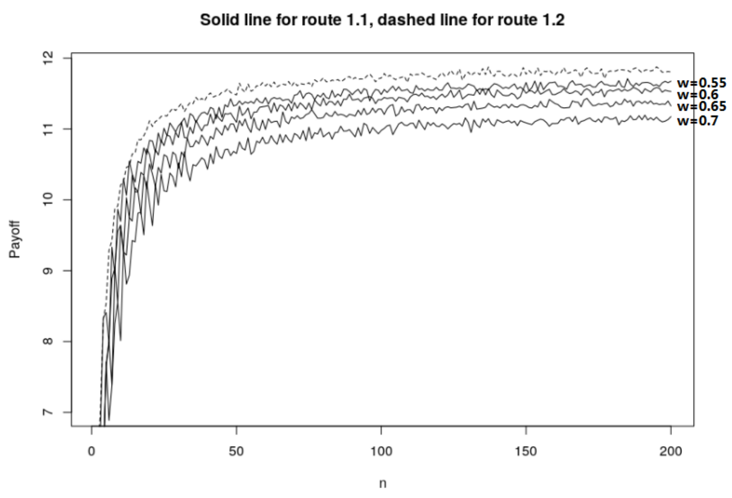


(a)

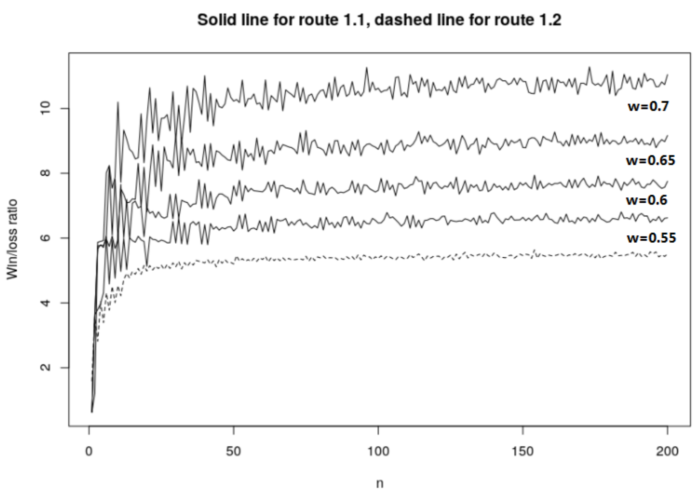


(b)

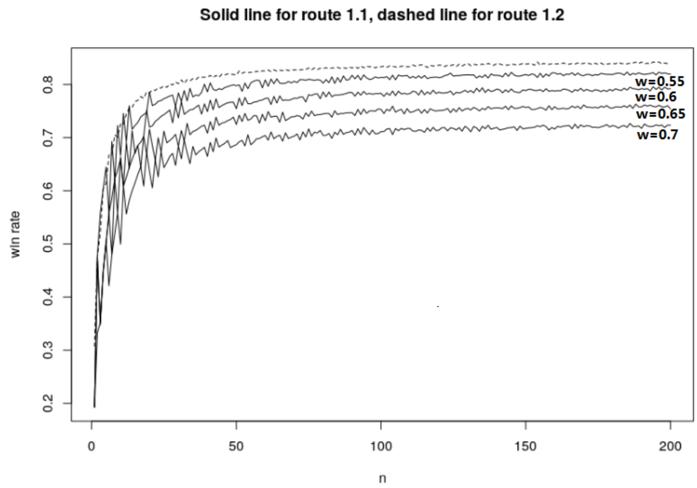
Figure 2: Route 1.1 ($w = 0.6$) and Route 1.2 average present payoff with n and T from 1 to 200



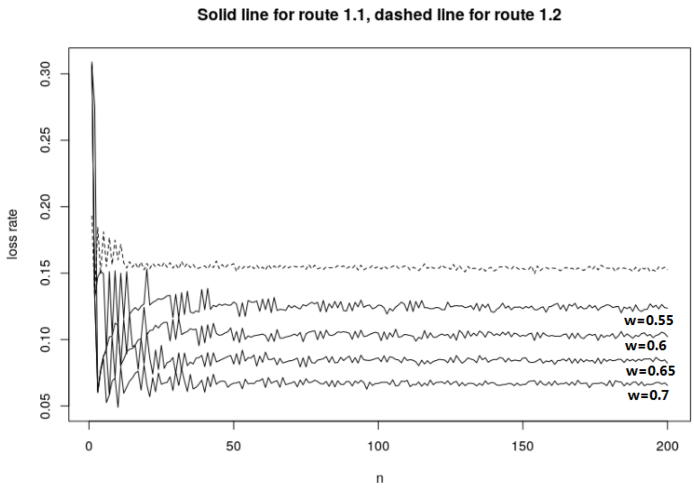
(a) Payoff performance



(b) Win-loss ratio performance



(c) Win rate performance



(d) Loss rate performance

Figure 3: Comparison of Route 1.1 and Route 1.2 with respect to different performance index with $T = 10$ and n from 1 to 200

Figure 2 shows that both Route 1.1 and 1.2 yield positive average payoff for all different pairs of n and T. Simulations for other parameter threshold values $w > 0.5$ for Route 1.1 also conducted, the shape of the average present value payoff is similar to the pattern for $w = 0.6$. It should be noticed that, with small amounts of historical data n available, Route 1.2 performs better in term of average present value payoff. This is due to the fact that Route 1.2 takes all probability mass assigned to all node in binomial tree in T time step into account while Route 1.1 only takes probability mass assigned to part of the nodes into account. As small amount of data most of time could not well present the underlying distribution, taking partially the nodes in T time step could lead to ineffective learning in Route 1.1 (This is indicated by the fan shape happen when n is small in Figure 2(a)).

Figure 3 demonstrates the performance of decision Routes 1.1 and 1.2 in terms of performance index $\overline{f_i^A}$,R^i_{wlA},R^i_{wrA} and R^i_{lrA} respectively as number of data n increases in $T = 10$ future time steps for asset trading.

Figure 3(a) shows that both NPI based asset trading routes have very quick learning speed. The average payoff from both routes increases when a small number of data become available. Both of them stabilise after 12 data point are available. ($\underset{i,w,n}{max}\overline{f_i^A} = 11.8774$; $\overline{f_{1.1}^A} = 10.0295$ at $n = 12$ $w = 0.60$; $\overline{f_{1.2}^A} = 10.4582$ at $n = 12$). Overall, in terms of long run payoff, decision Route 1.2 outperforms Route 1.1. However, one should notice that although decision Route 1.1 seems to be a worse choice than Route 1.2 in the long run, it still yields positive expected payoff. From Figure 3(b)-(d), although Route 1.2 is still dominating in terms of win rate, one should note that with adjustment of threshold parameter w, Route 1.1 actually provides more flexibility for investors to reduce the loss rate. From Figure 3(a), one can confirm that if a small amount of data is available, in terms of average present value payoff,imprecise expectation routes outperform imprecise probability routes. When only a small amount of data is available, sometimes the data do not reflect the true distribution or reflect the “opposite” distribution. In those cases, the imprecise probability routes will assign mass only to part of nodes in the wrong way. Although the imprecise expectation route also assigns the mass value wrongly, it takes all the nodes values into account which improves its performance.

4.2 Performance of Route 2.1 and 2.2

Let $N \in (1, 100000)$ be the index of simulation trial. The performances of the European option trading decision routes are measured by four statistics of present value payoff function $f_i^C(n, T, w, route_i; N)$ in 100000 simulations, $f_i^C(n, T, w, route_i; N)$ is defined as follow:

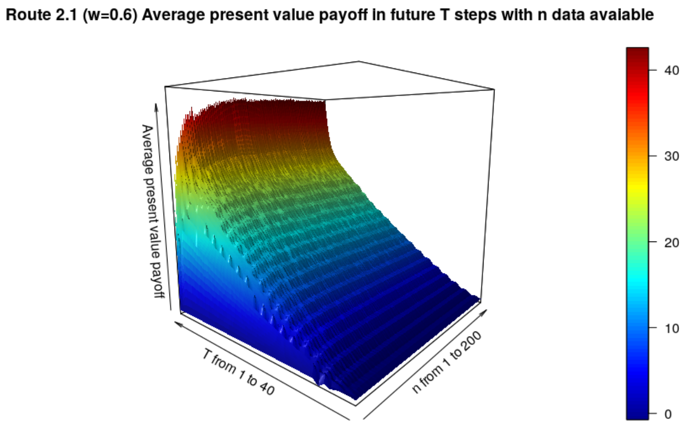
$$f_i^C(n, T, w, route_i; N)$$
$$= \begin{cases} B(T)^{-1}(A_T(s_T^N) - K_c)^+ - \Lambda_c^{\mathbb{Q}}(a_0, K_c) & \text{if choose to buy} \\ \Lambda_c^{\mathbb{Q}}(a_0, K_c) - B(T)^{-1}(A_T(s_T^N) - K_c)^+ & \text{if choose to short sell and invest in risk free rate} \\ 0 \quad \text{if invest in risk free rate} \end{cases}$$

where the input n is the length of data from the asset price history one could learn; T is the future time that the this function is evaluate; w is the thresold value of trading route if existed; $route_i$ is the trading route this function take; N is the index of simulation trial.

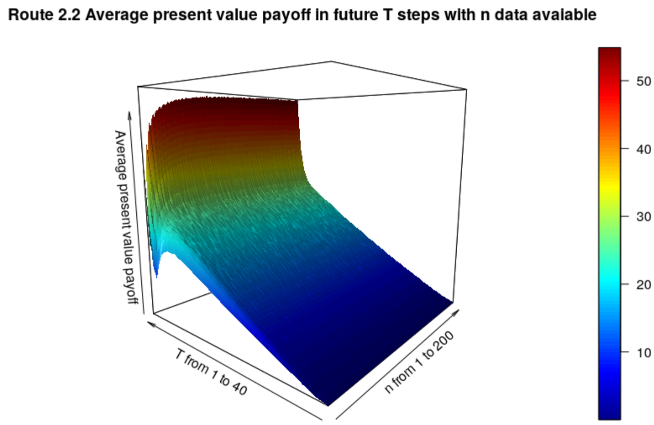
Four statistics of this function measure from 100000 simulations are:

Average present value payoff $\overline{f_i^C} = \frac{\sum_N f_i^C}{100000}$ win-loss ratio $R^i_{wlC} = \frac{|\{N: f_i^C > 0\}|}{|\{N: f_i^C < 0\}|}$

win rate $R^i_{wrC} = \frac{|\{N: f_i^C > 0\}|}{100{,}000}$ loss rate $R^i_{lrC} = \frac{|\{N: f_i^C) < 0\}|}{100{,}000}$

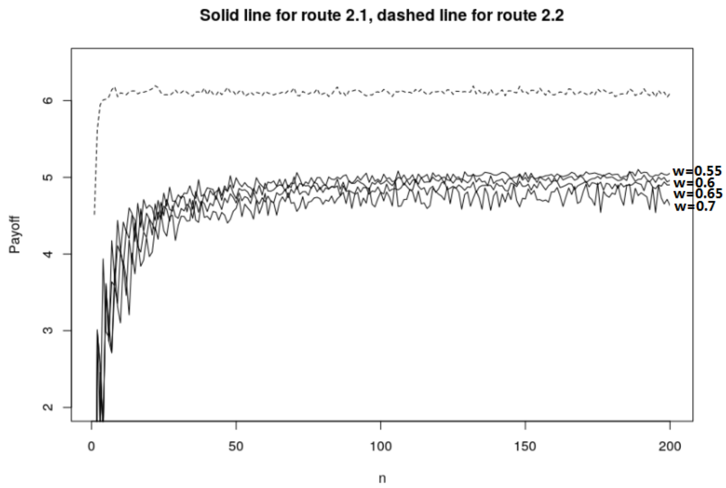


(a)

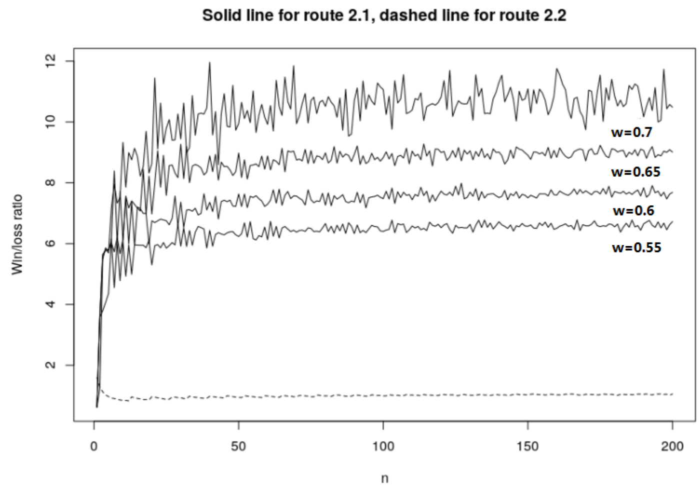


(b)

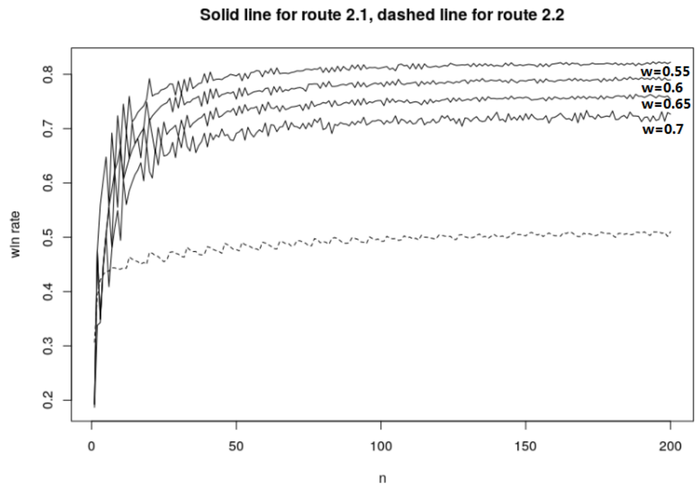
Figure 4: Route 2.1 ($w = 0.6$) and route 2.2 average present payoff with n from 1 to 200 and T from 1 to 40



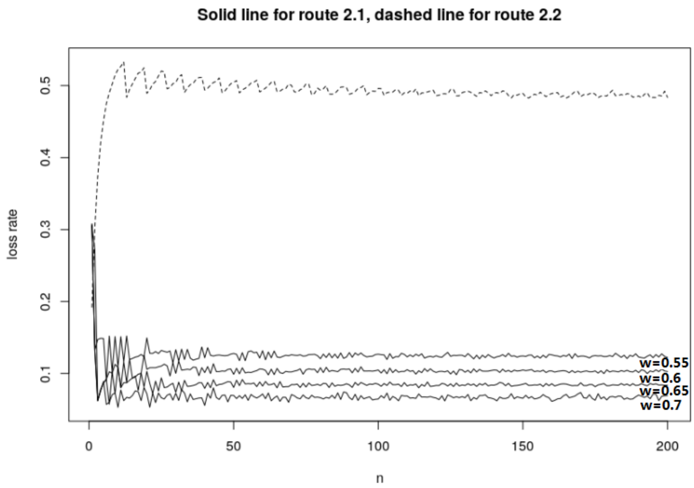
(a) Payoff performance



(b) Win-loss ratio performance



(c) Win rate performance



(d) Loss rate performance

Figure 5: Comparison of Route 2.1 and Route 2.2 with respect to different performance index with $T = 10$ and n from 1 to 200

Figure 4 shows that both Routes 2.3 and 2.4 generate positive average present payoff for all pairs n (from 1 to 200) and T (from 1 to 40). Other threshold values $w > 0.5$ for Route 2.3 were also used in simulation, they all follow the same pattern as in Figure 4(a)

Figure 5 demonstrates the performance of decision Routes 2.1 and 2.2 in terms of performance index $\overline{f_i^C}$,R^i_{wlC},R^i_{wrC} and R^i_{lrC}, respectively, as number of data n increases for trading European call option in future $T = 10$ time step.

Figure 5(a) shows that Route 2.1 and 2.2 have similar behaviour as Routes 1.1 and 1.2 in asset trading with Route 2.2 surpassing Route 2.1 in average payoff. Both Routes 2.1 and 2.2 also pick up information from data quickly. The payoff from both these routes tends to stabilise after 7 data points are available. ($\underset{i,w,n}{max}\overline{f_i^C} = 6.1936$; $\overline{f_{2.1}^C} = 4.1749$ at $n = 7$ $w = 0.60$; $\overline{f_{2.2}^C} = 6.1410$ at $n = 7$). Overall, in terms of long run payoff, decision Route 2.2 outperforms Route 2.1.

From Figure 5(b)-(d), one can see a completely different picture from parallel routes in asset trading. Route 2.2 although has better average pay off performance in the long run, its R^i_{wlC},R^i_{wrC} and R^i_{lrC} are worse than Route 2.1 significantly. Reason for this is that for European call option, imprecise expectation trading route prioritize long run average payoff over loss rate. When using imprecise expectation trading route in European call option, one make more loss in terms of number of times than imprecise probability route but one make more profit in each time that success happens and make less loss in each time that failure happens. Therefore, for a risk-averse investor, Route 2.1 seems to be a more reasonable choice for European call option trading.

4.3 Performance of Route 2.3 and 2.4

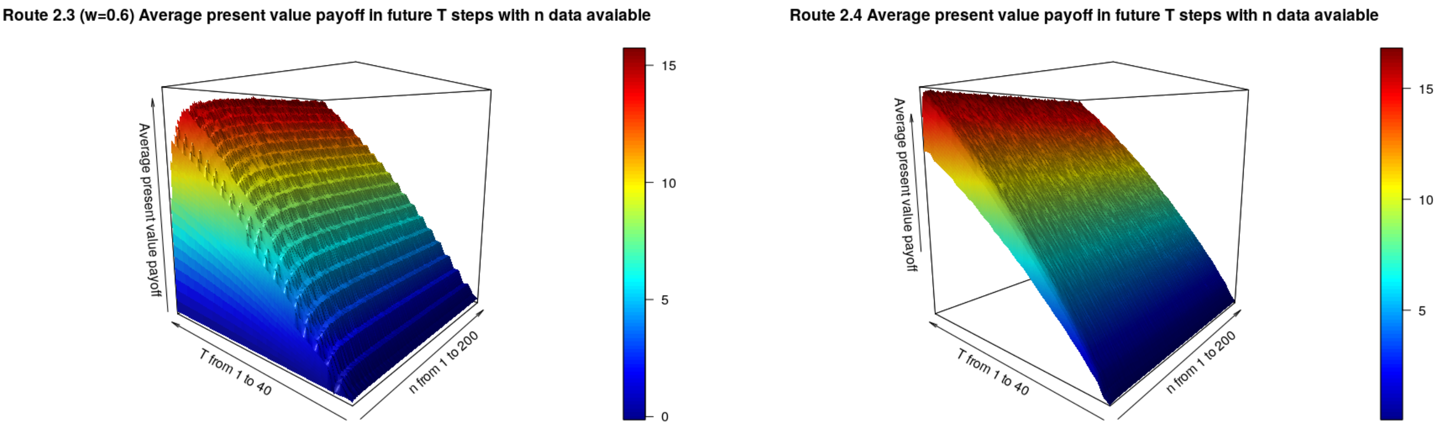
Let $N \in (1, 100000)$ be the index of simulation trial. The performances of European option trading decision routes are measured by four statistics of the present value payoff function $f_i^P(n, T, w, route_i; N)$ in 100000 simulations. $f_i^P(n, T, w, route_i; N)$ is defined as follow:

$$f_i^P(n, T, w, route_i, N)$$
$$= \begin{cases} B(T)^{-1}(K_p - A_T(s_T^N))^+ - \Lambda_p^{\mathbb{Q}}(a_0, K_p) & \text{if choose to buy} \\ \Lambda_p^{\mathbb{Q}}(a_0, K_p) - B(T)^{-1}(K_p - A_T(s_T^N))^+ & \text{if choose to short sell and invest in risk free rate} \\ 0 \quad \text{if invest in risk free rate} \end{cases}$$

where the input n is the length of data from the asset price history one could learn; T is the future time that the this function is evaluate; w is the thresold value of trading route if existed; $route_i$ is the trading route this function take; N is the index of simulation trial.

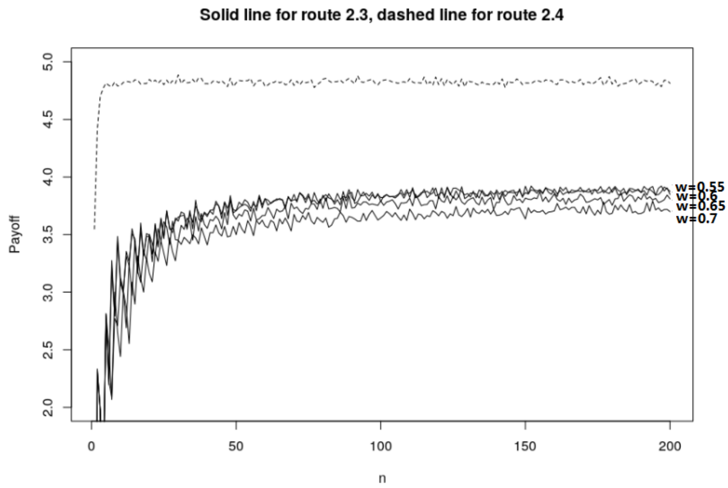
Four statistics of this function measure from 100000 simulations are:

$$\text{Average present value payoff } \overline{f_i^P} = \frac{\sum_N f_i^P}{100000} \qquad \text{win-loss ratio } R^i_{wlP} = \frac{|\{N: f_i^P > 0\}|}{|\{N: f_i^P < 0\}|}$$
$$\text{win rate } R^i_{wrP} = \frac{|\{N: f_i^P > 0\}|}{100{,}000} \qquad \text{loss rate } R^i_{lrP} = \frac{|\{N: f_i^P) < 0\}|}{100{,}000}$$

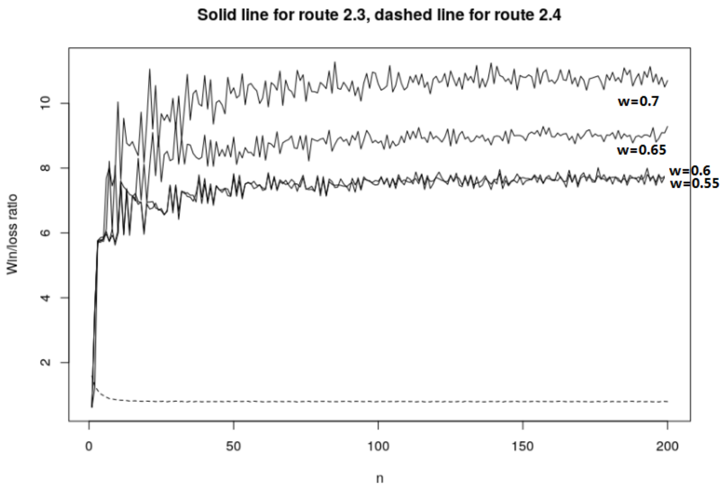


(a) (b)

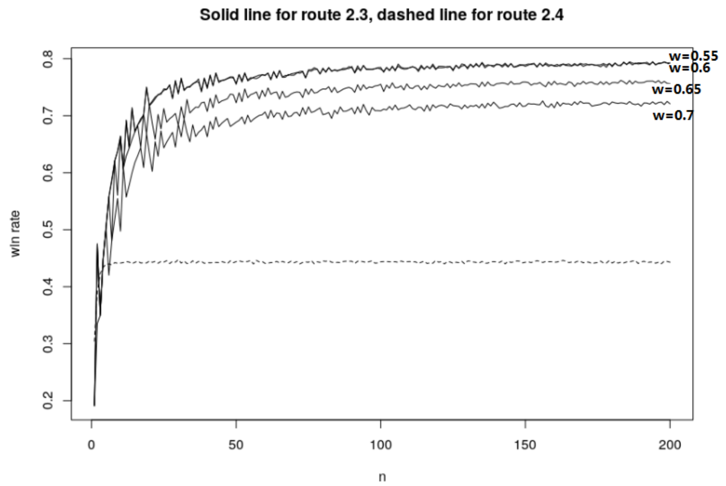
Figure 6: Route 2.3 ($w = 0.6$) and Route 2.4 average present payoff with n from 1 to 200 and T from 1 to 40



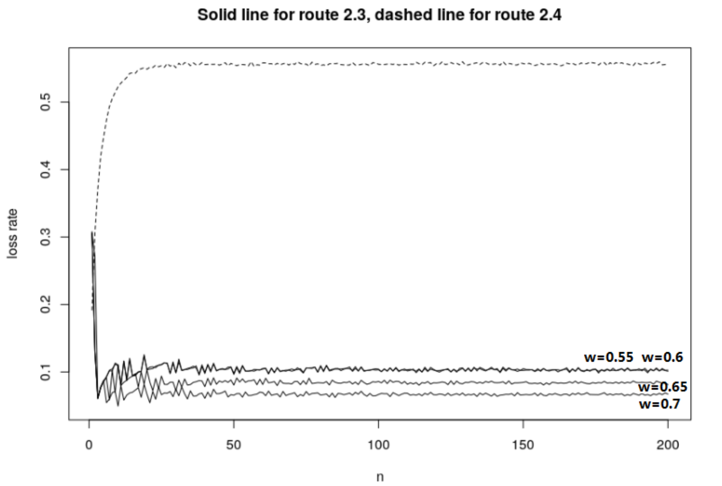
(a) Payoff performance



(b) Win-loss ratio performance



(c) Win rate performance



(d) Loss rate performance

Figure 7: Comparison of Route 2.3 and Route 2.4 with respect to different performance index with $T = 10$ and n from 1 to 200

Figure 6 shows that both trading Routes 2.3 and 2.4 have positive average present value payoff for all combinations of n from 1 to 200 and T from 1 to 40. Again, simulations for other threshold value $w > 0.5$ in Route 2.3 were also conducted, they follow the same pattern as Figure 6. Figure 7 demonstrates performance of decision Route 2.3 and 2.4 in terms of performance index $\overline{f_i^P}$,R^i_{wlP},R^i_{wrP} and R^i_{lrP}, respectively, as number of data n increases in future $T = 10$ time step for European put option trading.

From all figures above, it is seen that changing threshold value in Route 2.3 within the interval $[0.55, 0.6]$ affects all the performances insignificantly. Both Routes 2.3 and 2.4 lead to the similar pattern as Route 2.1 and 2.2 in payoff performance. Quick learning speed is confirmed. The payoff from both Routes tends to stabilise after 12 data point are available. ($\underset{i,w,n}{max}\overline{f_i^P} = 4.8868$. $\overline{f_{2.3}^P} = 3.3539$ at $n = 12$ $w = 0.60$. $\overline{f_{2.4}^P} = 4.8268$ at $n = 12$). As expected, decision Route 2.3 outperforms Route 2.4 in terms of average pay off.

Figure 7(b)-(d), it is indicate that Route 2.4 is a risky trading route for put option with win rate stabilise at 0.4442 and lose rate stabilise at 0.55518. Although Route 2.3 has less average payoff in the long run, it has better win-loss profile. ($R^i_{wlP} > 6.5$,$R^i_{wrP} > 0.65$ and $R^i_{lrP} < 0.12$ for all threshold values simulated.) Therefore, for a risk-averse or well considered investor, Route 2.4 appears to be a more attractive choice for European put option trading.

4.4 Overall conclusions from simulations

The conducted simulations indicated that all suggested NPI routes have positive long run average payoff. They also show quick learning speed, which is an important property for trading, especially if a small amount of data is available. One should noticed the imprecise probability routes and imprecise expectation routes have different advantages in trading. While imprecise probability routes prioritize better win-loss rate profiles, imprecise expectation routes prioritize maximum long run average payoff. Depending on an investor's personal interest, the investor can choose between them. When choosing imprecise probability routes, with adjustment of the threshold value w in decision making, investors can trade off between long run average payoff and win-loss rate in trading.

5 Concluding remarks

Imprecise probability is a relatively new stream in the study of uncertainties of random phenomena. It has increasing exposure to the literature in finance [3, 4, 16, 17].

This paper applies NPI methods in asset, European call option and European put option trading in binomial tree model. Using statistical software R, simulations are conducted to investigate the performance of all suggested trading routes. The property of quick learning from available data is validated. It is confirmed all the trading routes based on NPI have good predictive performance and yield positive results after a few data become available. With adjustment of the threshold value w in decision making, depending on individual risk preference, an investor can trade off between long run average payoff and win-loss rate in decision route selection.

The positive results suggest several topics for future research: Investigating trading routes which combine imprecise probability and imprecise expectation; testing trading routes suggested in this paper upon a financial portfolio; founding optimal length of historical data which is required to have good predictive result for given length of future prediction.

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