TRIVALENT EXPANDERS AND HYPERBOLIC SURFACES

I. IVRISSIMTZIS, N. PEYERIMHOFF AND A. VDOVINA

Abstract. We construct a new family of trivalent expanders tessellating hyperbolic surfaces with large isometry groups. These graphs are obtained from a family of Cayley graphs of nilpotent groups via $(\Delta - Y)$-transformations. We study combinatorial, topological and spectral properties of our trivalent graphs and their associated hyperbolic surfaces. We compare this family with Platonic graphs and their associated hyperbolic surfaces and see that they are generally very different with only one hyperbolic surface in the intersection. Finally, we provide a number theory free proof of the Ramanujan property for Platonic graphs and a special family of subgraphs.

1. Introduction

In this article we introduce a new family T_k of trivalent surface tessellations derived via $\Delta - Y$ transformations from Cayley graphs of groups G_k associated to a Euclidean building and we investigate combinatorial topological properties of their underlying surfaces $\mathcal{S}_k$. These graphs T_k form a family of trivalent expanders and give rise to another family of associated surfaces (tubes around them constructed via Y-pieces) with a uniform lower bound on their first positive Laplace-Beltrami eigenvalue.

Another prominent in the literature example where such an interplay between groups, graphs and hyperbolic surfaces has been utilized, are finite quotients of $\mathrm{PSL}(2, \mathbb{Z})$ and co-compact arithmetic lattices in $\mathrm{PSL}(2, \mathbb{R})$ (see, e.g., Buser [9], Brooks [6], and Lubotzky [19] and the references therein). While many finite quotients of these examples are *simple*, all our finite groups G_k are *nilpotent* and very different in nature.

A well known family of such surface tessellations associated to finite quotients of $\mathrm{PSL}(2, \mathbb{Z})$ are the Platonic graphs Π_N (see [13, 17]). We show that, while our graph T_2 in the surface $\mathcal{S}_2$ is dual to the Platonic

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graph Π_8, there is no direct relation between our family of graphs T_k and these Platonic graphs Π_N from $k \geq 3$ onwards. Finally, we provide an alternative number theory free proof of Gunnell's Theorem of the spectra of Platonic graphs Π_p, p prime, and a family of induced subgraphs Π'_p of them.

1.1. **Statement of results.** In Section 2, we present the construction of our family of trivalent graphs T_k. They are $(\Delta - Y)$-transformations of 6-valent Cayley graphs X_k of increasing nilpotent 2-groups G_k. The details of the construction of these groups are given in Section 2.1. The $(\Delta - Y)$-transformations are standard operations to simplify electrical circuits, and were also used in [2] in connection with Colin de Verdiére's graph parameter.

The graphs T_k can be naturally embedded as tessellations into both closed hyperbolic surfaces $\mathcal{S}_k$ and complete non-compact finite area hyperbolic surfaces $\mathcal{S}_k^\infty$. The edges of the tessellation of $\mathcal{S}_k$ are geodesic arcs and the vertices are their end points. The details of these embeddings are presented in Subsection 2.2 via covering theory. (An alternative direct construction is given in Subsection 2.4.) The following proposition describes the combinatorial properties of the tessellations $T_k \subset \mathcal{S}_k$.

Proposition 1.1. *Let $k \geq 2$. Then the generators x_0, x_1, x_3 of G_k have all the same order 2^{n_k} with*

$$n_k = \lfloor \log_2 k \rfloor + 1. \tag{1}$$

Let $|G_k| = 2^{N_k}$ and V_k, E_k and F_k denote the sets of vertices, edges, and faces of the tesselation $T_k \subset \mathcal{S}_k$, respectively. Then the isometry group of $\mathcal{S}_k$ has order $\geq 2^{N_k}$, we have

$$|V_k| = 2^{N_k+1}, \quad |E_k| = 3 \cdot 2^{N_k} \quad \text{and} \quad |F_k| = 3 \cdot 2^{N_k - n_k}, \tag{2}$$

and all faces of $T_k \subset \mathcal{S}_k$ are regular hyperbolic 2^{n_k+1}-gons with interior angles $2\pi/3$. Moreover, the genus of $\mathcal{S}_k$ is given by

$$g(\mathcal{S}_k) = 1 + 2^{N_k - n_k - 1}(2^{n_k} - 3).$$

A lower bound for the order 2^{N_k} of the groups G_k was given in [21, Cor. 2.3]:

$$N_k \geq 8\lfloor k/3 \rfloor + 3 \cdot (k \bmod 3) - 1, \tag{3}$$

where $k \bmod 3 \in \{0, 1, 2\}$.

It is easily checked via Euler's polyhedral formula that any triangulation X of a compact oriented surface $\mathcal{S}$ satisfies

$$|E(X)| = 3(|V(X)| - 2) + 6g(\mathcal{S}),$$

i.e., the number of edges $|E(X)|$ of every triangulation X with at least two vertices is $\geq 6g(\mathcal{S})$. Therefore, the ratio

$$\frac{6g(\mathcal{S})}{|E(X)|} \leq 1$$

measures the non-flatness of such a triangulation, i.e., how efficiently the edges of X are chosen to generate a surface of high genus. Note that, for every $k \geq 2$, the dual graph T_k^* can be viewed as a triangulation of $\mathcal{S}_k$ and that the number of edges of T_k and T_k^* coincide. Then we have the following asymptotic result, proved in Section 2.5.

Proposition 1.2. *We have*

$$\lim_{k\to\infty} \frac{6g(\mathcal{S}_k)}{|E(T_k^*)|} = 1, \tag{4}$$

where $E(T_k^*)$ denotes the set of edges of T_k^*.

In Section 3 we investigate spectral properties. Our first result is that both families of graphs T_k and X_k are expanders:

Theorem 1.3. *Let $k \geq 2$. Then every eigenvalue $\lambda \in [-3, 3]$ of T_k gives rise to an eigenvalue $\mu = \lambda^2 - 3 \in [-3, 6]$ on X_k. In particular, there exists a positive constant $C < 6$ such that*

(i) *the graphs X_k are 6-valent expanders with spectrum in $[-3, C] \cup \{6\}$,*
(ii) *the bipartite graphs T_k are trivalent expanders with spectrum in $[-\sqrt{C+3}, \sqrt{C+3}] \cup \{\pm 3\}$.*

The spectral properties of the graphs T_k carry over to the Laplace-Beltrami operator of associated hyperbolic surfaces. Using the Brooks-Burger transfer principle it can be shown that the first positive eigenvalues $\lambda_1(\mathcal{S}_k)$ of the underlying closed surfaces $\mathcal{S}_k$ have a uniform lower bound:

Theorem (see [14, Theorem 1.3])**.** *There is a positive constant $C_1 > 0$ such that we have for the compact hyperbolic surfaces $\mathcal{S}_k$ ($k \geq 2$),*

$$\lambda_1(\mathcal{S}_k) \geq C_1.$$

In Section 3.3 of this paper we introduce another family $\widehat{\mathcal{S}_k}$ of closed surfaces associated to T_k by glueing together special Y-pieces whose boundary curves have all length 2, as described in [8, Section 3]. It follows from Buser's results [8] that their smallest positive Laplace eigenvalue has also a uniform positive lower bound:

Proposition 1.4. *The compact hyperbolic surfaces $\widehat{\mathcal{S}}_k$ ($k \geq 2$) have genus $1 + |V_k|/2$ and isometry groups of order $\geq |V_k|/2$. They form a tower of coverings*

$$\cdots \longrightarrow \widehat{\mathcal{S}}_{k+1} \longrightarrow \widehat{\mathcal{S}}_k \longrightarrow \widehat{\mathcal{S}}_{k-1} \longrightarrow \cdots,$$

where all the covering indices are powers of 2. There is a positive constant $C_2 > 0$ such that we have, for all k,

$$\lambda_1(\widehat{\mathcal{S}}_k) \geq C_2.$$

Proposition 1.4 is proved in Section 3.3. There is a well-known classical result by Randol [22] which is, in some sense, complementary to this proposition, namely, there exist finite coverings $\widetilde{\mathcal{S}}$ of every closed hyperbolic surface $\mathcal{S}$ with arbitrarily small first positive Laplace eigenvalue.

In Section 4 we compare our tessellations $T_k \subset S_k$ to the well studied tessellations of hyperbolic surfaces by *Platonic graphs* Π_N. It turns out that both families agree in one tessellation (up to duality) but are, otherwise, very different. Let us first define the graphs Π_N. Let N be a positive integer ≥ 2. The vertices of Π_N are equivalence classes $[\lambda, \mu] = \{\pm(\lambda, \mu)\}$ with

$$\{(\lambda, \mu) \in \mathbb{Z}_N \times \mathbb{Z}_N \mid \gcd(\lambda, \mu, N) = 1\}.$$

Two vertices $[\lambda, \mu]$ and $[\nu, \omega]$ are connected by an edge if and only if

$$\det \begin{pmatrix} \lambda & \nu \\ \mu & \omega \end{pmatrix} = \lambda\omega - \mu\nu = \pm 1.$$

As mentioned earlier, Π_8 is isomorphic to the dual of T_2 in $\mathcal{S}_2$. Since the valence of the dual graph T_k^* is a power of 2, any isomorphism of T_k^* with a Platonic graph Π_N would imply $N = 2^{n_k+1}$ with n_k given in (1). However, this leads to a contradiction *for all* $k \geq 3$. The next proposition summarizes these results.

Proposition 1.5. *The Platonic graph Π_8 is isomorphic to the dual of T_2 in the* unique *compact genus 5 hyperbolic surface $\mathcal{S}_2$ with maximal automorphism group of order 192. For $k \geq 3$, there is no graph isomorphism between T_k^* and Π_N, for any $N \geq 2$.*

In Section 5 we derive spectral properties of the graphs Π_p, p prime, and special induced subgraphs Π_p'. Π_p' is obtained from Π_p by removing the set of vertices $[\lambda, 0]$ with vanishing second coordinate and all their adjacent edges. We will give an alternative number theory free proof of the following result for Π_p:

Theorem (Gunnells [13, Theorem 4.2]). *Let p be an odd prime. The graph Π_p has the following spectrum:*

(i) *p with multiplicity one,*
(ii) *-1 with multiplicity p, and*
(iii) *$\pm\sqrt{p}$ with multiplicity $(p-1)^2/4-1$, each.*

In particular, the graph Π_p is Ramanujan.

To our knowledge, all proofs for the *Ramanujan property* of the graphs Π_p in the literature (see, e.g., [13, 18, 12]) are based on some amount of number theory (characters of representations). We think it is remarkable that there is also an easy proof for the Ramanujan properties of the graphs Π_p and Π'_p with no reference to number theory other than the irrationality of $\sqrt{p}$ (see Section 5.3). Moreover, our number theory free arguments apply also to the graphs Π'_p:

Theorem 1.6. *Let p be an odd prime. Then the graphs Π_p and Π'_p have diameter 3 and maximal vertex connectivity p and $p-1$, respectively. Moreover, the spectrum of the graph Π'_p is given by*

(i) *$p-1$ with multiplicity one,*
(ii) *-1 with multiplicity $p-1$,*
(iii) *0 with multiplicity $(p-3)/2$, and*
(iv) *$\pm\sqrt{p}$ with multiplicity $(p-1)(p-3)/4$, each.*

In particular, the graph Π'_p is Ramanujan.

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2. Combinatorial properties of the tessellations $\mathcal{T}_k \subset \mathcal{S}_k$

Let $\widetilde{G}$ be the infinite group of seven generators and seven relations given by

$$\widetilde{G} = \langle x_0, \dots, x_6 \mid x_i x_{i+1} x_{i+3} \text{ for } i = 0, \dots, 6 \rangle, \tag{5}$$

where the indices are taken modulo 7. As explained in [10, Thm 3.4], this group acts simply transitively on the vertices of a thick Euclidean building of type $\tilde{A}_2$. Let $S = \{x_0^{\pm 1}, x_1^{\pm 1}, x_3^{\pm 1}\}$. We consider the index two subgroup $G \leq \widetilde{G}$, generated by S. (Note that $x_3 = x_1^{-1} x_0^{-1}$.) G is explicitly given by $G = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle$ with

$$\begin{aligned} r_1(x_0,x_1) &= (x_1x_0)^3x_1^{-3}x_0^{-3}, \\ r_2(x_0,x_1) &= x_1x_0^{-1}x_1^{-1}x_0^{-3}x_1^2x_0^{-1}x_1x_0x_1, \\ r_3(x_0,x_1) &= x_1^3x_0^{-1}x_1x_0x_1x_0^2x_1^2x_0x_1x_0. \end{aligned} \tag{6}$$

For more details we refer the readers to [21].

2.1. **A faithful matrix representation of G.** Let us first recall the faithful representation of G by infinite upper triangular Toeplitz matrices, given in [21] and based on representations introduced in [10]. In fact, every $x \in G$ has a representation of the form

$$x = \begin{pmatrix} 1 & a_{11} & a_{21} & \dots & a_{k1} & 0 & 0 & \dots & \dots \\ 0 & 1 & a_{12} & a_{22} & \dots & a_{k2} & 0 & \ddots & \\ 0 & 0 & 1 & a_{13} & a_{23} & \dots & a_{k3} & 0 & \ddots \\ \vdots & \ddots & 0 & 1 & a_{11} & a_{21} & \dots & a_{k1} & \ddots \\ \vdots & & \ddots & \ddots & 1 & a_{12} & a_{22} & \dots & \ddots \\ \vdots & & & \ddots & \ddots & 1 & a_{13} & a_{23} & \ddots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{7}$$

where each element a_{ij} is in the set $M(3, \mathbb{F}_2)$ of (3×3)-matries with entries in $\mathbb{F}_2$, and 0 and 1 stand for the zero and identity matrix in $M(3, \mathbb{F}_2)$. Note the periodic pattern in the upper diagonals of the matrix, i.e., the j-th upper diagonal is uniquely determined by the first three entries $a_j = (a_{j1}, a_{j2}, a_{j3})$, which can be understood as a (3×9)-matrix with values in $\mathbb{F}_2$. We use the short-hand notation $M_0(a_1, a_2, \dots, a_k)$ for the matrix in (7). If the first l upper diagonals in (7) vanish, we also write $M_l(a_{l+1}, \dots, a_k)$. Let G^k be the subgroup of all elements $x \in G$ with vanishing first k upper diagonals in their matrix representation. It follows from the structure of these matrices that G^k is normal and that the quotient group $G_k = G/G^k$ is a 2-group, i.e., nilpotent.

Recall that we use the same notation for the generators x_0, x_1, x_3 of G and their images in the quotient G_k. We will see later that the faces of the tessellation $T_k \subset \mathcal{S}_k$ are determined by the orders of these generators in G_k. We will now determine these orders. Let

$$\alpha_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&1&1&0&1&1&0&1&1 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&1&1&0&1&1&0&1&1 \\ 0&1&0&0&1&0&0&1&0 \end{pmatrix},$$

$$\alpha_1 = \begin{pmatrix} 0&0&0&0&1&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&1&1&0&0&0&0&1&0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0&0&0&0&1&1&0&1&0 \\ 0&1&0&1&0&0&0&0&1 \\ 1&1&1&0&0&0&0&1&0 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0&0&0&0&1&1&0&1&0 \\ 0&1&1&1&0&1&0&0&0 \\ 1&0&0&0&1&1&0&0&1 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0&0&0&0&0&1&0&1&1 \\ 1&1&0&0&1&1&0&0&0 \\ 0&1&1&0&0&1&1&0&0 \end{pmatrix}.$$

Then we have $x_i = M_0(\alpha_i, \dots)$ for $i = 0, 1, 3$, and we obtain the following fact about the leading diagonal of their 2-powers.

Lemma 2.1. *We have for $i \in \{0, 1, 3\}$ and $l \geq 0$:*

$$x_i^{2^l} = \begin{cases} M_{2^l-1}(\alpha_i, \dots), & \text{if } l \text{ is even,} \\ M_{2^l-1}(\beta_i, \dots), & \text{if } l \text{ is odd.} \end{cases} \tag{8}$$

This implies, in particular, for $k \in \mathbb{N}$ that the order of x_i in G_k is 2^{n_k} with n_k given in (1)*.*

Proof. Since G_k is a 2-group, $\mathrm{ord}_{G_k}(x_i)$ has to be a power of 2. The formulas (8) follow via a straightforward calculation using Prop. 2.5 in [21]. This implies that $\mathrm{ord}_{G_k}(x_i) = 2^l$ if and only if $2^{l-1} \leq k < 2^l$, i.e., $l = \lfloor \log_2 k \rfloor + 1 = n_k$. □

2.2. **The surfaces $\mathcal{S}_k$ and $\mathcal{S}_k^\infty$ via covering theory.** Let X_k be the Cayley graph $\mathrm{Cay}(G_k, S)$. We will now explain how to construct the closed hyperbolic surfaces $\mathcal{S}_k$: We start with an orbifold $\mathcal{S}_0$ by gluing together two *compact* hyperbolic triangles $\mathcal{T}_1, \mathcal{T}_2 \subset \mathbb{H}^2$ with angles $\pi/\mathrm{ord}_{G_k}(x_0), \pi/\mathrm{ord}_{G_k}(x_1)$ and $\pi/\mathrm{ord}_{G_k}(x_3)$ along their corresponding sides. Both triangles are equilateral since, by Lemma 2.1, we have $\mathrm{ord}_{G_k}(x_0) = \mathrm{ord}_{G_k}(x_1) = \mathrm{ord}_{G_k}(x_3) = 2^{n_k}$. It is useful to think of the two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathcal{S}_0$ to be coloured black and white, respectively. Let $P_0, P_1, P_2 \in \mathcal{S}_0$ be the singular points (i.e., the identified vertices of the triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathcal{S}_0$) and $Q \in \mathcal{S}_0$ be the center of the white triangle $\mathcal{T}_1$. Note that $\mathcal{S}_0 \backslash \{P_0, P_1, P_2\}$ carries a hyperbolic metric induced by the triangles $\mathcal{T}_1, \mathcal{T}_2$. Choose a geometric basis $\gamma_0, \gamma_1, \gamma_2$ of the fundamental group $\pi_1(\mathcal{S}_0 \backslash \{P_0, P_1, P_2\}, Q)$ such that γ_i is a simple loop (starting and ending at Q) around the singular point $P_i \in \mathcal{S}_0$ and $\gamma_0\gamma_1\gamma_2 = e$. Note that $\pi_1(\mathcal{S}_0 \backslash \{P_0, P_1, P_2\}, Q)$ is a free group in the generators γ_0, γ_1. The surjective homomorphism

$$\Psi : \pi_1(\mathcal{S}_0 \backslash \{P_0, P_1, P_2\}, Q) \to G_k,$$

given by $\Psi(\gamma_0) = x_0$, $\Psi(\gamma_1) = x_1$ and $\Psi(\gamma_2) = x_3$, induces a branched covering $\pi : \mathcal{S}_k \to \mathcal{S}_0$, by Riemann's existence theorem (see [25, Thms. 4.27 and 4.32] or [4, (17)]) with all ramification indices equals 2^{n_k} and, therefore, the closed surface $\mathcal{S}_k$ carries a hyperbolic metric such that the restriction

$$\pi : \mathcal{S}_k \backslash \pi^{-1}(\{P_0, P_1, P_2\}) \to \mathcal{S}_0 \backslash \{P_0, P_1, P_2\}$$

is a Riemannian covering. The surface $\mathcal{S}_k$ is a Belyi surface since it is a branched covering over $\mathcal{S}_0$ ramified at the three points P_0, P_1, P_2. Moreover, $\mathcal{S}_k$ is tessellated by $2|G_k| = 2^{N_k+1}$ equilateral hyperbolic triangles, half of them black and the others white. Hurwitz's formula yields

$$g(\mathcal{S}_k) = 1 + \frac{1 - \mu_k}{2} |G_k|, \tag{9}$$

where

$$\mu_k = \frac{1}{\mathrm{ord}(x_0)} + \frac{1}{\mathrm{ord}(x_1)} + \frac{1}{\mathrm{ord}(x_3)} = \frac{3}{2^{n_k}}. \tag{10}$$

Recall that the orders 2^{n_k} of x_i $(i = 0, 1, 3)$ were given in Lemma 2.1. In the case $k = 2$ we have $|G_2| = 32$ and $\mathrm{ord}(x_0) = \mathrm{ord}(x_1) = \mathrm{ord}(x_3) = 4$, which leads to

$$g(\mathcal{S}_2) = 1 + \frac{1}{8} \cdot 32 = 5.$$

G_k acts simply transitive on the black triangles of $\mathcal{S}_k$. Let

$$V = \pi^{-1}(Q) \tag{11}$$

and $V_{\mathrm{white}}, V_{\mathrm{black}} \subset V$ be the sets of centers of white and black triangles, respectively. Choose a reference point $Q_0 \in V_{\mathrm{white}}$, and identify the vertices of the Cayley graph $X_k = \mathrm{Cay}(G_k, S)$ with the points in V_{white} by $G_k \ni h \mapsto hQ_0 \in V_{\mathrm{white}}$. Then two adjacent vertices in X_k are the centers of two white triangles which share a black triangle as their common neighbour. The corresponding edge in X_k can then be represented by the minimal geodesic passing through these three triangles and connecting these two vertices. Moreover, G_k acts on the surface $\mathcal{S}_k$ by isometries and we have $\mathcal{S}_0 = \mathcal{S}_k / G_k$, i.e., the isometry group of $\mathcal{S}_k$ has order $\geq |G_k| = 2^{N_k}$.

We could also start the process by gluing together two *ideal* hyperbolic triangles $\mathcal{T}_1^\infty$ and $\mathcal{T}_2^\infty$, coloured black and white, along their corresponding edges. Each edge of $\mathcal{T}_i^\infty$ $(i = 1, 2)$ has a unique intersection point *(tick mark)* with the incircle of the triangle, and these tick-marks of corresponding edges of $\mathcal{T}_1^\infty$ and $\mathcal{T}_2^\infty$ are identified under the gluing. The resulting surface $\mathcal{S}_0^\infty$ is topologically a 3-punctured sphere, carrying a complete hyperbolic metric of finite volume, and the same arguments as above then lead to an embedding of the graphs X_k into complete non-compact finite area hyperbolic surfaces $\mathcal{S}_k^\infty$, triangulated by ideal black and white triangles, where the vertices of X_k correspond to the centers of the white ideal triangles in $\mathcal{S}_k^\infty$, and we have $\mathcal{S}_0^\infty = \mathcal{S}_k^\infty / G_k$.

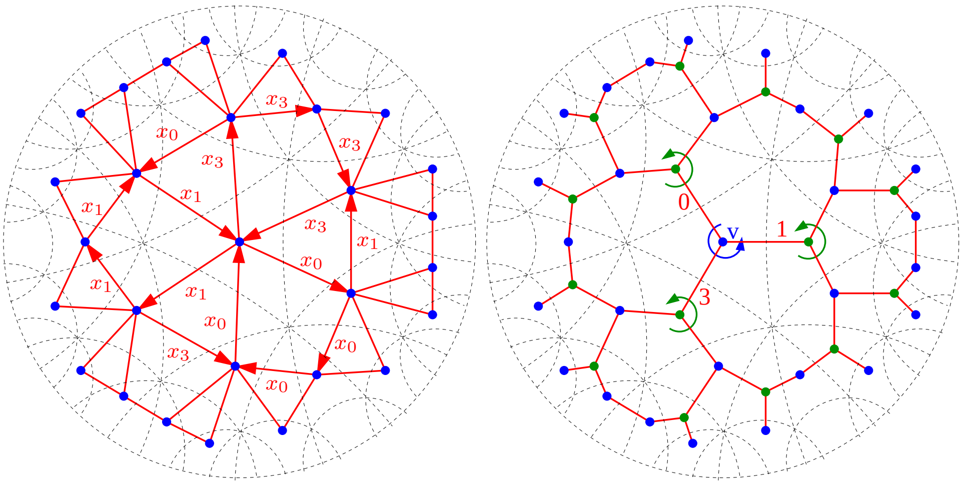

FIGURE 1. The lifts of the Cayley graph X_2 (left) and of the $(\Delta - Y)$-transformation T_2 (right) to the Poincaré unit disc $\mathbb{D}^2$

2.3. **From the Cayley graphs X_k to the trivalent graphs T_k.** For the transition from X_k to T_k we use the $(\Delta - Y)$-transformation. In this transformation, we add a new vertex v for every combinatorial triangle of the original graph, remove the three edges of this triangle and replace them by three edges connecting v with the vertices of this triangle. We apply this rule to our graph X_k and obtain a graph T_k, which we can view again as an embedding in $\mathcal{S}_k$ with the following properties: The vertex set of T_k coincides with V given in (11), and there is an edge (minimal geodesic segment) connecting every black/white vertex in V with the vertices in the three neighbouring white/black triangles. The best way to illustrate this transformation is to present it in the universal covering of the surface $\mathcal{S}_k$, i.e., the Poincaré unit disc $\mathbb{D}^2$ (see Figure 1, the new vertices replacing every triangle are green). Note that T_k has twice as many vertices as X_k, which shows that the isometry group of the above compact surface $\mathcal{S}_k$ has order $\geq |G_k| = |V_k|/2$, where V_k denotes the vertex set of T_K. Moreover, T_k is the dual of the triangulation of $\mathcal{S}_k$ by the above-mentioned compact black and white triangles.

2.4. **A direct construction of $\mathcal{S}_k$ and $\mathcal{S}_k^\infty$ from T_k.** There is another method to obtain the hyperbolic surfaces $\mathcal{S}_k$ and $\mathcal{S}_k^\infty$ using the construction in [7, Section 4] (see also [20, Chapter 1]). The start data are our trivalent graphs T_k with a suitable orientation.

Definition 2.2. *An* orientation $\mathcal{O}$ *on a trivalent graph T is a choice, at each vertex v of T, of a cyclic ordering of the three edges emanating from it.*

Let us first introduce an orientation $\mathcal{O}_k$ on T_k. We start with the Cayley graph X_k and orient its edges such that they only carry the Cayley graph labels x_0, x_1 and x_3, and not their inverses (see the Figure 1 on the left). Every triangle in X_k forms then an oriented cycle with consecutive labels x_0, x_1, x_3. This orientation induces an orientation on the new green vertex in T_k corresponding to this triangle, as illustrated by the oriented green circular arcs in the Figure 1 on the right. A blue vertex v of T_k stems from a vertex of X_k, and we can give the labels $0, 1, 3$ to the three edges in T_k emanating from v, agreeing with the label of the edge in the corresponding triangle of X_k not adjacent to v (see again Figure 1 for illustration). The orientation of the three edges emanating from v in T_k (illustrated by an oriented blue circular arc) is then given by the cyclic ordering $0, 3, 1$.

Now we follow the explanations in [20, Sections 1.1-1.4] closely. Let $\mathcal{T} \subset \mathbb{D}^2$ be an oriented compact equilateral hyperbolic triangle with interior angles $\pi/2^{n_k}$. We refer to the mid-points of the sides of $\mathcal{T}$ as *tick-marks*. The orientation of $\mathcal{T} \subset \mathbb{D}^2$ induces a cyclic order on these tick-marks. Connect the center of $\mathcal{T}$ with the three tick-marks by geodesic arcs and assume that these arcs are coloured red. Then we paste a copy of $\mathcal{T} \subset \mathbb{D}^2$ on each vertex v of T_k such that its center agrees with v, its tick-marks agree with the mid-points of the edges of T_k emanating from v, and that the cyclic orders of these egdes and of the tick-marks agree. Observe that, even though the mid-points of adjacent sides of triangles meet up at mid-points of edges of T_k, we have not yet identified the sides of these triangles. This identification is made in such a way that the orientations of adjacent triangles match up. The resulting hyperbolic surface $\mathcal{S}_k$ carries then a global orientation and the union of the red geodesic arcs from their mid-points to their tick-marks in the triangles provide an embedding of the graph T_k into this surface such that the faces are regular $2n_k$-gons.

The complete non-compact finite area hyperbolic surfaces $\mathcal{S}_k^\infty$ are obtained in the same way by starting instead with an oriented ideal hyperbolic triangle $\mathcal{T} \subset \mathbb{D}^2$ with tick-marks. As explained at the end of [20, Section 1.4], the cusps of $\mathcal{S}_k^\infty$ are then in bijection with the left-hand-turn pathes in $(T_k, \mathcal{O}_k)$. This construction is useful for the proof of Theorem 1.3 in [14].

2.5. **Proofs of Propositions 1.1 and 1.2.**

Proof of Proposition 1.1. The orders of $x_0, x_1, x_3 \in G_k$ were given in Lemma 2.1. It was explained in Section 2.2 that the isometry group of $\mathcal{S}_k$ has order $\geq |G_k| = 2^{N_k}$, and in Section 2.3 that $|V_k| = 2|G_k| = 2^{N_k+1}$. Lemma 2.1 implies that the faces of the triangulation $T_k \subset \mathcal{S}_k$ are regular 2^{n_k+1}-gons, which yields $2|E_k| = 3|V_k| = 2^{n_k+1}|F_k|$, proving (2). The genus $g(\mathcal{S}_k)$ can be derived either from Hurwitz's formula (9) or from the Euler characteristic $\chi(\mathcal{S}_k) = |V_k| - |E_k| + |F_k|$. This finishes the proof of Proposition 1.1. $\square$

Remark 2.3. *The number of cusps of the surface $\mathcal{S}_k^\infty$ agrees with the number of faces of the tessellation $T_k \subset \mathcal{S}_k$. For example, the genus five surface $\mathcal{S}_2$ mentioned in Section 2.2 is tessellated into 24 octagons, and the surface $\mathcal{S}_2^\infty$ has, therefore, 24 cusps.*

Proof of Proposition 1.2. We conclude from (9), $|V_k| = 2|G_k|$, $|E(T_K^*)| = |E_k|$, and the trivalence of T_k that

$$\frac{6g(\mathcal{S}(T_k))}{|E(T_k^*)|} = 6\,\frac{1 + (1-\mu_k)|V_k|/4}{3|V_k|/2}.$$

Note that $|V_k| = 2|G_k| = 2^{N_k+1} \to \infty$ because of (3), which implies that

$$\lim_{k\to\infty} \frac{6g(\mathcal{S}_k)}{|E(T_k^*)|} = 1 - \lim_{k\to\infty} \mu_k.$$

Recall from (10) and (1) that $\mu_k = 3/2^{n_k} \to 0$ as $k \to \infty$, finishing the proof of (4). $\square$

3. Spectral properties of the graphs X_k and T_k

In this section, we establish the expander properties of X_k and T_k and relations between their eigenfunctions and eigenvalues, which proves Theorem 1.3. We also investigate Ramanujan properties of these families of graphs.

3.1. **Precise relations between eigenfunctions and eigenvalues.** As before, let $\widetilde{G}$ be the group defined in (5) and G be the index two subgroup generated by S. Then both groups $\widetilde{G}$ and G have Kazhdan property (T) (see [21, Section 3]). Using [19, Prop. 3.3.1], we conclude that the Cayley graphs $X_k = \mathrm{Cay}(G_k, S)$ are expanders.

The adjacency operator A_X, acting on functions on the vertices of a graph X, is defined as

$$A_X f(v) = \sum_{w \sim v} f(w),$$

where $w \sim v$ means that the vertices v and w are adjacent. It is easy to see that the eigenvalues of the adjacency operator of a finite n-regular graph lie in the interval $[-n, n]$.

Recall that the set $V(X_k)$ of vertices of our 6-valent graph X_k is a subset of the vertex set V_k of the trivalent graph T_k. We have the following relations between the eigenfunctions of the adjacency operators on X_k and T_k.

Theorem 3.1. (a) *Every eigenfunction F on T_k to an eigenvalue $\lambda \in [-3, 3]$ gives rise to an eigenfunction f to the eigenvalue $\mu = \lambda^2 - 3 \in [-3, 6]$ on X_k (with $f(v) = F(v)$ for all $v \in V(X_k)$).*

(b) *Every eigenfunction f on X_k to an eigenvalue $\mu \in [-6, 6] - \{-3\}$ gives rise to two eigenfunctions $F_\pm$ to the eigenvalues $\pm\sqrt{\mu + 3}$ on T_k with*

$$F_\pm(v) = \begin{cases} f(v) & \text{if } v \in V(X_k), \\ \pm\frac{1}{\sqrt{\mu+3}} \sum_{w \sim v} f(w) & \text{if } v \in V_k - V(X_k). \end{cases}$$

(c) *An eigenfunction f on X_k to the eigenvalue -3 gives rise to an eigenfunction F to the eigenvalue 0 of T_k with*

$$F(v) = \begin{cases} f(v) & \text{if } v \in V(X_k), \\ 0 & \text{if } v \in V_k - V(X_k), \end{cases}$$

if and only if we have, for all triangles $\Delta \subset V(X_k)$, $\sum_{v \in \Delta} f(v) = 0$.

In the following proof, we use $\sim$ for adjacency in T_k and $\sim_{X_k}$ for adjacency in X_k. Moreover, d_{T_k} denotes the combinatorial distance function on the vertex set V_k of T_k.

Proof. (a) Let f and F be two functions on X_k and T_k, related by $f(v) = F(v)$ for all $v \in V(X_k)$. Then

$$A_{X_k} f(v) = \sum_{w \sim_{X_k} v} f(w) = \sum_{d_{T_k}(w,v)=2} F(w) = (A_{T_k})^2 F(v) - 3F(v),$$

which can also be written as $A_{X_k} = (A_{T_k})^2 - 3$. This implies immediately the connection between the eigenfunctions and eigenvalues.

(b) Let $A_{X_k} f = \mu f$ and $F_\pm$ be defined as in the theorem. Let $\lambda = \pm\sqrt{\mu + 3}$. Then we have for $v \in V(X_k)$:

$$\begin{aligned} A_{T_k} F_\pm(v) &= \sum_{w \sim v} F_\pm(w) = \frac{1}{\lambda} \sum_{w \sim v} \sum_{x \sim w} F_\pm(x) \\ &= \frac{1}{\lambda} \left(\sum_{w \sim_{X_k} v} f(w) + 3f(v) \right) = \frac{\mu + 3}{\lambda} f(v) = \lambda F_\pm(v), \end{aligned}$$

and for $v \in V_k - V(X_k)$:

$$A_{T_k} F_\pm(v) = \sum_{w \sim v} F_\pm(w) = \lambda \left(\frac{1}{\lambda} \sum_{w \sim v} f(w) \right) = \lambda F_\pm(v).$$

Note that $1/\lambda$ is well defined since $\mu \neq -3$ and, therefore, $\lambda = \pm\sqrt{\mu + 3} \neq 0$.

(c) In the case of $\mu = -3$ we have $\lambda = 0$, and

$$A_{T_k} F(v) = \sum_{w \sim v} F(w) = 0$$

holds trivially for $v \in V(X_k)$. For all vertices $v \in V_k - V(X_k)$, the conditions

$$0 = A_{T_k} F(v) = \sum_{w \sim v} f(v)$$

translate into the condition that the summation of f over the vertices of every triangle in X_k must vanish. □

An immediate consequence of Theorem 3.1 is that the expander property of the family X_k carries over to the graphs T_k (with the spectral bounds given in Theorem 1.3). Moreover, the spectrum of X_k cannot contain eigenvalues in the interval $[-6, -3)$, since this would lead to non-real eigenvalues of T_k. Therefore, these arguments also complete the proof of Theorem 1.3.

Remark 3.2. *It would be interesting to find an explicit value for the constant $C > 0$ in Theorem 1.3. This would be possible if we were able to estimate the Kazdhan constant of the index two subgroup G*

of $\widetilde{G}$ *(with respect to some choice of generators). While the Kazhdan constant of* $\widetilde{G}$ *with respect to the the standard set of seven generators was explicitly computed in* [11]*, it seems to be a difficult and challenging question to obtain an explicit estimate for the Kazhdan constant of the subgroup* G*.*

3.2. **Ramanujan properties.** Recall that a finite n-regular graph X is Ramanujan if all non-trivial eigenvalues $\lambda \neq \pm n$ lie in the interval $[-2\sqrt{n-1}, 2\sqrt{n-1}]$. Since the 6-regular graphs X_k are Cayley graphs of quotients of the group G with property (T), [19, Prop. 4.5.7]) implies that not all of these graphs are Ramanujan. The following results imply that, in fact, only two of them are Ramanujan. We obtain via MAGMA computations:

graph	number of vertices	largest non-trivial eigenvalue
X_2	32	2.828427...
X_3	128	4.340172...
X_4	1024	4.475244...
X_5	8192	5.160252...

This implies that only X_2 and X_3 are Ramanujan; their largest non-trivial eigenvalue needs to be $< 2\sqrt{5} = 4.472135\ldots$, which is no longer true for $k = 4$. Moreover, since X_{k+1} is a lift of X_k, the spectrum of X_k is contained in the spectrum of X_{k+1}.

Next, we consider Ramanujan properties of the graphs T_k. Theorem 3.1 implies that T_k is Ramanujan if and only if the largest non-trivial eigenvalue of X_k is ≤ 5. Therefore, the above numerical results imply that only T_2, T_3, T_4 are Ramanujan. Here are the numerical results for the largest *non-trivial* eigenvalues $\lambda_1(T_k)$ of the first T_k's:

graph	number of vertices	$\lambda_1(T_k)$
T_2	64	2.414213...
T_3	256	2.709275...
T_4	2048	2.734089...
T_5	16384	2.856615...

Note that the spectrum of T_k is symmetric around the origin since the graphs T_k are bipartite.

3.3. **A lower eigenvalue estimate for the surfaces $\widehat{\mathcal{S}}_k$.** It remains to prove Proposition 1.4 from the Introduction. The explicit construction of the surfaces $\widehat{\mathcal{S}}_k$ is explained in Buser [8, Section 3.2].

Proof of Proposition 1.4. The identity $2g-2 = |V_k|$ between the genus of the surface $\widehat{\mathcal{S}}_k$ and the number of vertices of the trivalent graph T_k is easily checked. Moreover, every automorphism of the graph T_k induces an isometry on $\widehat{\mathcal{S}}_k$. Since the graphs T_k form a power of coverings with powers of 2 as covering indices, the same holds true for the associated surfaces $\widehat{\mathcal{S}}_k$.

Next we prove the uniform lower eigenvalue bound of the family $\widehat{\mathcal{S}}_k$. From a classical result by Tanner [24] or Alon-Milman [1] we know that

$$\frac{3-\lambda_1(T_k)}{2} \le h(T_k) := \inf_E \frac{\#(E)}{\min\{\#(A), \#(A')\}},$$

where $E \subset E_k$ runs through all collection of edges such that $T_k \backslash E_k$ disconnects into two components with disjoint vertex sets $A \subset V_k$ and $A' \subset V_k$. This implies together with Theorem 1.3(ii) that the combinatorial Cheeger constants $h(T_k)$ have the following uniform positive lower bound

$$h_0 = \frac{3-\sqrt{C+3}}{2} \le h(T_k). \tag{12}$$

Moreover, we know from [8, (4.1)] that

$$\lambda_1(\widehat{\mathcal{S}}_k) \ge \frac{1}{144\pi^2} h(T_k).$$

Combining this with (12) leads to

$$\lambda_1(\widehat{\mathcal{S}}_k) \ge \frac{3-\sqrt{C+3}}{288\pi^2}$$

with the constant $C > 0$ from Theorem 1.3. This finishes the proof of Proposition 1.4. □

4. Comparison with Platonic graphs

4.1. **Basics about Platonic graphs.** The Platonic graphs Π_N were already introduced combinatorially in the Introduction. These graphs can also be viewed as triangulations of finite area surfaces $\mathcal{S}^\infty(\Pi_N) = \mathbb{H}^2/\Gamma(N)$ by ideal hyperbolic triangles, where $\mathbb{H}^2$ denotes the hyperbolic upper half plane and

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \mid \gamma \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} \tag{13}$$

is the principal congruence subgroup of the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ which is normal in Γ. The vertices of this triangulation are the cusps of $\mathcal{S}^\infty(\Pi_N)$. These and related graphs have been thoroughly investigated by several different communities. For example, in the general framework of *regular maps*, they were studied by D. Singerman and co-authors (see [16, 23, 15]).

Let us now recall a few useful facts about these graphs Π_N and the surfaces $\mathcal{S}^\infty(\Pi_N)$. For more details see, e.g., [15]. Let $\mathcal{F}$ be the Farey tessellation of the hyperbolic upper half plane $\mathbb{H}^2$, and let $\Omega(\mathcal{F})$ be the set of oriented geodesics in $\mathcal{F}$. Recall that the *Farey tessellation* is a triangulation of $\mathbb{H}^2$ with vertices on the line at infinity $\mathbb{R}\cup\{\infty\}$, namely, the subset of extended rationals $\mathbb{Q}\cup\{\infty\}$. Two rational vertices with reduced forms a/c and b/d are joined by an edge, a geodesic of $\mathbb{H}^2$, if and only if $ad - bc = \pm 1$ (see [15, Fig. 1] for an illustration of the Farey tessellation). The group of conformal transformations of $\mathbb{H}^2$ that leave $\mathcal{F}$ invariant is the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, which acts transitively on $\Omega(\mathcal{F})$.

It is well known (see, e.g, [15]) that $\mathcal{F}/\Gamma(N)$ and Π_N are isomorphic, and $\mathcal{F}/\Gamma(N)$ is a triangulation of the surface $\mathcal{S}^\infty(\Pi_N) = \mathbb{H}^2/\Gamma(N)$ by ideal triangles (the vertices are, in fact, the cusps of $\mathcal{S}^\infty(\Pi_N)$). The tessellation $\Pi_N \subset \mathcal{S}^\infty(\Pi_N)$ can be interpreted as a *map* $\mathcal{M}_N$ in the sense of Jones/Singerman [16]. The group $\mathrm{Aut}(\mathcal{M}_N)$ of automorphisms of $\mathcal{M}_N$ is the group of orientation preserving isometries of $\mathcal{S}^\infty(\Pi_N)$ preserving the triangulation. As $\Gamma(N)$ is normal in Γ, we have that the map $\mathcal{M}_N$ is *regular*, meaning that $\mathrm{Aut}(\mathcal{M}_N)$ acts transitively on the set of directed edges of Π_N (see [16, Thm 6.3]). Moreover, by [16, Thm 3.8],

$$\mathrm{Aut}(\mathcal{M}_N) \cong \Gamma/\Gamma(N) \cong \mathrm{PSL}(2,\mathbb{Z}_N).$$

(Note that in the case of a prime power $N = p^r$, $\mathrm{PSL}(2,\mathbb{Z}_N)$ is the group defined over the ring $\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z})$ and not over the field $\mathbb{F}_q$ with $q = p^r$ elements.) Let $N \geq 7$. Noticing that all vertices of Π_N have degree N, we obtain a smooth compact surface $\mathcal{S}(\Pi_N)$ by substituting every ideal triangle in $\Pi_N \subset \mathcal{S}^\infty(\Pi_N)$ by a compact equilateral hyperbolic triangle with interior angles $2\pi/N$, and glueing them along their edges in the same way as the ideal triangles of $\mathcal{S}^\infty(\Pi_N)$. The group of orientation preserving isometries of $\mathcal{S}(\Pi_N)$ preserving this triangulation is, again, isomorphic to $\mathrm{PSL}(2,\mathbb{Z}_N)$. Hence, the automorphism group of the triangulation $\Pi_8 \subset \mathcal{S}(\Pi_8)$ is $\mathrm{PSL}(2,\mathbb{Z}_8)$ of order 192. This implies that $\mathcal{S}(\Pi_8)$ is the unique compact hyperbolic surface of genus 5 with maximal automorphism group (see [3]).

4.2. **Duality between T_2 and Π_8 in $\mathcal{S}_2$.** The Π_8-triangulation of $\mathcal{S}(\Pi_8)$ is illustrated in Figure 2; the black-white pattern on the triangles is a first test whether this triangulation can be isomorphic to the T_2^*-triangulation of $\mathcal{S}_2$. (The Π_N-triangulations for $3 \leq N \leq 7$ can be found in Figs. 3 and 4 of [15].) $\mathrm{PSL}(2,\mathbb{Z}_8)$ acts simply transitively

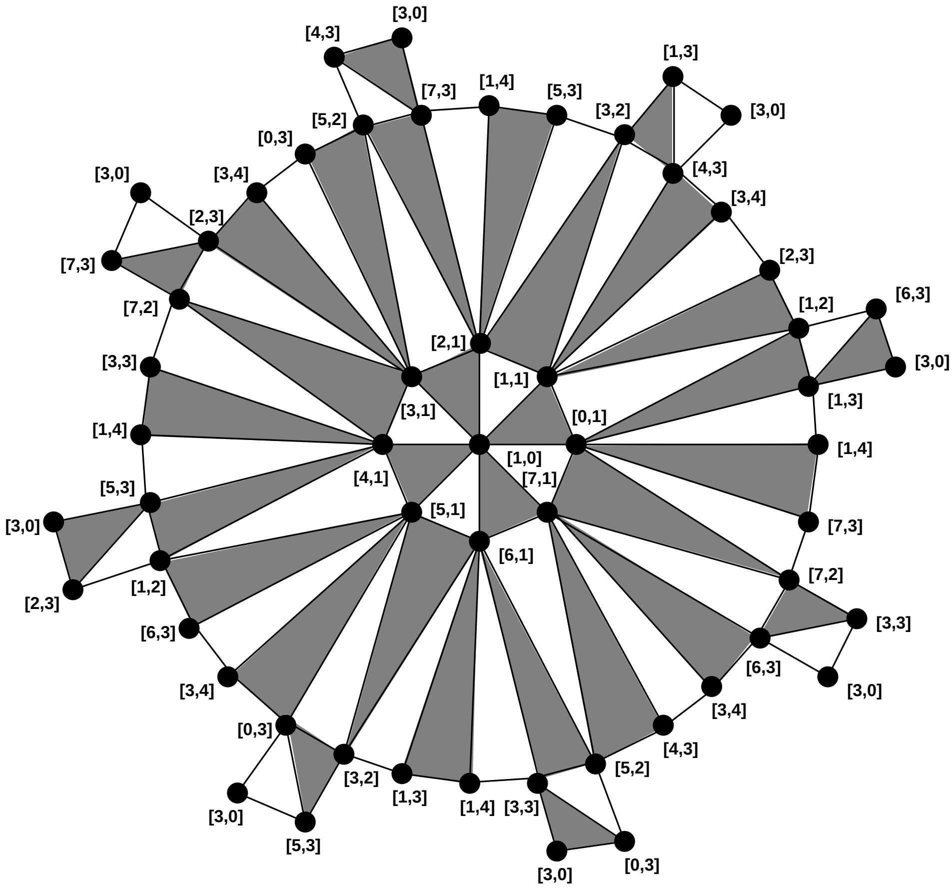

FIGURE 2. The Platonic graph Π_8: Each triangle corresponds to a hyperbolic $(\pi/4, \pi/4, \pi/4)$-triangle of the tessellation of $\mathcal{S}(\Pi_8)$. The edges along the boundary path are pairwise glued to obtain $\mathcal{S}(\Pi_8)$.

on the directed edges of this triangulation. Consider now a refinement of this triangulation by subdividing each $(\pi/4, \pi/4, \pi/4)$-triangle into six $(\pi/2, \pi/3, \pi/8)$-triangles. It is easily checked that the smaller $(\pi/2, \pi/3, \pi/8)$-triangles admit also a black-white colouring such that the neighbours of all smaller black triangles are white triangles and vice versa (see Figure 3). Each black $(\pi/2, \pi/3, \pi/8)$-triangle is in 1-1 correspondence to a half-edge of Π_8 which, in turn, can be identified with a directed edge of Π_8. Consequently, the orientation preserving isometries of the surface $\mathcal{S}(\Pi_8)$ corresponding to the elements in $\mathrm{PSL}(2, \mathbb{Z}_8)$ act simply transitively on the black $(\pi/2, \pi/3, \pi/8)$-triangles. In fact, $\mathrm{PSL}(2, \mathbb{Z}_8)$ can be interpreted as a quotient of the triangle group $\Delta^+(2, 3, 8)$, namely,

$$\mathrm{PSL}(2,\mathbb{Z}_8) \cong \langle x^2, y^3, z^8 \mid xyz, (xz^2xz^5)^2 \rangle,$$

where x, y, z correspond to rotations by $\pi, 2\pi/3, \pi/4$ about the three vertices of a given $(\pi/2, \pi/3, \pi/8)$-triangle.

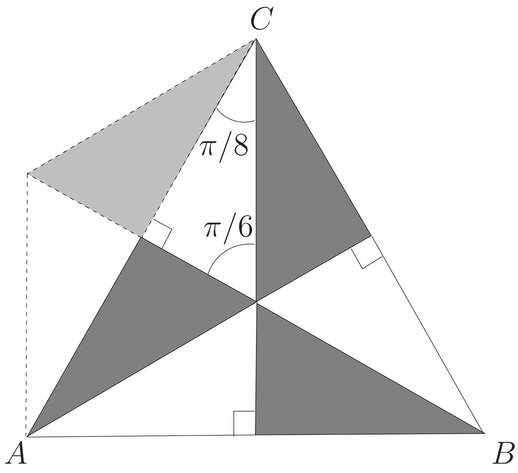

FIGURE 3. Subdivision of the $(\pi/4, \pi/4\, \pi/4)$-triangle ΔABC into six $(\pi/2, \pi/3, \pi/8)$-triangles with black-white colouring.

MAGMA computations show that $\mathrm{PSL}(2,\mathbb{Z}_8)$ has a unique normal subgroup N of index 6, generated by the elements $X = x^{-1}z^2x$, $Y = y^{-1}z^2y$ and $Z = z^2$, which is isomorphic to the triangle group quotient $\Delta^+(4,4,4)/P_2(\Delta^+(4,4,4))$ via the explicit isomorphism

$$X \mapsto \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad Z \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \tag{14}$$

where $P_2(\Delta^+(4,4,4))$ is a group in the lower exponent-2 series of $\Delta^+(4,4,4)$. Note that the matrices in (14), viewed as elements in $\mathrm{PSL}(2,\mathbb{Z})$, generate a group acting simply transitively on the black triangles of the Farey tessellation in $\mathbb{H}^2$, as illustrated in Figure 4. The images of a black triangle $\mathcal{T}$ with vertices $0, 1, \infty$ under $\{X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}\}$ are the six black triangles each sharing a common white triangle with $\mathcal{T}$.

MAGMA computations also show that we have the explicit isomorphism

$$N = \langle X, Y, Z \rangle \cong G_2 = \langle x_0, x_1, x_3 \rangle,$$

given by $X \mapsto x_0, Y \mapsto x_1, Z \mapsto x_3$. The normal group $N \triangleleft \mathrm{PSL}(2,\mathbb{Z}_8)$ is of order 32 and the quotient $\mathcal{S}_0 = \mathcal{S}(\Pi_8)/N$ is an orbifold consisting of two hyperbolic $(\pi/4, \pi/4, \pi/4)$-triangles (one of them black and the other white). We conclude from the explicit isomorphism $N \cong G_2$

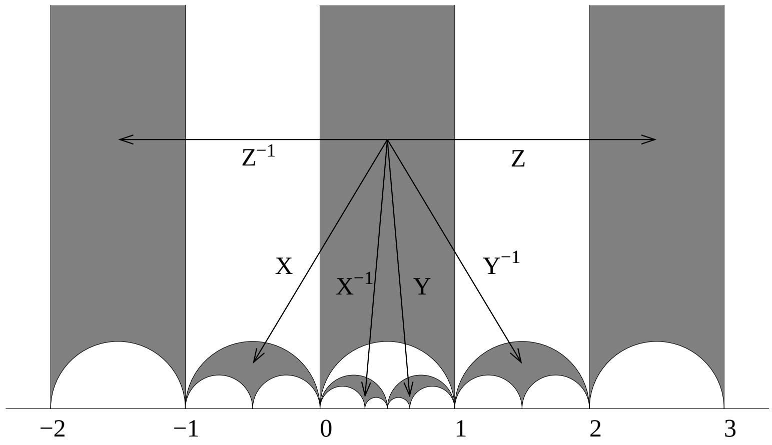

FIGURE 4. The action of the elements $X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1} \in PSL(2, \mathbb{Z})$ on a triangle $\mathcal{T}$ with vertices $0, 1, \infty$ of the Farey tessellation.

that the covering procedure discussed in Section 2.2 leads to isometric surfaces $\mathcal{S}(\Pi_8) \cong \mathcal{S}_2$, and that the tessellation $\Pi_8 \subset \mathcal{S}(\Pi_8)$ is dual to the tessellation $T_2 \subset \mathcal{S}(T_2)$ via this isometry of surfaces. This confirms the first statement of Proposition 1.5.

Remark 4.1. *For $k = 2$, the $(\Delta - Y)$-transformation $X_2 \to T_2$ has a group theoretical interpretation. There exists a group extension $\widetilde{G_2}$ of G_2 by $\mathbb{Z}_2$, generated by involutions A, B, C satisfying $X = AB$, $Y = BC$ and $Z = CA$, and T_2 is the Cayley graph of $\widetilde{G_2}$ with respect to the generators A, B, C. This group theoretic interpretation of the $(\Delta - Y)$-transformation* fails *for $k \geq 5$. In fact, the group*

$$T = \langle A, B, C \mid A^2, B^2, C^2, r_1(AB, BC), r_2(AB, BC), r_3(AB, BC) \rangle$$

with r_1, r_2, r_3 given in (6) *is finite and of order 6144. If the introduction of the above involutions A, B, C would lead to a group extension $\widetilde{G_k}$, then $\widetilde{G_k}$ would have to be of order $2|G_k|$ and a quotient of T and, therefore, of order ≤ 6144. However, we have $2|G_5| = 16384$ in contradiction to the second condition. Thus we do not obtain a Cayley graph representation of the graphs T_k for $k \geq 5$ via this procedure.*

4.3. **Non-duality of $T_k \subset \mathcal{S}_k$ and Platonic graphs for $k \geq 3$.** Let $V(\Pi_N)$ denote the vertex set of Π_N. Then we have

$$|V(\Pi_N)| = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

where the product runs over all primes p dividing N. This formula can be found in [15, p. 441], where Π_N is viewed as a triangular map and denoted by $\mathcal{M}_3(N)$. An isomorphism $T_k^* \cong \Pi_N$ leads to $N = 2^{n_k+1}$, since all vertices of T_k^* have degree 2^{n_k+1} (see Proposition 1.1) and all vertices of Π_N have degree N. In this case, the formula for the number of vertices of Π_N simplifies to

$$|V(\Pi_{2^{n_k+1}})| = \frac{2^{2n_k+2}}{2}\left(1 - \frac{1}{4}\right) = 3 \cdot 2^{2n_k-1}.$$

On the other hand, if $V(T_k^*)$ denotes the vertex set of T_k^*, we conclude from Proposition 1.1,

$$|V(T_k^*)| = 3 \cdot 2^{N_k - n_k}.$$

Hence, an isomorphism $T_k^* \cong \Pi_N$ leads to the identity $2n_k - 1 = N_k - n_k$, i.e.,

$$3\lfloor \log_2 k \rfloor + 3 = 3n_k = N_k + 1 \geq 8\lfloor k/3 \rfloor + 3 \cdot (k \bmod 3),$$

by (1) and (3). But one easily checks that this inequality holds only for $k = 1, 2$. (In the case $k = 1$, we have $\Pi_4 = T_1^*$, since T_1 is combinatorially the cube and Π_4 is the octagon.) This shows that the graph family Π_N cannot contain any of the dual graphs T_k^*, for indices $k \geq 3$. This finishes the proof of Proposition 1.5.

5. The Platonic graphs and their modifications

In this section we restrict our considerations to the case that N is a prime number p.

5.1. **Algebraic description of vertices, axes, and Π_p'.** Let us briefly recall some algebraic facts from [15]. Both groups $\Gamma = PSL(2, \mathbb{Z})$ and $PSL(2, \mathbb{Z}_p)$ act on $V(\Pi_p)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [\lambda, \mu] = [a\lambda + b\mu, c\lambda + d\mu],$$

and there is a 1-1 correspondence between the vertex set $V(\Pi_p)$ and the cosets $\Gamma/\Gamma_1(p)$. Here, $\Gamma_1(p)$ is the congruence subgroup given by

$$\Gamma_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p \right\}.$$

In [15], the set of vertices was partitioned into *axes*. Two vertices belong to the same axis if they have the same stabilizer in $PSL(2, \mathbb{Z}_p)$. Since $PSL(2, \mathbb{Z}_p)$ acts transitively on $V(\Pi_p)$, all axes have the same number of vertices. An interesting observation is that if an element of $PSL(2, \mathbb{Z}_p)$ leaves a vertex $[\lambda, \mu]$ invariant, then any vertex $[\nu, \omega]$ with

$\lambda\omega - \mu\nu = 0$ is also invariant under the same element. Thus the axis containing $[1, 0]$ is given by

$$\mathcal{A}_{\text{princ}} = \{[\lambda, 0] \mid \gcd(\lambda, p) = 1\}, \tag{15}$$

and we call this axis the *principal axis* of Π_p. There is a 1-1 correspondence between the axes of Π_p and the cosets $\Gamma/\Gamma_0(p)$, where $\Gamma_0(p)$ is the congruence subgroup

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p \right\}.$$

Note that the graph Π'_p was defined in the Introduction as the induced subgraph of Π_p with vertex sets $V(\Pi_p) - \mathcal{A}_{\text{princ}}$. Alternatively, Π'_p can also described as the Cayley graph $\text{Cay}(U_p, S)$ with

$$U_p = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in PSL(2, \mathbb{Z}_p) \right\} \cong \Gamma_0(p)/\Gamma(p) \text{ and } S = \left\{ \begin{pmatrix} * & 1 \\ 0 & * \end{pmatrix} \in U_p \right\}.$$

The vertices $[\lambda, \mu] \in V(\Pi'_p)$ (with non-vanishing second coordinate μ) are then identified with the matrices $\begin{pmatrix} \mu^{-1} & \lambda \\ 0 & \mu \end{pmatrix} \in U_p$.

5.2. **Vertex connectivity of Π_p and Π'_p.** Let p be a fixed odd prime and $n = (p-1)/2$. The *wheel structure* of Π_p was already discussed in [17, Thm 2.1]. Let us present this and other geometric facts in our terminology. The principal axis of Π_p is given by

$$\mathcal{A}_{\text{princ}} = \{[i, 0] \in V(\Pi_p) \mid 1 \le i \le n\}.$$

The vertices of $\mathcal{A}_{\text{princ}}$ and their 1-ring neighbours form a partition of $V(\Pi_p)$ into n components with $p+1$ vertices each. We call these components the *wheels* of Π_p, see Figure 5. The wheel with center $[i, 0]$ $(1 \le i \le n)$ is denoted by W_i and is a subgraph of Π_p with $p+1$ vertices and $2p$ edges. We also use the notation ∂W_i for the induced subgraph with vertex set $V(\partial W_i) = V(W_i) - \{[i, 0]\}$. We call ∂W_i the boundary of the i-th wheel. Note that ∂W_i is isomorphic to the cyclic graph of p vertices.

Every vertex that is not in $\mathcal{A}_{\text{princ}}$ is adjacent to exactly two vertices of the boundary of any given wheel W_i, $1 \le i \le n$. Indeed, because $PSL(2, \mathbb{Z}_p)$ acts transitively on $V(\Pi_p)$, we may consider, w.l.o.g., the vertex $[0, 1] \in \partial W_1$. The $p-1$ vertices adjacent to $[0, 1]$ that are not in $\mathcal{A}_{\text{princ}}$ are $[1, x]$ with $x \in \{1, 2, \ldots, p-1\}$. To find the vertices $[1, x]$ in ∂W_i, we need to solve

$$\det \begin{pmatrix} 1 & i \\ x & 0 \end{pmatrix} = \pm 1,$$

which has exactly two solution $x = \pm i^{-1}$ (where we think of $i \in \mathbb{Z}_p$) which correspond to two distinct vertices of Π_p.

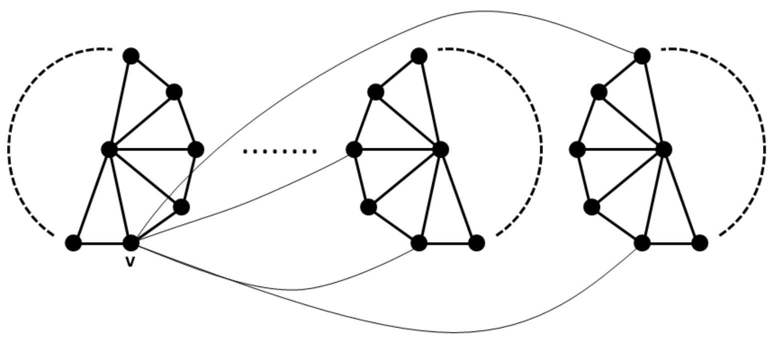

FIGURE 5. Π_p consists of $n = (p-1)/2$ wheels. Each vertex at the boundary of a wheel is connected with the center of its wheel and exactly two points on the boundary of any wheel (including itself).

Lemma 5.1. *Let $i, j \in \{1, 2, \dots, n\}$. Then we have the following facts.*

(a) *Let x_1, x_2 be two different vertices in ∂W_i and also y_1, y_2 be two different vertices in the same set ∂W_i ($\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$ is allowed). Then there exists a permutation $\sigma \in \mathrm{Sym}(2)$ and two vertex distinct paths p_1, p_2 in ∂W_i, such that p_1 connects x_1 with $y_{\sigma(1)}$ and p_2 connects x_2 with $y_{\sigma(2)}$.*

(b) *Every $x \in \partial W_i$ has precisely two neighbours in ∂W_j.*

(c) *Assume additionally that $i \neq j$. Then there exists a bijective map $\Phi : V(\partial W_i) \to V(\partial W_j)$ such that $v \sim \Phi(v)$ for all vertices $v \in \partial W_i$.*

Proof. Note that ∂W_i is isomorphic to the cyclic graph of p vertices. (a) is then a straighforward inspection of all possible cases. (b) is already proved by our previous arguments. It remains to prove (c): Think of $i, j \in \mathbb{Z}_p - \{0\}$. Then the vertices in ∂W_i are of the form $[\mu, i^{-1}]$ and the vertices in ∂W_j of the form $[\nu, j^{-1}]$ with $\mu, \nu \in \mathbb{Z}_p$. The map $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$, defined by $\phi(\mu) = i + ij^{-1}\mu$, is obviously a bijection, and we have $[\mu, i^{-1}] \sim [\phi(\mu), j^{-1}]$, finishing the proof. □

Note that the wheel structure is not confined to the choice of the principal axis. Since the group $PSL(2, \mathbb{Z}_p)$ maps axes to axes and acts transitively on them, we can choose any axis $\mathcal{A}$ as the centers of the n wheels, and Lemma 5.1 is still valid in this setting.

Now we prove that Π_p is p-vertex-connected. Notice that the arguments in this proof also give $\mathrm{diam}(\Pi_p) \leq 3$ as a by-product.

Proof. We will show that for any two vertices of Π_p, we can find p vertex disjoint paths connecting them. Then the result will follow from Menger's Theorem.

Since $PSL(2,\mathbb{Z}_p)$ acts transitively on $V(\Pi_p)$, we can assume that the start vertex is $[1,0] \in W_1$. Separating three cases, we will find p vertex distinct paths to

(i) the vertices in ∂W_1,
(ii) the vertices in any ∂W_j with $2 \le j \le n$,
(iii) the other vertices in $\mathcal{A}_p$.

Ad (i): Assume that the end vertex is $[\nu,1]$. Then we already have three vertex disjoint paths given by

$$[1,0] \to [\nu,1], \quad [1,0] \to [\nu \pm 1,1] \to [\nu,1].$$

We need to find vertex disjoint paths starting with $[1,0] \to [\nu \pm i,1]$ and ending at $[\nu,1]$, for $2 \le i \le n$. By Lemma 5.1(c), we can find two different vertices $x_1,x_2 \in \partial W_i$ such that $[\nu - i,1] \sim x_1$ and $[\nu + i,1] \sim x_2$. By Lemma 5.1(b), $[\nu,1]$ has two different neighbours $\{y_1,y_2\}$ in ∂W_i. We now use Lemma 5.1(a) to complete the paths.

Ad (ii): We assume $p \ge 5$, for otherwise there is nothing to prove. Let us assume that the end vertex is in ∂W_i with $2 \le i \le n$, and let us denote this vertex by $w \in \partial W_i$. Let $v_-,v_+ \in \partial W_1$ be the two neighbours of w in the first wheel. Choose three different vertices $v_1,v_2,v_3 \in \partial W_1 - \{v_-,v_+\}$, and use Lemma 5.1(c) to find three different vertices $w_1,w_2,w_3 \in \partial W_i - \{w\}$ such that $v_j \sim w_j$ for $1 \le j \le 3$. W.l.o.g., we can assume that the pair $\{w_1,w_3\}$ separates w_2 and w within ∂W_i. Let q_1,q_3 be the two vertex disjoint paths in $\partial W_i - \{w_2\}$ connecting w with w_1 and w_3, respectively. Then we already have five vertex disjoint paths given by

$$[1,0] \to v_\pm \to v, \quad [1,0] \to v_2 \to w_2 \to [i,0] \to w,$$

and

$$[1,0] \to v_1 \to w_1 \xrightarrow{q_1} w, \quad [1,0] \to v_3 \to w_3 \xrightarrow{q_3} w.$$

Notice that for any wheel W_j with $j \notin \{1,i\}$, we have not yet used any edges with one vertex in ∂W_j. We will see that every such wheel allows us to create two more vertex disjoint paths from $[1,0]$ to w, finishing this case. Let y_1,y_2 be the two different vertices in wheel ∂W_j adjacent to w. Choose two different vertices $v',v'' \in \partial W_1$ which have not been used yet and associate to them two different vertices $x_1,x_2 \in \partial W_j$ such that $v' \sim x_1$ and $v'' \sim x_2$, using Lemma 5.1(c). Then we can use Lemma 5.1(a) to complete the paths within ∂W_j.

Ad (iii): This is the easiest case. Assume that the end vertex is $[i,0] \in W_i$ with $2 \le i \le n$. We use the bijection $\Phi : V(\partial W_1) \to V(\partial W_i)$

in Lemma 5.1(c) to create the p vertex disjoint paths

$$[1,0] \to [0,\mu] \to \Phi([0,\mu]) \to [i,0]$$

with $1 \le \mu \le p$. □

Next, we present the proof that Π_p' is $(p-1)$-vertex-connected. In contrast to the previous proof, the arguments given here do not imply that $\operatorname{diam}(\Pi_p') \le 3$.

Proof. Let $v, w \in \Pi_p'$ be two different vertices with $v \in \partial W_i$ and $w \in \partial W_j$. We consider the two cases $i = j$ and $i \neq j$ separately:

Case $i = j$: Obviously, we can choose two vertex disjoint paths within ∂W_i to connect v and w. Next, we show that every wheel ∂W_j with $j \neq i$ gives rise to two additional vertex disjoint paths. Let $x_1, x_2 \in \partial W_j$ be the two distinct neighbours of v, and $y_1, y_2 \in \partial W_j$ be the two distinct neighbours of w. Then we can use Lemma 5.1(a) to complete the paths within ∂W_j.

Case $i \neq j$: Let $w_1, w_2 \in \partial W_j$ be the neighbours of v and $v_1, v_2 \in \partial W_i$ be the neighbours of w. Then, using only additional edges in $\partial W_i \cup \partial W_j$, we can find four vertex disjoint paths $v \to \cdots \to v_k \to w$, $v \to w_k \to \cdots \to w$ (for $k = 1, 2$). Again, every wheel W_l with $l \notin \{i, j\}$ will give rise to two more vertex disjoint paths. Let $x_1, x_2 \in \partial W_l$ be the neighbours of v, and $y_1, y_2 \in \partial W_l$ be the neighbours of w. Use Lemma 5.1(c) to complete the paths within ∂W_l. □

Finally, we prove $\operatorname{diam}(\Pi_p') = 3$.

Proof. Let us first confirm that any two different vertices in the same wheel can be connected by a path of length 2: Let $1 \le i \le n$ and $[\mu, i^{-1}], [\nu, i^{-1}] \in \partial W_i$ (thinking of $i \in \mathbb{Z}_p$) be the two vertices. The required path is then given by

$$[\mu, i^{-1}] \to [2(\mu-\nu)^{-1}\mu i - i, 2(\mu-\nu)^{-1}] \to [\nu, i^{-1}].$$

Now choose two vertices $v \in \partial W_i$ and $w \in \partial W_j$ on different wheels. Let $v' \in \partial W_i$ be one of the two neighbours of w in the i-th wheel. Connecting v and v' by a path of length 2 (as shown before) implies that $d(v,w) \le d(v,v') + 1 \le 3$. □

5.3. **Ramanujan properties "without number theory".** As before, we assume that p is a fixed odd prime and $n = (p-1)/2$. The considerations of the previous section show also that Π_p is a n-fold covering $\pi : \Pi_p \to K_{p+1}$ of the complete graph K_{p+1}, where the preimages $\pi^{-1}(v)$ correspond to the axes of Π_p. It is useful to think of the vertices in K_{p+1} as the points in the finite projective line over the field $\mathbb{Z}_p$, i.e.,

$V(K_{p+1}) = \{0, 1, \dots, p-1, \infty\}$ and the covering map is then given, algebraically, by

$$\pi([\lambda, \mu]) = \lambda\mu^{-1},$$

with the usual convention $\infty^{-1} = 0$ and $0^{-1} = \infty$. In particular, we have $\mathcal{A}_{\rm princ} = \pi^{-1}(\infty)$. Note that $PSL(2, \mathbb{Z}_p)$ acts also on the vertices of K_{p+1} via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z = (\alpha z + \beta)(\gamma z + \delta)^{-1}.$$

One easily checks that $\pi(gv) = g\pi(v)$ for all $g \in PSL(2, \mathbb{Z}_p)$ and $v \in V(\Pi_p)$.

Let us now explicitely derive the spectra of the graphs Π_p and Π'_p. We will use the following notation: For a linear operator T on a finite dimensional vector space, we denote the eigenspace of T to the eigenvalue λ by $\mathcal{E}(T, \lambda)$.

We start with a "number theory free" proof of Theorem 4.2 in [13], using the covering $\pi : \Pi_p \to K_{p+1}$.

Proof. Every eigenfunction f of K_{p+1} gives rise to an eigenfunction $F : V(\Pi_p) \to \mathbb{C}$ of the same eigenvalue via $F(v) = f(\pi(v))$. The spectrum of the adjacency operator on K_{p+1} is given by (see, e.g., [5, p. 17])

$$\sigma(K_{p+1}) = \{p, \underbrace{-1, \dots, -1}_{\text{p times}}\}.$$

This implies that $\sigma(\Pi_p)$ contains the eigenvalue p with multiplicity one and the eigenvalue -1 with multiplicity $\geq p$.

Our next aim is to prove that the eigenspace $\mathcal{E}(A^2, p)$ of the square of the adjacency operator on Π_p has dimension $(p+1)(p-3)/2$. Let $f : V(\Pi_p) \to \mathbb{C}$ be a function satisfying

$$A^2 f(v) = pf(v) \qquad \text{for all } v \in V(\Pi_p). \tag{16}$$

Note that (16) can be viewed as a homogenous system of $(p^2-1)/2$ linear equations. The key observation is that all linear equations corresponding to vertices of the same axis coincide, i.e., we end up with only $p+1$ linear independent homogeneous equations (since $p+1$ equals the number of axes), showing that the eigenspace has dimension at least

$$|V(\Pi_p)| - (p+1) = \frac{p^2-1}{2} - (p+1) = \frac{(p+1)(p-3)}{2}.$$

Indeed, since $PSL(2, \mathbb{Z}_p)$ acts transitively on the vertices, we only need to show that the linear equations of (16) corresponding to the vertices in the principal axis $\mathcal{A}_p$ coincide. Recall that Π_p has the wheel-structure given in Figure 5. Let $v \in \mathcal{A}_p$. Then we have

$$A^2 f(v) = pf(v) + 2\sum_{i=1}^{n} \sum_{w \in \partial W_i} f(w), \tag{17}$$

since there are exactly p paths of length 2 from v to itself, no paths of length 2 from the centers of all the other wheels to v, and for every $w \in \cup_i \partial W_i$ there are exactly 2 paths from w to v of length 2, because of Lemma 5.1(b). Note that the combination of (16) and (17) simplifies to

$$\sum_{i=1}^{n} \sum_{w \in \partial W_i} f(w) = 0,$$

independently of the choice of $v \in \mathcal{A}_p$. This shows that $\dim \mathcal{E}(A^2, p) \geq (p+1)(p-3)/2$. Adding up the multiplicities of all eigenvalues, we see that $\dim \mathcal{E}(A^2, p) = (p+1)(p-3)/2$.

If $f_1, \dots, f_K$ span the space $\mathcal{E}(A^2, p)$, then the $2K$ functions

$$\sqrt{p} f_1 \pm A f_1, \dots, \sqrt{p} f_K \pm A f_K$$

are eigenfunctions of A to the eigenvalues $\pm\sqrt{p}$, and they also span $\mathcal{E}(A^2, p)$. This shows that we have

$$\mathcal{E}(A^2, p) = \mathcal{E}(A, \sqrt{p}) \oplus \mathcal{E}(A, -\sqrt{p}).$$

Finally, the equality

$$\dim \mathcal{E}(A, \sqrt{p}) = \dim \mathcal{E}(A, -\sqrt{p}) = \frac{(p+1)(p-3)}{4}$$

follows from Lemma 5.2 below. □

Lemma 5.2. *Let T be a square matrix with rational entries and K be a positive integer which is not a square. Then we have*

$$\dim \mathcal{E}(T, \sqrt{K}) = \dim \mathcal{E}(T, -\sqrt{K}).$$

Proof. The proof is based on the fact that $\sqrt{K}$ is irrational. Let $p(z) \in \mathbb{Q}[z]$ be the characteristic polynomial of T. We split $p(z)$ into its even and odd part, i.e.,

$$p(z) = p_{even}(z) + p_{odd}(z) z,$$

with even polynomials $p_{even}(z), p_{odd}(z)$. Note that we have

$$p(\sqrt{K}) = p_{even}(\sqrt{K}) + p_{odd}(\sqrt{K})\sqrt{K}$$

and $p_{even}(\sqrt{K}), p_{odd}(\sqrt{K}) \in \mathbb{Q}$. Therefore, if $\sqrt{K}$ is a root of $p(z)$, then $\sqrt{K}$ is also a root of both polynomials $p_{even}(z)$ and $p_{odd}(z)$, separately. This implies that $-\sqrt{K}$ is also a root of $p(z)$. We can then split off the factor $z^2 - K$ from $p(z)$, and repeat the procedure with the remaining polynomial. $\square$

Next we derive the spectrum of the modified graph Π_p', using the n-fold covering map $\pi : \Pi_p' \to K_p$ and the wheel-structure, which partitions the vertex set $V(\Pi_p')$ into n disjoint sets ∂W_i of p vertices, each. This will finish the proof of Theorem 1.6.

Proof. The proof of the spectral statements in Theorem 1.6 proceeds in steps.

(i) Let $\mathcal{W}$ be the vector space of all functions which are constant on the wheels. We first introduce a basis of eigenfunctions of this vector space. Let $\zeta_n = e^{2\pi i/n}$ and, for $0 \leq j \leq n-1$, define

$$f_j(v) = \zeta_n^{ij} \quad \text{if } v \in \partial W_i.$$

Note that f_0 is the constant function to the eigenvalue $p-1$. It is easily checked that $Af_j = 0$ for $j \geq 1$. Since these functions are linearly independent, they form a basis of $\mathcal{W}$. Moreover, we have $\dim \mathcal{E}(A, 0) \geq n - 1 = (p-3)/2$.

(ii) Let $\mathcal{V}$ be the vector space of all functions which are constant along all axes. Every such function is a lift $F(v) = f(\pi(v))$ of a function f on K_p. Note that eigenfunctions of K_p are lifted to eigenfunctions to the same eigenvalue, so $\mathcal{V}$ can be viewed as the span of a constant function and $p-1$ linear independent eigenfunctions to the eigenvalue -1. In particular, we have $\dim \mathcal{E}(A, -1) \geq p - 1$.

(iii) Note that $\mathcal{W} \cap \mathcal{V} = \operatorname{span}(f_0)$. By the orthogonality of eigenfunctions, it only remains to study the eigenfunctions in the orthogonal complement $(\mathcal{W} + \mathcal{V})^\perp$ of dimension

$$|V(\Pi_p')| - (\dim \mathcal{W} + \dim \mathcal{V}) + 1 = \frac{(p-1)(p-3)}{2} = K.$$

Let $g_1, \dots, g_K$ be a basis of this orthogonal complement by eigenfunctions with $Ag_i = \lambda_i g_i$. We now extend each g_i trivially to a function $\widetilde{g}_i$ on Π_p by setting $\widetilde{g}_i(v) = 0$ for all $v \in \mathcal{A}_p$. Note that these extensions are eigenfunctions of the Platonic graph Π_p to the same eigenvalue, i.e., $A\widetilde{g}_i = \lambda_i \widetilde{g}_i$. Therefore, we must have $\lambda_i \in \{p+1, -1, \pm\sqrt{p}\}$. As discussed in the previous proof, the span of the eigenfunctions of Π_p to the eigenvalues -1 and $p+1$ is obtained via lifting the eigenfunctions of K_{p+1}, and the restriction of these functions to Π_p' must therefore lie in $\mathcal{V}$. This shows that we must have $\lambda_i = \pm\sqrt{p}$.

(iv) Adding up the multiplicities of all eigenvalues, we conclude that

$$\begin{aligned} \dim \mathcal{E}(A, \sqrt{p}) \oplus \mathcal{E}(A, -\sqrt{p}) &= (p-1)(p-3)/2, \\ \dim \mathcal{E}(A, 0) &= (p-3)/2, \\ \dim \mathcal{E}(A, -1) &= p-1. \end{aligned}$$

We finally obtain

$$\dim \mathcal{E}(A, \sqrt{p}) = \dim \mathcal{E}(A, -\sqrt{p}) = \frac{(p-1)(p-3)}{4},$$

by applying, again, Lemma 5.2. □

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Durham University, DH1 3LE, Great Britain
E-mail address: ioannis.ivrissimtzis@durham.ac.uk
E-mail address: norbert.peyerimhoff@durham.ac.uk

University of Newcastle, NE1 7RU, Great Britain
E-mail address: alina.vdovina@ncl.ac.uk