Ricci curvature and eigenvalue estimates for the magnetic Laplacian on manifolds

Michela Egidi, Shiping Liu, Florentin Münch, and Norbert Peyerimhoff

October 3, 2018

In this paper, we present a Lichnerowicz type estimate and (higher order) Buser type estimates for the magnetic Laplacian on a closed Riemannian manifold with a magnetic potential. These results relate eigenvalues, magnetic fields, Ricci curvature, and Cheeger type constants.

1 Introduction

In recent decades, the spectral theory of the magnetic Laplacian has attracted a lot of attention on various spaces: on domains (see, e.g., [8, 9]), on noncompact Riemannian manifolds (see, e.g., [20, 25]), on discrete graphs (see, .e.g, [24, 26]), and on fractals (see, e.g., [13]), just to name a few. In this article, we are particularly interested in eigenvalue estimates of the magnetic Laplacian on a closed (i.e. compact without boundary) connected Riemannian manifold. In contrast to the Laplace-Beltrami operator (Laplacian, for short), whose smallest eigenvalue is equal to zero and simple, the smallest eigenvalue of the magnetic Laplacian can be positive and of higher multiplicity. Most of the existing eigenvalue estimates are concerned with the smallest eigenvalue. Shigekawa [23] proved a comparison result for the smallest eigenvalue and studied the asymptotic behaviour of eigenvalues of the magnetic Laplacian. Paternain [22] obtained an upper bound of the smallest eigenvalue in terms of a harmonic value and a critical value of the corresponding Lagrangian. The magnetic Laplacian fits into the more general framework of the connection Laplacian. Ballmann, Brüning and Carron [1] proved lower bound estimates of the smallest eigenvalue of the connection Laplacian for Hermitian vector bundles over a closed Riemannian manifold in terms of a holonomy constant. Recently, Cheeger type estimates for all eigenvalues of the magnetic Laplacian were established in [15]. While our paper is concerned with closed manifolds, there has also been work in recent years on eigenvalue estimates for magnetic Laplacians and Schrödinger operators on manifolds with boundary. The papers [4, 5, 6] are concerned with Neumann boundary conditions and provide lower and upper bounds of eigenvalues in terms of the mean value of the scalar potential (in the case of Schrödinger operators), the norm of the magnetic field and distances of the magnetic potential to the lattice of integral harmonic 1-forms. In [6], they also provide a sharp lower bound for the first eigenvalue of the magnetic Laplacian on a 2-dimensional Riemannian cylinder and characterize the case of equality. Eigenvalues of magnetic Laplacians on manifolds with Robin boundary conditions are studied in [12] using similar techniques to the ones developed in this paper (in particular our Bochner type formula in Theorem 4.1). In addition to our ingredients, their integrated formula involves also the second fundamental form of the boundary.

In this article, we are interested in the interaction between eigenvalues of the magnetic Laplacian, Ricci curvature of the underlying closed Riemannian manifold, magnetic Cheeger constants, and the magnetic field. Building upon a Bochner formula involving both the classical Laplacian and the magnetic Laplacian (see Theorem 4.1 below) and inspired by earlier investigations of the discrete counterpart in [19], we obtain two eigenvalue estimates: the first result provides information about the first two eigenvalues and, in particular, a spectral gap in the case of positive Ricci curvature and small magnetic fields, and the other one is concerned with all eigenvalues of the magnetic Laplacian in terms of a non-positive lower Ricci curvature bound and the magnetic Cheeger constants. These estimates extend the classical Lichnerowicz and Buser estimates for eigenvalues of the Laplacian and provide new insights.

Let us now fix some notation. Let (M, g) be a closed n-dimensional Riemannian manifold. Throughout this paper, we assume that M is connected. Let α be a smooth real differential 1-form on M, which is called the *magnetic potential*. Let $0 \leq \lambda_1^\alpha \leq \lambda_2^\alpha \leq \cdots \nearrow \infty$ be the eigenvalues of the magnetic Laplacian Δ^α and $0 = \lambda_1 < \lambda_2 \leq \cdots \nearrow \infty$ be the eigenvalues of the classical Laplacian, both ordered increasingly and counted with multiplicity.

The classical Lichnerowicz estimate states that $\lambda_2 \geq (n/(n-1))K$, whenever $K > 0$ is a lower bound of the Ricci curvature of the manifold M. Recall that $\lambda_1 = 0$ and that λ_1 is simple. In other words, the Lichnerowicz estimate establishes a spectral gap between $\lambda_1 = 0$ and $\lambda_2 > 0$ of size at least $(n/(n-1))K$. As already mentioned above, the smallest eigenvalue λ_1^α of the magnetic Laplacian Δ^α can be positive and can even be equal to λ_2^α. Using the Bochner type formula in Theorem 4.1, we show in the case of positive Ricci curvature that there is also an interval of positive length between the first two eigenvalues λ_1^α and λ_2^α of the magnetic Laplacian, provided the *magnetic field* $d\alpha$ is not too large. More explicitly, we have the following result.

Theorem 1.1 (Magnetic Lichnerowicz Theorem). *Let M be a closed Riemannian manifold* *of dimension $n \geq 2$ with a magnetic potential $\alpha \in \Omega^1(M)$. If*

$$\mathrm{Ric} \geq K > 0 \qquad \textit{and} \qquad \|d\alpha\|_\infty \leq \left(1 + 2\sqrt{\frac{n-1}{n}}\right)^{-1} K, \tag{1}$$

then we have

$$0 \leq \lambda_1^\alpha \leq a_-(K, \|d\alpha\|_\infty, n) \qquad \textit{and} \qquad \lambda_2^\alpha \geq a_+(K, \|d\alpha\|_\infty, n), \tag{2}$$

where

$$a_\pm(K, \|d\alpha\|_\infty, n) = n \cdot \frac{(K - \|d\alpha\|_\infty) \pm \sqrt{(K - \|d\alpha\|_\infty)^2 - 4(\frac{n-1}{n})\|d\alpha\|_\infty^2}}{2(n-1)}.$$

In particular, there is a spectral gap

$$\lambda_2^\alpha - \lambda_1^\alpha \geq \frac{n}{n-1}\sqrt{(K - \|d\alpha\|_\infty)^2 - \frac{4(n-1)}{n}\|d\alpha\|_\infty^2}. \tag{3}$$

Note that when the magnetic potential vanishes or, more generally, when α can be gauged away (for example, when α is exact), the above result reduces to the classical Lichnerowicz estimate. Concerning the lower bound (3) for a gap between the first and second eigenvalue of the magnetic Laplacian, it would be interesting whether this lower bound is assumed for certain choices of manifolds and magnetic potentials and, if so, to characterize these situations. In the classical case of a vanishing magnetic potential, the gap is assumed if and only if the manifold is the round sphere (Obata's Theorem [21]). If the strength of the magnetic field $\|d\alpha\|_\infty$ assumes or exceeds the upper bound given in (1), our lower bound on the eigenvalue gap shrinks to zero and the first eigenvalue may possibly have higher multiplicity. While it would be very interesting to shed more light on these questions, it is difficult to find explicit examples allowing to calculate the spectrum of a Laplacian with non-trivial magnetic potential. At present, we do not have such examples illustrating these possibilities.

Cheeger's isoperimetric constant provides an important geometric lower bound for λ_2 of Δ, which is well-known as Cheeger's inequality [3]. Later, Buser [2] showed an upper bound of λ_2 in terms of Cheeger's constant, with a constant depending on the dimension and the Ricci curvature of M. Ledoux [16] established a dimension-free Buser inequality. Cheeger inequalities were extended to the magnetic Laplacian on a closed Riemannian manifold in [15]. In particular, a *k-way Cheeger type constant h_k^α* was introduced for the magnetic Laplacian Δ^α. The constant h_k^α is based on a mixture of the classical isoperimetric area/volume ratios of domains $\Omega \subset M$ and the *frustration index* of the magnetic potential α. The frustration index measures, in some sense, the non-triviality of α over Ω. In particular, the frustration index vanishes if and only if α can be gauged away on Ω. The readers can find the precise definitions in Subsection 2.2. It was shown in [15] that $h_k^\alpha \leq Ck^3\sqrt{\lambda_k^\alpha}$ for some absolute dimension-independent constant $C > 0$.

Building upon the Bochner type formula in Theorem 4.1 and techniques developed in Ledoux [16] and in [19], we prove the following upper estimate for λ_k^α in the case of $d\alpha = 0$ in terms of h_k^α and a time parameter restricted by the the lower Riccci curvature bound $-K$. Note that a potential α satisfying $d\alpha = 0$ can still be non-trivial (see Example 6.1 and Remark 6.2 for an explanation).

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold with a magnetic potential α such that $d\alpha = 0$. Let $-K$, $K \geq 0$, be a lower bound of the Ricci curvature of M. Then for any $0 \leq t \leq \frac{1}{2K}$ and any $k \in \mathbb{N}$, we have*

$$2\sqrt{t} \cdot h_k^\alpha \geq \frac{1}{k} - e^{-t\lambda_k^\alpha}. \tag{4}$$

In the special case $K = 0$ in Theorem 1.2, non-trivial magnetic potentials (i.e., potentials which cannot be gauged away) exist only in the case of *Ricci-flat manifolds*, due to the classical Bochner vanishing theorem for the first cohomology (see, e.g., [14, Thm. 3.5.1] and the remark thereafter). Note, however, in contrast to Theorem 1.1, that Theorem 1.2 holds also for Riemannian manifolds with negative Ricci curvature.

Theorem 1.2 can be considered as a higher order Buser inequality for every order $k \in \mathbb{N}$. In particular, when $k = 1$, it implies immediately the following (dimension-free) Buser type inequality.

Theorem 1.3 (Magnetic Buser inequality)**.** *Let (M, g) be a closed Riemannian manifold with a magnetic potential α such that $d\alpha = 0$. Let $-K$, $K \geq 0$ be a lower bound of the Ricci curvature of M. Then we have*

$$\lambda_1^\alpha \leq \max\left\{4\sqrt{2K}h_1^\alpha, \frac{4e^2}{(e-1)^2}(h_1^\alpha)^2\right\}. \tag{5}$$

Proof of Theorem 1.3. Theorem 1.2 states that, for any $0 \leq t \leq \frac{1}{2K}$,

$$2\sqrt{t} \cdot h_1^\alpha \geq 1 - e^{-t\lambda_1^\alpha}.$$

When $\lambda_1^\alpha > 0$ and $\lambda_1^\alpha \geq 2K$, we choose $t = \frac{1}{\lambda_1^\alpha}$, and obtain $\sqrt{\lambda_1^\alpha} \leq \frac{2e}{e-1}h_1^\alpha$. If, otherwise, $\lambda_1^\alpha \leq 2K$, we choose $t = \frac{1}{2K}$ and obtain $\lambda_1^\alpha \leq 4\sqrt{2K}h_1^\alpha$. □

The above proof of Theorem 1.3 is an extension of the proof of Ledoux [16, Theorem 5.2], and the techniques to further establish the higher order estimate in Theorem 1.2 is a continuous analogue of the methods developed for discrete graphs in [19, Theorem 5.1]. We also like to mention that higher order Buser type inequalities for the classical Laplacian Δ were first established by Funano [10], and later improved in [18].

This paper is structured as follows: In the next section, we introduce the general setting and review some fundamental facts. In Sections 3 and 4, we establish a Bochner type formula which is the foundation for our eigenvalue estimates. Finally, Theorem 1.1 is proved in Section 5 and Theorem 1.2 is proved in Section 6. Some calculations in this paper are a bit shortened in comparison to our arXiv version [7] to improve their readability.

2 Preliminaries

2.1 Magnetic Laplacian and its spectrum

Let (M, g) be a closed Riemannian manifold of dimension n with Riemannian metric g. By abuse of notation, we denote by $\langle \cdot, \cdot \rangle$ the inner product induced by g on the tangent bundle TM or on the cotangent bundle T^*M. We extend the inner product $\langle \cdot, \cdot \rangle$ as a Hermitian inner product on the complexified tangent bundle $TM \otimes \mathbb{C}$ or on the complexified cotangent bundle $T^*M \otimes \mathbb{C}$. We will still use the same notation $\langle \cdot, \cdot \rangle$.

Let α be a smooth real differential 1-form on M. Given a function $f \in C^\infty(M, \mathbb{C})$, the operator d^α

$$d^\alpha f := df + if\alpha, \tag{6}$$

maps f to a smooth complex valued 1-form on M. The magnetic Laplacian Δ^α is defined as

$$\Delta^\alpha := (d^\alpha)^* d^\alpha, \tag{7}$$

where $(d^\alpha)^*$ is the formal adjoint of d^α with respect to the L^2-inner product of functions and 1-forms. That is, for any $f \in C^\infty(M, \mathbb{C})$ and any smooth complex valued 1-form η, we have

$$\int_M \langle d^\alpha f, \eta \rangle \, \mathrm{dvol} = \int_M f \overline{(d^\alpha)^* \eta} \, \mathrm{dvol} \,. \tag{8}$$

There is a natural one-to-one correspondence from $T^*M \otimes \mathbb{C}$ to $TM \otimes \mathbb{C}$ via the musical isomorphism $\sharp : T^*M \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ such that

$$w(X) = \langle X, \overline{w^\sharp} \rangle, \qquad \forall\, X \in TM \otimes \mathbb{C}, \;\; \forall\, w \in T^*M \otimes \mathbb{C}. \tag{9}$$

Using the musical isomorphism $\sharp$, we have the following natural definitions.

Definition 2.1. Let $f \in C^\infty(M, \mathbb{C})$. We define the *magnetic gradient* $\mathrm{grad}^\alpha f$ of f as

$$\mathrm{grad}^\alpha f := (d^\alpha f)^\sharp = \mathrm{grad}\, f + if\alpha^\sharp. \tag{10}$$

We define the *magnetic divergence* $\mathrm{div}^\alpha X$ of a vector field $X \in C^\infty(TM \otimes \mathbb{C})$ as

$$\mathrm{div}^\alpha X := \mathrm{div}\, X + i\langle X, \alpha^\sharp \rangle. \tag{11}$$

It is straightforward to check that $-\operatorname{div}^\alpha$ is the formal adjoint operator of $\operatorname{grad}^\alpha$. In fact,

$$\int_M \langle \operatorname{grad}^\alpha f, X\rangle \,\mathrm{dvol} = -\int_M f\overline{\operatorname{div} X + i\langle X, \alpha^\sharp\rangle}\,\mathrm{dvol}. \tag{12}$$

Proposition 2.2. *For all $f \in C^\infty(M,\mathbb{C})$, we have*

$$\Delta^\alpha f = -\operatorname{div}^\alpha \operatorname{grad}^\alpha f = \Delta f - 2i\langle \operatorname{grad} f, \alpha^\sharp\rangle + f(-i\operatorname{div}\alpha^\sharp + |\alpha^\sharp|^2), \tag{13}$$

where $\Delta := -\operatorname{div}\operatorname{grad}$ is the Laplace-Beltrami operator, and $|\alpha^\sharp|^2 := \langle \alpha^\sharp, \alpha^\sharp\rangle$.

Proof. By (8) and (12), we have for any smooth complex valued 1-form η,

$$(d^\alpha)^*\eta = -\operatorname{div}^\alpha \eta^\sharp.$$

Recalling (7), this leads to

$$\Delta^\alpha f = -\operatorname{div}^\alpha (d^\alpha f)^\sharp = -\operatorname{div}^\alpha \operatorname{grad}^\alpha f.$$

Expanding the above formula using (10) and (11) , we obtain

$$\Delta^\alpha f = \Delta f - 2i\langle \operatorname{grad} f, \alpha^\sharp\rangle + f(-i\operatorname{div}\alpha^\sharp + |\alpha^\sharp|^2). \tag{14}$$

□

We now recall basic spectral properties of the magnetic Laplacian (see, e.g., [23], [22]). Let $L^2(M,\mathbb{C})$ be the set of all complex valued square integrable functions with respect to the Riemannian volume measure. The densely defined operator Δ^α on $L^2(M,\mathbb{C})$ is essentially self-adjoint. In the sequel, we consider the self-adjoint extension of Δ^α and still denote it by Δ^α. The operator Δ^α has only discrete spectrum, and we list its eigenvalues with multiplicity as follows ([23, Theorem 2.1]):

$$0 \le \lambda_1^\alpha \le \lambda_2^\alpha \le \cdots \nearrow \infty. \tag{15}$$

Similarly, we list the eigenvalues of Δ with multiplicity as

$$0 = \lambda_1 < \lambda_2 \le \cdots \nearrow \infty. \tag{16}$$

As already mentioned in the introduction, the first eigenvalue λ_1 of Δ is zero and has multiplicity 1. However, the first eigenvalue λ_1^α of Δ^α can be positive and can have larger multiplicity. This can be seen explicitly in the following example, which was discussed in [23, Example 1].

Example 2.3. Let $S^1_L = \mathbb{R}/L\mathbb{Z}$ be the circle of length L. We consider the 1-form $\alpha := Adx$ with $A \in \mathbb{R}$. Then, for any $f \in C^\infty(S^1_L, \mathbb{C})$, we have

$$\Delta^\alpha f = -f'' - 2iAf' + A^2 f,$$

and we have, for all $k \in \mathbb{Z}$,

$$\Delta^\alpha e^{i\frac{2\pi k}{L}x} = \left(\frac{2\pi k}{L} + A\right)^2 e^{i\frac{2\pi k}{L}x}.$$

Since $\{e^{i\frac{2\pi k}{L}x} : k \in \mathbb{Z}\}$ is a Hilbert basis of $L^2(S^1_L, \mathbb{C})$, the spectrum of Δ^α is given by

$$\sigma(\Delta^\alpha) = \left\{ \left(\frac{2\pi k}{L} + A\right)^2 : k \in \mathbb{Z} \right\}.$$

In particular, we have $\lambda^\alpha_1 > 0$ for any choice of $A \notin \{2\pi k/L : k \in \mathbb{Z}\}$. In the case of $A = -\pi/L$, we have $\lambda^\alpha_1 = \pi^2/L^2$, whose eigenfunctions are 1 and $e^{i\frac{2\pi}{L}x}$. Therefore, the first eigenvalue has multiplicity 2.

2.2 Gauge transformation and Cheeger constants

In this subsection, we recall the Cheeger constants for magnetic Laplacians introduced in [15].

Let $U(1) := \{z \in \mathbb{C} : z\overline{z} = 1\}$ and $C^\infty(M, U(1))$ be the set of smooth maps from M to $U(1)$. A function $\tau \in C^\infty(M, U(1))$ can thus be viewed as a complex valued function on M, and we can define a smooth 1-form α_τ as follows:

$$\alpha_\tau := \frac{d\tau}{i\tau}. \tag{17}$$

Then every function $\tau \in C^\infty(M, U(1))$ gives rise to a *gauge transformation*

$$\alpha \mapsto \alpha + \alpha_\tau, \tag{18}$$

and the operators Δ^α and Δ^{α_τ} are unitarily equivalent. In fact, we have ([23, Proposition 3.2])

$$\overline{\tau}\Delta^\alpha \tau = \Delta^{\alpha+\alpha_\tau}. \tag{19}$$

Let $\mathcal{B} := \{\alpha_\tau : \tau \in C^\infty(M, U(1))\}$, that is, $\mathcal{B}$ is the set of magnetic potentials which can be "gauged away". If $\alpha \in \mathcal{B}$, then Δ^α is unitarily equivalent to Δ. The set $\mathcal{B}$ has the following characterization ([23, Proposition 3.1 and Theorem 4.2]):

Theorem 2.4 (Shigekawa). *The following are equivalent:*

(i) $\alpha \in \mathcal{B}$;

(ii) $\lambda_1^\alpha = 0$*;*

(iii) $d\alpha = 0$ *and* $\int_C \alpha \equiv 0$ *(mod* 2π*) for any closed curve* C*.*

Note that we have the following inclusions:

$$\{\text{exact 1-forms}\} \subseteq \mathcal{B} \subseteq \{\text{closed 1-forms}\}.$$

For any nonempty Borel subset $\Omega \subseteq M$, the *frustration index* $\iota^\alpha(\Omega)$ of Ω is defined as ([15, Definition 7.2])

$$\iota^\alpha(\Omega) = \inf_{\tau \in C^\infty(\Omega, U(1))} \int_\Omega |(d + i\alpha)\tau| \, \mathrm{dvol} = \inf_{\eta \in \mathcal{B}_\Omega} \int_\Omega |\eta + \alpha| \, \mathrm{dvol}, \tag{20}$$

where $\mathcal{B}_\Omega := \{\alpha_\tau : \tau \in C^\infty(\Omega, U(1))\}$. Note that the frustration index $\iota^\alpha(M)$ measures, in some sense, the distance of α from the set $\mathcal{B}$.

Definition 2.5 (Cheeger constant [15])**.** Let M be a closed Riemannian manifold with a smooth real differential 1-form α. For any Borel subset Ω of M, we denote

$$\phi^\alpha(\Omega) := \frac{\iota^\alpha(\Omega) + \mathrm{area}(\partial\Omega)}{\mathrm{vol}(\Omega)}, \tag{21}$$

where $\mathrm{vol}(\Omega)$ is the Riemannian volume of Ω. The boundary measure $\mathrm{area}(\partial\Omega)$ is given by

$$\mathrm{area}(\partial\Omega) := \liminf_{r \to 0} \frac{\mathrm{vol}(\Omega_r) - \mathrm{vol}(\Omega)}{r}, \tag{22}$$

where Ω_r is the open r-neighbourhood of Ω. Then the *one-way (magnetic) Cheeger constant* h_1^α is defined as

$$h_1^\alpha := \inf_{\Omega \subseteq M, \mathrm{vol}(\Omega) > 0} \phi^\alpha(\Omega). \tag{23}$$

Moreover, the *k-way (magnetic) Cheeger constant* h_k^α is defined as

$$h_k^\alpha := \inf_{\{\Omega_p\}_{[k]}} \max_{p \in [k]} \phi^\alpha(\Omega_p), \tag{24}$$

where the infimum is taken over all possible k disjoint subsets $\{\Omega_p\}_{[k]}$ with $\mathrm{vol}(\Omega_p) > 0$ for every $p \in [k] := \{1, 2, \dots, k\}$.

All magnetic Cheeger constants are invariant under gauge transformation of the potential α. In particular, when $\alpha \in \mathcal{B}$, h_2^α reduces to the classical Cheeger constant.

The following (higher order) Cheeger type inequalities were proved in [15, Theorems 7.4 and 7.7].

Theorem 2.6 ([15]). *Let α be a smooth real differential 1-form on a closed connected Riemannian manifold M. Then we have*

$$h_1^\alpha \leq 2\sqrt{2\lambda_1^\alpha}. \tag{25}$$

Moreover, there exists an absolute dimension-independent constant $C > 0$, such that for any closed connected Riemannian manifold M with α and $k \in \mathbb{N}$, we have

$$h_k^\alpha \leq Ck^3\sqrt{\lambda_k^\alpha}. \tag{26}$$

Example 2.7 (S_L^1 revisited)**.** Consider the circle S_L^1 with the real differential 1-form $\alpha = Adx$, where $A \in \mathbb{R}$. The set $\mathcal{B}$ of magnetic potentials on S_L^1 which can be gauged away is given by

$$\mathcal{B} = \{f(x)dx : f \in C^\infty([0,L],\mathbb{R}), f(0) = f(L), \int_0^L f(x)dx \equiv 0 \ (\mathrm{mod}\ 2\pi)\}.$$

We show now that the frustration index $\iota^\alpha(S_L^1)$ of S_L^1 is

$$\iota^\alpha(S_L^1) = \min_{k\in\mathbb{Z}} |2k\pi - AL|. \tag{27}$$

Let $k_0 \in \mathbb{Z}$ be the integer attaining the minimum of the expression at the right hand side of (27), and set $A_0 := |2k_0\pi - AL|/L$. Note $A_0 \in [0, \pi/L]$. Then we have $\iota^\alpha(S_L^1) \leq A_0L$ since $\frac{2k\pi}{L}dx \in \mathcal{B}$.

Suppose now that $\iota^\alpha(S_L^1) < A_0L$. Then there exists $f \in C^\infty([0,L],\mathbb{R})$ satisfying

$$f(0) = f(L) \qquad \text{and} \qquad \int_0^L f(x)dx \equiv 0 \ (\mathrm{mod}\ 2\pi), \tag{28}$$

such that

$$\int_0^L |f(x) - A|dx < A_0L.$$

This implies that there exists $f_0 \in C^\infty([0,L],\mathbb{R})$ satisfying (28) such that

$$\int_0^L |f_0(x) - A_0|dx < A_0L. \tag{29}$$

In fact, we can set $f_0 := f - A + A_0$ when $AL \geq 2k_0\pi$ and $f_0 = -f + A - A_0$ when $AL < 2k_0\pi$. Then we have, by the triangle inequality,

$$A_0L > \int_0^L |f_0(x) - A_0|dx \geq \int_0^L |f_0(x)|dx - A_0L \geq \left|\int_0^L f_0(x)dx\right| - A_0L.$$

Since $2A_0L \leq 2\pi$, we must have

$$\int_0^L f_0(x)dx = 0, \tag{30}$$

by (28). Finally, (30) implies

$$\int_0^L |f_0(x) - A_0| dx \geq \left| \int_0^L (f_0(x) - A_0) dx \right| = A_0 L.$$

which contradicts (29). This proves (27).

On the other hand, for any proper subinterval $\Omega \subset S^1_L$, we have $\iota^\alpha(\Omega) = 0$. Therefore, by definition, the one-way magnetic Cheeger constant of S^1_L is

$$h_1^\alpha = \min \left\{ \frac{2}{L}, \min_{k \in \mathbb{Z}} \left| \frac{2k\pi}{L} - A \right| \right\}. \tag{31}$$

3 Commutator formulae

In this section, we derive the commutator formulae for the second order magnetic covariant derivative and the magnetic Hessian (see Definitions 3.3 and 3.5 below). They are particularly useful in the next section for the derivation of a Bochner type formula.

Since the divergence is the trace of the Levi-Civita connection on M, we have

$$\operatorname{div}^\alpha X = \sum_{j=1}^n \langle \nabla_{e_j} X, e_j \rangle + i \langle \sum_{j=1}^n \langle X, e_j \rangle e_j, \alpha^\sharp \rangle = \sum_{j=1}^n \langle \nabla_{e_j} X + i\alpha(e_j) X, e_j \rangle. \tag{32}$$

This suggests the following definition of magnetic covariant derivative.

Definition 3.1. Let $X, Y \in C^\infty(TM \otimes \mathbb{C})$. We define the *magnetic covariant derivative* of X with respect to Y as

$$\nabla^\alpha_Y X := \nabla_Y X + i\alpha(Y) X. \tag{33}$$

Note that both the magnetic divergence and the magnetic covariant derivative are complex linear operators in all entries.

A direct calculation using Definition 3.1 leads to the following lemma.

Lemma 3.2. *For all $X, Y, Z \in C^\infty(TM \otimes \mathbb{C})$ and for all $f \in C^\infty(M, \mathbb{C})$, we have the following properties:*

(i) (Riemannian property)

$$Z(\langle X, Y \rangle) = \langle \nabla^\alpha_Z X, Y \rangle + \langle X, \nabla^\alpha_{\overline{Z}} Y \rangle; \tag{34}$$

(ii) (Leibniz rule)

$$\nabla^\alpha_Y (fX) = Y(f) X + f \nabla^\alpha_Y X; \tag{35}$$

(iii)

$$\nabla^\alpha_X Y - \nabla^\alpha_Y X = [X, Y] + i(\alpha(X) Y - \alpha(Y) X). \tag{36}$$

Similarly to the classical case, we define the second order magnetic covariant derivative.

Definition 3.3. For all vector fields $X, Y, Z \in C^\infty(TM \otimes \mathbb{C})$, the *second order magnetic covariant derivative* is the operator

$$\nabla^\alpha_{X,Y} Z := \nabla^\alpha_X(\nabla^\alpha_Y Z) - \nabla^\alpha_{\nabla^\alpha_X Y} Z. \tag{37}$$

We now present a commutator formula that links the Riemannian curvature tensor with the second order magnetic covariant derivative.

For vector fields $U, V, W \in C^\infty(TM \otimes \mathbb{C})$, we extend the Riemannian curvature tensor as

$$R(U,V)W = \nabla_U \nabla_V \overline{W} - \nabla_V \nabla_U \overline{W} - \nabla_{[U,V]} \overline{W}, \tag{38}$$

such that R is complex linear in the first and second entry and complex anti-linear in the third entry. This implies that, for any $U, V \in C^\infty(TM \otimes \mathbb{C})$, $\langle R(U,V)V, U\rangle$ is real valued.

Lemma 3.4. *For all $X, Y, Z \in C^\infty(TM \otimes \mathbb{C})$, we have*

$$\nabla^\alpha_{X,Y} Z - \nabla^\alpha_{Y,X} Z = R(X,Y)\overline{Z} + i d\alpha(X,Y)Z - i\nabla_{\alpha(X)Y - \alpha(Y)X} Z. \tag{39}$$

Proof. Let us calculate explicitly the term $\nabla^\alpha_{X,Y} Z$ first. Applying (37), (33) and the Leibniz rule in Lemma 3.2, we check that

$$\begin{aligned}\nabla^\alpha_{X,Y} Z =& \nabla_X(\nabla_Y Z) + i\nabla_X(\alpha(Y)Z) - \nabla_{\nabla_X Y} Z - i\alpha(\nabla_X Y)Z \\ =& \nabla_{X,Y} Z + i\alpha(Y)\nabla_X Z + iD\alpha(X;Y)Z,\end{aligned} \tag{40}$$

where

$$D\alpha(X;Y) := X(\alpha(Y)) - \alpha(\nabla_X Y).$$

Recall that we have (see, e.g., [17, p. 366])

$$\begin{aligned}D\alpha(X;Y) - D\alpha(Y;X) &= X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \\ &= d\alpha(X,Y),\end{aligned} \tag{41}$$

and

$$\nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = R(X,Y)\overline{Z}. \tag{42}$$

Now (40), (41), and (42) together imply this lemma. □

We proceed with one last definition.

Definition 3.5. We define the *magnetic Hessian* by

$$\mathrm{Hess}^\alpha f(X,Y) := \langle \nabla^\alpha_X \operatorname{grad}^\alpha f, Y\rangle, \tag{43}$$

for all functions $f \in C^\infty(M, \mathbb{C})$ and for all vector fields $X, Y \in C^\infty(TM \otimes \mathbb{C})$.

The magnetic Hessian is not Hermitian as in the classical case. In fact, we have the following commutator formula.

Lemma 3.6. *For all $X, Y \in C^\infty(TM \otimes \mathbb{C})$ and $f \in C^\infty(M, \mathbb{C})$, we have*

$$\operatorname{Hess}^\alpha f(X, \overline{Y}) - \operatorname{Hess}^\alpha f(Y, \overline{X}) = i f d\alpha(X, Y). \tag{44}$$

Proof. We first calculate $\langle \nabla^\alpha_X \operatorname{grad}^\alpha f, \overline{Y} \rangle$ explicitly. Using the Riemannian property (34), and definitions (10) and (33), we obtain

$$\begin{aligned} \langle \nabla^\alpha_X \operatorname{grad}^\alpha f, \overline{Y} \rangle =& X(\langle \operatorname{grad} f, \overline{Y} \rangle) - \langle \operatorname{grad} f, \nabla_{\overline{X}} \overline{Y} \rangle \\ &+ iX(f\alpha(Y)) + i\alpha(X)Y(f) - if\alpha(\nabla_X Y) - f\alpha(X)\alpha(Y) \\ =& \langle \nabla_X \operatorname{grad} f, \overline{Y} \rangle + iX(f)\alpha(Y) + iY(f)\alpha(X) - f\alpha(X)\alpha(Y) \\ &+ ifD\alpha(X; Y). \end{aligned} \tag{45}$$

Observe that the entries in the first line of the last equality in (45) are symmetric w.r.t. X and Y. In particular, we have

$$\langle \nabla_X \operatorname{grad} f, \overline{Y} \rangle = \operatorname{Hess} f(X, \overline{Y}) = \operatorname{Hess} f(Y, \overline{X}).$$

Therefore, we conclude

$$\operatorname{Hess}^\alpha f(X, \overline{Y}) - \operatorname{Hess}^\alpha f(Y, \overline{X}) = if\left(D\alpha(X; Y) - D\alpha(Y; X)\right). \tag{46}$$

Recalling (41), we finish the proof. $\square$

4 A Bochner Type Formula for the Magnetic Laplacian

We first recall that the Hilbert-Schmidt norm of the magnetic Hessian of a function $f \in C^\infty(M, \mathbb{C})$ is

$$|\operatorname{Hess}^\alpha f|^2 = \sum_{i=1}^n |\nabla^\alpha_{e_i} \operatorname{grad}^\alpha f|^2, \tag{47}$$

where $e_1, \ldots, e_n$ is an orthonormal real basis of T_pM. In fact,

$$\begin{aligned} |\operatorname{Hess}^\alpha f|^2 &= \sum_{i,j=1}^n |\operatorname{Hess}^\alpha f(e_i, e_j)|^2 = \sum_{i,j=1}^n \langle \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, e_j \rangle \langle e_j, \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f \rangle \\ &= \sum_{i=1}^n \langle \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, \sum_{j=1}^n \langle \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, e_j \rangle e_j \rangle = \sum_{i=1}^n \langle \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f \rangle. \end{aligned}$$

Theorem 4.1 (Bochner type formula). *Let (M,g) be a complete Riemannian manifold of dimension n. Then, for all $f \in C^\infty(M,\mathbb{C})$, we have*

$$\begin{aligned}
-\frac{1}{2}\Delta(|\operatorname{grad}^\alpha f|^2) = |\operatorname{Hess}^\alpha f|^2 - \frac{1}{2}\big(\langle \operatorname{grad}^\alpha f, \operatorname{grad}^\alpha(\Delta^\alpha f)\rangle + \langle \operatorname{grad}^\alpha(\Delta^\alpha f), \operatorname{grad}^\alpha f\rangle\big) \\
+ \operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f) + i\big(d\alpha(\operatorname{grad}^\alpha f, \overline{\operatorname{grad}^\alpha f}) - d\alpha(\overline{\operatorname{grad}^\alpha f}, \operatorname{grad}^\alpha f)\big) \\
+ \frac{i}{2}\big(\langle \overline{f}\operatorname{grad}^\alpha f, (\delta d\alpha)^\sharp\rangle - \langle f\overline{\operatorname{grad}^\alpha f}, (\delta d\alpha)^\sharp\rangle\big), \quad (48)
\end{aligned}$$

where δ denotes the formal adjoint of the exterior derivative on (M,g).

Proof. Let $p \in M$ and consider a normal real basis $e_1,\dots,e_n$ at p, i.e., $|e_i|^2 = 1$ and $\nabla_{e_i} e_j = 0$ for all $i,j = 1,\dots,n$.

Using the Riemannian property of ∇ and ∇^α and the definition of magnetic Hessian, we calculate

$$\begin{aligned}
-\frac{1}{2}\Delta(|\operatorname{grad}^\alpha f|^2) &= \frac{1}{2}\operatorname{tr}\operatorname{Hess}(|\operatorname{grad}^\alpha f|^2) = \frac{1}{2}\sum_{i=1}^n \langle \nabla_{e_i}\operatorname{grad}(|\operatorname{grad}^\alpha f|^2), e_i\rangle \\
&= \frac{1}{2}\sum_{i=1}^n \Big[e_i\Big(\langle \operatorname{grad}(|\operatorname{grad}^\alpha f|^2), e_i\rangle\Big) - \langle \operatorname{grad}(|\operatorname{grad}^\alpha f|^2), \underbrace{\nabla_{e_i} e_i}_{=0}\rangle\Big] \\
&= \frac{1}{2}\sum_{i=1}^n e_i\Big(\langle \nabla^\alpha_{e_i}\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f\rangle + \langle \operatorname{grad}^\alpha f, \nabla^\alpha_{e_i}\operatorname{grad}^\alpha f\rangle\Big) \\
&= \frac{1}{2}\sum_{i=1}^n \Big[e_i\big(\operatorname{Hess}^\alpha f(e_i, \operatorname{grad}^\alpha f) + \overline{\operatorname{Hess}^\alpha f(e_i, \operatorname{grad}^\alpha f)}\big)\Big]. \quad (49)
\end{aligned}$$

It is now sufficient to analyse the first summand $\frac{1}{2}\sum_{i=1}^n e_i(\operatorname{Hess}^\alpha f(e_i, \operatorname{grad}^\alpha f))$, as the second one will directly give us its conjugate.

Using Lemma 3.6, (43), the Riemannian property (34), the fact $\nabla^\alpha_{e_i} e_i = i\alpha(e_i)e_i$ (since

$\nabla_{e_i} e_i = 0$), and the expansion of $\alpha^\sharp$ and $\operatorname{grad} f$ w.r.t. to the basis, we have

$$
\begin{aligned}
\frac{1}{2}\sum_{i=1}^n e_i(\operatorname{Hess}^\alpha f(e_i, \operatorname{grad}^\alpha f)) &= \frac{1}{2}\sum_{i=1}^n e_i\Big(\operatorname{Hess}^\alpha f(\overline{\operatorname{grad}^\alpha f}, e_i) + i f d\alpha(e_i, \overline{\operatorname{grad}^\alpha f})\Big) \\
&= \frac{1}{2}\sum_{i=1}^n \Big[\langle \nabla^\alpha_{e_i} \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, e_i\rangle + \langle \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, \nabla^\alpha_{e_i} e_i\rangle\Big] \\
&\qquad + \frac{i}{2}\sum_{i=1}^n \big[e_i(f) d\alpha(e_i, \overline{\operatorname{grad}^\alpha f}) + f e_i(d\alpha(e_i, \overline{\operatorname{grad}^\alpha f}))\big] \\
&= \left[\frac{1}{2}\sum_{i=1}^n \langle \nabla^\alpha_{e_i} \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, e_i\rangle\right] - \frac{i}{2}\langle \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, \alpha^\sharp\rangle \\
&\qquad + \frac{i}{2} d\alpha(\operatorname{grad} f, \overline{\operatorname{grad}^\alpha f}) + \frac{i}{2}\sum_{i=1}^n f e_i(d\alpha(e_i, \overline{\operatorname{grad}^\alpha f})). \qquad (50)
\end{aligned}
$$

The aim of the following calculations is to rewrite the first term of the RHS of (50) as the expression (55) below, involving $|\operatorname{Hess}^\alpha f|$ and Ricci curvature. We start by using the definition of magnetic second covariant derivative and Lemma 3.4 to bring the curvature tensor into the game:

$$
\begin{aligned}
\frac{1}{2}\sum_{i=1}^n \langle \nabla^\alpha_{e_i} \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, e_i\rangle &= \frac{1}{2}\sum_{i=1}^n \Big[\langle \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}, e_i} \operatorname{grad}^\alpha f, e_i\rangle \\
&\quad + \langle R(e_i, \overline{\operatorname{grad}^\alpha f})\overline{\operatorname{grad}^\alpha f}, e_i\rangle + i\langle d\alpha(e_i, \overline{\operatorname{grad}^\alpha f}) \operatorname{grad}^\alpha f, e_i\rangle \\
&\quad + \langle \nabla_{i\alpha(\overline{\operatorname{grad}^\alpha f})e_i - i\alpha(e_i)\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, e_i\rangle + \langle \nabla^\alpha_{\nabla^\alpha_{e_i}\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, e_i\rangle\Big]. \qquad (51)
\end{aligned}
$$

Undoing the magnetic second covariant derivative and using equation (36) leads then to

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^n \langle \nabla^\alpha_{e_i} \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \operatorname{grad}^\alpha f, e_i\rangle \\
&= \frac{1}{2}\operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f) + \frac{i}{2} d\alpha(\operatorname{grad}^\alpha f, \overline{\operatorname{grad}^\alpha f}) \\
&\qquad + \frac{1}{2}\sum_{i=1}^n \langle \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}} \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, e_i\rangle + \frac{1}{2}\sum_{i=1}^n \langle \nabla^\alpha_{[e_i, \overline{\operatorname{grad}^\alpha f}]} \operatorname{grad}^\alpha f, e_i\rangle \qquad (52)
\end{aligned}
$$

We now compute the last two terms of the RHS of (52). Using the Riemannian property of the magnetic covariant derivative, the fact that $\nabla^\alpha_{\operatorname{grad}^\alpha f} e_i = i\alpha(\operatorname{grad}^\alpha f) e_i$

due to choice of the basis, and the definition of magnetic gradient, we have

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}\langle\nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}}\nabla^\alpha_{e_i}\operatorname{grad}^\alpha f, e_i\rangle &= \frac{1}{2}\sum_{i=1}^{n}\left[\overline{\operatorname{grad}^\alpha f}(\langle\nabla^\alpha_{e_i}\operatorname{grad}^\alpha f, e_i\rangle) - \langle\nabla^\alpha_{e_i}\operatorname{grad}^\alpha f, \nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}}e_i\rangle\right]\\ &= -\frac{1}{2}\overline{\operatorname{grad}^\alpha f}(\Delta^\alpha f) + \frac{i}{2}\alpha(\overline{\operatorname{grad}^\alpha f})\Delta^\alpha f\\ &= -\frac{1}{2}\langle\operatorname{grad}^\alpha(\Delta^\alpha f), \operatorname{grad}^\alpha f\rangle. \end{aligned} \tag{53}$$

Then, using the fact that $\nabla_{\overline{\operatorname{grad}^\alpha f}}e_i = 0$ (due to the choice of the basis), Lemma 3.6 and (43), (47), and the representation of $\alpha^\sharp$ according to the basis, we have

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}\langle\nabla^\alpha_{[e_i,\overline{\operatorname{grad}^\alpha f}]}\operatorname{grad}^\alpha f, e_i\rangle = \frac{1}{2}\sum_{i=1}^{n}\langle\nabla^\alpha_{\nabla_{e_i}\overline{\operatorname{grad}^\alpha f}}\operatorname{grad}^\alpha f, e_i\rangle = \frac{1}{2}\sum_{i=1}^{n}\operatorname{Hess}^\alpha f(\nabla_{e_i}\overline{\operatorname{grad}^\alpha f}, e_i)\\ &= \frac{1}{2}\sum_{i=1}^{n}\left[\langle\nabla^\alpha_{e_i}\operatorname{grad}^\alpha f, \nabla^\alpha_{e_i}\operatorname{grad}^\alpha f - i\alpha(e_i)\operatorname{grad}^\alpha f\rangle + ifd\alpha(\nabla_{e_i}\overline{\operatorname{grad}^\alpha f}, e_i)\right]\\ &= \frac{1}{2}|\operatorname{Hess}^\alpha f|^2 + \frac{i}{2}\langle\nabla^\alpha_{\alpha^\sharp}\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f\rangle + \frac{i}{2}\sum_{i=1}^{n}fd\alpha(\nabla_{e_i}\overline{\operatorname{grad}^\alpha f}, e_i). \end{aligned} \tag{54}$$

Substituting equations (53), (54) into (52), we obtain

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}\langle\nabla^\alpha_{e_i}\nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}}\operatorname{grad}^\alpha f, e_i\rangle &= \frac{1}{2}\operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f) - \frac{1}{2}\langle\operatorname{grad}^\alpha(\Delta^\alpha f), \operatorname{grad}^\alpha f\rangle\\ &+ \frac{1}{2}|\operatorname{Hess}^\alpha f|^2 + \frac{i}{2}\langle\nabla^\alpha_{\alpha^\sharp}\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f\rangle\\ &+ \frac{i}{2}\left[\sum_{i=1}^{n}fd\alpha(\nabla_{e_i}\overline{\operatorname{grad}^\alpha f}, e_i)\right] + \frac{i}{2}d\alpha(\operatorname{grad}^\alpha f, \overline{\operatorname{grad}^\alpha f}). \end{aligned} \tag{55}$$

Consequently, substituting (55) into (50), we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}e_i(\operatorname{Hess}^\alpha f(e_i, \operatorname{grad}^\alpha f)) = \frac{1}{2}\operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f) - \frac{1}{2}\langle\operatorname{grad}^\alpha(\Delta^\alpha f), \operatorname{grad}^\alpha f\rangle\\ &+ \frac{1}{2}|\operatorname{Hess}^\alpha f|^2 + \frac{i}{2}d\alpha(\operatorname{grad}^\alpha f, \overline{\operatorname{grad}^\alpha f}) + \frac{i}{2}\sum_{i=1}^{n}f\left[d\alpha(\nabla_{e_i}\overline{\operatorname{grad}^\alpha f}, e_i) + e_i(d\alpha(e_i, \overline{\operatorname{grad}^\alpha f})\right]\\ &+ \frac{i}{2}\langle\nabla^\alpha_{\alpha^\sharp}\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f\rangle - \frac{i}{2}\langle\nabla^\alpha_{\overline{\operatorname{grad}^\alpha f}}\operatorname{grad}^\alpha f, \alpha^\sharp\rangle + \frac{i}{2}d\alpha(\operatorname{grad} f, \overline{\operatorname{grad}^\alpha f}). \end{aligned} \tag{56}$$

We now combine the last three terms using Lemma 3.6 and the definitions of magnetic

Hessian and magnetic gradient.

$$\begin{aligned}\frac{i}{2}\langle\nabla^{\alpha}_{\alpha^\sharp}\operatorname{grad}^\alpha f,\operatorname{grad}^\alpha f\rangle-\frac{i}{2}\alpha(\nabla^{\alpha}_{\overline{\operatorname{grad}^\alpha f}}\operatorname{grad}^\alpha f)+\frac{i}{2}d\alpha(\operatorname{grad} f,\overline{\operatorname{grad}^\alpha f})\\ =\frac{i}{2}d\alpha(\operatorname{grad} f+if\alpha^\sharp,\overline{\operatorname{grad}^\alpha f})=\frac{i}{2}d\alpha(\operatorname{grad}^\alpha f,\overline{\operatorname{grad}^\alpha f}).\end{aligned}\tag{57}$$

Moreover, the terms involving the sum give

$$\begin{aligned}&\frac{i}{2}\sum_{i=1}^{n} f\big[e_i(d\alpha(e_i,\overline{\operatorname{grad}^\alpha f}))+d\alpha(\nabla_{e_i}\overline{\operatorname{grad}^\alpha f},e_i)\big]\\ &\quad=\frac{i}{2}f\sum_{i=1}^{n}\big[e_i(d\alpha(e_i,\overline{\operatorname{grad}^\alpha f}))-d\alpha(e_i,\nabla_{e_i}\overline{\operatorname{grad}^\alpha f})-d\alpha(\nabla_{e_i}e_i,\overline{\operatorname{grad}^\alpha f})\big]\\ &\quad=\frac{i}{2}f\sum_{i=1}^{n}(\nabla_{e_i}d\alpha)(e_i,\overline{\operatorname{grad}^\alpha f})=-\frac{i}{2}f\delta d\alpha(\overline{\operatorname{grad}^\alpha f}).\end{aligned}\tag{58}$$

For the last equality of (58), see, e.g., [17, Def. 13.155 and Eq. (13.11)].

Substituting (57) and (58) into (56) we have

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n} e_i(\operatorname{Hess}^\alpha f(e_i,\operatorname{grad}^\alpha f))=\frac{1}{2}\operatorname{Ric}(\operatorname{grad}^\alpha f,\operatorname{grad}^\alpha f)-\frac{1}{2}\langle\operatorname{grad}^\alpha(\Delta^\alpha f),\operatorname{grad}^\alpha f\rangle\\ +\frac{1}{2}|\operatorname{Hess}^\alpha f|^2+id\alpha(\operatorname{grad}^\alpha f,\overline{\operatorname{grad}^\alpha f})-\frac{i}{2}f\langle\overline{\operatorname{grad}^\alpha f},(\delta d\alpha)^\sharp\rangle.\end{aligned}\tag{59}$$

Finally, summing the above with its conjugate and substituting into (49), we conclude

$$\begin{aligned}-\frac{1}{2}\Delta(|\operatorname{grad}^\alpha f|^2)=&-\frac{1}{2}\big(\langle\operatorname{grad}^\alpha(\Delta^\alpha f),\operatorname{grad}^\alpha f\rangle+\langle\operatorname{grad}^\alpha f,\operatorname{grad}^\alpha(\Delta^\sharp f)\rangle\big)\\ &+\operatorname{Ric}(\operatorname{grad}^\alpha f,\operatorname{grad}^\alpha f)+|\operatorname{Hess}^\alpha f|^2\\ &+i\big(d\alpha(\operatorname{grad}^\alpha f,\overline{\operatorname{grad}^\alpha f})-d\alpha(\overline{\operatorname{grad}^\alpha f},\operatorname{grad}^\alpha f)\big)\\ &+\frac{i}{2}\big(\langle\overline{f}\operatorname{grad}^\alpha f,(\delta d\alpha)^\sharp\rangle-\langle f\overline{\operatorname{grad}^\alpha f},(\delta d\alpha)^\sharp\rangle\big).\end{aligned}$$

□

We now derive an integrated version of the Bochner type formula.

Corollary 4.2. *Let M be a closed Riemannian manifold of dimension n. Then, for all $f\in C^\infty(M,\mathbb{C})$ we have*

$$\begin{aligned}\int_M|\operatorname{Hess}^\alpha f|^2\,\mathrm{dvol}+\int_M\operatorname{Ric}(\operatorname{grad}^\alpha f,\operatorname{grad}^\alpha f)\,\mathrm{dvol}\\ -\int_M\Re(\langle\operatorname{grad}^\alpha(\Delta^\alpha f),\operatorname{grad}^\alpha f\rangle)\,\mathrm{dvol}-\int_M|f|^2|d\alpha|^2\,\mathrm{dvol}\\ +\int_M\Re(id\alpha(\operatorname{grad}^\alpha f,\overline{\operatorname{grad}^\alpha f}))\,\mathrm{dvol}=0,\end{aligned}\tag{60}$$

where $\Re(\cdot)$ stands for the real part of the corresponding complex number.

Proof. Since M is closed, the LHS of the Bochner Formula (48) is zero under integration. Furthermore, we calculate

$$\int_M \frac{i}{2}\langle \overline{f}\operatorname{grad}^\alpha f, (\delta d\alpha)^\sharp\rangle \operatorname{dvol} = \int_M \frac{i}{2}\langle \overline{f} d^\alpha f, \delta d\alpha\rangle \operatorname{dvol} = \int_M \frac{i}{2}\langle d(\overline{f} d^\alpha f), d\alpha\rangle \operatorname{dvol},$$

where

$$\begin{aligned} d(\overline{f} d^\alpha f) &= d\overline{f} \wedge d^\alpha f + \overline{f} d(d^\alpha f) \\ &= d\overline{f} \wedge df + i f d\overline{f} \wedge \alpha + i\overline{f} df \wedge \alpha + i|f|^2 d\alpha \\ &= \overline{d^\alpha f} \wedge d^\alpha f + i|f|^2 d\alpha. \end{aligned}$$

That is, we have

$$\begin{aligned} \int_M \frac{i}{2}\langle \overline{f}\operatorname{grad}^\alpha f, (\delta d\alpha)^\sharp\rangle \operatorname{dvol} &= \int_M \frac{i}{2}\langle \overline{d^\alpha f} \wedge d^\alpha f, d\alpha\rangle \operatorname{dvol} - \int_M \frac{1}{2}|f|^2 |d\alpha|^2 \operatorname{dvol} \\ &= \int_M \frac{i}{2} d\alpha(\overline{\operatorname{grad}^\alpha f}, \operatorname{grad}^\alpha f) \operatorname{dvol} - \frac{1}{2}\int_M |f|^2 |d\alpha|^2 \operatorname{dvol}, \end{aligned} \tag{61}$$

and similarly for its conjugate. Therefore, integrating formula (48), we prove (60). $\square$

5 Lichnerowicz type estimates

In this section, we prove Theorem 1.1, namely an upper bound for λ_1^α and a lower bound for λ_2^α and a spectral gap between them in the case of a positive lower Ricci curvature bound K and small $\|d\alpha\|_\infty$.

Proof of Theorem 1.1. Let f be a normalized eigenfunction relative to λ_1^α, i.e. $\Delta^\alpha f = \lambda_1^\alpha f$. Then, $\int_M |\operatorname{grad}^\alpha f|^2 \operatorname{dvol} = \lambda_1^\alpha$, and Corollary 4.2 simplifies to

$$\begin{aligned} \int_M |\operatorname{Hess}^\alpha f|^2 \operatorname{dvol} + \int_M \operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f) \operatorname{dvol} - (\lambda_1^\alpha)^2 \\ - \int_M |f|^2 |d\alpha|^2 \operatorname{dvol} + \int_M \Re(i d\alpha(\operatorname{grad}^\alpha f, \overline{\operatorname{grad}^\alpha f})) \operatorname{dvol} = 0. \end{aligned} \tag{62}$$

We now bound all the terms from below. For an orthonormal basis $e_1, \ldots, e_n$, we have, using the Cauchy-Schwartz inequality,

$$\begin{aligned} |\operatorname{Hess}^\alpha f|^2 &= \sum_{i=1}^n |\nabla^\alpha_{e_i} \operatorname{grad}^\alpha f|^2 \geq \sum_{i=1}^n |\langle \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, e_i\rangle|^2 \\ &\geq \frac{1}{n}|\sum_{i=1}^n \langle \nabla^\alpha_{e_i} \operatorname{grad}^\alpha f, e_i\rangle|^2 = \frac{1}{n}|\Delta^\alpha f|^2, \end{aligned}$$

and therefore

$$\int_M |\operatorname{Hess}^\alpha f|^2 \,\mathrm{dvol} \geq \frac{1}{n}(\lambda_1^\alpha)^2.$$

The curvature condition gives

$$\int_M \operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f) \,\mathrm{dvol} \geq K \int_M |\operatorname{grad}^\alpha f|^2 \,\mathrm{dvol} = K\lambda_1^\alpha.$$

Moreover,

$$-\int_M |f|^2 |d\alpha|^2 \,\mathrm{dvol} \geq -\|d\alpha\|_\infty^2,$$

and

$$\int_M \Re(id\alpha(\operatorname{grad}^\alpha f, \overline{\operatorname{grad}^\alpha f})) \,\mathrm{dvol} \geq -\int_M \|d\alpha\|_\infty |\operatorname{grad}^\alpha f|^2 \,\mathrm{dvol} = -\lambda_1^\alpha \|d\alpha\|_\infty.$$

Substituting all of the above into (62), we obtain

$$\big(1 - \frac{1}{n}\big)(\lambda_1^\alpha)^2 - (K - \|d\alpha\|_\infty)\lambda_1^\alpha + \|d\alpha\|_\infty^2 \geq 0. \tag{63}$$

We now consider the magnetic field $\epsilon\alpha$ with $\epsilon \in [0,1]$. Then, the eigenvalues $\lambda_j^{\epsilon\alpha}$ of the magnetic Laplacian depend continuously on ϵ, and the above inequality becomes

$$\big(1 - \frac{1}{n}\big)(\lambda_1^{\epsilon\alpha})^2 - (K - \|\epsilon d\alpha\|_\infty)\lambda_1^{\epsilon\alpha} + \|\epsilon d\alpha\|_\infty^2 \geq 0. \tag{64}$$

When $\epsilon = \epsilon_0 = 0$, i.e., in absence of magnetic potential, the above inequality reduces to the classical Lichnerowicz Theorem giving the solutions $\lambda_1 = 0$ and $\lambda_2 \geq \frac{nK}{n-1}$. As ϵ starts to increase from $\epsilon_0 = 0$, $\lambda_1^{\epsilon\alpha}$ and $\lambda_2^{\epsilon\alpha}$ vary continuously but are still separated by an interval of positive length, as long as $\|d\alpha\|_\infty \leq \Big(2\sqrt{\frac{n-1}{n}} + 1\Big)^{-1} K$. Namely, inequality (63) gives the solutions

$$0 < \lambda_1^{\epsilon\alpha} \leq \frac{n(K - \|\epsilon d\alpha\|_\infty) - n\sqrt{(K - \|d\alpha\|_\infty)^2 - 4(\frac{n-1}{n})\|d\alpha\|_\infty^2}}{2(n-1)}, \tag{65}$$

$$\lambda_2^{\epsilon\alpha} \geq \frac{n(K - \|\epsilon d\alpha\|_\infty) + n\sqrt{(K - \|d\alpha\|_\infty)^2 - 4(\frac{n-1}{n})\|d\alpha\|_\infty^2}}{2(n-1)}. \tag{66}$$

Consequently, we obtain the spectral gap

$$\lambda_2^{\epsilon\alpha} - \lambda_1^{\epsilon\alpha} \geq \frac{n\sqrt{(K - \|d\alpha\|_\infty)^2 - 4(\frac{n-1}{n})\|d\alpha\|_\infty^2}}{n-1}. \tag{67}$$

□

6 Buser type estimates

In this section, we prove Theorem 1.2, namely the estimate

$$2\sqrt{t} \cdot h_k^\alpha \geq \frac{1}{k} - e^{-t\lambda_k^\alpha}$$

for all $k \in \mathbb{N}$ and $t \in [0, 1/2K]$ in the case of $d\alpha = 0$ and a non-positive lower Ricci curvature bound $-K$. Before we start with the proof, we recall the following example.

Example 6.1 (S_L^1 **revisited).** Consider the circle S_L^1 with the real differential 1-form $\alpha = Adx$, where $A \in \mathbb{R}$. Recall from Examples 2.3 and 2.7 that

$$\lambda_1^\alpha = \min_{k \in \mathbb{Z}} \left(\frac{2\pi k}{L} + A \right)^2, \quad h_1^\alpha = \min \left\{ \frac{2}{L}, \min_{k \in \mathbb{Z}} \left| \frac{2\pi k}{L} - A \right| \right\}.$$

Therefore, in the case $\min_{k \in \mathbb{Z}} \left| \frac{2\pi k}{L} - A \right| \leq \frac{2}{L}$ we have $\lambda_1^\alpha = (h_1^\alpha)^2$.

Remark 6.2. In the above example, we have $d\alpha = 0$. Note that in the case $A \notin \{2\pi k/L, k \in \mathbb{Z}\}$ we have $\alpha \notin \mathcal{B}$, i.e., α cannot be gauged away.

Now we present the proof of Theorem 1.2. First note that, in the case $d\alpha = 0$, the Bochner formula in Theorem 4.1 reduces as follows:

Lemma 6.3. *Let (M, g) be a closed Riemannian manifold with a magnetic potential α such that $d\alpha = 0$. Then, for all $f \in C^\infty(M, \mathbb{C})$, we have*

$$-\frac{1}{2}\Delta(|\operatorname{grad}^\alpha f|^2) = |\operatorname{Hess}^\alpha f|^2 - \frac{1}{2}\big(\langle \operatorname{grad}^\alpha f, \operatorname{grad}^\alpha(\Delta^\alpha f)\rangle + \langle \operatorname{grad}^\alpha(\Delta^\alpha f), \operatorname{grad}^\alpha f\rangle\big) + \operatorname{Ric}(\operatorname{grad}^\alpha f, \operatorname{grad}^\alpha f). \quad (68)$$

Let us denote by $(P_t^\alpha)_{t \geq 0}$ the heat semigroup corresponding to Δ^α. We write $(P_t)_{t \geq 0}$ for the classical heat semigroup.

Lemma 6.4. *Let (M, g) be a complete Riemannian manifold with a magnetic potential α such that $d\alpha = 0$. Let $-K$, $K \geq 0$, be a lower bound of the Ricci curvature of M. Then for any $f \in C^\infty(M, \mathbb{C})$, we have the pointwise inequalities*

(i) $|\operatorname{grad}^\alpha(P_t^\alpha f)|^2 \leq e^{2Kt} P_t(|\operatorname{grad}^\alpha f|^2), \ \forall\, t \geq 0$*;*

(ii) $P_t(|f|^2) - |P_t^\alpha f|^2 \geq \frac{1 - e^{-2Kt}}{K} |\operatorname{grad}^\alpha(P_t^\alpha f)|^2, \ \forall\, t \geq 0$, *where* $\left.\frac{1-e^{-2Kt}}{K}\right|_{K=0} := 2t$*;*

(iii) $\|f - P_t^\alpha f\|_1 \leq 2\sqrt{t} \|\operatorname{grad}^\alpha f\|_1, \ \forall\, 0 \leq t \leq \frac{1}{2K}$.

Remark 6.5. In fact, we will show $\operatorname{Ric} \geq -K \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof. Let f be any smooth complex valued functions on M. For $0 \le s \le t$, we define (at some point $x \in M$, which we suppress for the sake of readability)

$$F(s) := e^{2Ks} P_s(|\operatorname{grad}^\alpha P_{t-s}^\alpha f|^2).$$

Using the facts $\frac{\partial}{\partial s} P_s = -\Delta P_s = -P_s \Delta$ and $\frac{\partial}{\partial s} P_s^\sigma = -\Delta^\alpha P_s^\alpha = -P_s^\alpha \Delta^\alpha$, we calculate

$$\begin{aligned}\frac{d}{ds} F(s) =& 2e^{2Ks} P_s \Bigg(-K|\operatorname{grad}^\alpha(P_{t-s}^\alpha f)| - \frac{1}{2}\Delta(|\operatorname{grad}^\alpha f|^2) \\ & + \frac{1}{2}\left(\langle \operatorname{grad}^\alpha f, \operatorname{grad}^\alpha(\Delta^\alpha f)\rangle + \langle \operatorname{grad}^\alpha(\Delta^\alpha f), \operatorname{grad}^\alpha f\rangle\right)\Bigg).\end{aligned}$$

Now applying Lemma 6.3 and the fact $P_s \ge 0$, we conclude $\frac{d}{ds}F(s) \ge 0$. Note further that $F(0) = |\operatorname{grad}^\alpha(P_{t-s}^\alpha f)|^2$ and $F(t) = e^{2Kt} P_t(|\operatorname{grad}^\alpha f|^2)$. This leads to (i).

We then show $(i) \Rightarrow (ii)$. For $0 \le s \le t$, let $G(s) := P_s(|P_{t-s}^\alpha f|^2)$. Thus, we have

$$P_t(|f|^2) - |P_t^\alpha f|^2 = G(t) - G(0) = \int_0^t G'(s) ds,$$

where

$$\begin{aligned} G'(s) =& P_s\left(-\Delta(|P_{t-s}^\alpha f|^2) + \Delta^\alpha P_{t-s}^\alpha f \overline{P_{t-s}^\alpha f} + \overline{\Delta^\alpha P_{t-s}^\alpha f} P_{t-s}^\alpha f\right) \\ =& 2P_s(|\operatorname{grad}^\alpha(P_{t-s}^\alpha)|^2) \\ \ge& 2e^{-2Ks}|\operatorname{grad}^\alpha(P_s^\alpha P_{t-s}^\alpha f)|^2 = 2e^{-2Ks}|\operatorname{grad}^\alpha(P_t^\alpha f)|^2. \end{aligned}$$

In the above inequality, we used (i). In the case $K > 0$, we arrive at

$$G(t) - G(0) \ge \int_0^t 2e^{-2Ks} ds |\operatorname{grad}^\alpha(P_t^\alpha f)|^2 = \frac{1 - e^{-2Kt}}{K}|\operatorname{grad}^\alpha(P_t^\alpha f)|^2.$$

Note that in the case $K = 0$ we have $\int_0^t 2e^{-2Ks} ds = 2t$. This finishes the proof of $(i) \Rightarrow (ii)$.

It remains to show $(ii) \Rightarrow (iii)$. We will present the argument for the case $K > 0$. The case $K = 0$ can be shown similarly. Assuming (ii), we derive directly for $0 \le t \le \frac{1}{2K}$

$$\|\sqrt{P_t(|f|^2)}\|_\infty \ge \sqrt{t}\|\operatorname{grad}^\alpha(P_t^\alpha f)\|_\infty. \tag{69}$$

In the above, we used the inequality $1 - e^{-x} \ge \frac{x}{2}$ for $0 \le x \le 1$. For any $\phi \in C^\infty(M, \mathbb{C})$ with $\|\phi\|_\infty \le 1$, we calculate

$$\begin{aligned} \int_M (f - P_t^\alpha f)\phi \,\mathrm{dvol} =& -\int_M \phi \int_0^t \frac{\partial}{\partial s} P_s^\alpha f ds \,\mathrm{dvol} \\ =& \int_M \int_0^t (\Delta^\alpha(P_s^\alpha f))\phi ds \,\mathrm{dvol} \\ =& \int_0^t \int_M \Delta^\alpha f \cdot (P_s^\alpha \phi) \,\mathrm{dvol}\, ds \\ =& \int_0^t \int_M \langle \operatorname{grad}^\alpha f, \operatorname{grad}^\alpha(P_s^\alpha \phi)\rangle \,\mathrm{dvol}\, ds, \end{aligned}$$

where we used $\Delta^\alpha P_s^\alpha = P_s^\alpha \Delta^\alpha$ and the self-adjointness of P_t^α. Continuing the calculation, we arrive at

$$\int_M (f - P_t^\alpha f)\phi \,\mathrm{dvol} \le \|\operatorname{grad}^\alpha f\|_1 \int_0^t \|\operatorname{grad}^\alpha (P_s^\alpha \phi)\|_\infty ds.$$

For $0 < t \le \frac{1}{2K}$, we apply (69) to obtain

$$\int_M (f - P_t^\alpha f)\phi \,\mathrm{dvol} \le 2\sqrt{t}\|\operatorname{grad}^\alpha f\|_1, \tag{70}$$

where we used $\sqrt{P_s(|\phi|^2)} \le |\phi| \le 1$. Applying (70) to a sequence of smooth functions $\{\phi_k\}$ with $\|\phi_k\|_\infty \le 1$, approximating in the $L^2(M,\mathbb{C})$-norm the following function:

$$\phi_\infty = \begin{cases} \frac{\overline{f - P_t^\alpha f}}{|f - P_t^\alpha f|}, & \text{if } |f - P_t^\alpha f| \neq 0; \\ 0, & \text{otherwise,} \end{cases}$$

leads to the proof of (iii). □

Proof of Theorem 1.2. Let $\{\Omega_p\}_{[k]}$, $k \in \mathbb{N}$ be any k disjoint Borel subsets with $\mathrm{vol}(\Omega_p) > 0$ for each $p \in [k]$. For each $\Omega_p \subseteq M$, let $\tau_p : M \to \mathbb{C}$ be the function given by

$$\tau_p(x) = \begin{cases} \tau_p(x) \in U(1), & x \in \Omega_p; \\ 0, & \text{otherwise,} \end{cases} \tag{71}$$

such that $\tau_p|_{\Omega_p}$ is the minimizer in the definition of $\iota^\alpha(\Omega_p)$, i.e., $\int_{\Omega_p} |(d+i\alpha)(\tau_p|_{\Omega_p})| \,\mathrm{dvol} = \iota^\alpha(\Omega_p)$. Applying Lemma 6.4 (iii) to smooth complex valued functions approximating τ_p yields, for $0 \le t \le \frac{1}{2K}$,

$$\begin{aligned} 2\sqrt{t}\,(\iota^\alpha(\Omega_p) + \mathrm{area}(\partial\Omega_p)) \ge & \int_M |\tau_p - P_t^\alpha \tau_p| \,\mathrm{dvol} \\ \ge & \int_M |\tau_p - P_t^\alpha \tau_p| \cdot |\tau_p| \,\mathrm{dvol} \ge \int_M \Re\left(\tau_p \cdot \overline{\tau_p - P_t^\alpha \tau_p}\right) \\ = & \|\tau_p\|_2^2 - \|P_{\frac{t}{2}}^\alpha \tau_p\|_2^2. \end{aligned}$$

We remark that the corresponding estimate for Δ in [16, Theorem 5.2], although leading to an improved constant, seems not to be applicable here. Let $\{\psi_\ell\}_{\ell=1}^\infty$ be the orthonormal eigenfunctions corresponding to $\{\lambda_\ell^\alpha\}_{\ell=1}^\infty$. By the spectral theorem, we have

$$\|P_{\frac{t}{2}}^\alpha \tau_p\|_2^2 = \sum_{\ell=1}^\infty e^{-t\lambda_\ell^\alpha} |\langle \tau_p, \psi_\ell \rangle|^2.$$

Furthermore, observe that

$$\|\tau_p\|_2^2 = \sum_{\ell=1}^\infty |\langle \tau_p, \psi_\ell \rangle|^2 = \mathrm{vol}(\Omega_p).$$

Thus, we have, for $0 \le t \le \frac{1}{2K}$,

$$2\sqrt{t}\,(\iota^\alpha(\Omega_p) + \mathrm{area}(\partial\Omega_p)) \ge \mathrm{vol}(\Omega_p) - \sum_{\ell=1}^{\infty} e^{-t\lambda_\ell^\alpha} |\langle \tau_p, \psi_\ell\rangle|^2.$$

Therefore, for given $k \in \mathbb{N}$, we have

$$\begin{aligned} 2\sqrt{t}\phi^\alpha(\Omega_p) \ge & 1 - \sum_{\ell=1}^{k-1} \frac{|\langle \tau_p, \psi_\ell\rangle|^2}{\mathrm{vol}(\Omega_p)} - e^{t\lambda_k^\alpha} \sum_{\ell=k}^{\infty} \frac{|\langle \tau_p, \psi_\ell\rangle|^2}{\mathrm{vol}(\Omega_p)} \\ \ge & 1 - \sum_{\ell=1}^{k-1} \frac{|\langle \tau_p, \psi_\ell\rangle|^2}{\mathrm{vol}(\Omega_p)} - e^{t\lambda_k^\alpha}. \end{aligned} \tag{72}$$

Observing that the functions $\frac{\tau_p}{\sqrt{\mathrm{vol}(\Omega_p)}}$, $p \in [k]$, are orthonormal in $L^2(M, \mathbb{C})$, we obtain

$$\sum_{p=1}^{k} \frac{|\langle \tau_p, \psi_\ell\rangle|^2}{\mathrm{vol}(\Omega_p)} = \sum_{p=1}^{k} \left| \left\langle \frac{\tau_p}{\sqrt{\mathrm{vol}(\Omega_p)}}, \psi_\ell \right\rangle \right|^2 \le \|\psi_\ell\|^2 = 1.$$

Thus, we arrive at

$$\frac{1}{k} \sum_{p=1}^{k} \sum_{\ell=1}^{k-1} \frac{|\langle \tau_p, \psi_\ell\rangle|^2}{\mathrm{vol}(\Omega_p)} \le 1 - \frac{1}{k}.$$

This implies that there exists a $p_0 \in [k]$ such that

$$\sum_{\ell=1}^{k-1} \frac{|\langle \tau_{p_0}, \psi_\ell\rangle|^2}{\mathrm{vol}(\Omega_{p_0})} \le 1 - \frac{1}{k}.$$

Applying (72) to the set Ω_{p_0}, we obtain

$$2\sqrt{t} \max_{p \in [k]} \phi^\alpha(\Omega_p) \ge 2\sqrt{t}\phi^\alpha(\Omega_{p_0}) \ge \frac{1}{k} - e^{-t\lambda_k^\alpha}.$$

This completes the proof. □

We finish this section with the following consequence of Theorem 1.2.

Corollary 6.6. *Let M be a closed Riemannian manifold whose Ricci curvature is bounded from below by $-K$, $K \ge 0$. Let α be a magnetic potential such that $d\alpha = 0$. Then we have for any $k \in \mathbb{N}$,*

$$\lambda_k^\alpha \le 2\log(2k) \max\left\{K, 2k^2 (h_k^\alpha)^2\right\}. \tag{73}$$

Proof. Applying Theorem 1.2, we obtain

$$2\sqrt{t} \cdot h_k^\alpha \ge \frac{1}{k} - e^{-t\lambda_k^\alpha}.$$

If $\lambda_k^\alpha > 0$ and $\lambda_k^\alpha \ge 2\log(2k)K$, we choose $t = \frac{\log(2k)}{\lambda_k^\alpha} \le \frac{1}{2K}$ and obtain $\sqrt{\lambda_k^\alpha} \le 2k\sqrt{\log(2k)} h_k^\alpha$. Hence, we have (73). □

Acknowledgement

SL and NP acknowledge the financial support of the EPSRC Grant EP/K016687/1 "Topology, Geometry and Laplacians of Simplicial Complexes".

References

[1] W. Ballmann, J. Brüning, G. Carron, *Eigenvalues and holonomy*, Int. Math. Res. Not. (2003), no. 12, 657–665.

[2] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. **15** (1982), no. 2, 213-230.

[3] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, in: Problems in analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 1970, pp. 195–199.

[4] B. Colbois, A. Savo, *Eigenvalue bounds for the magnetic Laplacian*, arXiv:1611.01930, November 2016.

[5] B. Colbois, A. El Soufi, A. Savo, *Eigenvalue upper bounds for the magnetic Schrödinger operator*, arXiv:1709.09482, September 2017.

[6] B. Colbois, A. Savo, *Lower bounds for the first eigenvalue of the magnetic Laplacian*, J. Funct. Anal. **274** (2018), no. 10, 2818–2845.

[7] M. Egidi, S. Liu, F. Münch, N. Peyerimhoff, *Ricci curvature and eigenvalue estimates for the magnetic Laplacian on manifolds*, arXiv:1608.01955, August 2016.

[8] L. Erdős, *Rayleigh-type isoperimetric inequality with a homogeneous magnetic field*, Calc. Var. Partial Differential Equations **4** (1996), no. 3, 283–292.

[9] R. L. Frank, A. Laptev, S. Molchanov, *Eigenvalue estimates for magnetic Schrödinger operators in domains*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4245–4255.

[10] K. Funano, *Eigenvalues of Laplacian and multi-way isoperimetric constants on weighted Riemannian manifolds*, arXiv:1307.3919, July 2013.

[11] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, Springer-Verlag, Berlin, 1990.

[12] G. Habib, A. Kachmar, *Eigenvalue bounds of the Robin Laplacian with magnetic field*, Arch. Math. (Basel) **110** (2018), no. 5, 501–513.

[13] M. Hinz, A. Teplyaev, *Dirac and magnetic Schrödinger operators on fractals*, J. Funct. Anal. **265** (2013), no. 11, 2830–2854.

[14] J. Jost, *Riemannian geometry and geometric analysis*, Fifth edition, Universitext, Springer-Verlag, Berlin, 2005.

[15] C. Lange, S. Liu, O. Post, N. Peyerimhoff, *Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 4165–4196.

[16] M. Ledoux, *Spectral gap, logarithmic Sobolev constant, and geometric bounds*, Surveys in differential geometry, Vol. **IX**, 219-240, Surv. Differ. Geom. **IX**, Int. Press, Somerville, MA, 2004.

[17] J. M. Lee, *Manifolds and Differential Geometry*, Graduate Studies in Mathematics, Vol. **107**, Amer. Math. Soc., 2009.

[18] S. Liu, *An optimal dimension-free upper bound for eigenvalue ratios*, arXiv:1405.2213, May 2014.

[19] S. Liu, F. Münch, N. Peyerimhoff, *Curvature and higher order Buser inequalities for the graph connection Laplacian*, arXiv:1512.08134.

[20] A. Morame, F. Truc, *Counting function of the embedded eigenvalues for some manifold with cusps, and magnetic Laplacian*, Math. Res. Lett. **19** (2012), no. 2, 417–429.

[21] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340.

[22] G. P. Paternain, *Schrödinger operators with magnetic fields and minimal action functionals*, Israel J. Math. **125** (2001), 1–27.

[23] I. Shigekawa, *Eigenvalue problems for the Schrödinger operator with the magnetic field on a compact Riemannian manifold*, J. Funct. Anal. **75** (1987), no. 1, 92–127.

[24] M. A. Shubin, *Discrete magnetic Laplacian*, Comm. Math. Phys. **164** (1994), no. 2, 259–275.

[25] M. A. Shubin, *Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds*, J. Funct. Anal. **186** (2001), no. 1, 92–116.

[26] T. Sunada, *A discrete analogue of periodic magnetic Schrödinger operators*, Geometry of the spectrum (Seattle, WA, 1993), 283–299, Contemp. Math. **173**, Amer. Math. Soc., Providence, RI, 1994.

AMS Subject Classification: 58J50, 53C21, 58J35

Michela Egidi
Technische Universität Dortmund
Fakultät für Mathematik
Vogelpothsweg 87
44227 Dortmund
Germany
e-mail: michela.egidi@math.tu-dortmund.de

Shiping Liu
School of Mathematical Sciences
University of Science and Technology of China
96 Jinzhai Road
Hefei 230026
Anhui Province
China
e-mail: spliu@ustc.edu.cn

Florentin Münch
Universität Potsdam
Institut für Mathematik
Campus Golm, Haus 9
Karl-Liebknecht-Strae 24-25
14476 Potsdam
e-mail: chmuench@uni-potsdam.de

Norbert Peyerimhoff
Durham University
Department of Mathematical Sciences
Science Laboratories
Sout Road
Durham, DH1 3LE
United Kingdom
e-mail: norbert.peyerimhoff@durham.ac.uk