

Scattering Amplitudes and Wilson Loops in Twistor Space

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Abstract

This article reviews the recent progress in twistor approaches to Wilson loops, amplitudes and their duality for $\mathcal{N} = 4$ super Yang-Mills. Wilson loops and amplitudes are derived from first principles using the twistor action for maximally supersymmetric Yang-Mills theory. We start by deriving the MHV rules for gauge theory amplitudes from the twistor action in an axial gauge in twistor space, and show that this gives rise to the original momentum space version given by Cachazo, Svrček and Witten. We then go on to obtain from these the construction of the momentum twistor space loop integrand using (planar) MHV rules and show how it arises as the expectation value of a holomorphic Wilson loop in twistor space. We explain the connection between the holomorphic Wilson loop and certain light-cone limits of correlation functions. We give a brief review of other ideas in connection with amplitudes in twistor space: twistor-strings, recursion in twistor space, the Grassmannian residue formula for leading singularities and amplitudes as polytopes. This article is an invited review for a special issue of *Journal of Physics A* devoted to ‘Scattering Amplitudes in Gauge Theories’.

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1 Introduction

In recent studies of scattering amplitudes and correlation functions in $\mathcal{N} = 4$ super Yang-Mills (SYM), twistor variables have become a powerful tool. There are technical reasons for this: the twistor data for both scattering amplitudes and null polygonal Wilson loops are unconstrained, twistor space makes manifest symmetry under the superconformal group and, in particular, maximal supersymmetry is most naturally and straightforwardly expressed off-shell in twistor space. As a consequence, twistors have emerged in many approaches to scattering amplitudes and Wilson loops. Twistors play an important role in the Grassmannian integral formulae of [1, 2], various formulations of the correspondence between Wilson loops and scattering amplitudes [3–7], the computations of [8] using symbols to simplify the 2-loop 6-particle MHV remainder function, the Y-system for the amplitude at strong coupling [9], the all-loop extension of the BCFW recursion relation [10], and even in studying certain correlation functions [11, 12].

Although twistors are clearly a useful set of variables for these problems, in this review we will take the position that the reason scattering amplitudes in $\mathcal{N} = 4$ SYM look simplest when expressed in terms of twistors is because *the theory itself* is simple there. That is, rather than computing amplitudes in momentum space or null Wilson loops in space-time and then trying to discover hidden structures and simpler expressions by merely *translating* the results into twistor space, we instead seek to understand how to describe quantum field theory itself in twistor space. We shall see that this can indeed be done, and that twistor space scattering amplitudes and Wilson loops are beautiful objects *in their own right*, and are much more readily computable in twistor space than on space-time. The calculation from first principles of elementary multiparticle tree amplitudes in twistor space either directly or via the holomorphic Wilson loop formulation is essentially a straightforward combinatoric one or two line computation. The corresponding calculations on space-time require a tour de force of computation or computer algebra, and indeed their quantum consistency remains controversial [13]. This can therefore be taken as a nontrivial model for Penrose’s original proposal that twistor space should provide a more natural arena for physics than space-time [14].

Maximally supersymmetric Yang-Mills (SYM) theory is expressed on twistor space via an action functional on twistor space that, for $\mathcal{N} = 4$ SYM, was first introduced and studied in [15–17]¹. This action was first introduced as a field theory explanation for the MHV formalism, providing a bridge between the standard, space-time action when expressed in one gauge and the MHV formalism when expressed in an axial gauge that is inaccessible from space-time [17, 20]. However, this early work was only able to arrive at the MHV formalism by using twistor wave-functions corresponding to momentum eigenstates. These obscure the superconformal symmetry and so much of the power of the twistor representation is lost. Recent work [4, 21, 22] has now led to significant advances that make it possible to develop the

¹See also [18, 19] for a tentative twistor action for supergravity.

resulting quantum field theory directly in twistor space. The twistor space MHV formalism is much more efficient than that in momentum space, as it fully exploits the superconformal invariance of $\mathcal{N} = 4$ SYM.²

The main focus of the review will be to illustrate the use of the twistor action by computing scattering amplitudes and correlation functions of certain Wilson loops. The interest in these particular objects comes from a conjecture of Alday & Maldacena [23] at strong coupling, and of Drummond, Korchemsky & Sokatchev [24] at weak coupling, claiming that, suitably interpreted, the correlation function of a piecewise null Wilson loop is equal to the ratio of the planar all-loop MHV scattering amplitude, divided by the MHV tree amplitude³. On space-time this conjecture has been checked for a number of low-lying examples, both at weak and strong coupling [24, 28–31].

Reformulating the Wilson loop in twistor space [4] led to a conjecture relating a supersymmetric and holomorphic version of the Wilson loop in twistor space to the full super-amplitude (*i.e.*, with arbitrary external helicities). This conjecture has now been proved at the level of the loop integrand in [6] using a holomorphic version of the Migdal-Makeenko loop equations to recover the all-loop BCFW recursion. The extension of these ideas to the correspondence between Wilson loops and other correlators [32] can also be realized and extended supersymmetrically in twistor space [11]. There were almost simultaneous proposals for space-time versions of these ideas [5, 12], but it is considerably more difficult to calculate examples of amplitudes with this approach, nor has it been possible to develop clear proofs. On twistor space, the calculations of examples from first principles is as straightforward as any available technique and the proofs are now clear.

Although we still do not have a completely systematic regularization procedure for divergent integrals on twistor space, clear approaches certainly exist. In particular the Coulomb branch mass regularization of [33–36] is compatible with the twistor framework. For this reason, we confine ourselves in this review to studying the *integrand* [10] of the scattering amplitude or Wilson loop. This object is well-defined in a planar theory, and allows us to ‘freeze’ the loop momenta at some generic points. For generic external momenta, the loop integrand is finite and superconformally invariant data associated with the scattering amplitude. It is the sum of all Feynman diagrams before the loop integrations are actually carried out. The familiar infrared divergences of $\mathcal{N} = 4$ super Yang-Mills only arise when these integrations are performed, and it is at that point that one must move out along the Coulomb branch in order to regulate the amplitudes. We remark that it has been argued that there are quantum inconsistencies in the *space-time* definition even of the loop integrand (before integration) for the supersymmetric Wilson loop [13]. These arise from ambiguities in the definition of the

²As we will see, the twistor space MHV formalism gives very efficient computations for tree-level and IR finite loop amplitudes, but general (divergent) loop amplitudes still require a regularization mechanism; see §4.

³The relation between null Wilson lines and scattering amplitudes of course has a long history, and is important for example in understanding the exponentiation properties of infra-red divergences of scattering amplitudes [25–27].

space-time Wilson loop as it cannot be defined off-shell. On the other hand, we will see that the twistor space holomorphic Wilson-loop is defined completely off-shell so that issue does not arise in twistor space. Indeed, as we shall see, direct concrete calculation gives the correct answer for the loop integrand as proved by other means [37, 38].

The review is structured as follows. After providing a brief introduction to twistor geometry and the Penrose transform, in section 2 we describe the distributional twistor wavefunctions which provide a useful calculus that has led to many of the recent advances. In section 3 we then introduce the twistor action and develop the Feynman rules that arise when an axial gauge condition is imposed. In section 4 these are used to construct amplitudes in twistor space. Upon transforming to momentum space, the twistor action thus provides a *derivation* of the original momentum space MHV rules. (See also [39] for a review of the MHV formalism in momentum space.) The relationship between Wilson loops and amplitudes follows by encoding the momenta of particles taking part in a scattering process into edges of a polygon; the fact that the polygon is closed reflects momentum conservation, while the fact that the particles are massless means the polygon's edges are null. This momentum space data can then be translated into integrands for the scattering amplitudes on momentum twistor space, as covered in section 5. In section 6 we then review how to construct a supersymmetric and holomorphic Wilson loop in twistor space, and explain why its expectation value (evaluated with respect to the twistor action) is equivalent to the amplitude integrands in section 6. We also explain the extension of these ideas to a correspondence between correlation functions of local, gauge invariant operators and Wilson loops [11, 12, 32, 40, 41].

Recent developments in the study of amplitudes using twistor methods have moved quite rapidly, and it is beyond the scope of this review to give each aspect of these advances a fair treatment. Nevertheless, we also provide short (and very incomplete) summaries of a number of key topics: the current status of twistor-string theory [42–53], BCFW recursion in twistor space [53–55], the Grassmannian integral representation [1, 2, 48, 56–61], the relationship between amplitudes, polytopes and local forms. Some of these ideas are covered from a different perspective in other chapters of this review [62].

We hope that this review will be sufficient, not only to impress the reader with the power of recent twistor methods, but also with their elegance and simplicity. There are some sacrifices made for this simplicity. Twistor theory is chiral, non-local and unitarity is not manifest. Nevertheless, the simplicity is suggestive that we are making progress towards Penrose's original goal of reformulating physics on twistor space to the point where the twistor formulation supercedes that on space-time. To fulfill this goal, many more insights will be required, but the rapid progress of the last few years suggests that this may yet appear in the foreseeable future.

Brief guide to the background literature

There are now a number of textbooks now on twistor theory [63–67] of which [65] represents a first primer. Supertwistors are covered in the textbook [68]. An early review of twistor-strings

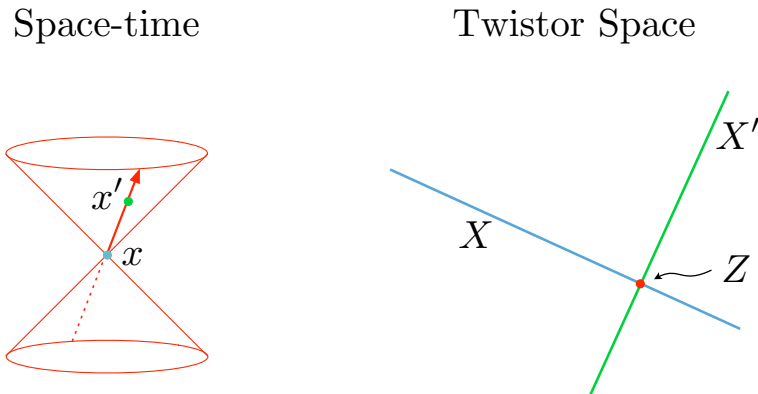


Figure 1: Points in space-time correspond to complex lines in twistor space. Two space-time points are null separated if and only if their corresponding twistor lines intersect.

and amplitudes is [69] whereas the more recent lecture notes [70] give a detailed introduction to some of the material in this review. Witten’s original twistor-string paper [42] also provides much excellent background and more.

2 Twistor Theory Basics

In this review, twistor space will mean the Calabi-Yau supermanifold $\mathbb{PT} \cong \mathbb{CP}^{3|4}$. When we wish to refer to the bosonic twistor space only, we will use the name $\mathbb{PT}_b \cong \mathbb{CP}^3$. \mathbb{PT} may be described by homogeneous coordinates

$$Z^I = (Z^\alpha, \chi^a) = (\lambda_A, \mu^{A'}, \chi^a), \quad (2.1)$$

where λ_A and $\mu^{A'}$ are 2-component complex Weyl spinors, and χ^a is an anti-commuting Grassmann coordinate⁴, with $a = 1, \dots, 4$ indexing the $\mathcal{N} = 4$ R-symmetry. Being homogeneous coordinates, the Z^I s are defined only up to the equivalence relation $Z^I \sim rZ^I$ for any non-zero complex number r that rescales all components equally.

The basic geometric correspondence with (complexified) chiral super Minkowski space $\mathbb{M}^{4|8}$ is that a point $(x, \theta) \in \mathbb{M}^{4|8}$ corresponds to a complex line⁵ X in twistor space. Furthermore, two space-time points (x, θ) and (x', θ') are null separated if and only if the corresponding twistor lines X and X' intersect, as illustrated in figure 1. In this way, the complex structure of \mathbb{PT} (i.e., knowledge of where the complex lines are) determines and is determined by the conformal structure (i.e., knowledge of the null cones) in space-time.

⁴These conventions, first adopted in [42], are not far off the *dual* twistor space conventions of standard twistor textbooks such as [64, 65].

⁵That is, X is a linearly embedded Riemann sphere.

This geometric correspondence is encapsulated in the *incidence relations*

$$\mu^{A'} = ix^{AA'}\lambda_A, \quad \chi^a = \theta^{Aa}\lambda_A, \quad (2.2)$$

where we can interpret λ_A as homogeneous coordinates on the Riemann sphere X , and (x, θ) then tell us how this Riemann sphere is embedded in \mathbb{PT} . If two lines X and X' intersect at the point $Z = (\lambda, \mu, \chi)$, then as well as (2.2) we have

$$\mu^{A'} = ix'^{AA'}\lambda_A, \quad \chi^a = \theta'^{Aa}\lambda_A, \quad (2.3)$$

and subtracting gives $(x - x')^{AA'}\lambda_A = 0$ and $(\theta - \theta')^{Aa}\lambda_A = 0$ so that $(x - x')^{AA'} = \tilde{\lambda}^{A'}\lambda^A$ and $(\theta - \theta')^{Aa} = \eta^a\lambda^A$ for some $\tilde{\lambda}$ and some η . In complex chiral superspace, as we vary the possible choices of $(\tilde{\lambda}, \eta)$, the possible vectors $(\tilde{\lambda}\lambda, \eta\lambda)$ span a totally null complex 2|4-dimensional plane known as a (super) α -plane. Thus, for every point $Z \in \mathbb{PT}$, the incidence relation assigns an α -plane in $\mathbb{M}^{4|8}$.

One of the reasons twistor space is useful when describing $\mathcal{N} = 4$ SYM is that it carries a particularly natural action of the (complexified) superconformal group $\text{PSL}(4|4, \mathbb{C})$. Acting on the homogeneous coordinates Z^I , this group is generated by

$$J^I{}_J = Z^I \frac{\partial}{\partial Z^J}, \quad (2.4)$$

except that the overall homogeneity operator $\sum_I Z^I \partial / \partial Z^I$ and the fermionic homogeneity operator $\sum_a \chi^a \partial / \partial \chi^a$ should each be removed⁶. In particular, the super Poincaré group is generated by

$$\begin{aligned} P_{AA'} &= \lambda_A \frac{\partial}{\partial \mu^{A'}} & J_{AB} &= \frac{1}{2} \left(\lambda_A \frac{\partial}{\partial \lambda^B} + \lambda_B \frac{\partial}{\partial \lambda^A} \right) & J_{A'B'} &= \frac{1}{2} \left(\mu_{A'} \frac{\partial}{\partial \mu^{B'}} + \mu_{B'} \frac{\partial}{\partial \mu^{A'}} \right) \\ Q_{Aa} &= \lambda_A \frac{\partial}{\partial \chi^a} & \tilde{Q}_{A'}^a &= \chi^a \frac{\partial}{\partial \mu^{A'}} & R^a{}_b &= \chi^a \frac{\partial}{\partial \chi^b}, \end{aligned} \quad (2.5)$$

while the superconformal generators also include

$$\begin{aligned} K^{AA'} &= \mu^{A'} \frac{\partial}{\partial \lambda_A} & D &= \frac{1}{2} \left(\lambda_A \frac{\partial}{\partial \lambda_A} - \mu^{A'} \frac{\partial}{\partial \mu^{A'}} \right) \\ S^{Aa} &= \chi^a \frac{\partial}{\partial \lambda_A} & \tilde{S}_a^{A'} &= \mu^{A'} \frac{\partial}{\partial \chi^a}. \end{aligned} \quad (2.6)$$

These show that λ is inert under a translation in chiral superspace, while (μ, χ) transform as

$$\mu^{A'} \rightarrow \mu^{A'} + iy^{AA'}\lambda_A \quad \chi^a \rightarrow \chi^a + \theta^{aA}\lambda_A. \quad (2.7)$$

⁶Any object in $\mathcal{N} = 4$ SYM should have overall homogeneity zero in each twistor, while the amplitudes are graded by their fermionic homogeneity, known as MHV degree.

Thus (μ, χ) have a rather different status to λ under the super Poincaré group, and for this reason, (μ, χ) are sometimes known as the ‘primary’ part of the supertwistor, while λ is the ‘secondary’ part. Their roles are interchanged under a special conformal transformation.

So far, we have explained the correspondence for complexified space-time. We now wish to impose various reality conditions on our space-time and twistor space. In Minkowski signature, the conformal group is $SU(2,2)$ as a real form of $SL(4; \mathbb{C})$. This real subgroup preserves a pseudo-Hermitian metric $g_{\alpha\bar{\beta}}$ of signature $(2,2)$ on non-projective twistor space, that we may write as

$$g_{\alpha\bar{\beta}} Z^\alpha \bar{Z}^\beta = \lambda_A \bar{\mu}^A + \mu^{A'} \bar{\lambda}_{A'} \quad (2.8)$$

where $\bar{\lambda}_{A'}$ is the Lorentzian complex conjugate of the spinor λ_A (similarly for $\bar{\mu}^A$ and $\mu^{A'}$). If we define

$$\bar{Z}_\alpha \equiv g_{\alpha\bar{\beta}} \bar{Z}^\beta = (\bar{\mu}^A, \bar{\lambda}_{A'}) \quad (2.9)$$

then we can equivalently view Lorentzian complex conjugation as an anti-holomorphic map from twistor space to dual twistor space. On the projective space, the value of $Z \cdot \bar{Z}$ is meaningless, but the sets

$$\mathbb{PT}^+ := \{Z \mid Z \cdot \bar{Z} > 0\} \quad \mathbb{PN} := \{Z \mid Z \cdot \bar{Z} = 0\} \quad \mathbb{PT}^- := \{Z \mid Z \cdot \bar{Z} < 0\} \quad (2.10)$$

are preserved under the scaling $Z \sim rZ$. In particular, if a twistor line X lies entirely in \mathbb{PN} , then from (2.2) we have

$$0 = i(x - x^\dagger)^{AA'} \lambda_A \bar{\lambda}_{A'} \quad \text{for all } \lambda, \quad (2.11)$$

which is possible if and only if the matrix

$$x^{AA'} = \sigma_\mu^{AA'} x^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \quad (2.12)$$

is Hermitian. Thus the corresponding point x lies in real Minkowski space. Conversely, a point $Z \in \mathbb{PN}$ corresponds to a unique real null ray (the intersection of the complex α -plane with the Minkowski real slice).

In Euclidean signature, we instead equip twistor space with an anti-holomorphic map $Z^\alpha \rightarrow \hat{Z}^\alpha$ that satisfies $\hat{\hat{Z}} = -Z$. This conjugation may be given explicitly by $Z^\alpha = (\hat{\lambda}_A, \hat{\mu}^{A'}) = (-\bar{\lambda}_1, \bar{\lambda}_0, -\bar{\mu}^1, \bar{\mu}^0)$. The fact that the conjugation squares to -1 shows that there are no (non-zero) real twistors in Euclidean signature. A point in real Euclidean space corresponds to a twistor line that is mapped to itself by this conjugation (with the conjugation acting as the antipodal map on $X \cong \mathbb{CP}^1$). Thus, given the Euclidean conjugation, there is a ‘preferred’ twistor line through any point Z – namely the line joining Z to \hat{Z} . This line is clearly preserved under the conjugation, so to any twistor Z we can always associate a unique point in Euclidean space. Explicitly, this point is

$$x^{AA'} = \frac{\mu^{A'} \hat{\lambda}^A - \hat{\mu}^{A'} \lambda^A}{\langle \lambda \hat{\lambda} \rangle} = \hat{x}^{AA'}. \quad (2.13)$$

Said differently, in Euclidean signature, there is a non-holomorphic fibration $\mathbb{CP}^3 \rightarrow S^4$ whose fibres are the twistor lines ($Z\hat{Z}$). The Euclidean structure was introduced by Atiyah, Hitchin & Singer [71] and was used in the ADHM approach to the construction of instantons [72].

Finally, in $(++--)$ space-time signature, the superconformal group is $\mathrm{PSL}(4|4; \mathbb{R})$, so we simply take all the twistors to be real (and drop the factors of i from the incidence relations). This signature was exploited by Witten [42] in his ‘half-Fourier transform’ to readily transform scattering amplitudes on $(2,2)$ -signature momentum space on to twistor space.

2.1 The Penrose transform and cohomology

The Penrose transform relates helicity h solutions of the zero-rest-mass (z.r.m.) free field equations on a region $U' \subset \mathbb{M}$ to *cohomology classes* of functions of homogeneity degree $2h - 2$ over a corresponding region $U \subset \mathbb{PT}_b$, where U is the region swept out by the twistor lines corresponding to the points of U' .

Cohomology classes can be represented in a variety of ways. In this review we will use the Dolbeault representation in which the cohomology classes are described by $\bar{\partial}$ -closed $(0, 1)$ -forms modulo $\bar{\partial}$ -exact ones. The Penrose transform is then expressed as the isomorphism

$$H^1(U, \mathcal{O}(2h - 2)) \cong \{\text{On-shell z.r.m. fields on } U' \text{ of helicity } h\}, \quad (2.14)$$

where

$$H^1(U, \mathcal{O}(n)) := \frac{\{f \in \Omega^{0,1}(n) \mid \bar{\partial}f = 0\}}{\{f \mid f = \bar{\partial}g\}}. \quad (2.15)$$

and $\Omega^{0,1}(n)$ is the space of smooth $(0,1)$ -forms⁷ on U that are homogeneous of degree n (i.e., $f(rZ) = r^n f(Z)$).

This transform is most easily realized by the integral formula

$$\phi_{A_1 A_2 \dots A_{|2h|}} = \frac{1}{2\pi i} \int_X \lambda_{A_1} \lambda_{A_2} \dots \lambda_{A_{|2h|}} f(ix^{AA'} \lambda_A, \lambda_A) \wedge D\lambda, \quad (2.16)$$

when $h \leq 0$, and

$$\phi_{A'_1 A'_2 \dots A'_{2h}} = \frac{1}{2\pi i} \int_X \frac{\partial}{\partial \mu^{A'_1}} \frac{\partial}{\partial \mu^{A'_2}} \dots \frac{\partial}{\partial \mu^{A'_{2h}}} g(ix^{AA'} \lambda_A, \lambda_A) \wedge D\lambda, \quad (2.17)$$

when $h > 0$. In these formulae, $D\lambda = \lambda_C d\lambda^C$ and in (2.17) the derivatives are really Lie derivatives acting on the forms. The fact that these integral formulae give rise to solutions of the massless field equations can be seen by directly differentiating under the integral sign and using the fact that

$$\frac{\partial f}{\partial x^{AA'}} = i\lambda_A \frac{\partial f}{\partial \mu^{A'}}. \quad (2.18)$$

⁷A (p, q) -form has degree p in the differentials of the holomorphic coordinates and degree q in the differentials of anti-holomorphic coordinates.

The Penrose isomorphism (2.14) states that all solutions of the z.r.m. field equations arise this way. See for example [64] for a proof⁸.

The beauty of the construction is that it transforms the differential equations that on-shell fields satisfy in space-time into pure holomorphy in twistor space. As well as the clean representation of the superconformal group, this provides a second reason that twistors are useful in describing scattering amplitudes: the constraint that the external states be on-shell is automatically satisfied by using *arbitrary* holomorphic wave-functions in twistor space.

We can easily construct an action on twistor space whose field equations yield the above cohomology classes. Let (f, \tilde{f}) be a pair of smooth $(0, 1)$ -forms on U of respective homogeneities $2h - 2$ and $-2h - 2$, and consider the action

$$S[f, \tilde{f}] = \int_U \tilde{f} \wedge \bar{\partial}f \wedge D^3Z \quad (2.19)$$

where $D^3Z = \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta dZ^\gamma dZ^\delta / 4!$. The field equations are $\bar{\partial}f = 0 = \bar{\partial}\tilde{f}$, while the fields are defined only up to the gauge freedom $f \rightarrow f + \bar{\partial}g$, etc. Thus the on-shell fields of (2.19) correspond to elements of the cohomology group $H^1(U, \mathcal{O}(\pm 2h - 2))$ and therefore to z.r.m. fields of helicity $\pm h$ on space-time. Notice that even when $h = 0$ (space-time scalars), the twistor action is still subject to gauge redundancy. This is because we are describing a four-dimensional theory in terms of a six (real) dimensional space.

Standard choices for U are the sets \mathbb{PT}^\pm introduced in (2.10). In complex space-time, these correspond to the future/past tubes \mathbb{M}^\pm : the sets on which the imaginary part of $x^{AA'}$ is past or future pointing time-like, respectively. This follows from the fact that if we take $x = u + iv$ and substitute into the incidence relation, then $Z \cdot \bar{Z} = -v^{AA'} \bar{\lambda}_{A'} \lambda_A$, which has a definite sign when v is time-like. The sign itself depends on whether v is future or past pointing. The significance of this is that a field of positive frequency, whose Fourier transform is supported on the future light-cone in momentum space, automatically extends over the future tube because $e^{ip \cdot x}$ is rapidly decreasing there, bounded by its values on the real slice. Another frequently used set is $U = \mathbb{PT}'$ on which $\lambda_A \neq 0$. This corresponds to excluding the light-cone of the ‘point at infinity’ in space-time and includes all momentum eigenstates.

⁸Traditionally, twistor theorists have used the Čech cohomology, which involves choosing an open cover of U and describing cohomology representatives in terms of holomorphic functions defined on overlaps. The integral formulae (2.16)-(2.17) are then taken to be contour integrals. We have not used that formulation because it is difficult to express the full invariances of the theory, and the combinatorics of the open covers becomes complicated.

The supersymmetric extension

On $\mathcal{N} = 4$ supertwistor space, the transform has a straightforward supersymmetrization to the superfield

$$\mathcal{A}(Z, \bar{Z}, \chi) = a(Z, \bar{Z}) + \chi^a \tilde{\psi}_a(Z, \bar{Z}) + \frac{\chi^a \chi^b}{2} \phi_{ab}(Z, \bar{Z}) + \frac{\epsilon_{abcd}}{3!} \chi^a \chi^b \chi^c \psi^d(Z, \bar{Z}) + \frac{\epsilon_{abcd}}{4!} \chi^a \chi^b \chi^c \chi^d g(Z, \bar{Z}) \quad (2.20)$$

where a , $\tilde{\psi}$, ϕ , ψ , and g have homogeneity 0 , -1 , -2 , -3 and -4 respectively, corresponding on-shell to zero-rest mass fields $(F_{A'B'}, \tilde{\Psi}_{aA'}, \Phi_{ab}, \Psi_A^a, G_{AB})$ on space-time. This is the supermultiplet appropriate to $\mathcal{N} = 4$ SYM. The supersymmetric action is an Abelian Chern-Simons action

$$S[\mathcal{A}] = \int \mathcal{A} \wedge \bar{\partial} \mathcal{A} \wedge D^{3|4} Z \quad (2.21)$$

where $D^{3|4} Z = D^3 Z d^4 \chi$. It is easily seen that integrating out the fermionic coordinates gives the appropriate actions for each component field. Thus \mathcal{A} is an off-shell representation of the complete $\mathcal{N} = 4$ supermultiplet on twistor space.

On-shell, the field equations modulo gauge redundancy of (2.21) show that \mathcal{A} represents an element of the cohomology group $H^1(U, \mathcal{O})$ on a region of supertwistor space. The integral formulae (2.17)-(2.16) extend directly to this supersymmetric context to give on-shell superfields on space-time that incorporate derivatives of the component fields. Specifically, one has

$$\begin{aligned} \mathcal{F}_{A'B'} &:= \int_X \frac{\partial^2}{\partial \mu^{A'} \partial \mu^{B'}} \mathcal{A}(ix^{AA'} \lambda_A, \lambda_A, \theta^{Aa} \lambda_A) \wedge D\lambda \\ &= F_{A'B'} + \theta^{Aa} \partial_{AA'} \left[\tilde{\Psi}_{aB'} + \theta^{Bb} \partial_{BB'} \left(\frac{\Phi_{ab}}{2} + \theta^{Cc} \epsilon_{abcd} \left(\frac{\Psi_C^d}{3!} + \theta^{Dd} \frac{G_{CD}}{4!} \right) \right) \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_{ab} &:= \int_X \frac{\partial^2}{\partial \chi^a \partial \chi^b} \mathcal{A}(ix^{AA'} \lambda_A, \lambda_A, \theta^{Aa} \lambda_A) \wedge D\lambda \\ &= \Phi_{ab} + \theta^{Cc} \epsilon_{abcd} (\Psi_C^d + \theta^{Dd} \frac{G_{CD}}{2}). \end{aligned} \quad (2.22)$$

These fields together $\mathcal{F}_{aA'}$ (which has a formula as above with a mixed μ and χ derivative) have the interpretation as being the non-zero parts of the curvature

$$\mathcal{F} = \mathcal{F}_{A'B'} \epsilon_{AB} dx^{AA'} \wedge dx^{BB'} + \mathcal{F}_{aA'} \epsilon_{AB} dx^{AA'} \wedge d\theta^{Ba} + \mathcal{F}_{ab} \epsilon_{AB} d\theta^{Aa} \wedge d\theta^{Bb} \quad (2.23)$$

of the on-shell space-time superconnection

$$\begin{aligned} \mathcal{A} &= \left[A_{AA'} + \theta_A^a \left(\tilde{\Psi}_{aA'} + \theta^{Bb} \partial_{BA'} \left(\frac{\Phi_{ab}}{2} + \epsilon_{abcd} \theta^{Cc} \left(\frac{\Psi_C^d}{3!} + \theta^{Dd} \frac{G_{CD}}{4!} \right) \right) \right) \right] dx^{AA'} \\ &\quad + \left[\Phi_{ab} + \epsilon_{abcd} \theta^{Bc} \left(\frac{\Psi_B^d}{2} + \theta^{Cd} \frac{G_{BC}}{3!} \right) \right] \theta_A^a d\theta^{Ab}. \end{aligned} \quad (2.24)$$

This superconnection can be obtained directly and geometrically from \mathcal{A} via the Penrose-Ward transform, which treats \mathcal{A} geometrically as a deformation of the $\bar{\partial}$ -operator on a line bundle and obtains \mathcal{A} as a (super)-connection on a corresponding line bundle on space-time. This interpretation will follow from our treatment of the Yang Mills equations in subsequent sections.

2.2 Distributional forms for twistor wave functions

In this section, we describe twistor representatives for various commonly used wave-functions. In our Dolbeault framework, it will be convenient to work with distribution-valued forms. We first describe twistor wave-functions for momentum eigenstates. We then describe the *elementary state* on twistor space that corresponds to the fundamental solution of the wave equation. Finally, we describe *elemental states* that are supported at a point in twistor space (i.e., they are twistor eigenstates). These elemental states form the main basis of the calculus that we actually use in the rest of the review.

On the complex plane with coordinate $z = x + iy$, the delta function supported at the origin is naturally a $(0, 1)$ -form which we denote by

$$\bar{\delta}^1(z) := \delta(x)\delta(y) d\bar{z} = \frac{1}{2\pi i} d\bar{z} \frac{\partial}{\partial \bar{z}} \frac{1}{z}, \quad (2.25)$$

the second equality being a consequence of the standard Cauchy kernel for the $\bar{\partial}$ -operator. This second representation makes clear the homogeneity property $\bar{\delta}(rz) = r^{-1}\bar{\delta}(z)$. (Note that there is no absolute-value sign here, in contrast to the case of real δ -functions.)

We can extend this to the Riemann sphere by defining

$$\bar{\delta}_m^1(\lambda, p) := \int \frac{ds}{s^{m+1}} \bar{\delta}^2(s\lambda_A + p_A). \quad (2.26)$$

This has support only when $p_A \propto \lambda_A$ with some constant of proportionality that we integrate over. Thus it has support only when p and λ coincide projectively. One can check that $\bar{\delta}_m^1(\lambda, p)$ has homogeneity m in λ and $-m - 2$ in p , so that

$$g(p) = \int_{\mathbb{CP}^1} \bar{\delta}_m^1(\lambda, p) g(\lambda) \wedge D\lambda \quad (2.27)$$

for any function g of homogeneity $-m - 2$.

This idea can be used to give the twistor cohomology class for an on shell momentum eigenstate with momentum $p_{AA'} = p_A \tilde{p}_{A'}$

$$f_{p, -m-2}(\mu, \lambda) = \int \frac{ds}{s^{m+1}} e^{s\mu^{A'} \tilde{p}_{A'}} \bar{\delta}^2(s\lambda_A + p_A). \quad (2.28)$$

It is easily seen that this evaluates via (2.16) and (2.17) to give the appropriate momentum eigenstates

$$p_{A_1} \dots p_{A_m} e^{ip \cdot x}, \quad \tilde{p}_{A'_1} \dots \tilde{p}_{A'_m} e^{ip \cdot x} \quad (2.29)$$

for the zero rest mass fields of momentum p and helicity $m/2$, depending on the sign of m .

In $\mathcal{N} = 4$ SYM, an on-shell supermultiplet with definite momentum $p_{AA'} = p_A \tilde{p}_{A'}$ and definite supermomentum $p_A \eta_a$ may be represented by the supertwistor space cohomology class

$$f_{p;\eta}(Z; \chi) := \int \frac{ds}{s} e^{s(\mu^{A'} \tilde{p}_{A'} + \eta_a \chi^a)} \bar{\delta}^2(s\lambda_A + p_A). \quad (2.30)$$

Inserting this into (2.22) gives the chiral Minkowski space wave-function

$$F_{A'B'}(x, \theta) = \tilde{p}_{A'} \tilde{p}_{B'} e^{ip \cdot x + i\eta_a \theta^{Aa} p_A} \quad (2.31)$$

for the on-shell supermultiplet. As in (2.22), in expanding out the θ s, all except the helicity -2 parts of the multiplet appear in a differentiated form; the helicity -1 part is differentiated once, and the higher ones twice.

Our second class of examples are ‘elementary states’. These are states that are singular on a line in twistor space and correspond to fields that are singular on the lightcone of the corresponding point in space-time. The simplest of these is for homogeneity -2 (space-time scalar), where we can set

$$\phi = \frac{1}{\mu^{0'}} \bar{\delta} \left(\frac{1}{\mu^{1'}} \right). \quad (2.32)$$

Evaluation on space-time via (2.16) can be seen to give the the fundamental solution $\Phi = 1/x^2$ to the wave equation. The general elementary state is obtained from this example by differentiating with respect to Z or the basis parameters and multiplying by monomials in the twistor coordinates.

While the above representatives are useful for converting twistor expressions directly to on-shell momentum space, part of the power of the twistor representation is its underlying conformal invariance and this is not manifest if we use momentum eigenstates as external wavefunctions. We therefore introduce a version of an ‘elemental state’ which may be viewed as ‘eigenstates of definite twistors’.

Following the strategy of (2.26), to obtain δ -functions on projective twistor space, we first introduce the Dolbeault δ -functions on⁹ $\mathbb{C}^{4|4}$

$$\bar{\delta}^{4|4}(Z) = \prod_{\alpha=0}^3 \bar{\delta}(Z^\alpha) \prod_{a=1}^4 \chi^a. \quad (2.33)$$

⁹Recall that a δ -function for a Grassmann variable χ is simply $\delta(\chi) = \chi$, as follows from the Berezinian integration rules $\int \chi d\chi = 1$ and $\int 1 d\chi = 0$, so that $\int f(\chi) \chi d\chi = f(0)$.

This is a $(0, 4)$ -form on $\mathbb{C}^{4|4}$ of homogeneity zero, having support only where $Z^I = 0$. We now define projective δ -functions by

$$\bar{\delta}^{3|4}(Z_1, Z_2) := \int_{\mathbb{C}} \frac{ds}{s} \bar{\delta}^{4|4}(Z_1 + sZ_2) . \quad (2.34)$$

This is a $(0, 3)$ -form on $\mathbb{P}\mathbb{T}_1 \times \mathbb{P}\mathbb{T}_2$, homogeneous of degree zero in each entry and antisymmetric under $Z_1 \leftrightarrow Z_2$. It satisfies

$$f(Z_1) = \int_{\mathbb{P}\mathbb{T}} f(Z_2) \bar{\delta}^{3|4}(Z_1, Z_2) \wedge D^{3|4} Z_2$$

as befits a delta function.

The elemental state will be taken to be the part of $\bar{\delta}^{3|4}(Z, Z_1)$ that is a $(0, 1)$ -form in Z and a $(0, 2)$ -form in Z_1 . Thus, an elemental state does not correspond to an ordinary real or complex valued z.r.m. field, but rather one that takes values in $(0, 2)$ -forms on the twistor space of the auxilliary Z_1 variable. It is peculiar in a number of ways that are perhaps best illustrated by evaluating it on space-time via (2.16). Applying this to the coefficient of $(\chi)^2(\chi_1)^2$ in (2.34) we obtain the space-time field

$$\phi_{Z_1}(x) = \bar{\delta}^2(\mu_1^{A'} - ix^{AA'} \lambda_{1A}) \quad (2.35)$$

which we understand as a distribution on complexified Minkowski space \mathbb{M} with values in the $(0, 2)$ -forms in the $Z_1 = (\lambda_1, \mu_1)$ variables. It should be emphasized that because it is a δ -function, $\phi_{Z_1}(x)$ is clearly not holomorphic on \mathbb{M} . It nevertheless satisfies the holomorphic wave equation made from just the holomorphic derivatives, and similarly the anti-holomorphic wave equation made from anti-holomorphic derivatives. However, when restricted to a real space-time slice, $\phi_{Z_1}(x)$ does not satisfy any the wave equation as the cross terms between holomorphic and anti-holomorphic derivatives in the real wave equation do not vanish.

2.2.1 Elemental states and amplitudes

By the Penrose transform, finite norm on-shell external wave-functions are represented on twistor space by cohomology classes $H^1(\mathbb{P}\mathbb{T}', \mathcal{O}(2h - 2))$. Since amplitudes are multilinear functionals of these external wave functions, on twistor space the kernel for an n -particle amplitude should be dual to the n -fold product of such H^1 s. Although we could use the Hilbert space structure on $H^1(\mathbb{P}\mathbb{T}', \mathcal{O}(n))$, this is actually a rather non-trivial duality that is non-local on twistor space and depends crucially on the choice of space-time signature. Instead, we will use the natural duality between $(1, 0)$ -forms and $(2, 0)$ -forms with compact support that is manifested by

$$(\phi, \alpha) \mapsto \int_{\mathbb{P}\mathbb{T}'} D^{3|4} Z \wedge \phi \wedge \alpha , \quad (\phi, \alpha) \in (\Omega^{(0,1)}, \Omega_c^{(0,2)}) .$$

Thus, we would like to say that an n -particle amplitude is represented by the twistor space kernel

$$A(Z_1, \dots, Z_n) \in \Omega_c^{(0,2n)}(\times_{i=1}^n \mathbb{P}\mathbb{T}'_i, \mathcal{O}) , \quad (2.36)$$

a $(0, 2n)$ -form with compact support on n copies of $\mathbb{P}\mathbb{T}'$. The fact that this is to be paired with cohomology classes means that it should make no difference if we add an exact form to $A(1, \dots, n)$, and for the amplitude to not depend on the choice of representative for the wave functions we should have that the amplitude is $\bar{\partial}$ -closed. Thus in an ideal world, the amplitude would be an element of $H^{2n}(\times_{i=1}^n \mathbb{P}\mathbb{T})$. However this picture is not attainable for a number of reasons. Firstly the relevant cohomology groups vanish. Secondly, collinear divergences of massless amplitudes mean (even at tree-level) that if we wedge such an $A(Z_1, \dots, Z_n)$ with finite-norm external wave-functions and integrate over n copies of twistor space, the resulting object will diverge. In the explicit formulae that we have for $A(Z_1, \dots, Z_n)$, we can compute its exterior derivative, and we find that it fails to vanish where at the diagonals where pairs of adjacent twistors come together. In effect, it is an element of $H^{2n}(\times_{i=1}^n \mathbb{P}\mathbb{T}_i)$ that has simple poles at these adjacent diagonals (see [73] for a discussion of such objects). In practice, at tree-level we simply treat $A(Z_1, \dots, Z_n)$ as a $(0, 2n)$ -form and choose ‘generic’ external twistors to avoid these singularities in the same manner as one usually chooses generic external momenta.

The effect of integrating a momentum eigenstate form as in (2.30) against such an amplitude kernel will be the same as inserting that momentum eigestate into the formula for an amplitude in the first place. As explained above, finite-norm twistor wave-functions do not have compact support in twistor space, being defined for example either on $\mathbb{P}\mathbb{T}^+$ or $\mathbb{P}\mathbb{T}^-$. Using the above elemental states, we will obtain amplitudes that are supported on $\mathbb{P}\mathbb{N}$ and hence valid when integrated against a twistor wave function defined on $\mathbb{P}\mathbb{T}^+$ or $\mathbb{P}\mathbb{T}^-$. This is how crossing symmetry becomes manifest on twistor space.

2.2.2 A calculus of distributional forms and the propagator

The $\delta^{3|4}(Z_1, Z_2)$ defined above are just the first example of a family of projective δ -functions. By integrating against a further parameter, we can obtain the δ -function

$$\begin{aligned} \bar{\delta}^{2|4}(Z_1, Z_2, Z_3) &:= \int_{\mathbb{C}\mathbb{P}^2} \frac{D^2c}{c_1 c_2 c_3} \bar{\delta}^{4|4}(c_1 Z_1 + c_2 Z_2 + c_3 Z_3) \\ &= \int_{\mathbb{C}^2} \frac{ds dt}{s t} \bar{\delta}^{4|4}(Z_3 + s Z_1 + t Z_2) \\ &= \int_{\mathbb{C}} \frac{ds}{s} \bar{\delta}^{3|4}(Z_1, Z_2 + s Z_3) , \end{aligned} \quad (2.37)$$

that has support when Z_1, Z_2 and Z_3 are collinear in projective space. This object is manifestly superconformally invariant, weightless in each twistor and antisymmetric under exchange of any two. It has simple poles where any pair of twistors coincide. We will see below that it can be used to define the propagator for the action (2.21) in an axial gauge.

We can similarly define a coplanarity delta function

$$\begin{aligned}
\bar{\delta}^{1|4}(Z_1, Z_2, Z_3, Z_4) &:= \int_{\mathbb{CP}^3} \frac{D^3 c}{c_1 c_2 c_3 c_4} \bar{\delta}^{4|4}(c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4) \\
&= \int_{\mathbb{C}^3} \frac{ds dt du}{s t u} \bar{\delta}^{4|4}(Z_4 + s Z_3 + t Z_2 + u Z_1) \\
&= \int_{\mathbb{C}} \frac{ds}{s} \bar{\delta}^{2|4}(Z_1, Z_2, Z_3 + s Z_4)
\end{aligned} \tag{2.38}$$

and the rational object

$$\begin{aligned}
\bar{\delta}^{0|4}(Z_1, Z_2, Z_3, Z_4, Z_5) &:= \int_{\mathbb{CP}^4} \frac{D^4 c}{c_1 c_2 c_3 c_4 c_5} \bar{\delta}^{4|4} \left(\sum_{i=1}^5 c_i Z_i \right) \\
&= \frac{((1234)\chi_5 + \text{cyclic})^4}{(1234)(2345)(3451)(4512)(5123)} ,
\end{aligned} \tag{2.39}$$

where the second formula is obtained by integration against the delta functions and $(1234) \equiv \epsilon_{\alpha\beta\gamma\delta} Z_1^\alpha Z_2^\beta Z_3^\gamma Z_4^\delta$ (see [2] for full details). The δ -functions in (2.38)-(2.39) enforce that their arguments lie on a common $\mathbb{CP}^2 \subset \mathbb{CP}^{3|4}$ or a common $\mathbb{CP}^3 \subset \mathbb{CP}^{3|4}$, respectively. They are each totally antisymmetric and of homogeneity zero in their arguments.

We will frequently abbreviate $\bar{\delta}^{0|4}(Z_1, Z_2, Z_3, Z_4, Z_5)$ by $[1, 2, 3, 4, 5]$. It is the most elementary superconformal invariant that one can form. In the context of momentum twistors it is the standard dual superconformal invariant of [74], sometimes denoted by $R_{5;13}$. Our definition makes it clear that these ‘R-invariants’ depend (antisymmetrically) on five arbitrary supertwistors (or momentum supertwistors).

These projective δ -functions are not generally $\bar{\partial}$ -closed, but rather

$$\bar{\partial} \bar{\delta}^{r|4}(Z_1, \dots, Z_{5-r}) = (2\pi i) \sum_{i=1}^{5-r} (-1)^{i+1} \bar{\delta}^{r+1|4}(Z_1, \dots, \widehat{Z}_i, \dots, Z_{5-r}) , \tag{2.40}$$

where \widehat{Z}_i is omitted (see [22] for a proof). The right hand side necessarily vanishes for $r = 3$. We will use these relations for $r = 2$, with $Z_3 = Z_*$ a fixed reference twistor, to obtain the propagator associated to the free field action (2.21):

$$\Delta(Z_1, Z_2) = \bar{\delta}^{2|4}(Z_1, *, Z_2). \tag{2.41}$$

This is easily seen to have the correct properties via the relation

$$\bar{\partial} \Delta(Z_1, Z_2) = 2\pi i (\bar{\delta}^{3|4}(Z_1, Z_2) + \bar{\delta}^{3|4}(Z_1, Z_*) + \bar{\delta}^{3|4}(Z_2, Z_*)) .$$

For $Z_1, Z_2 \in \mathbb{PT}'$ and $Z_* \notin \mathbb{PT}'$, the last two terms on the right hand side of this expression vanish. We are then left with

$$\bar{\partial} \Delta(Z_1, Z_2) = \bar{\delta}^{3|4}(Z_1, Z_2) , \tag{2.42}$$

which is the equation we require of the propagator for the action (2.21). This form of the propagator is in an axial gauge associated to the reference twistor; it satisfies the condition

$$\bar{Z}_*^{\bar{\alpha}} \frac{\partial}{\partial \bar{Z}^{\bar{\alpha}}} \lrcorner \Delta = 0 \quad (2.43)$$

that says the (0,2)-form $\Delta(Z_1, Z_2)$ vanishes when contracted into directions tangent to the lines through Z_* . It follows from the various definitions that when the integration in (2.37) with respect to s is performed, a \bar{Z}_* is skewed into the anti-holomorphic form indices.

As a final remark, note that many integrals can be performed algebraically when these distributional delta functions are involved. Useful examples that we will frequently use are

$$\begin{aligned} \bar{\delta}^{1|4}(Z_1, Z_2, Z_3, Z_4) &= \int \bar{\delta}^{2|4}(Z_1, Z_2, Z) \wedge \bar{\delta}^{2|4}(Z, Z_3, Z_4) \wedge D^{3|4} Z \\ [1, 2, 3, 4, 5] &= \int \bar{\delta}^{2|4}(Z_1, Z_2, Z) \wedge \bar{\delta}^{1|4}(Z, Z_3, Z_4, Z_5) \wedge D^{3|4} Z . \end{aligned} \quad (2.44)$$

Such relations form a key part of the calculus that enables us to readily evaluate Feynman diagrams or recursion relations on twistor space.

3 The Twistor Yang-Mills Action

The twistor action is our starting point for studying amplitudes and correlation functions so we provide a thorough introduction to its construction and properties here. Beginning from the Chalmers-Siegel [75] action for $\mathcal{N} = 4$ SYM on space-time, we proceed to show how the twistor action is constructed and discuss some of its features which are apparent in different choices of gauge.

The space-time action

We begin by recalling the Chalmer-Siegel action for ordinary Yang-Mills theory with no supersymmetry; this allows one to work with an action that manifestly allows for expansion around the self-dual sector of Yang-Mills theory. Consider a bundle $E' \rightarrow \mathbb{M}$ with a connection 1-form $A(x)$ taking values in $\text{End}(E')$ (the Lie algebra of the complexified gauge group) and curvature $F = dA + A \wedge A$. Rather than consider the ordinary Yang-Mills action for such a connection, introduce an auxilliary ASD 2-form $G \in \Omega^{2-}(\mathbb{M}, \text{End}(E'))$ coupled to the connection via the action [75]:

$$S[A, G] = \int_{\mathbb{M}} \text{tr} \left(G \cdot F - \frac{\varepsilon}{2} G \cdot G \right) d^4x, \quad (3.1)$$

for a parameter ε (here \cdot denotes the natural inner product on 2-forms).¹⁰ Splitting the curvature into its SD and ASD parts, $F = F^+ + F^-$, this action has field equations

$$F^- = \varepsilon G, \quad \nabla \wedge G = 0, \quad (3.2)$$

where $\nabla = d + A$ is the connection corresponding to A . We can easily see that when $\varepsilon = 0$, we have the SD Yang-Mills equations, and generally

$$\nabla \wedge F^* = \nabla \wedge (F^+ - F^-) = \nabla \wedge (F - 2F^-) = 0$$

by the Bianchi identity and field equations. Hence, the Chalmers-Siegel action of (3.1) has field equations which are equivalent to the full Yang-Mills equations, and ε has the natural interpretation of an expansion parameter away from the SD sector of the theory.

This action can be extended to $\mathcal{N} = 4$ SYM by including the additional fields of the multiplet, which (besides $A(x)$ and G) are: the ASD and SD gluino fermions Ψ_A^a and $\tilde{\Psi}_{aA'}$ respectively, and the scalars $\Phi_{ab} = \frac{1}{2}\epsilon_{abcd}\Phi^{cd}$. Once again, the action can be split into its two parts:

$$S[A, \Psi, \Phi, \tilde{\Psi}, G] = S_{\text{SD}}[A, \tilde{\Psi}, \Phi, \Psi, G] - \frac{\varepsilon}{2}I[A, \tilde{\Psi}, \Phi, \Psi, G], \quad (3.3)$$

where

$$\begin{aligned} S_{\text{SD}}[A, \tilde{\Psi}, \Phi, \Psi, G] &= \int_{\mathbb{M}} \text{tr} \left(\frac{1}{2}G \cdot F + \Psi_A^a \nabla^{AA'} \tilde{\Psi}_{aA'} - \frac{1}{8}(\nabla\Phi)^2 + \tilde{\Psi}_{aA'} \tilde{\Psi}_b^{A'} \Phi^{ab} \right) d^4x \\ I[A, \tilde{\Psi}, \Phi, \tilde{\Psi}, G] &= \frac{1}{2} \int_{\mathbb{M}} \text{tr} \left(G \cdot G + \Psi_A^a \Psi^{bA} \Phi^{ab} + \frac{1}{4} \Phi^{ac} \Phi_{ab} \Phi^{bd} \Phi_{cd} \right) d^4x. \end{aligned} \quad (3.4)$$

The second term contains all the interactions of the full theory that are absent in the self-dual theory. As in the bosonic case, (3.3) manifestly allows us to expand around the SD sector via the expansion parameter ε . In the next subsection, we express this action on twistor space.

3.1 The SYM Twistor Action

The Ward construction establishes a correspondence between self-dual Yang-Mills fields on a region U' in space-time and certain holomorphic vector bundles $E \rightarrow U \subset \mathbb{CP}^3$ where U is the corresponding region in bosonic twistor space (c.f., [64, 65]). Such bundles can be expressed in terms of a smooth topologically trivial bundle $E \rightarrow U$ with complex structure given by a deformed $\bar{\partial}$ -operator:

$$\bar{\partial}_a = \bar{\partial} + a, \quad a \in \Omega^{0,1}(\mathbb{PT}'_b, \text{End}(E)). \quad (3.5)$$

In the self-dual case, the space-time bundle is flat on β -planes, and so E_Z can be defined to be the parallel sections of E' on the β -plane corresponding to Z . This varies holomorphically

¹⁰The parameter ε is naturally proportional to the 't Hooft coupling, λ , of the theory. For a $SU(N)$ theory, $\lambda = \frac{g^2 N}{8\pi^2}$, and $\varepsilon = -\lambda$.

with Z (assuming the space-time field to vary holomorphically on space-time). The more remarkable fact of the Ward transform is that a determines the bundle with connection (E', A) up to gauge on space-time.

The supersymmetric Ward correspondence [64, 68] similarly gives a one-to-one correspondence between integrable complex structures on $E \rightarrow U \subset \mathbb{CP}^{3|4}$, now given in terms of a homogeneous $(0, 1)$ -form \mathcal{A} on \mathbb{PT}' (i.e., $\bar{\partial}_{\mathcal{A}}^2 = 0$), and SD solutions to the field equations of $\mathcal{N} = 4$ SYM on space-time. As $\bar{\partial}_{\mathcal{A}}^2 = 0$ are precisely the field equations for a holomorphic Chern-Simons action, we can account for the SD portion of the theory on twistor space with the functional

$$S_{\text{SD}}[\mathcal{A}] = \frac{i}{2\pi} \int_{\mathbb{PT}'} D^{3|4} Z \wedge \text{tr} \left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \quad (3.6)$$

Here we require that \mathcal{A} depends holomorphically on the fermionic coordinates χ^a and has no components in the $d\bar{\chi}$ -directions (c.f., [16, 42]). As in the abelian case, we can expand \mathcal{A} in powers of χ^a :

$$\mathcal{A} = a + \chi^a \tilde{\psi}_a + \frac{1}{2} \chi^a \chi^b \phi_{ab} + \epsilon_{abcd} \chi^a \chi^b \chi^c \left(\frac{1}{3!} \psi^d + \frac{1}{4!} \chi^d g \right), \quad (3.7)$$

with a , $\tilde{\psi}$, ϕ , ψ , and g of weights 0, -1 , -2 , -3 , and -4 respectively. We have already seen that in the $U(1)$ theory these give the standard $\mathcal{N} = 4$ super-Yang-Mills multiplet on space-time.

Just as with the Chalmers-Siegel action, we must add an extra term to the self-dual action to obtain the remaining interactions of the full theory. This can be expressed as the functional

$$I[\mathcal{A}] = \int_{\mathbb{M}_{\mathbb{R}}} d^{4|8} x \log \det (\bar{\partial}_{\mathcal{A}}|_X), \quad (3.8)$$

where X is the $\mathbb{CP}^1 \subset \mathbb{PT}$ corresponding to the point $(x, \theta) \in \mathbb{M}_{\mathbb{R}}$; $\mathbb{M}_{\mathbb{R}}$ is a real slice of the complexified space-time; $\bar{\partial}_{\mathcal{A}}|_X$ is the restriction of the deformed complex structure (3.5) to the line X ; and $d^{4|8} x$ is the natural supersymmetric volume form on space-time:

$$d^{4|8} x = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge d^8 \theta.$$

A motivation for this specific form of this interaction comes from twistor-string theory, see [15, 16] for discussion. The integrand of $I[\mathcal{A}]$ is not *a priori* well-defined because the determinant $\det(\bar{\partial}_{\mathcal{A}}|_X)$ is not an honest function, but in fact a section of a determinant line bundle over the space of connections on $E \rightarrow \mathbb{PT}'$; this determinant will pick up anomalous terms under gauge transformations of the SD portion of the action. However, such anomalous transformations can be seen to produce terms that necessarily vanish upon performing the fermionic integrals.¹¹

¹¹This follows essentially because the variations in $\log \det$ are additive and depend on the θ s only through the χ s which only have four components [16].

Hence, (3.8) is a well-defined functional, and we combine it with (3.6) to yield the full twistor action:

$$S[\mathcal{A}] = S_{\text{SD}}[\mathcal{A}] - \frac{\varepsilon}{2} I[\mathcal{A}]. \quad (3.9)$$

The first and second terms have precursors respectively in the work of [42, 76] and [77].

The gauge freedom $\bar{\partial}_{\mathcal{A}} \rightarrow h\bar{\partial}_{\mathcal{A}}h^{-1}$ of this action depends essentially arbitrarily on the full six real bosonic variables of \mathbb{PT} and so is much greater than that on space-time. It can be reduced in various ways. As shown in [16, 20], it can be reduced to the four variables of the space-time action (3.4) in Euclidean signature by requiring that the restriction of \mathcal{A} to any Euclidean real \mathbb{CP}^1 be ‘harmonic’:

$$\bar{\partial}^*|_X \mathcal{A}_0 = 0,$$

where \mathcal{A}_0 is the component of \mathcal{A} in the direction tangent to the Euclidean real \mathbb{CP}^1 s (which fibre \mathbb{CP}^3 over Euclidean space-time). It restricts the remaining gauge freedom to that of the standard space-time gauge transformations, as first shown by Woodhouse [78]. In this gauge the twistor action $S[\mathcal{A}]$ reduces to precisely the space-time action (3.4), [16]. Thus, the twistor action is non-perturbatively and classically equivalent to the space-time Chalmers-Siegel action in this gauge.

We will see, however, that there are other gauge choices available on twistor space which are inaccessible from space-time. The main example that we will use is an axial gauge. In general, for any holomorphic 1-dimensional distribution $D \subset T^{1,0}\mathbb{PT}$, this is the condition that $\mathcal{A}|_D = 0$. As already discussed, we will implement this by choice of a reference twistor Z_* and let D be the span of $Z_* \cdot \partial$. We will see that in this gauge the twistor action yields Feynman rules that have some surprising properties on twistor space, and, when transformed to momentum space, precisely reproduce the momentum MHV formalism of [79].

3.2 CSW Gauge and Twistor Space Feynman Rules

To obtain twistor space Feynman rules, we impose a simple axial gauge by choosing a twistor at infinity $Z_* = (0, \hat{t}^{A'}, 0)$ and imposing the condition

$$\overline{Z_* \cdot \frac{\partial}{\partial Z} \lrcorner \mathcal{A}} = 0, \quad (3.10)$$

where \mathcal{A} is the fundamental field in our twistor action, given by (3.7). This requires \mathcal{A} to vanish upon restriction to the leaves of the foliation of $\mathbb{PT} \setminus \{Z_*\}$ by the lines which pass through Z_* , and is referred to as the CSW gauge, since it was first introduced in [79].

Clearly, (3.10) reduces the number of components of \mathcal{A} from three down to two, so the cubic Chern-Simons vertex in (3.6) vanishes. This leaves the twistor action in the CSW gauge as:

$$\frac{i}{2\pi} \int_{\mathbb{PT}'} D^{3|4} Z \wedge \text{tr} (\mathcal{A} \wedge \bar{\partial} \mathcal{A}) - \frac{\varepsilon}{2} \int_{\mathbb{M}_{\mathbb{R}}} d^{4|8} x \log \det (\bar{\partial} \mathcal{A}|_X).$$

As far as the Feynman rules are concerned, the propagator will arise from the first term. This kinetic term is now the same as for the $U(1)$ theory, and so as discussed in §2.2.2 the propagator will be given by

$$\Delta(Z_1, Z_2) = \bar{\delta}^{2|4}(Z_1, Z_*, Z_2),$$

the $(0, 2)$ -form on \mathbb{PT} that imposes the collinearity of its three arguments. By our earlier remarks, this propagator satisfies the CSW gauge condition $(\ast \cdot \partial_1) \lrcorner \Delta = (\ast \cdot \partial_2) \lrcorner \Delta = 0$.

Vertices

We will write out the vertices in such a way as to manifest as much conformal symmetry as possible. Thus, we write the measure on the real contour $\mathbb{M}_{\mathbb{R}}$ as

$$d^{4|8}x = \frac{d^{4|4}Z_A \wedge d^{4|4}Z_B}{\text{vol}(\text{GL}(2, \mathbb{C}))}. \quad (3.11)$$

where we have supposed that X is the line joining twistors Z_A and Z_B and the quotient by $\text{GL}(2, \mathbb{C})$ is that of the choices of such Z_A and Z_B .

The vertices can be made explicit by perturbatively expanding the logarithm of the determinant, yielding [16, 17]:

$$\log \det (\bar{\partial}_A|_X) = \text{tr} (\log \bar{\partial}|_X) + \sum_{n=2}^{\infty} \frac{1}{n} \int_{X^n} \text{tr} (\bar{\partial}|_X^{-1} \mathcal{A}_1 \bar{\partial}|_X^{-1} \mathcal{A}_2 \cdots \bar{\partial}|_X^{-1} \mathcal{A}_n), \quad (3.12)$$

where $\bar{\partial}|_X$ is the restriction of the $\bar{\partial}$ -operator from \mathbb{PT} to $X \cong \mathbb{CP}^1$, and \mathcal{A}_i is a field inserted at a point $Z(\sigma_i) \in X$ where we have introduced the (inhomogeneous) coordinate σ on X by

$$Z(\sigma) = Z_A + \sigma Z_B. \quad (3.13)$$

In terms of this coordinate, the $\bar{\partial}|_X^{-1}$ are the Green's functions for the $\bar{\partial}$ -operator restricted to X given by integration against the Cauchy kernel

$$(\bar{\partial}|_X^{-1} \mathcal{A})(\sigma_{i-1}) = \frac{1}{2\pi i} \int \frac{\mathcal{A}(Z(\sigma_i)) \wedge d\sigma_i}{\sigma_i - \sigma_{i-1}},$$

Thus, the n th term in our expansion yields the vertex

$$\frac{1}{n} \left(\frac{1}{2\pi i} \right)^n \int_{\mathbb{M}_{\mathbb{R}}} d^{4|8}x \int_{X^n} \text{tr} \left(\prod_{i=1}^n \frac{\mathcal{A}_i(Z(\sigma_i)) \wedge d\sigma_i}{\sigma_i - \sigma_{i-1}} \right). \quad (3.14)$$

Here the index i is understood cyclically with $\sigma_i = \sigma_{n+i}$ and $\mathbb{M}_{\mathbb{R}}$ denotes a real slice of complexified space-time.

In order to obtain the dualized amplitude as discussed in §2.2.1, we use for the our external wave functions

$$\mathcal{A}_i = \bar{\delta}^{3|4}(Z_i, Z(\sigma_i)), \quad i = 1, \dots, n, \quad Z(\sigma_i) = Z_A + \sigma_i Z_B \quad (3.15)$$

thought of as a $(0, 1)$ -form in $Z(\sigma_i)$ and a $(0, 2)$ -form in Z_i .

Integrating over the insertion points on the line $Z_A \wedge Z_B$ in (3.14) reduces each external particle's contribution to a $(0, 2)$ -form, so the n -valent vertex takes values in the n -fold product of $\Omega_c^{0,2}$ (as required by (2.36)) and is supported on a line in $\mathbb{P}\mathbb{T}'$. This is precisely the n -particle MHV vertex in twistor space.

Thus, with (3.11) we can write the n -particle MHV vertex on twistor space as

$$V(Z_1, \dots, Z_n) = \int_{\mathbb{M}_{\mathbb{R}}} \frac{d^{4|4} Z_A \wedge d^{4|4} Z_B}{\text{vol}(\text{GL}(2, \mathbb{C}))} \int_{(X_{AB})^n} \prod_{i=1}^n \frac{\bar{\delta}^{3|4}(Z_i, Z(\sigma_i)) d\sigma_i}{(\sigma_i - \sigma_{i-1})}. \quad (3.16)$$

This manifests the conformal symmetry and cyclic invariance of the amplitude and corresponds to the twistor-string path-integral formulation of the tree-level MHV amplitude [42, 80, 81].

These formulae for the MHV vertices can be recursively built from lower order vertices and superconformally invariant delta functions. An $n + 1$ -particle vertex can be written in terms of a n -particle vertex and a delta function by transforming from the variable σ_i to $s = (\sigma_i - \sigma_{i-1})/(\sigma_i - \sigma_{i+1})$ and observing that the s integral decouples from the others to yield

$$V(Z_1, \dots, Z_{n+1}) = V(Z_1, \dots, \widehat{Z}_i, \dots, Z_{n+1}) \bar{\delta}^{2|4}(Z_{i-1}, Z_i, Z_{i+1}). \quad (3.17)$$

where \widehat{Z}_i is omitted from the right-hand-side. This operation has become known as the inverse soft limit [54, 55]. Proceeding inductively along these lines, one arrive at many expressions of the form [22]

$$V(Z_1, \dots, Z_n) = V(Z_1, Z_2) \prod_{i=2}^n \bar{\delta}^{2|4}(Z_1, Z_{i-1}, Z_i), \quad (3.18)$$

where $V(Z_1, Z_2)$ is the two point vertex. The two-point vertex can be written as

$$V(Z_1, Z_2) = \int_{\mathbb{M} \times (\mathbb{CP}^1)^2} D^3 Z_A D^3 Z_B \bar{\delta}_{0,-4}^3(Z_1, Z_A) \bar{\delta}_{0,-4}^3(Z_2, Z_B). \quad (3.19)$$

The expression for the MHV vertex given by (3.18) shows clear superconformal invariance and has the minimal number of residual integrals, but of course no longer manifests the cyclic invariance of the original twistor-string expression (this is due to our choice of representation of the inverse soft limit). The equivalence between cyclic permutations can be recovered by repeatedly using the identity

$$V(Z_1, Z_2, Z_3) \bar{\delta}^{2|4}(Z_1, Z_3, Z_4) = V(Z_2, Z_3, Z_4) \bar{\delta}^{2|4}(Z_2, Z_4, Z_1) \quad (3.20)$$

for the 4-point vertex.

4 From the Twistor Action to the MHV Formalism

An important output of Witten's twistor-string theory [42] is the MHV formalism, a set of momentum-space Feynman rules for calculating scattering amplitudes in $\mathcal{N} = 4$ SYM which are significantly simpler than the ordinary Feynman rules that arise from a space-time action [79, 82, 83]. In contrast to the full twistor-string theory, the MHV formalism can, on its own, be extended to loop calculations. Indeed, it has been shown to compute the correct amplitudes at 1-loop explicitly [84], and after being expressed on momentum twistor space (see [21] and §5 of this review) was shown to be correct to all loop-orders for planar $\mathcal{N} = 4$ SYM [37] using an all-line recursion relation building on the work of [85–87].

In [17] it was shown that the momentum space MHV formalism arises naturally as the Feynman rules of the twistor action (3.9) when a very simple axial gauge (the CSW gauge) is chosen, but it was not until recently that the MHV formalism was derived from the twistor action in a manner that is entirely self-contained on twistor space [22]. This section reviews the construction of the twistor space MHV formalism, emphasizing how amplitudes are represented on twistor space cohomologically and how tree-level N^k MHV amplitudes are calculated on twistor space. We also make some remarks about the treatment of loop amplitudes by the theory. The Feynman rules developed here will also be those used when we come to discuss correlation functions later.

On momentum space, an amplitude is said to be N^k MHV if it has homogeneity degree $4(k+2)$ in the fermionic portion of the super-momenta $P = (p_{AA'}, \eta_a)$ (it must be a multiple of 4 because the amplitude must be invariant under the $SU(4)$ R-symmetry acting on the a -index). Such an amplitude contains the gluon amplitude with $k+2$ negative helicity particles, with the rest positive. It is easily seen by inserting momentum eigenstate wave functions into a twistor amplitude that such an N^k MHV amplitude is obtained from one on twistor space that is a polynomial in the fermionic χ coordinates of homogeneity degree $4(n-k-2)$ (as far as the fermionic variables are concerned, it is a Fourier transform). By counting the contributions from each propagator and vertex, it can be seen that an l -loop diagram made from MHV vertices and propagators as above will contribute to an N^k MHV amplitude if it has $k+l+1$ vertices.

4.1 The MHV formalism in twistor space

Using the axial gauge Feynman rules for the twistor action developed in §3.2, we will see that the integrals corresponding to generic diagrams can be performed explicitly and algebraically against the delta functions. Here generic is meant in the sense of fixed k and large n . The reason for this is essentially geometric: each vertex corresponds to a line in twistor space, and if two of these vertices are connected by a propagator, then the delta function in the propagator forces the insertion points to be collinear with the reference twistor. However, it is easily seen that when the lines are in general position, there is a unique transversal line through Z_* to the pair of lines corresponding to the vertices. Thus geometrically, each

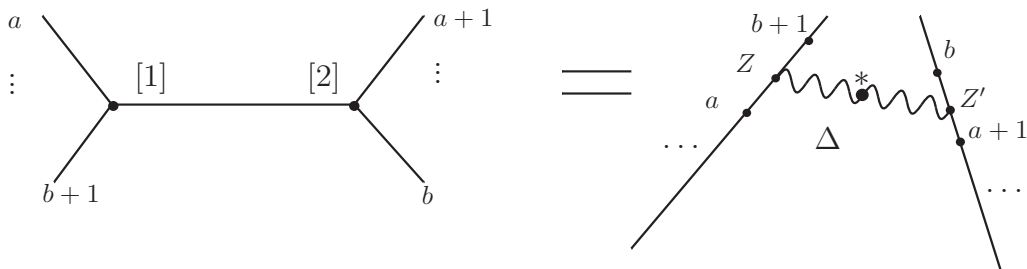


Figure 2: Twistor support of a typical NMHV tree diagram

diagram will correspond to $k + l + 1$ lines for the vertices connected by one line through Z_* for each propagator.

We will divide the diagrams into three categories. The *generic* diagram will be one with no adjacent propagator insertions on any of its vertices. The *boundary*¹² diagrams are those in which one or more vertices have two or more adjacent propagator insertions, but each vertex has at least two external particles attached to it. The *boundary-boundary* diagrams will be those in which there are MHV vertices with less than two external legs; in this last case we will not be able to perform all the integrals against delta functions. We deal with the generic case in some detail and refer the reader to [22] for details of the other cases and the particulars of all calculations, summarizing briefly the results.

Example: NMHV tree

For tree level NMHV amplitudes, the only diagrams that contribute are those appearing in Figure 2 with two vertices connected by one propagator. Our twistor space Feynman rules tell us that each term in this amplitude is computed as

$$\int_{\mathbb{PT} \times \mathbb{PT}} D^{3|4} Z D^{3|4} Z' V(b+1, \dots, a, Z) \bar{\delta}^{2|4}(Z, *, Z') V(a+1, \dots, b, Z').$$

We can separate out the dependence of the vertices on the internal twistors using (3.17) to obtain

$$V(b+1, \dots, a) V(a+1, \dots, b) \times \int_{\mathbb{PT} \times \mathbb{PT}} D^{3|4} Z D^{3|4} Z' \bar{\delta}^{2|4}(a, b+1, Z) \bar{\delta}^{2|4}(Z, *, Z') \bar{\delta}^{2|4}(b, a+1, Z').$$

These integrals can be performed algebraically against the delta functions as in (2.44) to yield

$$V(b+1, \dots, a) V(a+1, \dots, b) [b+1, a, *, b, a+1].$$

¹²The terminology arises from the fact that in the summation over diagrams to form the amplitude, the boundary diagrams arise at the boundary of the range of summation over the indices specifying the location of the external particles on the different vertices.

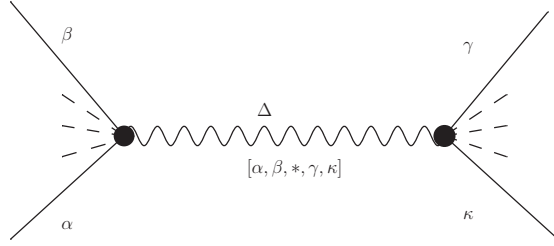


Figure 3: *Propagator contributions*

Geometrically this diagram corresponds to the lines for each vertex, and the R -invariant then can be thought of as being associated to the transversal to these two lines through Z_* .

The full NMHV amplitude is then the sum over $a < b$ of such contributions.

Generic diagrams

The NMHV calculation above extends directly to each propagator of a generic diagram. For such diagrams, there are no adjacent propagator insertions at any vertex, and so the neighbourhood of a propagator can be depicted as in Figure 3. We can use (3.17) to strip off a $\bar{\delta}^{2|4}$ at the propagator insertion point on each vertex, leaving MHV vertices that no longer depends on the propagator insertion points Z_1 and Z_2 . Thus a propagator leads to a factor of

$$\int D^{3|4} Z_1 D^{3|4} Z_2 \bar{\delta}^{2|4}(Z_1, *, Z_2) \bar{\delta}^{2|4}(Z_1, \alpha, \beta) \bar{\delta}^{2|4}(Z_2, \gamma, \kappa) = [\alpha, \beta, *, \gamma, \kappa] \quad (4.1)$$

multiplied by the diagram with that propagator removed. Here α and β are the two nearest external particles on one side of the propagator, while γ and κ are the closest on the other side (see Figure 3) and we have performed the integral algebraically against the delta functions using (2.44) as before.

Proceeding inductively, we see that a tree-level generic N^k MHV diagram gives a product of k R -invariants, one for each propagator depending on Z_* and each adjacent external twistor. These are multiplied by the $k+1$ MHV vertices that now depend only on the external particles on the corresponding original vertex in the initial MHV diagram.

Boundary Terms

If two or more propagators are adjacent at a vertex, we can still use (3.17) to pull out a $\bar{\delta}^{2|4}$ factor containing the only dependence on the insertion points of the propagator so long as there are at least two external particles at each vertex. However, the dependence of the $\bar{\delta}^{2|4}$ will yield a slightly more complicated integral. This can nevertheless be performed algebraically against the delta functions to yield an R -invariant, but now the R -invariant that corresponds

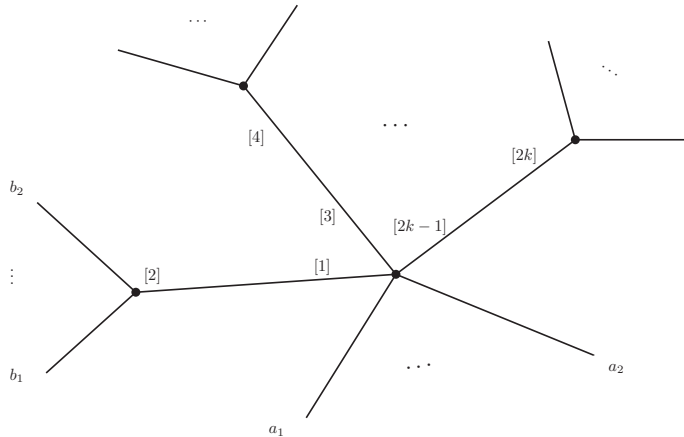


Figure 4: N^k MHV boundary term with k adjacent propagators

to a propagator will depend on shifted external twistors. For a boundary diagram such as figure 4 we end up with the rules [22]:

- Each vertex in the diagram gives rise to a factor of an MHV vertex in the answer that depends only on the external legs at that vertex.
- Each propagator corresponds to an R-invariant $[\widehat{a}_1, a_2, *, \widehat{b}_1, b_2]$ where a_1 and a_2 are the nearest external twistors with $a_1 < a_2$ in the cyclic ordering on the vertex at one end of the propagator, and similarly for $b_1 < b_2$ on the vertex at the other end. Let p be the insertion point on the vertex containing a_1 and a_2 . We have that \widehat{a}_1 is shifted according to the rule

$$Z_{\widehat{a}_1} = \begin{cases} Z_{a_1} & \text{if } p \text{ is next to } a_1 \\ X_{a_1, a_2} \cap \langle b_1, b_2, * \rangle & \text{if } p \text{ is next to the propagator on the } a_1 \text{ side} \\ & \text{that connects to the line } (b_1, b_2), \end{cases} \quad (4.2)$$

where X_{a_1, a_2} is the line defined by Z_{a_1} , Z_{a_2} and $\langle b_1, b_2, * \rangle$ is the plane defined by Z_{b_1} , Z_{b_2} and Z_* . The rule for \widehat{b}_1 follows by $a \leftrightarrow b$.

Boundary-Boundary Terms

The final class of diagrams are the boundary-boundary contributions when there are fewer than two external legs on some vertices. In this case, the above integration procedure breaks down since the location of the line corresponding to each MHV vertex is no longer fixed by the external particles. There will in general be some residual integrations associated to those in the MHV vertex to be performed. Although many of the integrations can be performed (or indeed all by the introduction of signature-dependent machinery [22]), we do not obtain as attractive a formalism as we do for the other diagrams.

Examination of these diagrams on momentum space suggests that there is nothing particularly special about these diagrams there.

Loop diagrams

The rules given above extend readily to loop diagrams leaving no residual integrals in the generic and boundary cases. Although all planar loop diagrams are boundary diagrams, in the non-planar theory there are many straightforwardly finite loop diagrams, even at 1-loop MHV. Indeed in the planar theory, there are straightforwardly finite (boundary) diagrams at NMHV. These presumably lead to polylogs on transform back to momentum space. Although most of the planar MHV diagrams ought to be finite, the procedure above leads to 0/0 and a good procedure has not yet been arrived at to evaluate these [22].

4.2 Derivation of the momentum space MHV formalism

As a reality check, we would like to see that this process indeed gives rise to the MHV formalism when transformed to momentum space. Since momentum space representations break conformal invariance, we will use a version of the MHV vertex in which the volume form on \mathbb{M} is $d^{4|8}x$ and the $GL(2)$ symmetry has been fixed by coordinatizing the line X by its projection to homogeneous coordinates λ_A as in (2.2). Up to an irrelevant constant factor, this yields the formula

$$\int_{\mathbb{M}_{\mathbb{R}}} d^{4|8}x \int_{X^n} \text{tr} \left(\prod_{i=1}^n \frac{\mathcal{A}_i(-ix^{AA'}\lambda_{A i}, \lambda_{A i}, \theta^{Aa}\lambda_{A i}) \wedge D\lambda_i}{\langle \lambda_i \lambda_{i-1} \rangle} \right), \quad (4.3)$$

where as usual $\langle \rangle$ denotes the unprimed spinor inner product. Most straightforwardly, we can check that the MHV vertices give the standard MHV amplitudes by taking the \mathcal{A}_i to be on-shell momentum eigenstates with super momentum $P_i = (p_{i A}, \tilde{p}_{i A'}, \eta_{i a})$

$$\mathcal{A}_i(P_i; Z) = \int_{\mathbb{C}} \frac{ds}{s} e^{s(\mu^{A'} \tilde{p}_{i A'} + \chi^a \eta_{i a})} \delta^2(s\lambda_A - p_{i A}), \quad (4.4)$$

multiplied by some generator of the Lie algebra of the gauge group. It is now easily seen that the delta functions simply enforce¹³ $s\lambda_i = p_i$ and the integral $d^{4|8}x$ gives the super-momentum conserving delta function to end up with the Parke-Taylor formula [80, 88] for the MHV tree amplitude extended to $\mathcal{N} = 4$ SYM:

$$A_{\text{MHV}}^0(P_1, \dots, P_n) = \frac{\delta^{4|8}(\sum_{i=1}^n P_i)}{\prod_{i=1}^n \langle p_i p_{i+1} \rangle}, \quad (4.5)$$

¹³ In this subsection, in order to be able to perform this integration that gives this identification between λ_A and p_A , we must distinguish between the value of the λ -coordinate at the twistor insertion point and the corresponding spinor part p_A of the momenta; in the future we will exploit the delta function to use the same λ_A notation for both where no ambiguity can arise.

Here we have stripped off an overall color trace factor together with an irrelevant constant factor, and the supermomentum conserving delta-function is

$$\delta^{4|8} \left(\sum P_i \right) = \delta^{4|0} \left(\sum p_i \tilde{p}_i \right) \delta^{0|8} \left(\sum \eta_i p_i \right),$$

where

$$\delta^{0|8} \left(\sum_i \eta_i p_i \right) = \prod_{a,A} \left(\sum_i \eta_i \lambda_{iA} \right).$$

We need to also check that our rule for the insertion of the propagator $\Delta(Z, Z') = \bar{\delta}^{2|4}(Z, *, Z')$ leads to the rules for the propagator found in the MHV formalism [79]. In order to do this we pull Δ back to the spin bundle (i.e., express it as a function of $(x, \theta, \lambda, x', \theta', \lambda')$ using (2.2)) and Fourier transform over the (x, x') variables. This requires the choice of a real slice of complex Minkowski space to integrate over for the Fourier transform. In order to obtain the correct Feynman $i\epsilon$ prescription, we do so over the Euclidean slice, and then analytically continue the momentum space formula to the Minkowski slice that we have focused on for the vertices. Observing that Δ only depends on $x - x'$ allows us to express the Fourier transform $\tilde{\Delta}(p, p', \dots) = \delta^4(p_{AA'} - p'_{AA'}) \tilde{\Delta}(p, \dots)$ where, after some calculation that can be found in [22],

$$\tilde{\Delta}(p_{AA'}, \chi, \lambda, \chi' \lambda') = \frac{1}{p^2} \int \frac{ds dt}{st} \bar{\delta}^2(\lambda_A + sp_A) \bar{\delta}^2(\lambda_A - t\lambda'_A) \delta^{0|4}(\chi - t\chi'). \quad (4.6)$$

Here, for reference twistor $Z_* = (0, \hat{i}^{A'})$, the spinor part of the off-shell momenta is given by

$$p_A = \iota^{A'} p_{AA'}. \quad (4.7)$$

If we now use a Fourier representation of the fermionic delta function, substitute in $\chi = \theta^{iA} \lambda_A$ etc., and use the support of the delta functions, we obtain the formula

$$\tilde{\Delta}(p_{AA'}, \chi, \lambda, \chi' \lambda') = \frac{1}{p^2} \int d^4 \eta \bar{\delta}_0^1(\lambda_A, p_A) \bar{\delta}_0^1(\lambda'_A, p_A) e^{i\eta \cdot (\theta - \theta') | p} \quad (4.8)$$

where $\bar{\delta}_0^1(\cdot, \cdot)$ are the spinor projective delta functions defined in (2.26). Fourier transforming back, we finally obtain the Fourier representation for the pullback of Δ to the spin bundle

$$\Delta(x, \chi, \lambda, x', \chi' \lambda') = \int \frac{d^4 p}{p^2} d^4 \eta \bar{\delta}_0^1(\lambda_A, p_A) \bar{\delta}_0^1(\lambda'_A, p_A) e^{ip \cdot (x - x') + i\eta \cdot (\theta - \theta') | p}. \quad (4.9)$$

This now yields the Fourier representation of Δ whose ends can be substituted into the twistor MHV vertices (4.3). The delta functions lead to the CSW prescription in which the spinor p_A associated to an off-shell momentum $p_{AA'}$ is given by $p_A = \iota^{A'} p_{AA'}$ where now $\iota^{A'}$ is the ‘reference spinor’ introduced in that formalism [79], although we see from the above that it

is also the Euclidean conjugate of the primary part of the reference twistor. The exponential factors guarantee that the ends introduce the appropriate supermomenta $P = \pm(p_{AA'}, p_A \eta^i)$.

Amplitudes are then constructed as a sum of Feynman-like diagrams built out of such propagators and MHV vertices. At N^k MHV there will be $k + 1$ vertices in each diagram. One can prove that dependence on the spinor $\hat{\iota}^{A'}$ drops out of the final sum of contributions and this corresponds precisely to the n -particle N^k MHV tree-amplitude [85–87]. The N^k MHV amplitudes in this context are then those of homogeneity $4(k + 2)$ in the η s.

Hence, the Feynman rules for the twistor action in the CSW gauge are precisely those of the MHV formalism: for a N^k MHV amplitude, connect $k + 1$ MHV vertices (given by (3.16) or (3.18)) together with k propagators (given by (2.41)), and integrate over propagator insertion points on each MHV line in twistor space.

5 The Momentum Twistor MHV Formalism

In their AdS/CFT approach to scattering amplitudes, Alday and Maldacena introduced a non-compact ‘T-duality’ in order to simplify their calculations [23]. This had the effect of replacing space-time by region momenta space, an affine version of momentum space with no fixed origin. Their work led to the conjecture [24] that the n -particle planar amplitude, at least at MHV, should be given by the correlation function of a certain Wilson loop in region momentum space, calculated in $\mathcal{N} = 4$ super Yang-Mills theory, but now defined on region momentum space as opposed to the original space-time. In particular their conjecture implied that there should be a completely unexpected *dual* (super) conformal symmetry: that acting on region momentum space. First introduced by Hodges [89], momentum twistors are the twistors associated to this region momentum space and serve to make explicit the invariance of the formalism, but now under *dual* (super) conformal symmetry which acts linearly on momentum twistor space. Momentum twistor space is still the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$, and has all the properties described in §2; the only difference is that the space-time is now *region momentum space* rather than the physical chiral Minkowski space-time.

We have illustrated how the momentum space MHV formalism can be derived from the Feynman rules of the twistor action in CSW axial gauge. In this section we show how the momentum-space MHV formalism can be re-expressed as Feynman rules in terms of momentum twistor data that will be locally related to the momentum space data. Whereas the twistor formalism brought out the superconformal invariance of the amplitudes and performed all integrals; we shall see that the momentum twistor formalism emphasizes *dual* superconformal invariance, and computes the *integrand* of scattering amplitudes. In the next section we will see that this momentum twistor version of the MHV formalism arises as the Feynman diagrams for the calculation of the correlation function of a Wilson loop in twistor space.

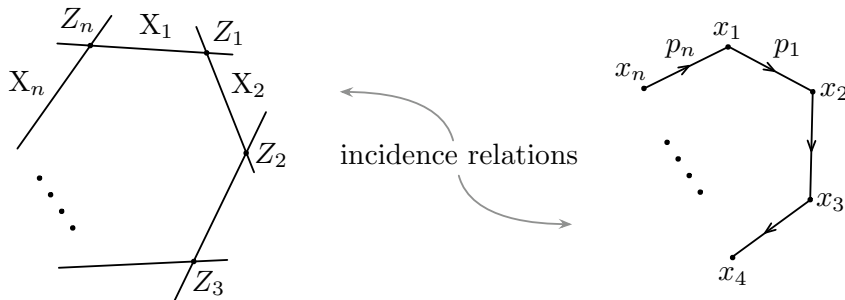


Figure 5: *The momentum twistor polygon and its associated null polygon in spacetime.*

5.1 Momentum Twistor Space

Momentum twistors encode the kinematic data for scattering amplitudes of massless particles in a way that builds in momentum conservation. The region momentum space and corresponding momentum twistor space are constructed in the following fashion.

As we have seen, the scattering of incoming massless particles is described by momenta $p_i^{AA'}$ that are null $p_i^2 = 0$ and obey the momentum conservation condition $\sum_i p_i^{AA'} = 0$. The null condition is solved by expressing the four-momenta in terms of spinors¹⁴

$$p_{i AA'} = \lambda_{i A} \tilde{\lambda}_{i A'} . \quad (5.1)$$

For planar amplitudes, the four-momenta involved in a scattering process may be expressed as the differences of region momenta

$$p_i^{AA'} = x_{i+1}^{AA'} - x_i^{AA'} . \quad (5.2)$$

The conditions $p_i^2 = 0$ and $\sum_i p_i = 0$ imply that the region momenta x_i form the vertices of a null polygon. Since the whole polygon may be translated without changing the kinematic data, these vertices $x_i^{AA'}$ live in affine Minkowski space, to which we can associate a twistor space as described in §2. This associated twistor space has become known as *momentum twistor space*. It is simply the standard twistor space of region momentum space.

Via the incidence relation (2.2), the x_i correspond to complex lines $X_i \cong \mathbb{CP}^1$ in (bosonic) momentum twistor space. Since the points x_i and x_{i+1} are null separated in the region momentum space, the associated lines X_i and X_{i+1} intersect in twistor space at some point Z_i . Therefore we have a sequence of intersecting lines called the momentum twistor polygon - see figure 5.

We may now turn the construction around and freely specify any n momentum twistors

¹⁴We denote on shell momenta in this fashion rather than $p_{AA'} = p_A \tilde{p}_{A'}$ because in momentum twistor space, the homogeneous coordinate λ_A is directly identified with the un-primed portion of the momentum.

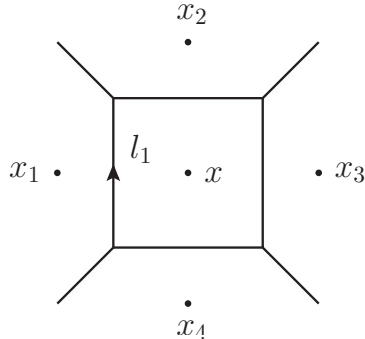


Figure 6: Region momenta may be assigned to internal regions.

with components $Z_i^\alpha = (\lambda_{iA}, \mu_i^{A'})$; this determines a sequence of intersecting lines

$$X_i \leftrightarrow X_i^{\alpha\beta} = \frac{Z_{i-1}^\alpha Z_i^\beta - Z_i^\alpha Z_{i-1}^\beta}{\langle \lambda_{i-1} \lambda_i \rangle}, \quad (5.3)$$

which in turn determines a null polygon with cusps

$$x_i^{AA'} = \frac{\lambda_{i-1}^A \mu_i^{A'} - \lambda_i^A \mu_{i-1}^{A'}}{\langle \lambda_{i-1} \lambda_i \rangle}$$

in region momentum space. Note that *any* choice of n momentum twistors defines a polygon with n null edges in region momentum space, so the specification of the kinematic data in terms of momentum twistors is completely unconstrained and therefore solves the momentum conservation condition trivially.

For planar loop amplitudes, the region momenta and momentum twistors play an important role in defining the *loop integrand* [10]: the ambiguity in assigning loop momenta between separate Feynman diagrams is absorbed into the overall freedom in choice of origin for the dual spacetime. Region momenta may then be assigned consistently to all internal regions. For example, in figure 6 the internal region is assigned coordinate x and the four-momenta through internal propagators are expressed as differences, e.g. $l_1 = x - x_1$. The loop integrand is then a rational function $f(x, x_i)$ of the internal and external region momenta.

In momentum twistor space, internal region momenta x are associated with complex lines X , so loop integration is equivalent to integration over these lines in twistor space. We have already seen a convenient way to express this integral in terms of lines through two auxiliary twistors Z_A and Z_B via (3.11):

$$d^4x = \frac{d^4Z_A \wedge d^4Z_B}{\text{vol}(\text{GL}(2, \mathbb{C})) \langle AB \rangle^4} \quad (5.4)$$

and l -loop integrands $I_n^{(l)}$ become rational functions of momentum twistors

$$I_n^{(l)} = I(Z_1, \dots, Z_n; (AB)_1, \dots, (AB)_l) \quad (5.5)$$

The notation (AB) means that the integrand depends only on the skew product $Z_A \wedge Z_B$ or in other words on the line X . It is important for consistency that for $l > 1$ the integrand is symmetrised over the loop momentum twistors $(AB)_1, \dots, (AB)_l$.

The extension of this picture to superamplitudes in planar $\mathcal{N} = 4$ SYM follows naturally, again along the lines mentioned earlier in §2. The momentum twistors now have an additional four fermionic components $Z_i^I = (Z_i^\alpha, \chi_i^a)$ and determine a chiral null polygon with cusps (x_i, θ_i) through the additional relations

$$\theta^{Aa} = \frac{\lambda_{i-1}^A \chi_i^a - \lambda_i^A \chi_{i-1}^a}{\langle \lambda_{i-1} \lambda_i \rangle}. \quad (5.6)$$

Once again, any set of n momentum (super-)twistors corresponds to a null polygon in region momentum space, with $\theta_i - \theta_{i-1} = \lambda_i \eta_i$, where η_i is now interpreted as the fermionic portion of the super-momentum $P_i = (\lambda_i \tilde{\lambda}_i, \eta_i)$. Conformal transformations on this region momentum space are called *dual* superconformal transformations.

Internal regions are points in chiral superspace (x, θ) and again correspond (with a slight abuse of notation) to lines $X \cong \mathbb{CP}^1$ which are now described by two momentum supertwistors Z_A^I and Z_B^I . The loop integration measure is extended supersymmetrically as before

$$d^{4|8}x = \frac{d^{4|4}Z_A \wedge d^{4|4}Z_B}{\text{vol}(\text{GL}(2, \mathbb{C}))},$$

and again we may consider loop integrands as in equation (5.5) where all momentum twistors are now super-momentum twistors. The l -loop integrand may be expanded in powers of the fermionic components

$$I_n^{(l)} = I_{n,0}^{(l)} + I_{n,1}^{(l)} + \dots + I_{n,n-4}^{(l)}, \quad (5.7)$$

where $I_{n,k}^{(l)}$ has Grassmann degree $4k$ and corresponds to N^k MHV amplitudes.

It has been shown by use of BCFW recursion (for which see §7.2) [10, 90] that the loop integrands $I_{n,k}^{(l)}$ in planar $\mathcal{N} = 4$ SYM are annihilated by generators of superconformal and dual superconformal symmetries, which together generate a Yangian symmetry algebra. In terms of momentum twistors, the generators of the dual superconformal and standard superconformal are respectively [91]

$$J^I{}_J = \sum_{i=1}^n Z_i^I \frac{\partial}{\partial Z_i^J} \quad (5.8)$$

$$J^{(1)I}{}_J = \sum_{i < j} (-1)^K \left[Z_i^I \frac{\partial}{\partial Z_i^K} Z_j^K \frac{\partial}{\partial Z_j^J} - (i \leftrightarrow j) \right] \quad (5.9)$$

The Yangian invariant quantities appearing in amplitudes of planar $\mathcal{N} = 4$ SYM and the numerous non-trivial relations among them are captured by a Grassmannian contour integral [1, 49, 55, 58, 92], as we discuss briefly in §7.

5.2 MHV Diagrams in Momentum Twistor Space

We now explain how the momentum space MHV diagram formalism of [79], derived in §4.2, can be reformulated in momentum twistor space to compute the loop integrands $I_{n,k}^{(l)}$ [21]. As we shall see, the result is that vertices become unity and that each propagator contributes a single dual superconformal¹⁵ R-invariant $[,,,]$ to the loop integrand, whose arguments are determined by a simple rule. This allows expressions for loop integrands to be quickly generated in a form that manifests cyclic invariance. In addition, the independence of choice of reference twistor Z_* forms a highly non-trivial check on the result.

Because momentum twistors encode supermomentum conservation, we cannot of course encode the delta function part of the amplitude. In fact it turns out that, in order to obtain dual conformal invariant expressions, we factor out by the MHV amplitude. Thus, below, we rewrite the diagrams from the MHV formalism in momentum twistor variables, but with this MHV prefactor removed.

Tree Diagrams

As discussed in §4 and 4.2, the simplest diagrams have a single propagator connecting two vertices and contribute to the NMHV tree amplitude. For the particular diagram illustrated in figure 2, we have the following momentum space expression

$$\int d^4\eta A_{\text{MHV}}^{(0)}(i, \dots, j-1, \{\lambda, \eta\}) \frac{1}{(x_i - x_j)^2} A_{\text{MHV}}^{(0)}(\{-\lambda, \eta\}, j, \dots, i-1) \quad (5.10)$$

where $(x_i - x_j) = (p_1 + \dots + p_{j-1})$ is the four-momentum through the propagator and the Grassmann integration performs the sum over states propagating in the channel. The momentum spinor for the off-shell propagator momenta is $\lambda_A = p_{AA'}\iota^{A'}$ according to the CSW prescription [79] where $\iota^{A'}$ is the auxiliary CSW reference spinor of (4.7).

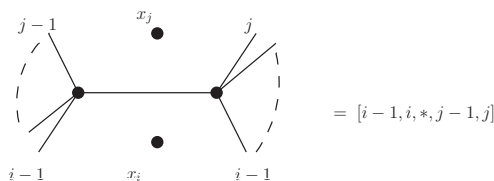


Figure 7: The MHV diagrams contributing to the NMHV tree amplitude.

We now extract an overall factor of the MHV amplitude $A^{\text{MHV}}(1, \dots, n)$ to expose the dual superconformal structure of the diagram, and introduce the momentum twistor $Z_*^I = (0, \iota^{A'}, 0)$ containing the auxiliary spinor as its only non-zero component. Once this has been done the contribution from the diagram becomes [21]

$$[i-1, i, *, j-1, j] \quad (5.11)$$

¹⁵R-invariants are now *dual* superconformal invariants because their arguments are momentum twistors.

where $[\cdot, \cdot, \cdot, \cdot]$ is the dual superconformal R-invariant introduced in §2.2.2. Summing over inequivalent diagrams, the NMHV tree amplitude may then be expressed as follows

$$I_{n,1}^{(0)} = \frac{1}{2} \sum_{i < j} [i - 1, i, *, j - 1, j]. \quad (5.12)$$

This expression manifests cyclic invariance in the external particles but breaks Lorentz invariance because of the dependence on the choice of reference momentum twistor Z_* . Note that as in §4, propagators correspond to R-invariants, but now they are dual conformal and can be thought of as linking the two pairs of external legs adjacent to the propagator via Z_* as in twistor space, but we will see in the next section that the propagator can also be regarded as linking the nearest regions via the reference twistor.

We can nevertheless demonstrate the dual superconformal symmetry of the tree amplitude. However, given any R-invariant $[a, b, c, d, e]$ we can perform a deformation $Z_a \rightarrow Z_a + tZ_f$ and apply Cauchy's theorem to derive the following linear identity

$$[a, b, c, d, e] + [b, c, d, e, f] + [c, d, e, f, a] + [d, e, f, a, b] + [e, f, a, b, c] + [f, a, b, c, d] = 0. \quad (5.13)$$

Performing the deformation $Z_* \rightarrow Z_* + tZ_{*'}$ on each term in equation (5.12), all dependence on Z_* cancels pairwise in the sum, so Z_* may be replaced by any other reference twistor $Z_{*'}$. In particular, when Z_* is an external momentum twistor, then equation (5.12) reduces immediately to the BCFW expression [10].

For N^2 MHV tree amplitudes we have a new phenomenon. The diagrams contain three vertices connected by two propagators, as illustrated in figure 8. For the generic diagram where $j < k$ we associate a dual superconformal R-invariant with each propagator, to obtain

$$[i - 1, i, *, j - 1, j] [k - 1, k, *, l - 1, l]. \quad (5.14)$$

However, when there are two propagators connected to adjacent sites on a vertex, we have a boundary diagram in the same sense as §4, one or more of the nearest external legs is the other side of an adjacent propagator and the arguments of the R-invariants are shifted. For the case when $j = k$ the correct expression is

$$[i - 1, i, *, j - 1, j] [k - 1, k, *, i - 1, \widehat{i}] \quad (5.15)$$

where the shifted momentum twistor is defined as

$$\widehat{Z}_i = (*, j - 1, j, i - 1) Z_i - (*, j - 1, j, i) Z_{i-1}. \quad (5.16)$$

Geometrically, the shifted twistor is the intersection of the line X_i with the plane through the line X_j and the auxiliary twistor Z_* . Note that this shifted geometry is different from the shifts required for boundary terms in the twistor space picture of §4.

A general tree-level N^k MHV amplitude corresponds to a product of k dual superconformal R-invariants built in this way, with boundary terms defined in terms of shifted momentum twistors [21]. Because the geometry of these shifts does not rely on our ability to define a line from the external particles at any one vertex, boundary-boundary terms are not a distinguished class of diagram in the momentum twistor formalism.

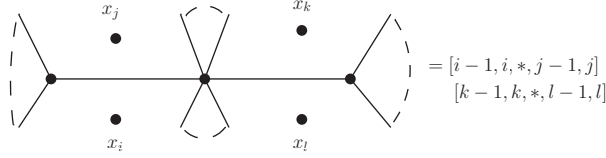


Figure 8: The MHV diagrams contributing to the N^2 MHV tree amplitude.

Loop integrands

Let $I_{n,0}^{(1)}$ denote the one-loop contribution to the planar MHV amplitude. The relevant MHV diagrams have two propagators connecting two vertices and are illustrated in figure 9. Once again we assign a R-invariant to each propagator; however, now the arguments involve the momentum twistors (AB) associated with the loop integration. The correct result for the diagram in figure is

$$[i-1, i, *, A, B'] [j-1, j, *, A, B''] \quad (5.17)$$

where

$$B' = (AB) \cap \langle *, j-1, j \rangle, \quad B'' = (AB) \cap \langle *, i-1, i \rangle. \quad (5.18)$$

We remark that although the B' and B'' are determined by the geometry, A is an arbitrary twistor on the line (AB) and the expression is actually independent of that choice. The fermionic integrations may be performed easily, leading to the following expression for the bosonic integrand [21]:

$$\frac{1}{2} \sum_{i < j} \frac{(*, i-1, i, [A] \langle B \rangle, j-1, j, *)^2}{(A, B, i-1, i)(A, B, j-1, j)(*, i-1, i, j-1)(*, i-1, i, j)(*, i-1, j-1, j)(*, i, j-1, j)}$$

Each term in this expansion contains four spurious propagators of the form $(*, , ,)$ and is highly chiral. However, the parity even part of this expression has been shown to agree with the standard box expansion at the level of the integrand, and the BCFW expression is again found immediately by choosing the auxiliary twistor Z_* to be an external momentum twistor.

Generic Diagrams

For a general diagram at arbitrary Grassmann degree and loop level, the vertices are unity and each propagator contributes a dual superconformal R-invariant $[\ , \ * \ , \]$. There is a simple local rule for assigning the momentum twistor arguments of this R-invariant for any propagator. Let the region momenta u, v, \dots , (which may be external or internal regions) correspond to lines U, V, \dots through pairs of momentum twistors $(U_1 U_2), (V_1 V_2), \dots$. The general assignment of

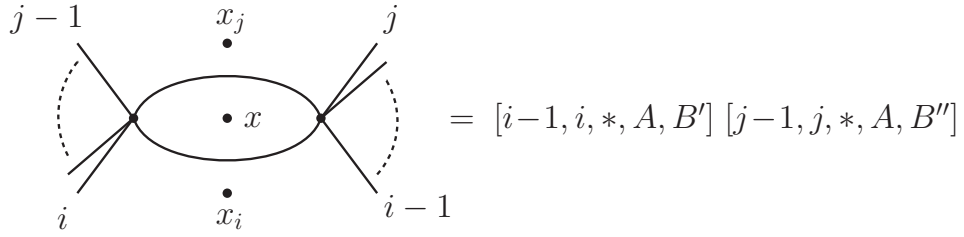


Figure 9: *The MHV diagrams contributing to the MHV one-loop integrand.*

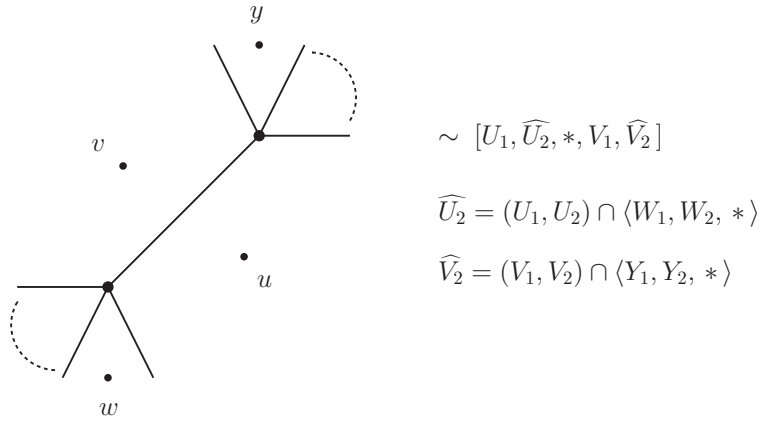


Figure 10: *Assignment of momentum twistors for a propagator in a generic diagram*

momentum twistors for the invariant may then be summarised by figure 10. (This prescription requires a choice of orientation of the diagram, but the expression arrived at turns out to be independent of the choice.)

6 Holomorphic Wilson Loops in Twistor Space

Although we have a fundamental derivation of the MHV diagrams as Feynman diagrams for the twistor action in ordinary twistor space, now that we have reformulated them in momentum twistor space, we would like to know how they arise from some construction in momentum twistor space directly. In this section we explain how the amplitude reformulated in momentum twistor space arises as the correlation function of the holomorphic Wilson loop in momentum twistor space defined by the polygon we have been using, computed in the theory defined by the twistor action. The Feynman integrals for the computation of this correlation function are precisely those for the MHV formalism that we have just reformulated in momentum twistor space. However, the Feynman diagrams for the amplitude are not those for the correlation function of the Wilson loop, but are related by planar duality. That this is meaningful relies on the fact that the contributions from the vertices is 1, and the only nontrivial contributions are from the propagators.

This correspondence was motivated by and extends the duality between space-time Wilson loops and the MHV sector of scattering amplitudes. This was first discovered at strong coupling [23] through AdS/CFT and subsequently observed at weak coupling [24,28–31]. The holomorphic Wilson loop in momentum twistor space gives the full amplitude, and not just the MHV sector. In the twistor formulation, the correspondence is clear, and easy to prove, whereas on space-time the integral representations for the amplitude are quite different. It leads directly to many approaches for computing the loops integrands, including the MHV diagram formalism [4] and the BCFW recursion relations [6] §7.2.

We will first consider the abelian theory where the expectation value can be computed exactly and then indicate the general structure of the Feynman diagrams in the non-abelian case.

6.1 Motivation and the abelian theory

To motivate the holomorphic Wilson loop, we return to the simplest MHV diagram with a single propagator. In momentum twistor space, this diagram contributes a single dual superconformal R-invariant $[i-1, i, *, j-1, j]$ from (5.12). By §2.2.2, we see that this R-invariant may be expressed as an integral of a collinearity delta-function:

$$[i-1, i, *, j-1, j] = \int \frac{ds}{s} \frac{dt}{t} \bar{\delta}^{2|4}(Z(s), *, Z(t)) \quad (6.1)$$

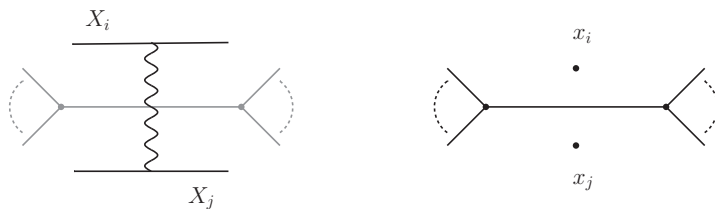


Figure 11: *The momentum twistor Feynman diagram and planar dual MHV diagram corresponding to $[i-1, i, *, j-1, j]$.*

for momentum twistors

$$\begin{aligned} Z(s) &= Z_{i-1} + sZ_i \\ Z(t) &= Z_{j-1} + tZ_j \end{aligned}$$

parameterising the complex lines X_i and X_j in the momentum twistor polygon. The delta-function appearing on the right hand side in equation (6.1) is the propagator for the twistor action in the axial gauge from (2.41):

$$\bar{\delta}^{2|4}(*, Z(s), Z(t)) = \Delta(Z(s), Z(t)). \quad (6.2)$$

Therefore, this contribution is that of a momentum twistor Feynman diagram where a single propagator connects the lines X_i and X_j ; this would arise from the expectation value

$$\left\langle \text{tr} \left(\int_{X_i} \omega_i \wedge \mathcal{A} \right) \left(\int_{X_j} \omega_j \wedge \mathcal{A} \right) \right\rangle \quad (6.3)$$

for forms ω_i, ω_j are meromorphic 1-forms defined on the lines X_i, X_j by

$$\omega_i = \int_{\mathbb{PT}} \bar{\delta}^{2|4}(i-1, Z, i) \wedge D^{3|4}Z. \quad (6.4)$$

The diagram for this correlator is the planar dual graph of the momentum space MHV diagram - see figure 11. We will see that this duality between momentum space MHV diagrams and momentum twistor Feynman diagrams remains true for all tree amplitudes and planar loop integrands.

The full tree-level NMHV amplitude is then given by

$$A_{\text{NMHV}}^0 = \left\langle \left(\int_C \omega_C \wedge \mathcal{A} \right)^2 \right\rangle. \quad (6.5)$$

where $C = \cup_i X_i$ is our twistor space polygon and the meromorphic 1-form ω_C is defined by

$$\omega_C = \int_{\mathbb{PT}} \bar{\delta}_C \wedge D^{3|4}Z, \quad \bar{\delta}_C(Z) = \sum_{i=1}^n \bar{\delta}^{2|4}(Z_{i-1}, Z, Z_i),$$

where $\bar{\delta}_C$ is dual to C , and clearly $\omega_C|_{X_i} = \omega_i$.

More geometrically, in the abelian theory, we can consider this NMHV amplitude to be the first nontrivial term in the expectation value of the holomorphic Wilson loop:

$$\langle W[C] \rangle = \left\langle \exp \int_C \omega_C \wedge \mathcal{A} \right\rangle. \quad (6.6)$$

This can be evaluated exactly in the abelian theory. Let us first concentrate on the self-dual abelian theory, where there are no MHV vertices and the cubic holomorphic Chern-Simons term vanishes. Our path integral expression becomes:

$$\begin{aligned} \langle W[C] \rangle &= \mathcal{Z}^{-1} \int \mathcal{D}\mathcal{A} e^{-S_{\text{SD}}[\mathcal{A}]} \exp \left(\int_{\mathbb{PT}} \mathcal{A} \wedge \bar{\delta}_C \wedge D^{3|4} Z \right) \\ &= \mathcal{Z}^{-1} \int \mathcal{D}\mathcal{A} \exp \left(- \int_{\mathbb{PT}} D^{3|4} Z \wedge \mathcal{A} \wedge (\bar{\partial}\mathcal{A} - \bar{\delta}_C) \right). \end{aligned} \quad (6.7)$$

This can be evaluated explicitly by introducing a background gauge field \mathcal{A}_0 which satisfies $\bar{\partial}\mathcal{A}_0 = \bar{\delta}_C$, and expanding about $\mathcal{A} = \mathcal{B} - \mathcal{A}_0/2$. The dependence on \mathcal{A}_0 can then be pulled outside the path integral to get

$$\langle W[C] \rangle = \exp \left(-\frac{1}{4} \int D^{3|4} Z \wedge \mathcal{A}_0 \wedge \bar{\partial}\mathcal{A}_0 \right). \quad (6.8)$$

Solving for the background field \mathcal{A}_0 using the twistor propagator, we find an expression for the *logarithm* of the momentum twistor Wilson loop:

$$\log \langle W(C) \rangle = \frac{1}{4} \int \Delta(Z, Z') \bar{\delta}_C(Z) \bar{\delta}_C(Z') D^{3|4} Z D^{3|4} Z'. \quad (6.9)$$

In axial gauge this becomes

$$\log \langle W(C) \rangle = \sum_{i < j} [i-1, i, *, j-1, j], \quad (6.10)$$

which is exactly the NMHV tree amplitude of (5.12).

To see what happens when MHV vertices are included, we can expand around the self-dual sector, to include the simplest 2 point MHV vertex in the abelian theory. Hence, the path integral is again Gaussian, and may be evaluated exactly with the result

$$\log \langle W(C) \rangle = \sum_{i < j} \left([i-1, i, *, j-1, j] + \int d^{4|8} x [i-1, i, *, A, B'] [j-1, j, *, A, B''] \right) \quad (6.11)$$

where X is the complex line through the momentum twistors Z_A and Z_B . In the following section we will see that the second term in equation (6.11) is precisely the MHV one-loop

amplitude. Hence, in the full abelian theory the *logarithm* of the momentum twistor Wilson loop is the sum of the NMHV tree and MHV one-loop amplitudes.

This already demonstrates a striking feature of the amplitude Wilson loop correspondence, that the tree level amplitude for the full $\mathcal{N} = 4$ SYM is computed in the self-dual sector of the Wilson loop, and the expansion around that self-dual sector will be the loop expansion for the amplitude. However, it is clear that the full tree amplitude is not simply the exponential of the NMHV amplitude (nor is the multiloop amplitude the exponential of the 1-loop one) and we will see that to extend this correspondence further, we will need to introduce nonabelian holomorphic Wilson loops. However, this is not something that can be done for a general curve C , as in the complex there is not in general a notion of parallel transport for sections of a holomorphic vector bundle on a complex curve, and we will see that we need to use its special structure as a nodal curve with rational components.

6.2 Holomorphic Wilson Loops

For a curve in space-time, a Wilson loop is simply (the expectation of) the trace of the holonomy around a closed curve, and we are in particular interested in the case of our polygon in region momentum space. A $\bar{\partial}$ -operator on a bundle over a complex curve doesn't generally lead to a notion of parallel transport around the curve. However, the Ward transform allows one to reformulate the space-time holonomy into twistor space as a functional of the $(0, 1)$ -form gauge field \mathcal{A} . These concepts can all be defined quite naturally in the bosonic setting, where the momentum twistor polygon becomes a nodal curve in \mathbb{CP}^3 and the gauge field is reduced to the leading component of \mathcal{A} , namely $a \in \Omega^{0,1}(\mathbb{PT}'_b, \text{End}(E))$ where \mathcal{A} (or a) defines a $\bar{\partial}$ -operator on the bundle $E \rightarrow \mathbb{PT}$ (or its bosonic 'body'). The supersymmetric Wilson loop is then found by passing to the immediate supersymmetric extension of these constructions. Here, we present only the final supersymmetric results as we are primarily concerned with $\mathcal{N} = 4$ SYM; see [4, 6] for full details.

The momentum twistor polygon C is a (nodal) complex curve whose components are Riemann spheres. It is a standard fact from complex geometry that holomorphic bundles on the Riemann sphere that are topologically trivial are also generically holomorphically trivial. In particular it will be trivial for any choice of $\bar{\partial}$ -operator $\bar{\partial}_{\mathcal{A}}$ that is not too far from the trivial one, as will be the case in the perturbative context in which we are working. Consider first a single component $X \cong \mathbb{CP}^1$ of the momentum twistor polygon and choose a local inhomogeneous coordinate σ . The first step is to introduce a global holomorphic frame. To emphasize the analogy with parallel propagation, we will work with an operator U that propagates from an initial point σ_0 to a final marked point σ on the Riemann sphere; see figure 12. U will be the identity in a global holomorphic frame on the Riemann sphere and can be made explicit by first solving for a frame of $E|_X$ that holomorphically trivializes $E|_X$. This is a gauge transformation h such that

$$h^{-1} \bar{\partial}_{\mathcal{A}}|_X h = \bar{\partial}|_X,$$

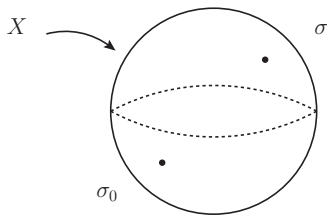


Figure 12: The monodromy operator $U(\sigma, \sigma_0)$ propagates from an initial point σ_0 to a final marked point σ along the line X .

or equivalently

$$(\bar{\partial}|_X + \mathcal{A}(\sigma))h(\sigma) = 0, \quad (6.12)$$

where the notation $\mathcal{A}(\sigma)$ indicates that the gauge field is pulled back from twistor space to the point $\sigma \in X$. Now, define

$$U(\sigma, \sigma_0) := h(\sigma)h^{-1}(\sigma_0). \quad (6.13)$$

Clearly $U(\sigma, \sigma_0) : E|_{\sigma_0} \rightarrow E|_{\sigma}$ and is subject to the initial condition $U(\sigma_0, \sigma_0) = \mathbb{I}$. Furthermore, (6.12) makes it clear that U satisfies

$$U(\sigma_3, \sigma_2)U(\sigma_2, \sigma_1) = U(\sigma_3, \sigma_1) \quad (6.14)$$

and under gauge transformations $\bar{\partial}_{\mathcal{A}} \rightarrow g\bar{\partial}_{\mathcal{A}}g^{-1}$ transforms according to

$$U(\sigma_2, \sigma_1) \longrightarrow g(\sigma_2)U(\sigma_2, \sigma_1)g(\sigma_1)^{-1}. \quad (6.15)$$

This will play the role in our holomorphic context of the parallel propagator along a real curve in real Chern-Simons or Yang-Mills theory. We can concatenate these together from node to node around the complex polygon in twistor space to obtain a holonomy around the polygon. This is precisely the holonomy that one finds for the gauge field obtained on space-time via the Ward correspondence, see [4]. The expectation value of its trace will be the holomorphic Wilson loop that we are after.

We now find an explicit perturbative expression for U by reformulating equations (6.12), (6.13) as the boundary value problem

$$(\bar{\partial} + \mathcal{A})U = 0, \quad U(\sigma_0) = \mathbb{I}. \quad (6.16)$$

Formally we can solve for this by rewriting the equation as $U = \mathbb{I} + \bar{\partial}^{-1}(\mathcal{A}U)$ and iterating. The operator $\bar{\partial}^{-1}$ is integration against the Green's function for the $\bar{\partial}$ -operator on the complex line X with zero boundary condition at σ_0 which is the meromorphic one-form

$$\omega(\sigma) = \frac{(\sigma_1 - \sigma_0)d\sigma}{(\sigma_1 - \sigma)(\sigma - \sigma_0)}, \quad \bar{\partial}\omega(\sigma) = (\delta^2(\sigma - \sigma_0) - \delta^2(\sigma - \sigma_1))d^2\sigma. \quad (6.17)$$

This leads to the perturbative solution for the monodromy operator:

$$\begin{aligned}
U_X(\sigma, \sigma_0) &= \mathbb{I} + \int_X \frac{(\sigma - \sigma_0)d\sigma_1 \wedge \mathcal{A}(\sigma_1)}{(\sigma - \sigma_1)(\sigma_1 - \sigma_0)} + \int_{X^2} \frac{(\sigma - \sigma_0)d\sigma_2 d\sigma_1 \wedge \mathcal{A}(\sigma_2)\mathcal{A}(\sigma_1)}{(\sigma - \sigma_2)(\sigma_2 - \sigma_1)(\sigma_1 - \sigma_0)} + \dots \\
&= \mathbb{I} + \sum_{n=1}^{\infty} \int_{X^n} \prod_{i=1}^n \omega(\sigma_i) \wedge \mathcal{A}(\sigma_i).
\end{aligned} \tag{6.18}$$

The meromorphic form $\omega(\sigma_i)$ has simple poles at the initial point σ_0 and the next point σ_{i+1} in the ordering (which is σ when $i = n$). Note that all of the intermediate points $\{\sigma_1, \dots, \sigma_n\}$ in each term are integrated over the whole line X . Equation (6.18) defines the holomorphic analogue of path-ordering in the sense that the matrix multiplication follows the ordering of the sequence of points σ_i $i = 0, 1, 2, \dots$ (although these points are of course not themselves ordered on the Riemann sphere) and we therefore write this perturbative solution as

$$U_X(\sigma, \sigma_0) = \text{P exp} \int_X \omega \wedge \mathcal{A} \tag{6.19}$$

by analogy with the parallel propagator along a real curve.

To form the Wilson loop, we now consider the whole momentum twistor polygon. This consists of a sequence of intersecting lines

$$C = X_1 \cup \dots \cup X_n$$

whose images intersect pairwise at the momentum twistor points $Z_i = X_i \cap X_{i+1}$. The monodromy operator on the component X_i which propagates from momentum twistor Z_{i-1} to momentum twistor Z_i is denoted by $U_{X_i}(\sigma_i, \sigma_{i-1})$, where σ_i labels the momentum twistor Z_i on the curve. The momentum twistor Wilson loop is now defined by propagating from point to point around the momentum twistor polygon C and taking the trace [4]:

$$\begin{aligned}
W[C] &= \text{tr} [U_{X_n}(\sigma_n, \sigma_{n-1}) \dots U_{X_2}(\sigma_2, \sigma_1) U_{X_1}(\sigma_1, \sigma_n)] \\
&= \text{trP exp} \int_C \omega \wedge \mathcal{A}.
\end{aligned} \tag{6.20}$$

The properties (6.14) and (6.15) of the monodromy ensure the gauge invariance of this operator and independence of the choice of initial point. This operator is our definition of the holomorphic Wilson loop in momentum twistor space. Classically, it is also a transcription of the corresponding Wilson loop around the space-time polygon.

6.3 The Expectation Value

We would now like to compute the quantum expectation value of the holomorphic Wilson loop with respect to the twistor action:

$$\langle W[C] \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\mathcal{A} W[C] \exp(-S[\mathcal{A}]), \tag{6.21}$$

where $S[\mathcal{A}]$ is the complete twistor action for $\mathcal{N} = 4$ SYM defined in (3.9), and \mathcal{Z} is its partition function.

Recall that the twistor action has two components: $S[\mathcal{A}] = S_{\text{SD}}[\mathcal{A}] - \frac{\varepsilon}{2}I[\mathcal{A}]$, where ε is a parameter expanding the theory around the SD sector. Hence, by expanding (6.21) in ε , the loop corrections will be equivalent to computing expectation values of $W[C]$ in a holomorphic Chern-Simons theory but with insertions from the interactions in $I[\mathcal{A}]$. From (3.8), we know that these interactions contain a space-time integral $\int d^{4|8}x$, which can be pulled outside to define the *Wilson loop integrand* in the same manner as described for scattering amplitudes earlier in §5.

The momentum twistor Wilson loop integrands may also be expanded in the fermionic momentum twistor components, and we will use the notation

$$W_{n,k}^{(l)}(Z_1, \dots, Z_n, (AB)_1, \dots, (AB)_l) \quad (6.22)$$

to denote the l -loop integrand with Grassmann degree $O(\chi^{4k})$. It has now been proven that the integrands $W_{n,k}^{(l)}$ are equal to the corresponding amplitude integrands $I_{n,k}^{(l)}$ by demonstrating that they satisfy an all-loop generalisation of BCFW recursion relations [6] §7.2.

6.3.1 Perturbative Expansion and MHV Diagrams

We now consider the non-abelian case and expand the expectation value (6.21) perturbatively around the self-dual sector of the theory in the large N (planar) limit. We will find that axial gauge Feynman diagrams for the expectation value (6.21) are in one-to-one correspondence with MHV diagrams for scattering amplitudes.

As before, in our axial gauge, the cubic term in the holomorphic Chern-Simons action vanishes, and the log det term expands to give the infinite sum of additional interaction terms given in (3.14) as

$$\int_{\mathbb{M}_{\mathbb{R}}} d^{4|8}x \sum_{m=2}^{\infty} \int_{X^m} \text{tr} \left(\prod_{i=1}^m \frac{\mathcal{A}(\sigma_i) \wedge d\sigma_i}{\sigma_i - \sigma_{i-1}} \right).$$

Each term corresponds to an interaction connecting the line X (which in this context will become a supersymmetric loop integration variable) to m insertions of the twistor gauge field \mathcal{A} either on other vertices, or on C .

We first start by working in the self-dual sector omitting the MHV vertices. The simplest Feynman diagram is a single propagator connecting two components X_i and X_j of the momentum twistor polygon; we have already seen that these diagrams are in one-to-one correspondence by planar duality with the MHV diagrams for the NMHV tree amplitude $I_{n,1}^{(0)}$.

The next simplest case is of two propagators connecting two pairs of lines, X_i to X_j and X_k to X_l . It is a standard fact that in the planar limit, these two lines must not cross (the crossing diagrams are suppressed by $1/N$). Thus we can order cyclically $i < j \leq k < l \leq i$. The generic

contribution will then be the product of R-invariants $[i-1, i, *, j-1, j][k-1, k, *, l-1, l]$. However, when $j = k$ or $l = i$, we must perform the calculation more carefully, and we find that the answer is again a product of R-invariants, but now the twistors are shifted according to the prescription given in (5.16). These diagrams are again easily seen to be the planar duals to the N^2 MHV tree-level MHV diagrams.

This pattern can easily be seen to extend to the full tree-level sector of the theory: the Feynman diagrams for the Wilson loop in the planar limit calculated in the self-dual sector of the theory are precisely the planar duals to the MHV tree diagrams for the scattering amplitude.

The pattern easily extends to the loop integrands when we include the MHV vertices. We start with a single, quadratic (MHV) vertex connected to two components X_i and X_j of the momentum twistor polygon. Such Feynman diagrams contribute to the Wilson loop integrand $W_{n,0}^{(1)}$. The contribution of the diagram illustrated in figure 13 is

$$\int \frac{ds dt}{s t} \frac{d\sigma_1 d\sigma_2}{(\sigma_1 - \sigma_2)^2} \Delta(Z_i(s), Z(\sigma_1, \sigma_2)) \Delta(Z_j(t), Z(\sigma_1, \sigma_2)) \quad (6.23)$$

where we have defined the momentum twistors $Z_i(s) = Z_{i-1} + sZ_i$ and $Z(\sigma_1, \sigma_2) = \sigma_1 Z_A + \sigma_2 Z_B$ which parametrise respectively the component X_i of the momentum twistor polygon and the interaction line X . The integrals are performed against the delta-functions to give

$$[i-1, i, *, A, B'] [j-1, j, *A, B''] \quad (6.24)$$

where

$$B' = (AB) \cap \langle *, j-1, j \rangle, \quad B'' = (AB) \cap \langle *, i-1, i \rangle.$$

This is identical to the contribution to the MHV one-loop integrand $I_{n,0}^{(1)}$ from an MHV diagram which is planar dual to the momentum twistor Feynman diagram, as illustrated in figure 13 and calculated in (5.17).

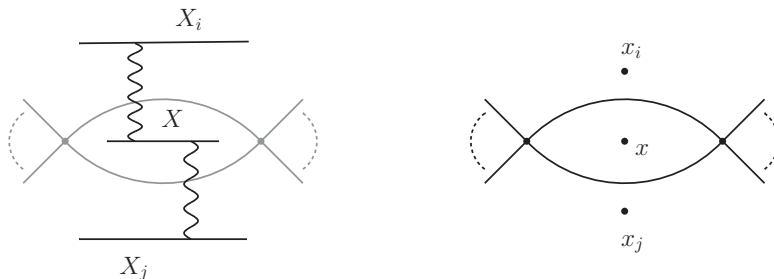


Figure 13: *The momentum twistor Feynman diagrams and planar dual MHV diagrams for MHV one-loop integrand.*

In [4] further computations of momentum twistor Feynman diagrams were performed. In all cases checked, the Feynman diagrams contributing to the integrand $W_{n,k}^{(l)}$ are identical to

the planar dual MHV diagrams for the amplitude integrand $I_{n,l}^{(l)}$, leading to the conjecture that the momentum twistor Wilson loop computes all planar integrands of scattering amplitudes in $\mathcal{N} = 4$ SYM.

The fact that this is consistent arises from the fact that the diagram rules only associate a nontrivial contribution to the propagators, and these are essentially the same for the dual diagram, except for being rotated by a right angle, but this too is a symmetry of the propagator contribution.

6.3.2 A proof via loop equations and BCFW recursion

We can understand why the expectation value of the momentum twistor Wilson loop should compute scattering amplitudes directly from its fundamental properties without resorting to a perturbative analysis. This follows by examining how the expectation value changes when the momentum twistor polygon C is deformed. We can then show that the correct singularities of scattering amplitudes as functions of the kinematic variables emerge from the properties of holomorphic Wilson loops. This allows us to show that the Wilson loop satisfies BCFW recursion relations (which are briefly described in §7.2) and therefore leads to a proof of the duality for the integrands between the amplitude and the Wilson loop.

We consider a one-parameter holomorphic deformations of the momentum twistor polygon C_t , labelled by a complex parameter t . In practice we will simply shift just one of the twistors using (6.26) although the following applies to general deformations. We would like to know whether the expectation value $\langle W(C_t) \rangle$ changes holomorphically. The action of the $\bar{\partial}$ -operator with respect to the complex parameter t on the classical holomorphic Wilson loop is given by [6]

$$\bar{\partial}_t W[C_t] = \int_{C_t} \omega(\sigma) \wedge \text{tr}(\text{Hol}_\sigma(C_t) \mathcal{F}^{0,2}(\sigma)) \quad (6.25)$$

where $\text{Hol}_\sigma(C_t)$ is the holonomy around the momentum twistor polygon C_t starting and finishing at the point σ , and $\mathcal{F}^{0,2}(\sigma)$ is the $(0,2)$ -form component of the curvature of \mathcal{A} pulled back to the infinitesimal surface swept out by C_t as t is perturbed. This is the holomorphic analogue of the result that the change in a real Wilson loop under a small deformation of the curve is given by the flux through the area swept out.

In holomorphic Chern-Simons theory, the equations of motion are $\mathcal{F}^{0,2} = 0$ and hence we conclude that $\bar{\partial}_t W = 0$ on-shell in the self-dual gauge theory. However, there are corrections to this picture both from quantum anomalies and from the additional MHV vertices in the twistor action. These corrections imply that the expectation value $\langle W[C] \rangle$ has the right singularity structure to compute tree amplitudes and loop amplitudes respectively. Here we summarise the results; details may be found in [6].

- There is an anomaly for the equation $\bar{\partial}_t \langle W[C_t] \rangle = 0$ when the momentum twistor polygon C_t intersects itself. This occurs when two components X_i and X_j intersect and corresponds to the momentum space factorisation channel $(p_i + \dots + p_{j-1})^2 \rightarrow 0$. This

intersection pinches the momentum twistor polygon into two nodal curves C_L and C_R which meet at the intersection. In the planar limit, we can deduce that the anomaly is proportional to the product $\langle W[C_L] \rangle \langle W[C_R] \rangle$. This implies that the integrands $W_{n,k}^{(l)}$ have the same factorisation behaviour along multi-particle channels as the integrands $I_{n,l}^{(l)}$ of scattering amplitudes.

- The interaction terms in the twistor action provide another correction to the equation $\bar{\partial}_t \langle W[C_t] \rangle = 0$. This occurs when a component X_i of the momentum twistor polygon intersects the interaction line X , and corresponds to an internal propagator in the loop integrand going on shell $(x - x_i)^2 \rightarrow 0$. These corrections imply that the integrands $W_{n,k}^{(l)}$ have the same factorisation behaviour along internal channels as the loop integrands $I_{n,k}^{(l)}$ of scattering amplitudes.

By directly substituting in the standard BCFW [93, 94] deformation in momentum space (7.2) into the coordinate transformation into momentum twistors, we arrive at the particular deformation

$$Z_n(t) = Z_n - tZ_{n-1}. \quad (6.26)$$

For this deformation, the loop equations can be integrated over the deformation parameter t . Once expanded in Grassmann degree and loop order, we find the all-loop BCFW recursion relations for the loop integrand of the momentum twistor Wilson loop:

$$\begin{aligned} W_{n,k}^{(l)}(1, \dots, n) &= W_{n-1,k}^{(l)}(1, \dots, n-1) \\ &+ \sum_j \sum_{n_i, k_1, l_1} [n-1, n, 1, j-1, j] W_{n_1, k_1}^{(l_1)}(1, \dots, j-1, I_j) W_{n_2, k_2}^{(l_2)}(I_j, j, \dots, n-1, \hat{n}_j) \\ &+ \int_{\mathbb{M}_{\mathbb{R}} \times S^1 \times S^1} D^{3|4} Z_A \wedge D^{3|4} Z_B [n-1, n, 1, A, B] W_{n+2, k+1}^{(l-1)}(1, \dots, n-1, \hat{n}_{AB}, \hat{A}, B), \end{aligned} \quad (6.27)$$

where the hatted momentum twistors are

$$\begin{aligned} \hat{A} &= (A, B) \cap (n-1, n, 1) & \hat{n}_{AB} &= (n-1, n) \cap (A, B, 1) \\ \hat{n}_j &= (n-1, n) \cap (j-1, j, 1) & I_j &= (j-1, j) \cap (n-1, n, 1), \end{aligned} \quad (6.28)$$

and the summation ranges are for $n_1 + n_2 = n + 2, k_1 + k_2 = k - 1$ and $l_1 + l_2 = l$. This recursion is precisely the same as that obtained for the amplitude integrands $I_{n,k}^{(l)}$ via BCFW recursion in momentum twistor space in [10]. Since this recursion determines the loop integrand, this gives a proof of the duality $I_{n,k}^{(l)} = W_{n,k}^{(l)}$ between the amplitude and momentum twistor Wilson loop integrands.

6.4 Correlation Functions

The duality between MHV scattering amplitudes and Wilson loops has recently been extended further to include light-cone limits of certain n -point correlation functions in work by Alday,

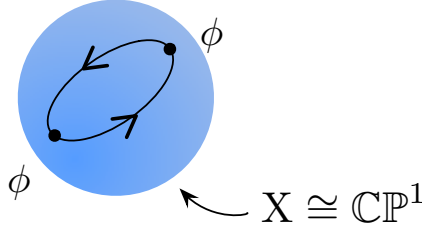


Figure 14: In a non-Abelian theory, the twistor space form of the local space-time operator $\text{Tr } \Phi^2$ involves holomorphic Wilson lines on the Riemann sphere X .

Eden, Korchemsky, Maldacena and Sokatchev [32, 40, 41]. This has recently been extended to the supersymmetric setting [11, 12] encoding the whole amplitude as opposed to just the MHV sector. This correspondence can be understood as arising from the presence of the operators $U_X(\sigma, \sigma')$ in the twistor definition of a field.

We consider super-correlation functions of local gauge invariant operators $\mathcal{O}(x, \theta)$ on chiral superspace. We can for example choose the Konishi multiplet, whose superconformal primary operator is

$$\mathcal{O}(x) = \frac{1}{2} \epsilon^{abcd} \text{Tr} (\Phi_{ab}(x) \Phi_{cd}(x)) , \quad (6.29)$$

or some 1/2 BPS relative. As we saw in the Abelian theory (2.22), the operators $\phi_{ab}(x)$ have an extension $\mathcal{F}_{ab}(x, \theta)$ to chiral superspace that is given by a straightforward integral formula in terms of \mathcal{A} over X . In the non-abelian case, we must trivialize the bundle E over X and the abelian integral formula generalizes to become

$$\mathcal{F}_{ab} = \int_X U_X(\sigma', \sigma) \frac{\partial^2 \mathcal{A}}{\partial \chi^a \partial \chi^b} U_X(\sigma, \sigma') d\sigma . \quad (6.30)$$

Thus, in twistor space, these superfields become non-local operators based on the complex line X . For the Konishi multiplet we have [11]

$$\mathcal{O}(x, \theta) = \frac{1}{2} \epsilon^{abcd} \int_{X \times X} d\sigma d\sigma' \text{Tr} \left(\frac{\partial^2 \mathcal{A}}{\partial \chi^a \partial \chi^b}(\sigma) U_X(\sigma, \sigma') \frac{\partial^2 \mathcal{A}}{\partial \chi^c \partial \chi^d}(\sigma') U_X(\sigma', \sigma) \right) . \quad (6.31)$$

This is illustrated schematically in figure 14. Correlation functions of such operators can then be computed in twistor space using the twistor action for $\mathcal{N} = 4$ SYM.

The duality is obtained by examining correlation functions of n such operators

$$G_n(x_1, \theta_1; \dots; x_n, \theta_n) = \langle \mathcal{O}(x_1, \theta_1) \dots \mathcal{O}(x_n, \theta_n) \rangle \quad (6.32)$$

in the limit as the chiral superspace points become null separated to form the vertices of our null polygon so that the lines X_{i+1} and X_i intersect in twistor space. Working with the twistor action and at the level of the loop *integrand*, the most singular contribution comes

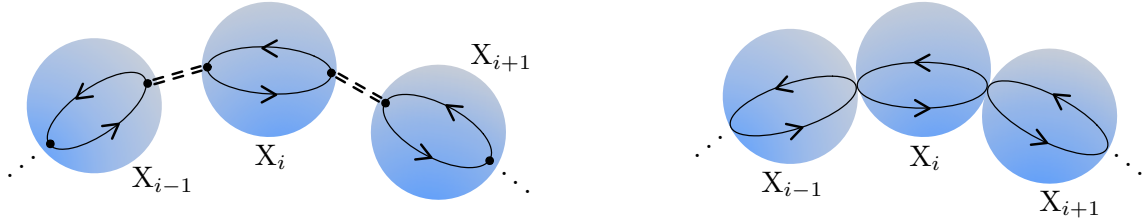


Figure 15: The only non-vanishing contribution to the integrand ratio in the null limit comes from direct contractions between ϕ s on adjacent Riemann spheres. The remaining operator is the supersymmetric momentum twistor Wilson loop, acting in the adjoint representation.

from the contraction of two scalar component fields ϕ in the expansion of $\partial^2 \mathcal{A} / \partial \chi^1 \partial \chi^2$ on adjacent lines X_i and X_{i+1} . Once the singular part $1/x_{12}^2 \dots x_{n1}^2$ has been extracted, the U operators on each line are now multiplied together around the polygon, so that the remainder is a momentum twistor Wilson loop in the *adjoint* representation, as illustrated schematically in figure 15 (so we have two copies of our former Wilson loop). The duality may be expressed as

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \frac{G_n}{G_{n,0}^{(0)}} = \langle W_n^{\text{adj}} \rangle = \langle W_n \rangle^2 \quad (6.33)$$

where the final equality is valid only in the planar limit.

We have already discussed in some detail the duality $I_{n,k}^{(l)} = W_{n,k}^{(l)}$ between scattering amplitude and Wilson loop integrands. The new duality with correlation functions implies further relationships with the integrands of the correlation function. Expanding equation (6.33) in the momentum twistor Grassmann degree and number of loops, we find relationships involving the correlation function integrands $G_{n,k}^{(l)}$. For example, in the self-dual theory we find the tree-level result

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \frac{G_{n,1}^{(0)}}{G_{n,0}^{(0)}} = 2W_{n,1}^{(0)} \quad \lim_{x_{i,i+1}^2 \rightarrow 0} \frac{G_{n,2}^{(0)}}{G_{n,0}^{(0)}} = 2W_{n,2}^{(0)} + \left(W_{n,1}^{(0)}\right)^2, \quad (6.34)$$

and similarly for integrands of Grassmann degree zero we find

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \frac{G_{n,0}^{(1)}}{G_{n,0}^{(0)}} = 2W_{n,0}^{(1)} \quad \lim_{x_{i,i+1}^2 \rightarrow 0} \frac{G_{n,2}^{(0)}}{G_{n,0}^{(0)}} = 2W_{n,0}^{(2)} + \left(W_{n,0}^{(1)}\right)^2, \quad (6.35)$$

corresponding to agreement with the all-loop integrand of the non-supersymmetric adjoint Wilson loop in space-time.

7 Further topics

There are many important twistor developments that we have not been able to cover in detail in this review. To make up for this, we give a brief introduction to some of the most important with pointers to the literature.

7.1 Twistor-string theory

Witten's twistor-string [42] was the breakthrough that stimulated these recent developments. Although still not yet satisfactorily realized, it remains one of the most ambitious visions for how scattering amplitudes should be understood. The idea is that the full l -loop N^k MHV amplitude should be expressed as a path-integral over the space of holomorphic maps from algebraic curves of degree $k + 1 + l$ and genus l into twistor space. Twistor-string theory, as first introduced by Witten, was a twisted B-model coupled to D1-instantons supported on holomorphic curves in twistor space. The B-model is more usually found in the study of Calabi-Yau compactifications and, through a worldsheet 'twisting' procedure, is sensitive only to the holomorphic structure. Soon after, Berkovits introduced a simpler model essentially of holomorphic strings in twistor space [43] and later a definition was given as a half-twisted heterotic model in [44]. All these models have essentially the same physical content and their spectra includes both $\mathcal{N} = 4$ SYM and also $\mathcal{N} = 4$ conformal supergravity. The conformal supergravity corrupts gauge theory amplitudes at loop level and there is currently no obvious mechanism for decoupling. None of these models has yet been shown to be fully consistent and there remain a number of interesting unanswered questions about their anomalies [43]. There has been little work on building twistor-string theories with different physical content: the one attempt to build a twistor-string adapted to Einstein supergravity [95] was shown to contain only self-dual interactions [96, 97].

As far as gauge theory amplitudes are concerned, then, the main direct impact of twistor-string theory thus far is to provide a remarkable formula for tree-level amplitudes. In many ways, this remains the best and most compact formula available, manifesting more symmetries than any other. It gives the n -point tree-level N^k MHV amplitude as an integral over the moduli space of degree $k + 1$ maps from the \mathbb{CP}^1 worldsheet to \mathbb{PT} . Letting σ be an affine coordinate on the world-sheet, a general degree d map can be written

$$Z(\sigma) = \sum_{r=0}^d Y_r \sigma^r ,$$

where the $Z_j \in \mathbb{C}^{4|4}$ are thought of as coordinates on a $4(d + 1)|4(d + 1)$ -dimensional space. As in the MHV case discussed earlier, fields are inserted at points σ_i on the worldsheet. If we insert elemental states $\mathcal{A}_i = \bar{\delta}^{3|4}(Z_i, Z(\sigma_i))$ we obtain the path integral formula for the tree-level amplitudes (generalizing a form used by Roiban, Spradlin and Volovich [81] in split

signature):

$$A^{\text{tree}}(1, \dots, n) = \int \frac{d^{4(d+1)} Y_r}{\text{vol}(\text{GL}(2, \mathbb{C}))} \int_{(\mathbb{CP}^1)^{\otimes n}} \prod_{i=1}^n \frac{\bar{\delta}^{3|4}(Z_i, Z(\sigma_i)) \wedge d\sigma_i}{\sigma_i - \sigma_{i-1}}. \quad (7.1)$$

where the $\text{vol}(\text{GL}(2; \mathbb{C}))$ that we must quotient by consists of Möbius transformations on the worldsheet together with an overall scaling of the Z_i s. In this Dolbeault framework, the integral should be taken over the full n copies of \mathbb{CP}^1 , but over a middle dimensional cycle in the space of holomorphic maps; the construction of this cycle is not yet properly understood except for $(++--)$ signature. This path integral formula is easily seen to reduce to the expression (3.16) for the MHV vertex by setting $d = 1$. In this case, $Z(\sigma_i) = Z_A + Z_B \sigma_i$, and the integral over the $(8|8)$ -dimensional moduli space of degree one maps (divided by the volume of $\text{GL}(2)$ transformations) becomes an integral over $D^{4|4} Z_A \wedge D^{4|4} Z_B$. At higher degree (7.1) was shown to satisfy the tree-level BCFW recursion relations in [53], building on earlier work [46–51].

At higher genus, some 1-loop amplitudes for the original twistor string (thus containing conformal supergravity modes running around the loop) were studied in [45]. It was shown in [48] that leading singularities of multi-loop amplitudes have exactly the twistor support expected from a putative higher-genus twistor string, with the cut momenta of the leading singularity corresponding to degenerations of the worldsheet. It was then conjectured [48] that it might be possible to use an extension of the Grassmannian formula (discussed below) to determine a meromorphic form on the space of genus g , degree d , n -pointed maps to \mathbb{PT} (analogous to the $g = 0$ measure in (7.1)) by requiring this measure to have poles corresponding to leading singularities. In a separate line, Yangian charges on the worldsheet of the twistor-string have been studied in [52].

7.2 Recursion relations

Recursion techniques have been a powerful tool for generating amplitudes and proving conjectures concerning momentum space amplitudes. They are reviewed in more detail in [39].

The starting point (is to deform the external momenta in such a way that total momentum is still conserved and each of the external particles remains on-shell. For BCFW recursion [93, 94, 98, 99], this is typically done by replacing

$$\begin{aligned} (\lambda_1, \tilde{\lambda}_1, \eta_1) &\rightarrow (\lambda_1 + t\lambda_n, \tilde{\lambda}_1, \eta_1) \\ (\lambda_n, \tilde{\lambda}_n, \eta_n) &\rightarrow (\lambda_n, \tilde{\lambda}_n - t\tilde{\lambda}_1, \eta_n - t\eta_1), \end{aligned} \quad (7.2)$$

in the momentum space amplitude, where t is a complex parameter. These shifts are generated by the vector

$$\lambda_n \frac{\partial}{\partial \lambda_1} - \tilde{\lambda}_1 \frac{\partial}{\partial \tilde{\lambda}_n} - \eta_1 \frac{\partial}{\partial \eta_n}, \quad (7.3)$$

which becomes

$$\lambda_n \frac{\partial}{\partial \lambda_1} + \mu_n \frac{\partial}{\partial \mu_1} + \chi_n \frac{\partial}{\partial \chi_1} = Z_n \frac{\partial}{\partial Z_1} \quad (7.4)$$

in twistor space, showing that the twistor amplitude is similarly deformed by replacing $Z_1 \rightarrow Z_1 + tZ_n$. In particular, this makes it manifest that the BCFW shift procedure is superconformally invariant.

Upon transforming to twistor space, the momentum space BCFW recursion formula

$$A(1, \dots, n) = \sum_i \frac{A(1(t_i); \dots, i, -P_i(t_i)) A(P_i(t_i); i+1; \dots; n(t_i))}{P_i^2}, \quad (7.5)$$

(where $P_i = p_1 + \dots + p_i$ and $P_i(t_i) = p_1(t_i) + \dots + p_i$, with $t_i = P_i^2/[1|P_i|n]$) becomes [54]

$$A(Z_1, \dots, Z_n) = \sum_i \int D^{3|4} Z \frac{dt}{t} A(Z_1 + tZ_n, \dots, Z_i, Z) A(Z, Z_{i+1}, \dots, Z_n). \quad (7.6)$$

in twistor space. The recursion needs to be seeded by the three point amplitudes. Clearly the 3- point MHV amplitude is simply $A_{\text{MHV}}(1, 2, 3) = V(1, 2, 3) = V(1, 2) \bar{\delta}^{2|4}(1, 2, 3)$ as in section 3.2, while the 3-point $\overline{\text{MHV}}$ amplitude is

$$\begin{aligned} A_{\overline{\text{MHV}}}(Z_1, Z_2, Z_3) &= \int_{\mathbb{PT}} D^{3|4} Z \wedge \bar{\delta}^{3|4}(Z, Z_1) \wedge \bar{\delta}^{3|4}(Z, Z_2) \wedge \bar{\delta}^{3|4}(Z, Z_3) \\ &= \bar{\delta}^{3|4}(Z_1, Z_2) \bar{\delta}^{3|4}(Z_1, Z_3), \end{aligned} \quad (7.7)$$

coming from the Chern-Simons vertex.

The BCFW representation of twistor space tree amplitudes was first studied in [54], working in split signature¹⁶, and twistor BCFW representations were further investigated in [48, 100]. The transform of BCFW recursion into twistor variables was actually first studied by Hodges in [101, 102] in a formalism based on the use of both twistors and dual twistors. It was used to make contact with twistor diagram theory and to define twistor diagrams for arbitrary tree amplitudes. This approach was taken further by [55], but now more systematically using Witten's half-Fourier transform. This led to a new representation – the *link representation* – that was a precursor of the Grassmannian formalism given below. Whereas the MHV formalism remains controversial for gravity amplitudes, BCFW recursion does work for gravity and can also be used to construct gravity amplitudes in twistor space [54, 55].

As well as being superconformally invariant, the BCFW procedure was also shown to be *dual* superconformally invariant in [99] where it was used to prove dual superconformal

¹⁶In the formalism of this paper, the expressions given in [54] simplify as all the conformal symmetry breaking sign functions are omitted. In split signature, they were required to give the formulae the correct symmetries, but here those symmetries are provided by the algebra of differential forms. However, in the present formalism, factors of the two-point MHV vertex (coming with the 3-point MHV vertex as above) are required instead; these were absent in the split signature formalism.

invariance of all tree amplitudes. This invariance can most easily be seen by reformulating the shift (7.2) in momentum twistor space where it becomes

$$Z_n \rightarrow Z_n - tZ_{n-1} \tag{7.8}$$

as in (6.26). (Note that the $\langle 1n \rangle$ shift in momentum space here translates Z_n along the momentum twistor line $(n-1, n)$). Using momentum twistors, BCFW recursion was extended to the all-loop integrand for planar supersymmetric gauge theories by Arkani-Hamed et. al. [10], yielding the recursion formula (6.27) for the scattering amplitude (see also [90] for a discussion in momentum space).

Another recursion procedure that is naturally expressed in momentum twistor space is that due to Risager [85]. This gave the first proof of MHV rules at tree level [86]. In momentum space it involved a choice of reference spinor $\iota^{A'}$ and in its most general form was the shift $\tilde{\lambda}_{iA'} \rightarrow \tilde{\lambda}_{iA'} + a_i t \iota_{A'}$ where the a_i are chosen so as to conserve momentum. In momentum twistor space it can be expressed easily as the shift $Z_i \rightarrow Z_i + c_i t Z_*$ with c_i arbitrary. In this form it was used to prove the MHV rules for the construction of the planar loop integrand in [37, 38].

7.3 The Grassmannian formula and leading singularities

The Grassmannian formula generates rational Yangian invariants of the kinematic data. These include not only all possible BCFW terms in the decomposition of tree amplitudes but also all possible leading singularities of multiloop amplitudes. Leading singularities are invariants of multiloop scattering amplitudes obtained by performing the $4l$ loop integrations not over the physical contour (l copies of momentum space), but by residues. This has the effect of putting propagators on-shell. The result is expressed as a product of on-shell tree amplitudes, glued together at the legs corresponding to the on-shell propagators.

The Grassmannian formula comes in two forms: the original one on twistor space [1] that makes superconformal invariance manifest, and a subsequent one on momentum twistor space [2] that manifests dual superconformal invariance. These were shown to be essentially equivalent in [57]. The first papers to show that all leading singularities live in the Grassmannian were [48, 56], while the Yangian invariance of the formula was investigated in [57, 60]. See also [62] in this journal issue for a review of the Grassmannian formula emphasising its symmetries. Our treatment here will emphasize the connection with twistor-string theory following [48].

The Grassmannian $G(k, n)$ is the space of k -planes $\mathbb{C}^k \subset \mathbb{C}^n$. It can be parametrized by a $k \times n$ matrix C_{ri} , (with $r = 1, \dots, k$ and $i = 1, \dots, n$) where we must quotient by the action of $GL(k; \mathbb{C})$ on the r -index. For an n -particle N^k MHV amplitude, the twistor space formula involves the Grassmannian $G(k+2, n)$ whereas the momentum twistor formula

involves $G(k, n)$. Both formulae take a similar form¹⁷

$$\begin{aligned}\mathcal{L}_{n,k}(Z_1, \dots, Z_n) &= \oint_{\Gamma} \frac{d^{(k+2)n}C}{\text{vol GL}(k+2)} \frac{\left(\prod_{r=1}^{k+2} d^{4|4}Y_r\right) \left(\prod_{i=1}^n \bar{\delta}^{4|4}(Z_i - C_{qi}Y_q)\right)}{(1\ 2 \dots k+2)(2\ 3 \dots k+3) \dots (n\ 1\ 2 \dots k+1)} \quad (7.9) \\ \mathcal{R}_{n,k}(Z_1, \dots, Z_n) &= \oint_{\tilde{\Gamma}} \frac{d^{kn}C}{\text{vol GL}(k)} \frac{\prod_{r=1}^k \bar{\delta}^{4|4}(C_{ri}Z_i)}{(1\ 2 \dots k)(2\ 3 \dots k+1) \dots (n\ 1\ 2 \dots k-1)},\end{aligned}$$

where the summation convention is understood on q and

$$(i\ i+1 \dots i+k) := \det(C_{ri}, C_{r(i+1)}, \dots, C_{r(i+k)}) \quad (7.10)$$

are the Plücker coordinates on the Grassmannian $G(k, n)$ that respect the cyclic ordering. In these formulae, Γ and $\tilde{\Gamma}$ are contours that allow integration by residues down to a cycle that should have dimension $2n-4$ in the case of Γ and dimension $4k$ in the case of $\tilde{\Gamma}$. We will abuse notation and denote the cycles that support these residues also by Γ and $\tilde{\Gamma}$, respectively.

Although the two formulae in (7.9) are superficially identical there are key differences. For the momentum twistor $\mathcal{R}_{n,k}$, once the contour integral over $\tilde{\Gamma}$ has been performed, there are precisely enough bosonic delta functions to saturate the integrals over the remaining parameters. Thus, $\mathcal{R}_{n,k}$ yields an algebraic function of the momentum twistors Z_1, \dots, Z_n with Grassman degree $4k$. This therefore gives an N^k MHV invariant that turns out to be some leading singularity; momentum twistor space is essentially a coordinate transformation of ordinary momentum space and all tree amplitudes and leading singularities are rational or algebraic functions on momentum space. In the momentum twistor formulation, the MHV amplitude is 1, so the simplest nontrivial case is at $k=1$, and then $G(1, n)$ is just projective space \mathbb{CP}^{n-1} . At $n=5$, the quotient by $\text{GL}(1)$ can be implemented by setting one of the C_i coordinates equal to 1, and the formula reduces to our original formula (2.39) for the basic R-invariant – the 5-point NMHV amplitude. For higher n our cycle must be a linear combination of cycles that set all but four components of C to zero (including the $\text{GL}(1)$ gauge fixing), so we obtain a linear combination of R-invariants.

The $\mathcal{L}_{n,k}$ formula gives the same leading singularities (or Yangian invariants) as $\mathcal{R}_{n,k}$, but expressed in original twistor space. In this case delta functions remain after all integrations have been performed, leading to the fact that leading singularities have restricted support in twistor space. The $\mathcal{L}_{n,k}$ formula makes it clear that this support occurs where $Z_i = C_{ri}Y_r$ for $C \in \Gamma$ and Y_r arbitrary. For example, when $k=0$, Γ is the whole of $G(2, n)$, and setting

$$Y_1 = Z_A, \quad Y_2 = Z_B, \quad C_{ri} = \sigma_i^{r-1} s_i, \quad (7.11)$$

where σ_i is a parameter on the line and s_i the parameter that we use to integrate a $\bar{\delta}^{4|4}$ down to give the projective delta function $\bar{\delta}^{3|4}(Z_i, Z(\sigma_i))$, the Grassmannian formula can be seen to give our twistor formula for the MHV amplitude (3.16).

¹⁷We have expressed $\mathcal{L}_{n,k}$ in a form that makes the relation to twistor-string theory (discussed below) more direct. $\mathcal{L}_{n,k}$ can be made to more closely resemble $\mathcal{R}_{n,k}$ (or its dual) by integrating out the Y s.

All the Yangian invariants generated by the Grassmannian correspond to ‘primitive’ leading singularities – those made up from gluing tree subamplitudes that are either MHV or $\overline{\text{MHV}}$. A priori, this may seem to be a restriction on the leading singularities that can be obtained. However, BCFW recursion decomposes $N^k\text{MHV}$ tree amplitudes into leading singularities of higher loop amplitudes whose components all have lower degree. (In fact, this was the original approach to the recursion relation by Britto, Cachazo and Feng [93].) Thus, all leading singularities can be expressed as residues of the Grassmannian.

In twistor space, primitive leading singularities are supported on sets of intersecting lines [48]. This was first observed for terms in the BCFW expansion of a tree amplitude by [100] and extended to all leading singularities and Grassmannian residues in [48]. The twistor geometry underlying leading singularities thus leads to a partial classification of Grassmannian residues. For example, at MHV we only ever obtain lines in twistor space for leading singularities, whatever the loop order. At NMHV, the BCFW terms are triangles (i.e., genus one), but the most general NMHV leading singularity is an arrangement of five lines with genus 3 and so on. These ideas have now been developed into a systematic classification of residues in the Grassmannian [51, 61, 103].

Individual residues of the Grassmannian correspond to individual leading singularities, and one would like to understand how to combine residues so as to compute the tree amplitudes themselves. One way to achieve this uses a correspondence between the Grassmannian and the twistor-string moduli space. The line bundle $\mathcal{O}(1)$ on twistor space, when pulled back to the string worldsheet Σ gives a line bundle $L \cong \mathcal{O}(k+1)$ whose holomorphic sections can be described by polynomials of degree $k+1$ in the worldsheet coordinates. Given trivialisations of L at each of the marked points $\sigma_i \in \Sigma$ (i.e., preferred coordinates on each of the fibres $L|_{\sigma_i}$), we can obtain n complex numbers from any holomorphic section of L by looking at its values at each marked point. Thus the space $H^0(\Sigma, L) \cong \mathbb{C}^{k+2}$ of holomorphic sections of L is naturally a subspace of $\bigoplus_i L|_{\sigma_i} \cong \mathbb{C}^n$. In other words, a choice of n marked points on the worldsheet, together with a degree $k+1$ line bundle L and trivialisations at each marked point, naturally corresponds to a point in $G(k+2, n)$. If we vary the location of our marked points on Σ (subject to $\text{SL}(2; \mathbb{C})$ transformations) and vary the choice of trivialisation at each marked point, we obtain a cycle in $G(k+2, n)$ of dimension

$$(n-3) + n - 1 = 2n - 4 , \tag{7.12}$$

where the -1 comes from an overall rescaling. The twistor-string moduli space thus naturally provides a cycle $\Gamma \subset G(k+2, n)$ of dimension $2n-4$, exactly as needed in $\mathcal{L}_{n,k}$. Explicitly, the embedding is given by the Veronese map

$$C_{ri} = \sigma_i^{r-1} s_i , \tag{7.13}$$

as observed by a number of authors [46–50]. (As at MHV above, the s_i scaling parameter can be interpreted as defining the trivialisation of $L|_{\sigma_i}$.) Although this defines a map from the twistor-string moduli space to cycles in the Grassmannian, the resulting cycles can be

obtained as residues of $\mathcal{L}_{n,k}$ only when $k = 0, -1$. Various tricks based on global residue theorems in the Grassmannian have been used to convert this embedding into the precise cycle that generates the tree amplitude [49–51]. It has also been seen how to make contact with the MHV formalism [58].

The construction above also extends to nodal curves, provided each component is rational [48], thus providing a ‘twistor-string theory for leading singularities’ based on higher genus, but nodal curves. It is hoped that this construction for leading singularities provides a good starting point for constructing a twistor-string theory proper at higher genus.

7.4 Spurious singularities, polytopes and local forms

Hodges’ paper introducing momentum twistors [89] also contained the suggestive insight that the momentum twistor formula for the R-invariants – the principle ingredients of the BCFW expansion of gauge theory NMHV tree amplitudes – could be interpreted as the volume of a simplex in a dual momentum twistor space defined by the arguments of the R-invariant. The sum of R-invariants that form an amplitude then correspond to fitting these basic simplices together into a polytope, whose volume is the amplitude. He further emphasised that the BCFW decomposition of tree amplitudes, whilst both superconformally and dual superconformally invariant, contains spurious singularities that could not have arisen from a physical propagator. The different choices of BCFW decomposition were interpreted as different decompositions of the polytope for the full amplitude, while the spurious singularities were associated to vertices of the constituent simplices that do not end up being vertices of the final polytope.

These ideas have been explored further in [92, 104]. In particular they find a new basis for NMHV tree and certain multi-loop MHV and NMHV amplitudes that, in contrast to the BCFW expressions, have only local poles – ones which correspond to a sum of momenta (compatible with the cyclic ordering) going on-shell. The MHV formulae are particularly striking, being suggestive of some kind of integrable structure underlying at least the MHV sector.

A similar idea was taken up in a one loop context in [36]. There, it was shown that Hodges’ procedure [35] for studying 1-loop box functions using momentum twistors has the geometric interpretation of computing 3-volumes of tetrahedra in AdS_5 . The vertices of these tetrahedra lie ‘at infinity’ on conformal Minkowski space, where their location defines the region momenta associated to the box integral. When the whole MHV amplitude is computed, these tetrahedra join together to give a closed 3-polytope in AdS_5 . Once again, the various spurious singularities or branch cuts of individual box integrals cancel when the whole polytope is considered.

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