Population model with immigration in continuous space

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April 16, 2019

*The study was funded by the Russian Science Foundation (project no. 17-11-01098)
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Abstract

In a population model in continuous space, individuals evolve independently as branching random walks subject to immigration. If the underlying branching mechanism is subcritical, the model has a unique steady state for each value of the immigration intensity. Convergence to the equilibrium is exponentially fast. The resulting dynamics are Lyapunov stable in that their qualitative behavior does not change under suitable perturbations of the main parameters of the model.

keywords: spatial population dynamics; branching random walk; immigration; correlation functions; steady state; Lyapunov stability

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Short title:

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1 Introduction

One of the simplest model with a steady state, also known as space-time equilibrium, is the contact model in $\mathbb{R}^d$ (Kondratiev and Skorokhod, 2006; Kondratiev et al., 2008). For this model, the corresponding point field has multiplicity one, so that the population dynamics can be described as a Markov process in the space of infinite but locally finite point configurations in $\mathbb{R}^d$ (Kondratiev and Skorokhod, 2006; Kondratiev et al., 2008). In contrast, the dynamics of lattice point fields of multiplicity one are not Markovian, which complicates their analysis (Liggett, 1985).

The contact model is instable with respect to small random perturbations, notably local ones, of the rates of splitting and death. We introduce a related model, where the steady state is stable in the strongest Lyapunov sense, which means that the stochastic equilibrium survives under sufficiently small (in L^∞-norm) perturbations of the rates. In section 1.1, we describe the time evolution of a population in $\mathbb{R}^d$, subject to immigration, and whose individuals evolve independently as branching random walks. We demonstrate that the qualitative behavior of this model persists under perturbations, possibly heterogeneous over space, of the key parameters.

We present the main results in section 1.3 with emphasis on the stationary case of rates constant in space and over time. In section 2, we derive equations for the correlation functions. As in Kondratiev and Skorokhod (2006) and Kondratiev et al. (2008), the space is continuous and the field of particles has multiplicity one. In section 3, the uniform estimates on the correlation functions and the Carleman condition allow us to prove the existence of a unique steady state. We show that the correlation functions converge to their limiting values exponentially fast and therefore the initial condition quickly loses influence on the current state.

Molchanov and Whitmeyer (2017) and Han et al. (2017) review several classes of population models on discrete graphs, including lattices. Our analysis applies to a large class of population models in $\mathbb{R}^d$, in particular isotropic models, which do not exist in the lattice setting.

1.1 Model

Populations in $\mathbb{R}^d$, $d \geq 1$, are realizations of a point field, where $\mathsf{n}(t,\Gamma)$ denotes the total number of particles in a region $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ at time $t \geq 0$. $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sigma-field in $\mathbb{R}^d$. Initially, the configuration $\mathsf{n}(0,\Gamma)$ is a realization of the Poisson point field in $\mathbb{R}^d$ of constant intensity $\lambda > 0$, that is,

$$\mathbb{P}\big(\mathsf{n}(0,\Gamma) = m\big) = \frac{(\lambda|\Gamma|)^m}{m!} e^{-\lambda|\Gamma|} \tag{1}$$

for integer $m \geq 0$, where $|\Gamma|$ is the Lebesgue measure of Γ. Each of the $\mathsf{n}(t,\Gamma)$ individual particles in Γ evolves independently as a branching random walk. Particles can:

immigrate: given a constant $\gamma > 0$, particles independently appear in $\mathbb{R}^d$ according to a Poisson point field of intensity γ, so that a particle appears infinitesimally close to x during a time interval $[t, t+dt)$ with probability asymptotically equal to $\gamma\, dxdt$.

move around: given a constant $\kappa > 0$ and a symmetric probability kernel $a(z)$, $z \in \mathbb{R}^d$, that is,

$$a(z) \geq 0\,, \qquad a(z) \equiv a(-z)\,, \qquad \int_{\mathbb{R}^d} a(z)\, dz = 1\,, \tag{2}$$

individual particles jump independently with generator

$$\kappa \mathcal{L}_a \psi(x) = \kappa \int_{\mathbb{R}^d} \big(\psi(x+z) - \psi(x)\big) a(z)\, dz\,. \tag{3}$$

The probability that a particle at x jumps out of its location during the time interval $[t, t+dt)$ approximately equals $\kappa\, dt$; the probability that the particle lands infinitesimally close to $x+z$ approximately equals $a(z)\, dz$. For simplicity, the Fourier transform $\hat{a}(k)$ of the kernel $a(z)$ is assumed integrable:

$$\widehat{a}(k) = \int_{\mathbb{R}^d} e^{ikz} a(z)\, dz \in L^1(\mathbb{R}^d)\,, \tag{4}$$

so that the kernel $a(z)$ is uniformly continuous.

split: the probability that a particle at x generates an offspring during the time interval

$[t, t+dt)$ approximately equals $\beta\, dt$, with fixed birth rate $\beta > 0$. The probability that offspring appears infinitesimally close to $x+z$ is $b(z)\, dz$, where $b(z)$, $z \in \mathbb{R}^d$, is a symmetric probability kernel with properties as in Eq. (2) and (4), that is,

$$\begin{gathered} b(z) \geq 0\,, \qquad b(z) \equiv b(-z)\,, \qquad \int_{\mathbb{R}^d} b(z)\, dz = 1\,, \\ \widehat{b}(k) = \int_{\mathbb{R}^d} e^{ikz} b(z)\, dz \in L^1(\mathbb{R}^d)\,. \end{gathered} \tag{5}$$

As in Eq. (3), we introduce the corresponding generator

$$\beta \mathcal{L}_b \psi(x) = \beta \int_{\mathbb{R}^d} \big(\psi(x+z) - \psi(x)\big) b(z)\, dz\,. \tag{6}$$

die: individual particles die independently at constant rate $\mu > 0$, that is, the probability that a given particle dies within the time interval $[t, t+dt)$ is asymptotically equal to $\mu\, dt$. We assume that $\mu > \beta$, so that the branching mechanism is subcritical.

Unlike in the lattice case, the local limit theorem for densities does not necessarily follow from the central limit theorem. Pestman et al. (2016) give an example of a density with compact support (and thus satisfying the central limit theorem) but with unbounded convolutions of all orders (and thus not satisfying the local limit theorem). The technical condition of multiplicity one in Kondratiev and Skorokhod (2006) and Kondratiev et al. (2008), as the condition that $\widehat{a}(k)$ and $\widehat{b}(k)$ are integrable, exclude cases where the local density of particles remains unbounded. Our densities $a(z)$ and $b(z)$ are uniformly continuous, and hence bounded in $\mathbb{R}^d$, implying that neither migration nor dispersal can lead to local accumulations of particles.

Sewastjanow (1974, Chap. X) studies diffusive branching random processes in bounded domains. His analysis does not apply to jump processes in the whole space.

1.2 Correlation functions

Correlation functions encode stochastic properties of the population dynamics. For integer $n \geq 1$ and a collection of distinct points $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$, the n-th correlation function $\mathsf{k}_t^{(n)}(x_1,\dots,x_n)$ is the density of the probability

$$\mathbb{P}\big(\mathsf{n}(t,x_1+dx_1)=1,\dots,\mathsf{n}(t,x_n+dx_n)=1\big) \tag{7}$$

that an infinitesimal neighborhood of each point x_1, ..., x_n contains a single particle. By the choice of the initial distribution,

$$\mathsf{k}_0^{(n)}(x_1,\dots,x_n)=\lambda^n\,, \qquad n\geq 1\,. \tag{8}$$

In the setting of the contact model, Kondratiev and Skorokhod (2006) and Kondratiev et al. (2008) define the correlation functions and construct the corresponding dynamics.

The first correlation function $\mathsf{k}_t^{(1)}(x)$ is the density of the particles at location x at time t,

$$\mathbb{P}\big(\mathsf{n}(t,x+dx)=1\big)=\mathsf{k}_t^{(1)}(x)\,dx\,. \tag{9}$$

Therefore, the average total number of particles in $\Gamma\in\mathcal{B}(\mathbb{R}^d)$ at time t is

$$\mathsf{m}_1(t,\Gamma)=\mathbb{E}\,\mathsf{n}(t,\Gamma)=\int_\Gamma \mathsf{k}_t^{(1)}(x)\,dx\,. \tag{10}$$

Write $(n)_l:=n(n-1)\dots(n-l+1)$ for the falling factorial of order $l\geq 1$. Then the l-th factorial moment of $\mathsf{n}=\mathsf{n}(t,\Gamma)$ is

$$\mathsf{m}_l(t,\Gamma)=\mathbb{E}\big(\mathsf{n}(t,\Gamma)\big)_l=\int_\Gamma\cdots\int_\Gamma \mathsf{k}_t^{(l)}(x_1,\dots,x_l)\,dx_1\dots dx_l\,. \tag{11}$$

For the initial configuration,

$$\mathsf{m}_l(0,\Gamma)=\big(\lambda|\Gamma|\big)^l\,, \qquad l\geq 1\,, \qquad \Gamma\in\mathcal{B}(\mathbb{R}^d)\,. \tag{12}$$

1.3 Results

We use the fact that the family of correlation functions $\mathsf{k}_t^{(n)}(x_1,\dots,x_n)$, $n\geq 1$, satisfies a system of parabolic equations with initial conditions $\mathsf{k}_0^{(n)}(x_1,\dots,x_n)=\lambda^n$. Recall the stability assumption $\mu > \beta$.

Theorem 1. *For each integer $n \geq 1$ and for all $(x_1, x_2, \ldots, x_n) \in (\mathbb{R}^d)^n$ with pairwise distinct x_i, there exists $\mathsf{k}_\infty^{(n)}(x_1, \ldots, x_n)$ such that, as $t \to \infty$,*

$$\mathsf{k}_t^{(n)}(x_1, \ldots, x_n) \to \mathsf{k}_\infty^{(n)}(x_1, \ldots, x_n)\,. \tag{13}$$

Moreover, there exists a positive constant $C = C(\lambda, \mu, \beta, \gamma)$ such that for each integer $n \geq 1$,

$$\left\|\mathsf{k}^{(n)}\right\| := \sup_{t\geq 0}\ \sup_{x_1,\ldots,x_n} \left|\mathsf{k}_t^{(n)}(x_1, \ldots, x_n)\right| \leq n!\, C^n\,. \tag{14}$$

The limiting correlation functions $\left\{\mathsf{k}_\infty^{(n)}(x_1, \ldots, x_n)\right\}$ can be computed in a recursive way using Eq. (72) and (73) below. The upper bound in Eq. (14) does not depend on κ, which is consistent with the heuristic argument that more intense diffusion mixes the configuration faster and prevents the local density of the field from growing too large.

An important corollary of Theorem 1 is that, for all $\kappa \geq 0$ and $\gamma \geq 0$, the model of a branching random walk with immigration, introduced in section 1.1, possesses a steady state:

Theorem 2. *For all Borel $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,*

$$\mathsf{n}(t, \Gamma) \to \mathsf{n}(\infty, \Gamma) \tag{15}$$

in law, as $t \to \infty$. The distribution of $\left\{\mathsf{n}(\infty, \Gamma) : \Gamma \in \mathcal{B}(\mathbb{R}^d)\right\}$ is the unique steady state for the population dynamics of section 1.1.

While deriving an explicit description of the steady state from the limiting correlation functions $\left\{\mathsf{k}_\infty^{(n)}(x_1, \ldots, x_n)\right\}$ might not be immediate, we show below that its first moment is constant in space and its second moment is invariant by translation. The latter property also persists to higher moments.

Under the key assumption $\mu > \beta$, by Eq. (11), the factorial moments $\mathsf{m}_n(t, \Gamma)$ also converge: for each $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathsf{m}_n(t,\Gamma) \to \mathsf{m}_n(\infty,\Gamma) \qquad \text{as } t \to \infty\,. \tag{16}$$

Moreover, the uniform estimate of Eq. (14) implies the bound on the factorial moments:

$$|\mathsf{m}_n(t,\Gamma)| \le n!\,\big(C|\Gamma|\big)^n\,, \qquad 0 \le t \le \infty\,, \tag{17}$$

which, by Carleman's condition (Feller, 1971, section VII.3):

$$\sum_{n\ge 1}\big(\mathsf{m}_{2n}(t,\Gamma)\big)^{-\frac{1}{2n}} = \infty\,, \tag{18}$$

implies the existence of a unique distribution $\big\{\mathsf{n}(t,\Gamma) : \Gamma \in \mathcal{B}(\mathbb{R}^d)\big\}$ for each $t \in [0,\infty]$. Alternatively, a slightly weaker condition (Feller, 1971, section XV.4, Eq. (4.15)),

$$\limsup_{n\to\infty} \frac{1}{n}\big(\mathsf{m}_n(t,\Gamma)\big)^{\frac{1}{n}} < \infty\,, \tag{19}$$

is also applicable here.

Each individual alive at $t = 0$ as well as each immigrant arriving at $t > 0$ generates a subpopulation, which evolves according to the rules of section 1.1 with $\gamma = 0$ (no immigration). Therefore each of the $\mathsf{n}(t,\Gamma)$ individuals in Γ at time $t > 0$ can be tracked back to its earliest ancestor, either present at $t = 0$ or arrived as an immigrant. Then $\mathsf{n}(t,\Gamma)$ is the sum of subpopulation sizes, where each subpopulation evolves as a (subcritical) branching random walk with migration governed by Eq. (3), with birth governed by Eq. (6) and mortality at fixed rate $\mu > 0$:

$$\mathsf{n}(t,\Gamma) = \sum_i \mathsf{n}(t - t_i, y_i, \Gamma)\,, \tag{20}$$

where the sum runs over all individual ancestors, with $(y_i, t_i) \in \mathbb{R}^d \times [0,t]$ denoting the location and the time of their individual arrivals, and where $\mathsf{n}(t - t_i, y_i, \Gamma)$ is the corresponding number of descendants in the Borel set Γ at time $t \ge 0$. The choice of the initial population and the immigration process guarantee that the total number of possible ancestors arriving during the time interval $[0,t]$ is countable.

When combined with stochastic monotonicity of the solution $\mathsf{n}(t,\Gamma)$, Theorems 1 and 2 imply stability of the evolution with respect to small perturbations of the rates (a random variable X is stochastically smaller than a random variable Y (denoted $X \preccurlyeq Y$) if $\mathbb{P}(X \geq z) \leq \mathbb{P}(Y \geq z)$ for all $z \in \mathbb{R}$). Indeed, if β_x and μ_x satisfy

$$\beta_x = \beta + \varepsilon\xi_x\,, \qquad \mu_x = \mu + \varepsilon\eta_x\,, \qquad \text{where} \qquad \sup_{x\in\mathbb{R}^d}\big(|\xi_x|, |\eta_x|\big) \leq 1\,, \tag{21}$$

with possibly random $(\xi_x,\eta_x)_{x\in\mathbb{R}^d}$, for the particle field $\mathsf{n}(t,\Gamma)$ corresponding to birth and death rates $(\beta_x,\mu_x)_{x\in\mathbb{R}^d}$, the particle field $\mathsf{n}^*(t,\Gamma)$ corresponding to the constant rates $(\beta+\varepsilon, \mu-\varepsilon)$, and the particle field $\mathsf{n}_*(t,\Gamma)$ corresponding to the constant rates $(\beta-\varepsilon,\mu+\varepsilon)$, we have Theorem 3:

Theorem 3. *If $\mu - \beta > 2\varepsilon > 0$ and the rates $(\xi_x,\eta_x)_{x\in\mathbb{R}^d}$ are given by Eq.* (21)*, then, for all $t \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, the stochastic order*

$$\mathsf{n}_*(t,\Gamma) \preccurlyeq \mathsf{n}(t,\Gamma) \preccurlyeq \mathsf{n}^*(t,\Gamma) \tag{22}$$

holds.

We verify the stochastic order of Eq. (22) by constructing the three processes n_*, n, and n^* on a common probability space; this procedure is known as coupling (Lindvall, 1992). Thanks to the decomposition in Eq. (20) into the sum of the subpopulations, it is sufficient to verify the stochastic comparison of Eq. (22) for individual subpopulations with common ancestor. Because

$$\beta - \varepsilon \leq \beta_x \leq \beta + \varepsilon \qquad \text{and} \qquad \mu + \varepsilon \geq \mu_x \geq \mu - \varepsilon\,, \tag{23}$$

this comparison is achieved as described in (Lindvall, 1992). This implies the Lyapunov stability of Theorem 3.

We present the construction on the example of n_* and n^* for a single subpopulation starting from $x \in \mathbb{R}^d$ at time $t = 0$. Then $\mathsf{n}_*(0, \Gamma) \preccurlyeq \mathsf{n}^*(0, \Gamma)$ and we show that the point field for the process n_* is a subset of the point field for the process n^* for all $t \geq 0$. Because, until extinction, the total number of particles $\mathsf{n}^*(t, \mathbb{R}^d)$ in a single subpopulation forms a linear continuous-time birth-and-death process, at every time $t \geq 0$ its size is almost surely finite, which implies that the processes n_* and n^* are well defined.

Assume that, for fixed $t \geq 0$, the configuration n_* is contained in that of n^* and that the next jump occurs at time $s > t$. If this jump occurs at a location belonging to n^* only, it follows the rules of section 1.1 with $\gamma = 0$. Otherwise, it originates at a location y common to both processes, and is determined by the smallest of the five independent exponential variables

$$\begin{gathered} \xi_1 \sim \mathsf{Exp}(\mu - \varepsilon)\,, \quad \xi_2 \sim \mathsf{Exp}(\beta - \varepsilon)\,, \quad \xi_3 \sim \mathsf{Exp}(\kappa)\,, \\ \xi_4 \sim \mathsf{Exp}(2\varepsilon)\,, \quad \xi_5 \sim \mathsf{Exp}(2\varepsilon)\,. \end{gathered} \tag{24}$$

If ξ_1 is the smallest, the particle at y dies in both processes n_* and n^*. If it is ξ_2, an offspring is created in both processes at location $y+z$, where z is generated by the kernel $b(\,\cdot\,)$. If it is ξ_3, the particle moves in both processes from y to $y+z$, where z is generated by the kernel $a(\,\cdot\,)$. If it is ξ_4, the particle dies in n_* (but not in n^*). If it is ξ_5, an offspring is created in n^* at location $y+z$, where z is generated by the kernel $b(\,\cdot\,)$. Then the changes in n_* have rates $(\beta - \varepsilon, \mu + \varepsilon)$ while the changes in n^* have rates $(\beta + \varepsilon, \mu - \varepsilon)$; after the jump, all particles are almost surely in distinct locations, and the configuration of n_* is still a subset of n^*. This construction goes further by induction until the subpopulation dies out in both processes. Because individual subpopulations evolve independently of one another, the full configuration of n_* is a subset of the full configuration of n^*, and therefore $\mathsf{n}_*(t, \Gamma) \preccurlyeq \mathsf{n}^*(t, \Gamma)$ for all $t \geq 0$. The argument for Eq. (22) is analogous.

The stochastic order in Eq. (22) also results from varying the immigration rate. Indeed, consider the particle field $\mathsf{n}(t, \Gamma)$ corresponding to birth, death and immigration rates $(\beta, \mu, \gamma_x)_{x \in \mathbb{R}^d}$, the particle field $\mathsf{n}^*(t, \Gamma)$ corresponding to the constant rates (β, μ, γ^*), and the particle field $\mathsf{n}_*(t,\Gamma)$ corresponding to the constant rates (β,μ,γ_*). Then

$$\gamma_* \leq \gamma_x \leq \gamma^*\,, \tag{25}$$

where (possibly random) γ_x can depend on $x \in \mathbb{R}^d$, implies the stochastic order of Eq. (22). Furthermore, the stochastic order of Eq. (22) is true if the birth and death rates satisfy Eq. (23) and the immigration rates satisfy Eq. (25).

The Lyapunov stability of Theorem 3 can fail at criticality, where $\mu = \beta$ (Kondratiev and Skorokhod, 2006; Kondratiev et al., 2008). Indeed, if the random rates β_x and μ_x in Eq. (21) satisfy the criticality assumption

$$\mathbb{E}\beta_x \equiv \beta = \mu \equiv \mathbb{E}\mu_x\,, \tag{26}$$

while the joint distribution of (β_x,μ_x) allows the existence of large enough regions Γ where $\beta_x - \mu_x > \varepsilon > 0$ with positive probability, then the population count $\mathsf{n}(t,\Gamma)$ may keep growing as $t \to \infty$. Kondratiev et al. (2017) use spectral analysis to derive this result for a general class of Schrödinger operators.

We now prove Theorem 1.

2 Time evolution of correlation functions

We derive parabolic equations for the family of the correlation functions $\mathsf{k}_t^{(n)}(x_1,\ldots,x_n)$, $n \geq 1$, defined in section 1.2, with initial conditions $\mathsf{k}_0^{(n)}(x_1,\ldots,x_n) = \lambda^n$. A key feature of the resulting system is that the equation for $\mathsf{k}_t^{(n)}(x_1,\ldots,x_n)$ includes correlation functions of lower orders.

To study the first correlation function $\mathsf{k}_t^{(1)}(x_1)$, consider the events

$$\begin{aligned} A^{(1)}_{t,t+dt} &= \left\{\mathsf{n}(t+dt,x+dx) = 1 \;\middle|\; \mathsf{n}(t,x+dx) = 1\right\}, \\ B^{(1)}_{t,t+dt} &= \left\{\mathsf{n}(t+dt,x+dx) = 1 \;\middle|\; \mathsf{n}(t,x+dx) = 0\right\}. \end{aligned} \tag{27}$$

Then, up to the errors of higher order,

$$\begin{aligned}\mathsf{k}^{(1)}_{t+dt}(x)\,dx &= \mathbb{P}\big(\mathsf{n}(t+dt,x+dx)=1\big)\\ &= \mathbb{P}\big(A^{(1)}_{t,t+dt}\big)\,\mathsf{k}^{(1)}_t(x)\,dx + \mathbb{P}\big(B^{(1)}_{t,t+dt}\big)\big(1-\mathsf{k}^{(1)}_t(x)\,dx\big).\end{aligned} \tag{28}$$

As the leading contribution to the event $A^{(1)}_{t,t+dt}$ comes from the trajectories in which the state of the infinitesimal neighborhood of x does not change during the time interval $[t,t+dt)$, at the first order:

$$\mathbb{P}\big(A^{(1)}_{t,t+dt}\big) = 1-(\kappa+\mu)\,dt\,. \tag{29}$$

The splitting move at x during the time interval $[t,t+dt)$ is not excluded, as the parental particle stays at its location. Likewise, the leading contribution to the event $B^{(1)}_{t,t+dt}$ comes from the arrival of a single particle in the infinitesimal neighborhood of x (due to either immigration, migration, or a splitting event at a different location). We thus get

$$\begin{aligned}\mathbb{P}\big(B^{(1)}_{t,t+dt}\big) &= \Big(\gamma+\kappa\int_{\mathbb{R}^d}\mathsf{k}^{(1)}_t(x-z)\,a(z)\,dz+\beta\int_{\mathbb{R}^d}\mathsf{k}^{(1)}_t(x-z)\,b(z)\,dz\Big)\,dt\,dx\\ &= \big(\gamma+\kappa\mathcal{L}_a\mathsf{k}^{(1)}_t(x)+\beta\mathcal{L}_b\mathsf{k}^{(1)}_t(x)+(\kappa+\beta)\mathsf{k}^{(1)}_t(x)\big)\,dt\,dx\,,\end{aligned} \tag{30}$$

where the last equality follows by symmetry of the kernels $a(\,\cdot\,)$ and $b(\,\cdot\,)$,

$$\int_{\mathbb{R}^d}\mathsf{k}^{(1)}_t(x-z)\,a(z)\,dz=\int_{\mathbb{R}^d}\mathsf{k}^{(1)}_t(x-z)\,a(-z)\,dz=\int_{\mathbb{R}^d}\mathsf{k}^{(1)}_t(x+z)\,a(z)\,dz\,. \tag{31}$$

Putting all this together, we deduce

$$\frac{\partial\mathsf{k}^{(1)}_t}{\partial t}(x)=\big(\kappa\mathcal{L}_a+\beta\mathcal{L}_b\big)\mathsf{k}^{(1)}_t(x)+(\beta-\mu)\mathsf{k}^{(1)}_t(x)+\gamma \tag{32}$$

with the initial condition $\mathsf{k}^{(1)}_0(x)\equiv\lambda$.

Higher-order correlation functions are derived similarly. Write $A^{(n)}_{t,t+dt}$ for the event that simple occupancy of infinitesimal neighborhoods of the locations in the collection $\mathrm{x}_n := (x_1,\dots,x_n)$ does not change during the infinitesimal time interval $[t,t+dt)$. Then, at the first order,

$$\mathbb{P}\big(A^{(n)}_{t,t+dt}\big) = 1 - n(\kappa + \mu)\, dt\,. \tag{33}$$

Denote by $B^{(n,i)}_{t,t+dt}$ the event that an initially unoccupied infinitesimal neighborhood of the location x_i receives a single particle during the time interval $[t, t+dt)$, while infinitesimal neighborhoods of all other locations in $\mathrm{x}_{n,i} := \{x_j\}_{j\neq i, j=1,\dots,n}$ remain simply occupied during $[t, t+dt)$. The new particle at x_i arrives either as an offspring of a single parent from $\mathrm{x}_{n,i}$ or from a location not in $\mathrm{x}_{n,i}$ (due to either migration or arrival of an offspring of a particle there). The former event $C^{(n,i)}_{t,t+dt}$ satisfies

$$\mathbb{P}\big(C^{(n,i)}_{t,t+dt}\big) = \sum_{j:j\neq i} \beta b(x_i - x_j)\, dt\, dx_i\,, \tag{34}$$

implying that

$$\begin{aligned}\mathbb{P}\big(B^{(n,i)}_{t,t+dt}\big) = \Big(\gamma &+ \kappa \int_{\mathbb{R}^d} \mathsf{k}^{(n)}_t(x_1,\dots,x_{i-1}, x_i - z, x_{i+1}\dots,x_n)\, a(z)\, dz \\ &+ \beta \int_{\mathbb{R}^d} \mathsf{k}^{(n)}_t(x_1,\dots,x_{i-1}, x_i - z, x_{i+1}\dots,x_n)\, b(z)\, dz\Big)\, dt\, dx_i \\ &+ \sum_{j:j\neq i} \beta b(x_i - x_j)\, dt\, dx_i\,.\end{aligned} \tag{35}$$

Up to higher-order terms, $k^{(n)}_{t+dt}(\mathrm{x}_n)\, dx_1 \dots dx_n$ equals the probability

$$\begin{aligned}&\mathbb{P}\big(\mathsf{n}(t+dt, x_1 + dx_1) = 1, \dots, \mathsf{n}(t+dt, x_n + dx_n) = 1\big) \\ &\quad = \mathbb{P}\big(A^{(n)}_{t,t+dt}\big)\, \mathsf{k}^{(n)}_t(\mathrm{x}_n) \prod_{j=1}^n dx_j + \sum_{i=1}^n \mathbb{P}\big(B^{(n,i)}_{t,t+dt}\big)\, \mathsf{k}^{(n-1)}_t(\mathrm{x}_{n,i}) \prod_{j:j\neq i} dx_j\,.\end{aligned} \tag{36}$$

The correlation function $\mathsf{k}^{(n)}_t(\mathrm{x}_n)$ solves the forward Kolmogorov equation

$$\begin{aligned}\frac{\partial \mathsf{k}^{(n)}_t}{\partial t}(\mathrm{x}_n) &= n(\beta - \mu)\, \mathsf{k}^{(n)}_t(\mathrm{x}_n) + \sum_{i=1}^n \big(\kappa \mathcal{L}^i_a + \beta \mathcal{L}^i_b\big) \mathsf{k}^{(n)}_t(\mathrm{x}_n) \\ &\quad + \sum_{i=1}^n \Big(\beta \sum_{j:j\neq i} b(x_i - x_j) + \gamma\Big)\, \mathsf{k}^{(n-1)}_t(\mathrm{x}_{n,i})\,,\end{aligned} \tag{37}$$

where we use the restricted operators $\mathcal{L}_a^i$ and $\mathcal{L}_b^i$:

$$\begin{aligned}\mathcal{L}_a^i \mathsf{k}_t^{(n)}(x_1,\ldots,x_n) = \int_{\mathbb{R}^d} \big(\mathsf{k}_t^{(n)}(x_1,\ldots,x_{i-1},x_i+z,x_{i+1}\ldots,x_n) \\ - \mathsf{k}_t^{(n)}(x_1,\ldots,x_n)\big)\, a(z)\, dz\end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathcal{L}_b^i \mathsf{k}_t^{(n)}(x_1,\ldots,x_n) = \int_{\mathbb{R}^d} \big(\mathsf{k}_t^{(n)}(x_1,\ldots,x_{i-1},x_i+z,x_{i+1}\ldots,x_n) \\ - \mathsf{k}_t^{(n)}(x_1,\ldots,x_n)\big)\, b(z)\, dz\,.\end{aligned} \tag{39}$$

3 Proofs

We derive the a priori bounds for the correlation functions $\mathsf{k}_t^{(n)}(x_1,\ldots,x_n)$ by analyzing Eq. (32) and (37). We fix

$$\nu := \mu - \beta > 0\,. \tag{40}$$

The uniform bounds of Eq. (14) follow from Lemma 4:

Lemma 4. *For an integer $n \geq 1$, define $\|\mathsf{k}^{(n)}\|$ as in Eq. (14). Then*

$$\|\mathsf{k}^{(1)}\| \leq \lambda + \frac{\gamma}{\nu}\,, \tag{41}$$

and, for $n > 1$,

$$\|\mathsf{k}^{(n)}\| \leq \lambda^n + \|\mathsf{k}^{(n-1)}\| \Big(\frac{\gamma}{\nu} + \frac{\beta B}{\nu}\,(n-1)\Big)\,, \tag{42}$$

where

$$B := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \big|\widehat{b}(k)\big|\, dk\,. \tag{43}$$

Using the bounds in Eq. (41) and (42), we deduce that, for all $n \geq 1$,

$$\|\mathsf{k}^{(n)}\| \leq n!\,\big(\lambda + (\gamma + \beta B)/\nu\big)^n\,, \tag{44}$$

which is the bound in Eq. (14). It is thus sufficient to verify Lemma 4.

3.1 First correlation function

We proceed by induction in n and start by considering the case $n = 1$. The first correlation function $\mathsf{k}_t^{(1)}(x)$ satisfies Eq. (32),

$$\frac{\partial \mathsf{k}_t^{(1)}}{\partial t}(x) = \mathcal{L}\,\mathsf{k}_t^{(1)}(x) - \nu\,\mathsf{k}_t^{(1)}(x) + \gamma\,, \qquad \mathsf{k}_0^{(1)}(x) = \lambda\,, \tag{45}$$

where

$$\mathcal{L} := \kappa\mathcal{L}_a + \beta\mathcal{L}_b \tag{46}$$

and ν is as in Eq. (40). For $\nu \neq 0$, the solution of Eq. (45) is

$$\mathsf{k}_t^{(1)}(x) = \frac{\gamma}{\nu} + \left(\lambda - \frac{\gamma}{\nu}\right) e^{-\nu t} \equiv \frac{\gamma}{\mu - \beta} + \left(\lambda - \frac{\gamma}{\mu - \beta}\right) e^{-(\mu-\beta)t}\,, \tag{47}$$

which, for $\nu = \mu - \beta > 0$, implies Eq. (41). By the maximum principle for parabolic equations (Vasy, 2015), $\mathsf{k}_t^{(1)}(x)$ given by Eq. (47) is the only solution to Eq. (45). Due to the spatial homogeneity of Eq. (45), this solution does not depend on the spatial variable x.

The asymptotics of the solution $\mathsf{k}_t^{(1)}(x)$ of Eq. (45) is such that:

1) if $\beta > \mu$, then $\mathsf{k}_t^{(1)}(x) \to \infty$ exponentially as $t \to \infty$;

2) if $\beta = \mu$, then $\mathsf{k}_t^{(1)}(x) \to \infty$ linearly as $t \to \infty$;

3) if $\beta < \mu$, then $\mathsf{k}_t^{(1)}(x) \to \gamma/(\mu - \beta)$ exponentially as $t \to \infty$.

The limit behavior of the solution does not depend on the initial condition $\mathsf{k}_0^{(1)}(x)$. When it is convenient, we assume that $\mathsf{k}_0^{(1)}(x)$ vanishes identically. The assumption $\mu > \beta$ characterizes the region of non-explosive behavior of the first correlation function $\mathsf{k}_t^{(1)}$.

3.2 Induction step

For $n > 1$, denote the single coordinate analogues of the operator in Eq. (46) by

$$\mathcal{L}^i := \kappa\mathcal{L}_a^i + \beta\mathcal{L}_b^i\,, \qquad i = 1, \ldots, n\,, \tag{48}$$

where $\mathcal{L}_a^i$ and $\mathcal{L}_b^i$ are defined as in Eq. (38). Consider the particular case $n = 2$.

3.2.1 Second correlation function

The second correlation function $\mathsf{k}_t^{(2)}$ satisfies the case $n = 2$ of Eq. (37):

$$\begin{aligned}\frac{\partial \mathsf{k}_t^{(2)}}{\partial t}(x_1, x_2) = -2\nu\, \mathsf{k}_t^{(2)}(x_1, x_2) + \big(\mathcal{L}^1 + \mathcal{L}^2\big)\, \mathsf{k}_t^{(2)}(x_1, x_2) \\ + 2\big(\beta\, b(x_1 - x_2) + \gamma\big)\, \mathsf{k}_t^{(1)}\,,\end{aligned} \tag{49}$$

where we used the fact that $b(\,\cdot\,)$ is symmetric and that, by Eq. (47), $\mathsf{k}_t^{(1)}(x) \equiv \mathsf{k}_t^{(1)}$ does not depend on the spatial variable. As the last term in Eq. (49) depends only on $x_1 - x_2$, we deduce that

$$\mathsf{k}_t^{(2)}(x_1, x_2) = f_t(x_1 - x_2) \equiv f_t(x_2 - x_1)\,, \tag{50}$$

with a symmetric function $f_t(\,\cdot\,)$ solving the forward Kolmogorov equation

$$\frac{\partial f_t}{\partial t}(z) = -2\nu\, f_t(z) + 2\mathcal{L}\, f_t(z) + 2(\beta b(z) + \gamma)\, \mathsf{k}_t^{(1)}\,, \qquad f_0(z) = \lambda^2\,. \tag{51}$$

By Duhamel's principle (Vasy, 2015), the solution to Eq. (51) is

$$f_t(z) = \lambda^2 e^{-2\nu t} + 2\int_0^t e^{-2\nu(t-s)} e^{2(t-s)\mathcal{L}}(\beta b(z) + \gamma)\, \mathsf{k}_s^{(1)}\, ds\,. \tag{52}$$

Our analysis of $f_t(z)$ is based on Lemma 5. Recall the generator $\mathcal{L}$ from Eq. (46),

Lemma 5. *The family* $\big\{e^{u\mathcal{L}} : u \geq 0\big\}$ *constitutes a positive semigroup of bounded linear operators. Moreover, if* $\widehat{\mathcal{L}}$ *is the Fourier transform of* $\mathcal{L}$*, then for each real* $u \geq 0$*,*

$$0 \leq \widehat{e^{u\mathcal{L}}} = e^{u\widehat{\mathcal{L}}} \leq 1\,. \tag{53}$$

Proof. With $\mathcal{I}$ denoting the identity operator, denote

$$\big(\mathcal{L} + (\kappa + \beta)\mathcal{I}\big)\psi(x) := \int_{\mathbb{R}^d} \psi(x + z)\, \big(\kappa a(z) + \beta b(z)\big)\, dz\,. \tag{54}$$

The assumptions in Eq. (2) and (5) imply that the right-hand side of Eq. (54) is a bounded positive operator. This property is inherited by the semigroup

$$e^{u\mathcal{L}} = e^{-u(\kappa+\beta)}\, e^{u(\mathcal{L}+(\kappa+\beta)\mathcal{I})}\,. \tag{55}$$

Because the assumptions of Eq. (2) imply $|\widehat{a}(k)| \leq 1$ for all $k \in \mathbb{R}^d$, the Fourier transform of the generator $\mathcal{L}_a$ in Eq. (3) satisfies $\widehat{\mathcal{L}_a} = \widehat{a} - 1 \in [-2, 0]$. Likewise, $\widehat{\mathcal{L}}_b = \widehat{b} - 1 \in [-2, 0]$. By symmetry of $a(\,\cdot\,)$ and $b(\,\cdot\,)$, the right-hand side of Eq. (54) is a convolution. For each function $\psi : \mathbb{R}^d \to \mathbb{R}$,

$$\widehat{\mathcal{L}\psi} = \big(\kappa\widehat{\mathcal{L}_a} + \beta\widehat{\mathcal{L}_b}\big)\widehat{\psi} = \big(\kappa(\widehat{a}-1) + \beta(\widehat{b}-1)\big)\widehat{\psi} \equiv \widehat{\mathcal{L}}\widehat{\psi}\,, \tag{56}$$

where $\widehat{\psi}$ is the Fourier transform of ψ. By induction, $\widehat{\mathcal{L}^n} = \widehat{\mathcal{L}}^n$ for every integer $n \geq 0$, and therefore, for every $u \geq 0$,

$$\widehat{e^{u\mathcal{L}}\psi} = \sum_{n\geq 0} \frac{u^n}{n!} \widehat{\mathcal{L}^n}\,\widehat{\psi} = e^{u\widehat{\mathcal{L}}}\,\widehat{\psi} = e^{u(\kappa(\widehat{a}-1)+\beta(\widehat{b}-1))}\,\widehat{\psi}\,, \tag{57}$$

from which we deduce Eq. (53). □

By Eq. (5), the Fourier transform $\widehat{b}(k)$ is integrable. Therefore, for every $z \in \mathbb{R}^d$ and $u \geq 0$,

$$e^{u\mathcal{L}} b(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{e^{u\mathcal{L}}}\,\widehat{b}(k) e^{-i(k,z)}\, dk \tag{58}$$

is well defined. Eq. (58) and Eq. (53) imply the uniform bound

$$\big|e^{u\mathcal{L}} b(z)\big| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \big|\widehat{e^{u\mathcal{L}}}\,\widehat{b}(k)\big|\, dk \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \big|\widehat{b}(k)\big|\, dk =: B\,. \tag{59}$$

To study the large-time behavior of the function $f_t(z)$ in Eq. (52), we use the fact that, as $t \to \infty$,

$$\begin{aligned} \int_0^t e^{-2\nu(t-s)}\,ds &= \int_0^t e^{-2\nu u}\,du = \frac{1}{2\nu}\left(1 - e^{-2\nu t}\right) \to \frac{1}{2\nu}\,, \\ \int_0^t e^{-2\nu(t-s)}\,e^{-\nu s}\,ds &= e^{-\nu t}\int_0^t e^{-\nu(t-s)}\,ds = O\bigl(e^{-\nu t}\bigr) \to 0\,. \end{aligned} \tag{60}$$

Eq. (53) and (59) imply that the absolute value of the integral in Eq. (52) is bounded by

$$\int_0^t e^{-2\nu(t-s)}\bigl(\beta\,\bigl|e^{2(t-s)\mathcal{L}}b(z)\bigr| + \gamma\bigr)\,\|\mathsf{k}_s^{(1)}\|\,ds \le \bigl(\beta B + \gamma\bigr)\bigl\|\mathsf{k}^{(1)}\bigr\|\int_0^t e^{-2\nu(t-s)}\,ds\,. \tag{61}$$

As the first term on the right-hand side of Eq. (52) decays exponentially, Eq. (60) and (61) imply the case $n = 2$ of the induction estimate in Eq. (42).

To derive the limit of $f_t(\,\cdot\,)$ as $t \to \infty$, we use the fact that Eq. (60) implies

$$\begin{aligned} &\Bigl|\int_0^t e^{-2\nu(t-s)}e^{2(t-s)\mathcal{L}}(\beta b(z) + \gamma)\,(\mathsf{k}_s^{(1)} - \gamma/\nu)\,ds\Bigr| \\ &\qquad \le (\beta B + \gamma)\,\bigl|\lambda - \gamma/\nu\bigr|\int_0^t e^{-2\nu(t-s)}e^{-\nu s}\,ds = O\bigl(e^{-\nu t}\bigr) \to 0\,. \end{aligned} \tag{62}$$

Therefore the large-time behavior of the integral in Eq. (52) comes from the constant term $\gamma/\nu \equiv \mathsf{k}_\infty^{(1)}$ in Eq. (47).

For a function $\psi : \mathbb{R}^d \to \mathbb{R}$ with integrable Fourier transform $\widehat{\psi}$, denote

$$(\mathcal{E}\psi)(z) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\widehat{\psi}(k)\,e^{-i(k,z)}}{\nu - \widehat{\mathcal{L}}(k)}\,dk\,, \tag{63}$$

which is well defined as $-\widehat{\mathcal{L}}(k) \ge 0$ for all $k \in \mathbb{R}^d$. Using the relation

$$\int_0^t e^{-2\nu(t-s)}e^{2(t-s)\mathcal{L}}b(z)\;ds = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{b}(k)\,e^{-i(k,z)}\int_0^t e^{-2(\nu-\hat{\mathcal{L}})u}\,du\,dk \tag{64}$$

and the inequality

$$\Bigl|\int_0^t e^{-2(\nu-\hat{\mathcal{L}})u}\,du - \frac{1}{2(\nu - \widehat{\mathcal{L}}(k))}\Bigr| \le \frac{1}{2\nu}\,e^{-2\nu t}\,, \tag{65}$$

following from the first property in Eq. (60), we obtain the bound

$$\Big|2\int_0^t e^{-2\nu(t-s)}e^{2(t-s)\mathcal{L}}b(z)\;ds-(\mathcal{E}b)(z)\Big|\le\frac{B}{\nu}\,e^{-2\nu t} \tag{66}$$

and deduce that, as $t\to\infty$,

$$\Big|\mathsf{k}_t^{(2)}(x_1,x_2)-\frac{\gamma^2}{\nu^2}-\frac{\beta\gamma}{\nu}\big(\mathcal{E}b\big)(x_1-x_2)\Big|=O\big(e^{-\nu t}\big)\,, \tag{67}$$

uniformly in $z\in\mathbb{R}^d$. Theorem 1 with $n=2$ follows.

3.2.2 Higher-order correlation functions

We solve the Kolmogorov Eq. (37) similarly. Denoting

$$\mathcal{L}_n:=\sum_{i=1}^n\mathcal{L}^i\equiv\sum_{i=1}^n\big(\kappa\mathcal{L}_a^i+\beta\mathcal{L}_b^i\big) \tag{68}$$

and applying Duhamel's principle (Vasy, 2015), we represent its solution as

$$\begin{aligned}\mathsf{k}_t^{(n)}(\mathrm{x}_n)&=\lambda^n\,e^{-n\nu t}\\&\quad+\int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)}\sum_{i=1}^n\Big(\beta\sum_{j:j\ne i}b(x_i-x_j)+\gamma\Big)\,\mathsf{k}_s^{(n-1)}(\mathrm{x}_{n,i})\,ds\,.\end{aligned} \tag{69}$$

As in the case $n=2$, we upper bound the absolute value of the last integral by

$$\begin{aligned}\big\|\mathsf{k}^{(n-1)}\big\|\int_0^t e^{-n\nu(t-s)}\big(n\gamma+\beta\sum_{i\ne j:i,j=1,\dots,n}\big|e^{(t-s)\mathcal{L}_n}b(x_i-x_j)\big|\big)\,ds\\ \le\big\|\mathsf{k}^{(n-1)}\big\|\big(\tfrac{\gamma}{\nu}+\tfrac{\beta B}{\nu}(n-1)\big)\,,\end{aligned} \tag{70}$$

where the estimate in Eq. (59) is used for each pair (i,j) with $i\ne j$, $i,j=1,\dots,n$. Together with the bound λ^n on the "initial condition" term in Duhamel's representation of Eq. (69), we deduce Eq. (42).

3.3 Convergence of the correlation functions

We extend the argument of the previous section to estimate the speed of convergence of the correlation functions. For every integer $n \geq 1$ and the non-positive operator $\mathcal{L}_n$ from Eq. (68), consider the resolvent

$$\mathcal{R}_n^\nu \equiv \mathcal{R}_n(\nu) := \left(n\nu - \mathcal{L}_n\right)^{-1}, \tag{71}$$

where, as in Eq. (40), $\nu = \mu - \beta > 0$. We recursively define

$$\mathsf{k}_\infty^{(1)}(\mathrm{x}_1) := \frac{\gamma}{\nu}, \qquad \mathrm{x}_1 \in \mathbb{R}^d, \tag{72}$$

and, using x_n and $\mathrm{x}_{n,i}$ defined in section 2,

$$\mathsf{k}_\infty^{(n)}(\mathrm{x}_n) := \mathcal{R}_n^\nu\Big(\sum_{i=1}^n \Big(\beta \sum_{j:j\neq i} b(x_i - x_j) + \gamma\Big)\mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\Big). \tag{73}$$

In terms of differences

$$\tilde{\mathsf{k}}_s^{(n)}(\mathrm{x}_n) := \mathsf{k}_s^{(n)}(\mathrm{x}_n) - \mathsf{k}_\infty^{(n)}(\mathrm{x}_n), \tag{74}$$

we have Proposition 6.

Proposition 6. *There exists a positive sequence $(C_n)_{n\geq 1}$ such that, for all $t \geq 0$,*

$$\sup_{\mathrm{x}_n \in (\mathbb{R}^d)^n} \left|\tilde{\mathsf{k}}_t^{(n)}(\mathrm{x}_n)\right| \leq C_n \left\|\mathsf{k}^{(n)}\right\| e^{-\nu t}. \tag{75}$$

This implies that, as $t \to \infty$, the correlation functions $\mathsf{k}_t^{(n)}(\,\cdot\,)$ converge exponentially to their limits $\mathsf{k}_\infty^{(n)}(\,\cdot\,)$ introduced in Eq. (73). In particular, the family $\{\mathsf{k}_\infty^{(n)}(\,\cdot\,)\}_{n\geq 1}$ satisfies the Carleman condition in Eq. (18) and thus corresponds to a unique steady state for the model of section 1.1.

Proof. Using Duhamel's formula in Eq. (69) and the decomposition of Eq. (74), we use mathematical induction to prove inequality (75). The argument defines the sequence $(C_n)_{n\geq 1}$ recursively.

For $n = 1$, the claim is true with $C_1 = 1$. To verify the induction step, we consider the contribution of immigration and birth in Eq. (69) separately.

Because each $\widetilde{\mathcal{L}}^i := \mathcal{L}^i + (\kappa+\beta)\mathcal{I}$ is a non-negative integral operator (as is $\widetilde{\mathcal{L}}_n := \mathcal{L}_n + n(\kappa+\beta)\mathcal{I}$), the difference

$$\begin{aligned}\int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)} \sum_{i=1}^n \mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\, ds - \mathcal{R}_n^\nu\Big(\sum_{i=1}^n \mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\Big) \\ = \int_t^\infty e^{-(n\nu-\mathcal{L}_n)v} \sum_{i=1}^n \mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\, dv\end{aligned} \tag{76}$$

is upper bounded in absolute value by

$$\begin{aligned}\int_t^\infty e^{-n(\nu+\kappa+\beta)v}\, e^{v\widetilde{\mathcal{L}}_n} \sum_{i=1}^n \big|\mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\big| &\le n \int_t^\infty e^{-n(\nu+\kappa+\beta)v}\, e^{v\widetilde{\mathcal{L}}_n} \big\|\mathsf{k}^{(n-1)}\big\|\, dv \\ &= n \int_t^\infty e^{-n\nu v}\, e^{v\mathcal{L}_n} \big\|\mathsf{k}^{(n-1)}\big\|\, dv \\ &= n\,\big\|\mathsf{k}^{(n-1)}\big\| \int_t^\infty e^{-n\nu v}\, dv = \frac{1}{\nu}\,\big\|\mathsf{k}^{(n-1)}\big\|\, e^{-n\nu t}\,.\end{aligned} \tag{77}$$

By the induction hypothesis,

$$\begin{aligned}\Big|\int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)} \sum_{i=1}^n \tilde{\mathsf{k}}_s^{(n-1)}(\mathrm{x}_{n,i})\, ds\Big| \\ &\le nC_{n-1} \int_0^t e^{-n(\nu+\kappa+\beta)(t-s)}\, e^{\widetilde{\mathcal{L}}_n(t-s)} \big\|\mathsf{k}^{(n-1)}\big\| e^{-\nu s}\, ds \\ &\le nC_{n-1}\big\|\mathsf{k}^{(n-1)}\big\| \int_0^t e^{-n\nu(t-s)-\nu s}\, ds \\ &\le \frac{nC_{n-1}}{(n-1)\nu}\big\|\mathsf{k}^{(n-1)}\big\| e^{-\nu t} \le \frac{2C_{n-1}}{\nu}\big\|\mathsf{k}^{(n-1)}\big\| e^{-\nu t}\,.\end{aligned} \tag{78}$$

Together with the bound of Eq. (77), this yields

$$\begin{aligned}\Big|\gamma \int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)} \sum_{i=1}^n \mathsf{k}_s^{(n-1)}(\mathrm{x}_{n,i})\, ds - \gamma\mathcal{R}_n^\nu\Big[\sum_{i=1}^n \mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\Big]\Big| \\ \le \frac{\gamma}{\nu}\big\|\mathsf{k}^{(n-1)}\big\|\big(2C_{n-1} + e^{-(n-1)\nu t}\big)e^{-\nu t} \le \frac{\gamma}{\nu}\big(2C_{n-1}+1\big)\big\|\mathsf{k}^{(n-1)}\big\| e^{-\nu t}\,.\end{aligned} \tag{79}$$

Likewise, with $\sum_{j\to i} := \sum_i \sum_{j:j\neq i}$ denoting the sum over all configurations where the particle at x_i is born by the particle at x_j,

$$\begin{aligned}\Big|\int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)} \sum_{j\to i} b(x_i-x_j)\tilde{\mathsf{k}}_s^{(n-1)}(\mathrm{x}_{n,i})\,ds\Big| & \\ \le n(n-1)C_{n-1}\|b\|\,\big\|\mathsf{k}^{(n-1)}\big\|\int_0^t e^{-n\nu(t-s)-\nu s}\,ds & \qquad (80)\\ \le \frac{nC_{n-1}}{\nu}\|b\|\,\big\|\mathsf{k}^{(n-1)}\big\|e^{-\nu t}\,, &\end{aligned}$$

where $\|b\|:=\sup_x|b(x)|$ is a finite constant, and

$$\begin{aligned}\Big|\int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)} \sum_{j\to i} b(x_i-x_j)\mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\,ds & \\ -\mathcal{R}_n^\nu\Big(\sum_{j\to i} b(x_i-x_j)\mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\Big)\Big| & \qquad (81)\\ \le \frac{2(n-1)}{\nu}\|b\|\,\big\|\mathsf{k}^{(n-1)}\big\|e^{-n\nu t}\,, &\end{aligned}$$

implying that

$$\begin{aligned}\Big|\beta\int_0^t e^{-(n\nu-\mathcal{L}_n)(t-s)} \sum_{j\to i} b(x_i-x_j)\mathsf{k}_s^{(n-1)}(\mathrm{x}_{n,i})\,ds & \\ -\beta\mathcal{R}_n^\nu\Big(\sum_{j\to i} b(x_i-x_j)\mathsf{k}_\infty^{(n-1)}(\mathrm{x}_{n,i})\Big)\Big| & \qquad (82)\\ \le \frac{\beta n}{\nu}(C_{n-1}+2)\|b\|\,\big\|\mathsf{k}^{(n-1)}\big\|e^{-\nu t}\,. &\end{aligned}$$

Finally, inequality (75) follows with $C_n=\frac{2(C_{n-1}+1)}{\nu}\big(\gamma+\beta\|b\|n\big)$. □

The relation of Eq. (73) allows a description of the limiting correlation functions $\big\{\mathsf{k}_\infty^{(n)}(x_1,\ldots,x_n)\big\}_{n\ge1}$ in terms of the family of all directed graphs on the vertices x_1, …, x_n, where the directed edges indicate parental relations. Such graphs are known in combinatorics as directed forests.

4 Conclusion

The population dynamics introduced in section 1.1 is Lyapunov stable in that its qualitative behavior is unchanged under suitable perturbations of the main parameters of the model. For each value of the immigration rate, the finite-time distribution of the model converges exponentially to a unique steady state. The density of this steady state increases with the immigration rate.

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