

CONVERGENCE OF SEQUENCES OF LINEAR OPERATORS AND THEIR SPECTRA

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ABSTRACT. We establish spectral convergence results of approximations of unbounded non-selfadjoint linear operators with compact resolvents by operators that converge in generalized strong resolvent sense. The aim is to establish general assumptions that ensure spectral exactness, i.e. that every true eigenvalue is approximated and no spurious eigenvalues occur. A main ingredient is the discrete compactness of the sequence of resolvents of the approximating operators. We establish sufficient conditions and perturbation results for strong convergence and for discrete compactness of the resolvents.

1. INTRODUCTION

The spectra of linear operators T , e.g. describing the time evolution of a physical system, are usually not known analytically and need to be computed numerically by approximating the operators and determining the eigenvalues of simpler operators. However, it is well-known that spectral computations may lead to *spectral pollution*, i.e. to numerical artefacts which do not belong to the spectrum of T , so-called *spurious eigenvalues*. Vice versa, not every eigenvalue or spectral point of T may be approximated; an approximation $(T_n)_{n \in \mathbb{N}}$ is called *spectrally inclusive* if this phenomenon does not occur. If spectral inclusion prevails and no spectral pollution occurs, then $(T_n)_{n \in \mathbb{N}}$ is said to be a *spectrally exact* approximation of T .

The existing spectral exactness results in the literature are restricted either to bounded operators or to particular classes of differential operators, or they are only local spectral exactness results, e.g. for spectral gaps of selfadjoint operators. On the other hand, many important applications in physics such as linear stability problems in fluid mechanics, magnetohydrodynamics, or elasticity theory require reliable knowledge on the spectra of unbounded non-selfadjoint linear operators.

The present paper aims at filling this gap. The novelty of the results established here lies in 1) their far-reaching generality covering wide classes of unbounded non-selfadjoint linear operators; 2) their simultaneous applicability to different approximation schemes such as the Galerkin (finite section) method and the domain truncation method; 3) their global nature which yields spectral exactness in the *entire* complex plane; and 4) a comprehensive analysis of necessary conditions and perturbation results for spectral exactness. We present applications to interval truncation of singular 2×2 differential operator matrices, to domain truncation of magnetic Schrödinger operators with complex-valued potentials on \mathbb{R}^d , and to the Galerkin method for operators of block-diagonally dominant form.

The first main theorem (Theorem 2.6) is the following global spectral convergence result: If $(T_n)_{n \in \mathbb{N}}$ converges in generalized strong resolvent (gsr) sense to T and the resolvents are compact and form a discretely compact sequence, then the approximation is spectrally exact. In the second main result (Theorem 2.7) we prove

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that under the additional assumptions that the operators act in Hilbert spaces and $(T_n^*)_{n \in \mathbb{N}}$ converges to T^* in gsr-sense, then the resolvents converge even in operator norm. The third group of important results comprise additive perturbation results: For a sequence $(S_n)_{n \in \mathbb{N}}$ of relatively bounded perturbations S_n of T_n with (uniform) relative bound < 1 , we establish perturbation results for gsr-convergence (Theorem 3.3) and discrete resolvent compactness (Theorem 4.2). A fourth group of results guarantee gsr-convergence and discrete resolvent compactness of a sequence of block operator matrices by means of easily verifiable assumptions that are formulated in terms of the matrix entries. First we prove results for unbounded *finite* operator matrices and then for *infinite* matrices (Theorems 3.15 and 4.9).

The notions of spectral inclusion and spectral exactness were introduced by Bailey et al. [2] for regular approximations of singular selfadjoint Sturm-Liouville problems via interval truncation. They were further studied, in particular, by Brown and Marletta for the domain truncation procedure of non-selfadjoint differential operators [7, 8, 9]. The notion of *generalized norm/strong resolvent convergence* developed in this paper for approximations of unbounded linear operators is closely related to norm or strong resolvent convergence studied by Kato [17, Sections IV.2, VIII.1], Reed-Simon [22, Theorems VIII.23-25] and Weidmann [28, Section 9.3]; in the latter two, only selfadjoint operators were considered. In general, the approximating operators T_n cannot be chosen to act in the same space as T , so we compare the projected resolvents $(T_n - \lambda)^{-1}P_n$ and $(T - \lambda)^{-1}P$ in a common larger space. Note that the meaning of “generalized” used in this paper is different from Kato’s *generalized convergence* (meaning resolvent convergence) where it indicates that the operators are unbounded. The spectral exactness result (Theorem 2.6) relies on gsr-convergence, however we are also interested in generalized *norm* resolvent convergence (see Theorem 2.7) since the latter is used to prove convergence of pseudospectra in Hausdorff metric (see [4, Theorem 2.1]).

To conclude spectral exactness, it is not enough to assume that the operators T and T_n , $n \in \mathbb{N}$, have compact resolvents and converge in gsr-sense. In fact, even in the selfadjoint case, if the operator T is unbounded below and above, then the Galerkin method may produce spurious eigenvalues *anywhere* on the real line (see [19, Theorem 2.1]). We prove spectral exactness under the additional assumption that the sequence $((T_n - \lambda)^{-1})_{n \in \mathbb{N}}$ is *discretely compact*. The latter notion was introduced by Stummel who established a spectral convergence theory for bounded operators in [25, 26]. Similar result were obtained by Anselone-Palmer and Osborn [1, 21] for the closely related notion of *collectively compact* sets of bounded operators. For relations between the various results and notions, we refer to Chatelin’s monograph [10] (see, in particular, Sections 3.1-3.6, 5.1-5.5).

We mention that if the assumptions of Theorem 2.6 are not satisfied (in particular if essential spectrum is present), a different approach is to establish *local* spectral exactness results, i.e. to identify regions in the complex plane where no spectral pollution occurs or to find enclosures for true eigenvalues. This was done for the Galerkin approximation of selfadjoint operators by means of *higher order relative spectra* (introduced by Davies in [11], see also the comprehensive overviews by Shargorodsky et al. [24, 19]), the closely related methods of Davies-Plum [12] and Mertins-Zimmermann [20], and the perturbation method of Hinchcliffe-Strauss [16]. For non-selfadjoint operators, we prove local spectral exactness results in terms of the *region of boundedness* (see Theorem 2.3). An alternative but computationally very expensive method to obtain reliable information on isolated eigenvalues uses *interval arithmetic* which yields eigenvalue enclosures with absolute certainty (see e.g. Brown et al. [6] and the references therein).

This paper is organized as follows. In Section 2 we prove convergence results for operators and their spectra under the assumptions that the operators have compact and discretely compact resolvents and converge in gsr-sense. In Section 3 and Section 4 we derive sufficient conditions and perturbation results for gsr-convergence and for discrete resolvent compactness, respectively. In both sections we complement the general theorems by results for finite and for infinite unbounded operator matrices. Applications to the domain truncation method for singular differential operators (and operator matrices) and to the Galerkin method are given in Section 5.

We use the following notation. The norm of a normed space E is denoted by $\|\cdot\|_E$. The convergence in E , i.e. $\|x_n - x\|_E \rightarrow 0$, is written as $x_n \rightarrow x$. In a Hilbert space H the scalar product is $\langle \cdot, \cdot \rangle_H$. Weak convergence in H , i.e. $\langle x_n, z \rangle_H \rightarrow \langle x, z \rangle_H$ for all $z \in H$, is denoted by $x_n \xrightarrow{w} x$. For two normed vector spaces D and E we denote by $L(D, E)$ the space of all bounded linear operators from D to E ; we write $L(E)$ if $D = E$. Analogously, the space of all closed operators in E is denoted by $C(E)$. The spectrum, point spectrum, approximate point spectrum and resolvent set of a linear operator T are denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{\text{app}}(T)$ and $\varrho(T)$, respectively, and the Hilbert space adjoint operator of T is T^* . For an operator $T \in C(E)$ the graph norm is $\|\cdot\|_T := \|\cdot\|_E + \|T \cdot\|_E$; then $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space. For bounded linear operators we write $T_n \rightarrow T$ and $T_n \xrightarrow{s} T$ for norm and strong convergence in $L(D, E)$. An identity operator in a Banach or Hilbert space is denoted by I ; scalar multiples λI are written as λ . Analogously, the operator of multiplication with a function m in some L^2 -space is also denoted by m . Given $\lambda \in \mathbb{C}$ and a subset $\Omega \subset \mathbb{C}$, the distance of λ to Ω is $\text{dist}(\lambda, \Omega) := \inf_{z \in \Omega} |\lambda - z|$, and $B_r(\lambda) := \{z \in \mathbb{C} : |z - \lambda| < r\}$ for $r > 0$. Let $\mathbb{N} := \{1, 2, 3, \dots\}$, in particular, $0 \notin \mathbb{N}$. Finally, for a subset $I \subset \mathbb{N}$, we denote by $\#I$ the number of elements in I .

2. CONVERGENCE OF OPERATORS AND THEIR SPECTRA

In this section we establish convergence results for operators acting in different spaces and spectral convergence results. In Subsection 2.1 are the main convergence results. Before we prove these results in Subsection 2.3, we recall the notions of discretely compact sequences or collectively compact sets of bounded operators and we analyze their effect on strong or norm operator convergence (see Subsection 2.2).

In the following, we assume that E_0 is a Banach space and $E, \tilde{E}_n \subset E_0$, $n \in \mathbb{N}$, are closed complemented subspaces, i.e. $E_0 = E + \tilde{E} = E_n + \tilde{E}_n$ with $E \cap \tilde{E} = E_n \cap \tilde{E}_n = \{0\}$ for $n \in \mathbb{N}$. Let $P : E_0 \rightarrow E$ be the projection on E along \tilde{E} and, for $n \in \mathbb{N}$, let $P_n : E_0 \rightarrow E_n$ be the projection on E_n along \tilde{E}_n converging strongly, $P_n \xrightarrow{s} P$; note that then $\|P\| \leq \liminf_{n \rightarrow \infty} \|P_n\| < \infty$ by [17, Equation III.(3.2)]. Throughout, in results for Hilbert spaces $H_0 := E_0$, $H := E$, $H_n := E_n$, $n \in \mathbb{N}$, we assume that $P, P_n, n \in \mathbb{N}$, are the *orthogonal* projections onto the respective subspaces; then $\|P\| = \|P_n\| = 1$, $n \in \mathbb{N}$.

2.1. Main convergence results for unbounded linear operators and their spectra. The following definition of *generalized* strong and norm resolvent convergence is due to Weidmann [28, Section 9.3], and the region of boundedness was introduced by Kato [17, Section VIII.1].

Definition 2.1. Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$.

- i) The sequence $(T_n)_{n \in \mathbb{N}}$ is said to *converge in generalized strong resolvent sense* to T , $T_n \xrightarrow{gsr} T$, if there exist $n_0 \in \mathbb{N}$ and $\lambda \in \bigcap_{n \geq n_0} \varrho(T_n) \cap \varrho(T)$ with

$$(T_n - \lambda)^{-1} P_n \xrightarrow{s} (T - \lambda)^{-1} P, \quad n \rightarrow \infty.$$

- ii) The sequence $(T_n)_{n \in \mathbb{N}}$ is said to *converge in generalized norm resolvent sense* to T , $T_n \xrightarrow{gnr} T$, if there exist $n_0 \in \mathbb{N}$ and $\lambda \in \bigcap_{n \geq n_0} \varrho(T_n) \cap \varrho(T)$ with

$$(T_n - \lambda)^{-1}P_n \longrightarrow (T - \lambda)^{-1}P, \quad n \rightarrow \infty.$$

- iii) The *region of boundedness* of the sequence $(T_n)_{n \in \mathbb{N}}$ is defined as

$$\Delta_b((T_n)_{n \in \mathbb{N}}) := \left\{ \lambda \in \mathbb{C} : \exists n_0 \in \mathbb{N} \text{ with } \begin{array}{l} \lambda \in \varrho(T_n), n \geq n_0, \\ (\|(T_n - \lambda)^{-1}\|)_{n \geq n_0} \text{ bounded} \end{array} \right\}.$$

The following notions of spectral inclusion and spectral exactness were introduced by Bailey et al. in [2]. We will also use local versions of these notions.

Definition 2.2. Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$.

- i) The approximation $(T_n)_{n \in \mathbb{N}}$ of T is called *spectrally inclusive* if for every $\lambda \in \sigma(T)$ there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of elements $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, with $\lambda_n \rightarrow \lambda$.
- ii) An element $\lambda \in \mathbb{C}$ for which there exist an infinite subset $I \subset \mathbb{N}$ and $\lambda_n \in \sigma(T_n)$, $n \in I$, with $\lambda_n \rightarrow \lambda$ but $\lambda \notin \sigma(T)$ is called *spurious eigenvalue*. The occurrence of such a point is known as *spectral pollution*.
- iii) The approximation $(T_n)_{n \in \mathbb{N}}$ of T is called *spectrally exact* if it is spectrally inclusive and no spectral pollution occurs.

The following result yields local spectral exactness in the region of boundedness.

Theorem 2.3. Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$. Suppose that $T_n \xrightarrow{gst} T$.

- i) For each $\lambda \in \sigma(T)$ such that for some $\varepsilon > 0$ we have

$$B_\varepsilon(\lambda) \setminus \{\lambda\} \subset \Delta_b((T_n)_{n \in \mathbb{N}}) \cap \varrho(T), \quad (1)$$

there exist $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$.

- ii) No spectral pollution occurs in $\Delta_b((T_n)_{n \in \mathbb{N}})$.

Next we generalize results that are known for the special case $E_0 = E = E_n$ and resolvent convergence to *generalized* resolvent convergence. Claim i) is a generalization of [17, Theorem IV.2.25] and [28, Satz 9.26 b)]; the latter result only applies to selfadjoint operators. Claim ii) is a generalization of [28, Satz 9.24 a)].

Theorem 2.4. Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$.

- i) If $T_n \xrightarrow{gnr} T$, then no spectral pollution occurs.
- ii) Let E_0, E, E_n , $n \in \mathbb{N}$, be Hilbert spaces, and let T, T_n , $n \in \mathbb{N}$, be selfadjoint with $T_n \xrightarrow{gst} T$. Then

$$\sigma(T) \subset \mathbb{C} \setminus \Delta_b((T_n)_{n \in \mathbb{N}}) = \left\{ \lambda \in \mathbb{C} : \liminf_{n \rightarrow \infty} \text{dist}(\lambda, \sigma(T_n)) = 0 \right\}, \quad (2)$$

and hence $(T_n)_{n \in \mathbb{N}}$ is a *spectrally inclusive approximation* of T .

To formulate the main results (Theorems 2.6 and 2.7 below), we use Stummel's notions of discrete compactness of a sequence of bounded operators (see [25, Definition 3.1.(k)]).

Definition 2.5. Let D_n , $n \in \mathbb{N}$, be arbitrary Banach spaces and $A_n \in L(D_n, E_n)$, $n \in \mathbb{N}$. The sequence $(A_n)_{n \in \mathbb{N}}$ is said to be *discretely compact* if for each infinite subset $I \subset \mathbb{N}$ and each bounded sequence of elements $x_n \in D_n$, $n \in I$, there exist $y \in E$ and an infinite subset $\tilde{I} \subset I$ so that $\|A_n x_n - y\|_{E_0} \rightarrow 0$ as $n \in \tilde{I}$, $n \rightarrow \infty$.

The following theorem is the main spectral convergence result of this section.

Theorem 2.6. *Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$. Assume that there exists an element $\lambda_0 \in \bigcap_{n \in \mathbb{N}} \varrho(T_n) \cap \varrho(T)$ such that $(T - \lambda_0)^{-1}$, $(T_n - \lambda_0)^{-1}$, $n \in \mathbb{N}$, are compact operators and the sequence $((T_n - \lambda_0)^{-1})_{n \in \mathbb{N}}$ is discretely compact. If $T_n \xrightarrow{gst} T$, then the following hold:*

i) *The region of boundedness coincides with the resolvent set of T ,*

$$\Delta_b((T_n)_{n \in \mathbb{N}}) = \varrho(T),$$

and, for any $\lambda \in \varrho(T)$,

$$(T_n - \lambda)^{-1} P_n \xrightarrow{s} (T - \lambda)^{-1} P, \quad n \rightarrow \infty. \quad (3)$$

ii) *The sequence $(T_n)_{n \in \mathbb{N}}$ is a spectrally exact approximation of T . More precisely, no spectral pollution occurs, and if $\lambda \in \mathbb{C}$ is an eigenvalue of T of algebraic multiplicity m , then, for n large enough, T_n has exactly m eigenvalues (repeated according to their algebraic multiplicities) in a neighbourhood of λ which converge to λ as $n \rightarrow \infty$ and the corresponding normalized elements of the algebraic eigenspaces converge (with respect to $\|\cdot\|_{E_0}$).*

Now we assume that the underlying spaces are Hilbert spaces. We establish sufficient conditions guaranteeing that generalized strong resolvent convergence implies generalized norm resolvent convergence.

Theorem 2.7. *Let $T \in C(H)$ and $T_n \in C(H_n)$, $n \in \mathbb{N}$. Assume that there exists an element $\lambda_0 \in \bigcap_{n \in \mathbb{N}} \varrho(T_n) \cap \varrho(T)$ such that $(T_n - \lambda_0)^{-1}$, $n \in \mathbb{N}$, are compact operators and the sequence $((T_n - \lambda_0)^{-1})_{n \in \mathbb{N}}$ is discretely compact. If*

$$(T_n - \lambda_0)^{-1} P_n \xrightarrow{s} (T - \lambda_0)^{-1} P, \quad (T_n^* - \overline{\lambda_0})^{-1} P_n \xrightarrow{s} (T^* - \overline{\lambda_0})^{-1} P, \quad n \rightarrow \infty,$$

then, for every $\lambda \in \varrho(T)$, the operator $(T - \lambda)^{-1}$ is compact and

$$(T_n - \lambda)^{-1} P_n \rightarrow (T - \lambda)^{-1} P, \quad n \rightarrow \infty.$$

2.2. Convergence and compactness concepts for bounded operators. In this subsection we study discretely compact operator sequences and the effect of this notion on strong operator convergence.

First we prove multiplicative and additive perturbation results on discrete compactness. Denote by D_n , $n \in \mathbb{N}$, arbitrary Banach spaces.

Lemma 2.8. i) *Let $A_n \in L(E_n)$, $B_n \in L(D_n, E_n)$, $n \in \mathbb{N}$. If $(A_n)_{n \in \mathbb{N}}$ is discretely compact and $(B_n)_{n \in \mathbb{N}}$ is a bounded sequence, then $(A_n B_n)_{n \in \mathbb{N}}$ is discretely compact.*

ii) *Let $A_n \in L(E_n)$, $B_n \in L(D_n, E_n)$, $n \in \mathbb{N}$. If $(B_n)_{n \in \mathbb{N}}$ is discretely compact and there exists $A \in L(E)$ with $A_n P_n \xrightarrow{s} AP$, then $(A_n B_n)_{n \in \mathbb{N}}$ is discretely compact.*

iii) *For $j = 1, \dots, k$, let $A_n^{(j)} \in L(D_n, E_n)$, $n \in \mathbb{N}$. If the sequences $(A_n^{(j)})_{n \in \mathbb{N}}$, $j = 1, \dots, k$, are discretely compact, then so is*

$$\left(\sum_{j=1}^k A_n^{(j)} \right)_{n \in \mathbb{N}}.$$

Proof. i) Let $I \subset \mathbb{N}$ be an infinite subset, $M > 0$ and $x_n \in D_n$, $n \in I$, with $\|x_n\|_{D_n} \leq M$ for all $n \in I$. Define $C := \sup_{n \in I} \|B_n\|$. Then $\|B_n x_n\|_{E_n} \leq CM$, $n \in I$. Now the discrete compactness of $(A_n)_{n \in \mathbb{N}}$ (see Definition 2.5) implies that a subsequence of $(A_n B_n x_n)_{n \in I}$ is convergent in E_0 with limit in E . So $(A_n B_n)_{n \in \mathbb{N}}$ is discretely compact.

ii) Let $I \subset \mathbb{N}$ be an infinite subset, $M > 0$ and $x_n \in D_n$, $n \in I$, with $\|x_n\|_{D_n} \leq M$, $n \in I$. The discrete compactness of $(B_n)_{n \in \mathbb{N}}$ implies that there exist $y \in E$

and an infinite subset $\tilde{I} \subset I$ such that $\|B_n x_n - y\|_{E_0} \rightarrow 0$ as $n \in \tilde{I}$, $n \rightarrow \infty$. By the assumptions, we have $A_n P_n \xrightarrow{s} AP$, so the element $z := Ay \in E$ satisfies $\|A_n B_n x_n - z\|_{E_0} \rightarrow 0$. Hence $(A_n B_n)_{n \in \mathbb{N}}$ is discretely compact.

iii) Let $I \subset \mathbb{N}$ be an infinite subset, $M > 0$ and $x_n \in D_n$, $n \in I$, with $\|x_n\|_{D_n} \leq M$, $n \in I$. By the discrete compactness of $(A_n^{(1)})_{n \in \mathbb{N}}$, there exists an infinite subset $I^{(1)} \subset I$ such that the sequence $(A_n^{(1)} x_n)_{n \in I^{(1)}}$ is convergent in E_0 with limit in E . Now, inductively for every $j = 2, \dots, k$, the discrete compactness of $(A_n^{(j)})_{n \in \mathbb{N}}$ implies that there exists an infinite subset $I^{(j)} \subset I^{(j-1)}$ such that $(A_n^{(j)} x_n)_{n \in I^{(j)}}$ is convergent in E_0 with limit in E . Therefore

$$\left(\sum_{j=1}^k A_n^{(j)} x_n \right)_{n \in I^{(k)}}$$

is convergent in E_0 with limit in E . \square

Analogously to the result that the limit of a sequence of compact operators is compact (see e.g. [17, Theorem III.4.7]), one can show the following result for discrete compactness. An application of Proposition 2.9 is given in Theorem 4.9 where infinite diagonal operator matrices are approximated by $k \times k$ matrices.

Proposition 2.9. *Let $E_0 = E$. For each $n \in \mathbb{N}$ let $A_n, A_n^{(k)} \in L(D_n, E_n)$, $k \in \mathbb{N}$, with*

$$\sup_{n \in \mathbb{N}} \|A_n^{(k)} - A_n\| \rightarrow 0, \quad k \rightarrow \infty. \quad (4)$$

If all sequences $(A_n^{(k)})_{n \in \mathbb{N}}$, $k \in \mathbb{N}$, are discretely compact, then so is $(A_n)_{n \in \mathbb{N}}$.

Proof. Consider an infinite subset $I \subset \mathbb{N}$ and a bounded sequence of elements $x_n \in D_n$, $n \in I$, i.e. there exists $M > 0$ such that $\|x_n\|_{D_n} \leq M$, $n \in I$. We show the existence of an infinite subset $\tilde{I} \subset I$ such that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ with

$$\|A_n x_n - A_m x_m\|_E < \varepsilon, \quad n, m \in \tilde{I}, \quad n, m \geq N_\varepsilon.$$

Then the claim follows from the completeness of E .

The sequence $(A_n^{(1)})_{n \in \mathbb{N}}$ is discretely compact by the assumptions, hence there exists an infinite subset $I^{(1)} \subset I$ such that $(A_n^{(1)} x_n)_{n \in I^{(1)}}$ is convergent in E . Inductively, for each $k \geq 2$, we find an infinite subset $I^{(k)} \subset I^{(k-1)}$ such that $(A_n^{(k)} x_n)_{n \in I^{(k)}}$ is convergent in E . Therefore, there exists an increasing sequence $(N^{(k)})_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $N^{(k)} \in I^{(k)}$, $k \in \mathbb{N}$, and

$$\|A_n^{(k)} x_n - A_m^{(k)} x_m\|_E < \frac{1}{k}, \quad n, m \in I^{(k)}, \quad n, m \geq N^{(k)}.$$

We define $\tilde{I} := \{N^{(k)} : k \in \mathbb{N}\}$. Let $\varepsilon > 0$ be fixed. By the assumption (4), we find $K_\varepsilon \in \mathbb{N}$ such that $K_\varepsilon \geq \frac{3}{\varepsilon}$ and

$$\|A_n^{(k)} - A_n\| < \frac{\varepsilon}{3M}, \quad n \in \mathbb{N}, \quad k \geq K_\varepsilon.$$

Altogether, for $k \geq K_\varepsilon$ (which yields $\frac{1}{k} \leq \frac{1}{K_\varepsilon} \leq \frac{\varepsilon}{3}$) and $l \geq k$, the elements $n = N^{(k)}$, $m = N^{(l)} \in \tilde{I}$ satisfy $n, m \geq N^{(K_\varepsilon)} =: N_\varepsilon$ and

$$\begin{aligned} & \|A_n x_n - A_m x_m\|_E \\ & \leq \|A_n - A_n^{(k)}\| \|x_n\|_{E_n} + \|A_m - A_m^{(k)}\| \|x_m\|_{E_n} + \|A_n^{(k)} x_n - A_m^{(k)} x_m\|_E \\ & < \frac{\varepsilon}{3M} M + \frac{\varepsilon}{3M} M + \frac{1}{k} \leq \varepsilon. \end{aligned} \quad \square$$

Now we assume that the underlying spaces are Hilbert spaces. It is well-known that an operator is compact if and only if so is its adjoint (see e.g. [17, Theorem III.4.10]). We prove an analogous result for discrete compactness of a sequence of operators.

Proposition 2.10. *Let $A \in L(H)$ and $A_n \in L(H_n)$, $n \in \mathbb{N}$. If*

$$A_n P_n \xrightarrow{s} AP, \quad A_n^* P_n \xrightarrow{s} A^* P, \quad n \rightarrow \infty,$$

then the following are equivalent:

- i) *the sequence $(A_n)_{n \in \mathbb{N}}$ is discretely compact;*
- ii) *the sequence $(A_n^* A_n)_{n \in \mathbb{N}}$ is discretely compact;*
- iii) *the sequence $(A_n^*)_{n \in \mathbb{N}}$ is discretely compact.*

Proof. The claim “i) \implies ii)” follows from Lemma 2.8 ii). To prove the reverse implication (and thus equivalence), let $M > 0$ and let $I \subset \mathbb{N}$ be an infinite subset and $y_n \in H_n$, $n \in I$, with $\|y_n\|_{H_n} \leq M$, $n \in I$. We show that there exists a convergent subsequence of $(A_n y_n)_{n \in I} \subset H_0$ with limit in H .

Since H_0 is weakly compact, there exists an infinite subset $I_2 \subset I$ such that $(y_n)_{n \in I_2} \subset H_0$ is weakly convergent; denote the weak limit by y . Since $P_n \xrightarrow{s} P$, it is easy to see that $y = Py \in H$. In addition, $A_n^* P_n \xrightarrow{s} A^* P$ implies

$$A_n y_n \xrightarrow{w} Ay, \quad n \in I_2, \quad n \rightarrow \infty.$$

Below we show that $(\|A_n y_n\|_{H_0})_{n \in I_3}$ converges to $\|Ay\|_{H_0}$ for some infinite subset $I_3 \subset I_2$; then we obtain the desired convergence $\|A_n y_n - Ay\|_{H_0} \rightarrow 0$ as $n \in I_3$, $n \rightarrow \infty$.

By the assumptions, the sequence $(A_n^* A_n)_{n \in \mathbb{N}}$ is discretely compact, thus there exist an infinite subset $I_3 \subset I_2$ and $x \in H$ such that $(A_n^* A_n y_n)_{n \in I_3}$ converges in H_0 to x . On the other hand, the strong convergences $A_n P_n \xrightarrow{s} AP$ and $A_n^* P_n \xrightarrow{s} A^* P$ imply the weak convergence

$$A_n^* A_n y_n \xrightarrow{w} A^* Ay, \quad n \in I_3, \quad n \rightarrow \infty.$$

By the uniqueness of the weak limit, we have $A^* Ay = x$. So we obtain, for $n \in I_3$,

$$\|A_n y_n\|_{H_0}^2 = \langle A_n^* A_n y_n, y_n \rangle_{H_0} \longrightarrow \langle x, y \rangle_{H_0} = \langle A^* Ay, y \rangle_{H_0} = \|Ay\|_{H_0}^2, \quad n \rightarrow \infty;$$

this proves “ii) \implies i)”.

The claim “iii) \implies ii)” follows from Lemma 2.8 i). Altogether we conclude “iii) \implies i)”. Now the reverse implication (and thus equivalence) follows from taking the adjoint operators. \square

Related to discrete compactness is the notion of collectively compact sets of bounded linear operators (see Anselone and Palmer [1]).

Definition 2.11. Let \mathcal{B} be the closed unit ball in E_0 . A subset $\mathcal{K} \subset L(E_0)$ is called *collectively compact* if the set

$$\mathcal{KB} = \{Kx : K \in \mathcal{K}, x \in \mathcal{B}\} \subset E_0$$

is relatively compact in E_0 .

Remark 2.12. i) Every operator of a collectively compact set is compact.
 ii) A set $\{A_n : n \in \mathbb{N}\}$ is collectively compact if and only if the operators A_n , $n \in \mathbb{N}$, are compact and the sequence $(A_n)_{n \in \mathbb{N}}$ is discretely compact.

The following result yields sufficient conditions on Hilbert space operators such that strong convergence implies norm convergence.

Proposition 2.13. *Let $A \in L(H)$ and $A_n \in L(H_n)$, $n \in \mathbb{N}$. Assume that A_n , $n \in \mathbb{N}$, are all compact operators and $(A_n)_{n \in \mathbb{N}}$ is a discretely compact sequence. If $A_n P_n \xrightarrow{s} AP$ and $A_n^* P_n \xrightarrow{s} A^* P$, then A is compact and $A_n P_n \rightarrow AP$.*

Proof. It is well-known that if A_n , $n \in \mathbb{N}$, are all compact operators, then so are $A_n P_n$, $A_n^* P_n$, $n \in \mathbb{N}$. Proposition 2.10 yields the discrete compactness of the sequence $(A_n^*)_{n \in \mathbb{N}}$. By Lemma 2.8 i), the sequences $(A_n P_n)_{n \in \mathbb{N}}$, $(A_n^* P_n)_{n \in \mathbb{N}}$ are discretely compact. Remark 2.12 ii) implies that $\{A_n P_n : n \in \mathbb{N}\}$, $\{A_n^* P_n : n \in \mathbb{N}\}$ are collectively compact sets. Then, by [1, Proposition 2.1 (a) \implies (b)], so are the sets $\{A_n P_n - AP : n \in \mathbb{N}\}$, $\{A_n^* P_n - A^* P : n \in \mathbb{N}\}$, and AP (and thus A) is a compact operator. Now the claim follows from [1, Theorem 3.4 (c)]. \square

2.3. Proofs of main results. In this subsection we prove the theorems in Subsection 2.1.

The following two elementary results will be used later on.

Lemma 2.14. *Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$. Assume that $T_n \xrightarrow{gsr} T$. Then for all $x \in \mathcal{D}(T)$ there exists a sequence of elements $x_n \in \mathcal{D}(T_n)$, $n \in \mathbb{N}$, such that*

$$\|x_n - x\|_{E_0} + \|T_n x_n - Tx\|_{E_0} \longrightarrow 0, \quad n \rightarrow \infty. \quad (5)$$

Proof. By Definition 2.1 i) of $T_n \xrightarrow{gsr} T$, there exist $n_0 \in \mathbb{N}$ and $\lambda \in \bigcap_{n \geq n_0} \varrho(T_n) \cap \varrho(T)$ such that

$$(T_n - \lambda)^{-1} P_n \xrightarrow{s} (T - \lambda)^{-1} P, \quad n \rightarrow \infty. \quad (6)$$

Let $x \in \mathcal{D}(T)$ and define

$$x_n := (T_n - \lambda)^{-1} P_n (T - \lambda)x \in \mathcal{D}(T_n), \quad n \geq n_0.$$

Then, using $P_n \xrightarrow{s} P$ and (6), it is easy to verify that (5) holds. \square

Lemma 2.15. *Let $T_n \in C(E_n)$, $n \in \mathbb{N}$, and let $K \subset \Delta_b((T_n)_{n \in \mathbb{N}})$ be a compact subset. Then there exist $M_K > 0$ and $n_K \in \mathbb{N}$ such that*

$$\forall \lambda \in K : \quad \lambda \in \varrho(T_n), \quad \|(T_n - \lambda)^{-1}\| \leq M_K, \quad n \geq n_K.$$

Proof. Assume that the claim is false, i.e. no such $M_K > 0$ exists. Then there exist an infinite subset $I_1 \subset \mathbb{N}$ and $(\lambda_n)_{n \in I_1} \subset K$ such that $\|(T_n - \lambda_n)^{-1}\| \rightarrow \infty$ as $n \in I_1$, $n \rightarrow \infty$. By the compactness of K , there are $\lambda \in K$ and an infinite subset $I_2 \subset I_1$ so that $(\lambda_n)_{n \in I_2}$ converges to λ . Since $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}})$, there exist $M_\lambda > 0$ and an infinite subset $I_3 \subset I_2$ such that $\lambda \in \varrho(T_n)$ and $\|(T_n - \lambda)^{-1}\| \leq M_\lambda$ for all $n \in I_3$. Then, for every $n \in I_3$ so large that $|\lambda_n - \lambda| \leq 1/(2M_\lambda)$, a Neumann series argument yields

$$\|(T_n - \lambda_n)^{-1}\| = \|(T_n - \lambda)^{-1} (I - (\lambda_n - \lambda)(T_n - \lambda)^{-1})^{-1}\| \leq 2M_\lambda.$$

The obtained contradiction proves the claim. \square

For generalized strong/norm resolvent convergence we assume the resolvents to converge for one particular λ . In the following result we investigate for which points the resolvents then converge as well.

Proposition 2.16. *Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$.*

- i) *Assume that $T_n \xrightarrow{gsr} T$. Then $\Delta_b((T_n)_{n \in \mathbb{N}}) \subset \mathbb{C} \setminus \sigma_{\text{app}}(T)$ and, for any $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}}) \cap \varrho(T)$,*

$$(T_n - \lambda)^{-1} P_n \xrightarrow{s} (T - \lambda)^{-1} P, \quad n \rightarrow \infty. \quad (7)$$

- ii) *Assume that $T_n \xrightarrow{gnr} T$. Then $\varrho(T) \subset \Delta_b((T_n)_{n \in \mathbb{N}})$ and, for any $\lambda \in \varrho(T)$,*

$$(T_n - \lambda)^{-1} P_n \longrightarrow (T - \lambda)^{-1} P, \quad n \rightarrow \infty. \quad (8)$$

Proof. i) By Definition 2.1 i) of $T_n \xrightarrow{gsr} T$, there exist $n_0 \in \mathbb{N}$ and an element $\lambda_0 \in \bigcap_{n \geq n_0} \varrho(T_n) \cap \varrho(T)$ such that

$$(T_n - \lambda_0)^{-1}P_n \xrightarrow{s} (T - \lambda_0)^{-1}P, \quad n \rightarrow \infty. \quad (9)$$

Let $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}})$. By Definition 2.1 iii) of the region of boundedness, there exists $n_1 \in \mathbb{N}$ (without loss of generality $n_1 \geq n_0$) such that $\lambda \in \varrho(T_n)$, $n \geq n_1$, and $M := \sup_{n \geq n_1} \|(T_n - \lambda)^{-1}\| < \infty$. First assume that λ belongs to the approximate point spectrum $\sigma_{\text{app}}(T)$. Then there exists $x \in \mathcal{D}(T)$ with $\|x\|_E = 1$ and $\|(T - \lambda)x\|_E < 1/(2M)$. By Lemma 2.14, there exists a sequence of elements $x_n \in \mathcal{D}(T_n)$, $n \in \mathbb{N}$, such that $\|x_n\|_{E_n} = 1$ and $\|(T_n - \lambda)x_n\|_{E_n} < 1/(2M)$ for all large enough $n \in \mathbb{N}$. The obtained contradiction to $M = \sup_{n \geq n_1} \|(T_n - \lambda)^{-1}\|$ implies $\lambda \notin \sigma_{\text{app}}(T)$.

Now assume that $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}}) \cap \varrho(T)$. If we set, for $n \geq n_1$,

$$S_n(\lambda) := I + (\lambda_0 - \lambda)(T_n - \lambda_0)^{-1}P_n = (T_n - \lambda)(T_n - \lambda_0)^{-1}P_n + (I - P_n),$$

then a straightforward application of the first resolvent identity yields

$$\begin{aligned} S_n(\lambda)((T_n - \lambda)^{-1}P_n - (T - \lambda)^{-1}P) \\ = ((T_n - \lambda_0)^{-1}P_n - (T - \lambda_0)^{-1}P)(I + (\lambda - \lambda_0)(T - \lambda)^{-1}P), \quad n \geq n_1. \end{aligned} \quad (10)$$

The operators $S_n(\lambda)$, $n \geq n_1$, are invertible,

$$S_n(\lambda)^{-1} = (T_n - \lambda_0)(T_n - \lambda)^{-1}P_n + (I - P_n) = I + (\lambda - \lambda_0)(T_n - \lambda)^{-1}P_n, \quad (11)$$

and the inverses are uniformly bounded since $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}})$. Now the claimed convergence (7) follows from (9) and (10).

ii) Let $\lambda \in \varrho(T)$. By Definition 2.1 ii) of $T_n \xrightarrow{gnr} T$, there exist $n_0 \in \mathbb{N}$ and an element $\lambda_0 \in \bigcap_{n \geq n_0} \varrho(T_n) \cap \varrho(T)$ such that

$$(T_n - \lambda_0)^{-1}P_n \longrightarrow (T - \lambda_0)^{-1}P, \quad n \rightarrow \infty. \quad (12)$$

This implies

$$S_n(\lambda) = I + (\lambda_0 - \lambda)(T_n - \lambda_0)^{-1}P_n \longrightarrow I + (\lambda_0 - \lambda)(T - \lambda_0)^{-1}P =: S, \quad n \rightarrow \infty.$$

Since S is boundedly invertible, [17, Theorem IV.1.16] yields the existence of some $n_1 \in \mathbb{N}$ such that the operators $S_n(\lambda)$, $n \geq n_1$, are uniformly boundedly invertible. Then $(T_n - \lambda)$, $n \geq n_1$, are uniformly boundedly invertible because, by (11),

$$(T_n - \lambda)^{-1} = (\lambda - \lambda_0)^{-1} (S_n(\lambda)^{-1} - I)|_{E_n}, \quad n \geq n_1.$$

Now, analogously as in i), the claim (8) follows from (12) and (10). \square

Now we are ready to prove the main results of Subsection 2.1.

Proof of Theorem 2.3. i) Let $\lambda \in \sigma(T)$ and $\varepsilon > 0$ satisfy (1). Choose $\delta > 0$ with $\delta < \varepsilon$. Assume that there exists an infinite subset $I \subset \mathbb{N}$ with $\text{dist}(\lambda, \sigma(T_n)) \geq \delta$, $n \in I$. For $\Gamma := \partial B_{\delta/2}(\lambda)$ define the contour integrals

$$P_\Gamma := \frac{1}{2\pi i} \int_\Gamma (T - z)^{-1} dz, \quad P_{\Gamma, n} := \frac{1}{2\pi i} \int_\Gamma (T_n - z)^{-1} dz, \quad n \in I.$$

The operator P_Γ is the spectral projection corresponding to $\lambda \in \sigma(T)$. However, since $z \mapsto (T_n - z)^{-1}$ is holomorphic in $B_\delta(\lambda)$, we have $P_{\Gamma, n} = 0$, $n \in I$. Let $x \in E_0$ be arbitrary. For $n \in I$ define the function $f_n : \Gamma \rightarrow [0, \infty)$ by $f_n(z) := \|(T - z)^{-1}Px - (T_n - z)^{-1}P_n x\|_{E_0}$. Then

$$\|P_\Gamma Px - P_{\Gamma, n} P_n x\|_{E_0} \leq \frac{1}{2\pi} \int_\Gamma f_n(z) d|z|, \quad n \in I.$$

Note that, by Proposition 2.16 i), $f_n(z) \rightarrow 0$, $n \rightarrow \infty$, for every $z \in \Gamma$. Moreover, f_n , $n \in \mathbb{N}$, are uniformly bounded by the compactness of $\Gamma \subset \Delta_b((T_n)_{n \in \mathbb{N}})$ and by Lemma 2.15. Lebesgue's dominated convergence theorem implies $\|P_\Gamma P x - P_{\Gamma,n} P_n x\|_{E_0} \rightarrow 0$ as $n \rightarrow \infty$. Hence $P_{\Gamma,n} P_n \xrightarrow{s} P_\Gamma P$, $n \rightarrow \infty$, and so we arrive at the contradiction $P_\Gamma = 0$. Therefore, there exists $n_\delta \in \mathbb{N}$ such that $\text{dist}(\lambda, \sigma(T_n)) < \delta$, $n \geq n_\delta$. Since δ can be chosen arbitrarily small, we finally obtain $\text{dist}(\lambda, \sigma(T_n)) \rightarrow 0$, $n \rightarrow \infty$.

ii) Let $\lambda \in \varrho(T) \cap \Delta_b((T_n)_{n \in \mathbb{N}})$. Definition 2.1 iii) of the region of boundedness implies that there exist $n_0 \in \mathbb{N}$ and $M > 0$ such that $\lambda \in \varrho(T_n)$, $n \geq n_0$, and $\|(T_n - \lambda)^{-1}\| \leq M$, $n \geq n_0$. As a consequence,

$$\text{dist}(\lambda, \sigma(T_n)) \geq \frac{1}{\|(T_n - \lambda)^{-1}\|} \geq \frac{1}{M}, \quad n \geq n_0,$$

so λ cannot be the limit of a sequence of points in the spectra of T_n , $n \geq n_0$. \square

Proof of Theorem 2.4. i) By Proposition 2.16 ii), we have $\varrho(T) \subset \Delta_b((T_n)_{n \in \mathbb{N}})$. Now the claim follows from Theorem 2.3 ii).

ii) Since T is assumed to be selfadjoint, it satisfies $\sigma(T) = \sigma_{\text{app}}(T)$ and thus Proposition 2.16 i) implies $\Delta_b((T_n)_{n \in \mathbb{N}}) \subset \varrho(T)$. In addition, since T_n is selfadjoint, we have $\text{dist}(\lambda, \sigma(T_n)) = \|(T_n - \lambda)^{-1}\|^{-1}$ for any $\lambda \in \varrho(T_n)$, which implies (2). \square

Proof of Theorem 2.6. i) Since T has compact resolvent, it satisfies $\sigma(T) = \sigma_p(T) = \sigma_{\text{app}}(T)$. Proposition 2.16 i) implies that $\Delta_b((T_n)_{n \in \mathbb{N}}) \subset \varrho(T)$.

Conversely, take $\lambda \in \mathbb{C} \setminus \Delta_b((T_n)_{n \in \mathbb{N}})$. Note that $\sigma(T_n) = \sigma_p(T_n)$ since T_n is assumed to have compact resolvent. Then there are an infinite subset $I \subset \mathbb{N}$ and $x_n \in \mathcal{D}(T_n)$, $n \in I$, with

$$\|x_n\|_{E_n} = 1, \quad n \in I, \quad \|(T_n - \lambda)x_n\|_{E_n} \rightarrow 0, \quad n \rightarrow \infty. \quad (13)$$

Define $y_n := (T_n - \lambda_0)x_n$ for $n \in I$. Then $(\|y_n\|_{E_n})_{n \in I}$ is a bounded sequence. Since $((T_n - \lambda_0)^{-1})_{n \in \mathbb{N}}$ is discretely compact by the assumptions, there exist $x \in E$ and an infinite subset $\tilde{I} \subset I$ so that $\|x_n - x\|_{E_0} \rightarrow 0$ as $n \in \tilde{I}$, $n \rightarrow \infty$. By (13), we have $\|x\|_E = 1$ and $\|y_n - (\lambda - \lambda_0)x\|_{E_0} \rightarrow 0$. However, $(T_n - \lambda_0)^{-1}P_n \xrightarrow{s} (T - \lambda_0)^{-1}P$ then yields

$$x_n = (T_n - \lambda_0)^{-1}y_n \rightarrow (\lambda - \lambda_0)(T - \lambda_0)^{-1}x \in \mathcal{D}(T), \quad n \in \tilde{I}, \quad n \rightarrow \infty.$$

By the uniqueness of the limit, we obtain $x \in \mathcal{D}(T)$ and $Tx = \lambda x$. Since $x \neq 0$, we have $\lambda \in \sigma(T)$.

The convergence (3) for all $\lambda \in \varrho(T)$ now follows from Proposition 2.16 i).

ii) Spectral exactness follows from claim i) and Theorem 2.3; note that all $\lambda \in \sigma(T)$ are isolated since T is assumed to have compact resolvent. In an analogous way as in the proof of 2.3 i), one may prove that the corresponding spectral projections converge strongly, $P_{\Gamma,n} P_n \xrightarrow{s} P_\Gamma P$. This implies that for $x = P_\Gamma P x$ in the algebraic eigenspace of λ there exists a sequence of elements $x_n := P_{\Gamma,n} P_n x \in \mathcal{R}(P_{\Gamma,n})$, $n \in \mathbb{N}$, with $\|x_n - x\|_{E_0} \rightarrow 0$, and the normalized elements converge as well. This proves that $m = \text{rank } P_\Gamma \leq \liminf_{n \rightarrow \infty} \text{rank } P_{\Gamma,n}$.

To prove that $\limsup_{n \rightarrow \infty} \text{rank } P_{\Gamma,n} \leq m$ (and thus $\text{rank } P_{\Gamma,n} = m$ for all sufficiently large n), let $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, such that $\lambda_n \rightarrow \lambda \in \sigma(T)$ as $n \rightarrow \infty$. For $n \in \mathbb{N}$ denote by k_n the ascent of λ_n , i.e. the smallest $k \in \mathbb{N}$ such that $(T_n - \lambda_n)^k P_{\Gamma,n} = (T_n - \lambda_n)^{k+1} P_{\Gamma,n}$; then there exist k_n orthonormal elements $x_n^{(k)} \in \mathcal{R}(P_{\Gamma,n})$ with $(T_n - \lambda_n)x_n^{(1)} = 0$ and $(T_n - \lambda_n)x_n^{(k)} = x_n^{(k-1)}$ for $k = 2, \dots, k_n$. By induction over $k \in \mathbb{N}$, we prove that if there exists an infinite subset $I^{(k)} \subset \mathbb{N}$ such that $k \leq k_n$, $n \in I^{(k)}$, then there exist $x^{(k)} \in \mathcal{R}(P_\Gamma)$ and an infinite subset

$\tilde{I}^{(k)} \subset I^{(k)}$ such that $\|x^{(k)}\|_E = 1$, $(T - \lambda)^k x^{(k)} = 0$ and $\|x_n^{(k)} - x^{(k)}\|_{E_0} \rightarrow 0$ as $n \in \tilde{I}^{(k)}$, $n \rightarrow \infty$; then we obtain $\limsup_{n \rightarrow \infty} \text{rank } P_{\Gamma, n} \leq m$.

If $k = 1$, then the claim follows from the proof of i) since $\lambda_n \rightarrow \lambda$ and thus $x_n^{(1)}$, $n \in I^{(1)}$, satisfy (13). If $k > 1$, set $y_n^{(k)} := (T_n - \lambda_0)x_n^{(k)} = x_n^{(k-1)} + (\lambda_n - \lambda_0)x_n^{(k)}$, $n \in I^{(k)}$. Since $\lambda_n \rightarrow \lambda$, the sequence $(\|y_n^{(k)}\|_{E_n})_{n \in I^{(k)}}$ is bounded. By proceeding as in i) and using the induction hypothesis, we obtain the claim. \square

Remark 2.17. Theorem 2.6 would also follow from Stummel's results [26, Sätze 2.2.(1), 3.2.(8)] applied to the bounded operators $A, B \in L(D, E)$ and $A_n, B_n \in L(D_n, E_n)$, $n \in \mathbb{N}$, where the Banach spaces $D := \mathcal{D}(T)$, $D_n := \mathcal{D}(T_n)$ are equipped with the graph norm of T, T_n , respectively, the operators $B : D \rightarrow E$, $B_n : D_n \rightarrow E_n$ are the natural embeddings, and A, A_n are defined by $Ax := Tx$, $A_n x_n := T_n x_n$. However, it is very technical to check that the assumptions of Stummel's results are satisfied (see [3, Chapter 1] for this approach).

Proof of Theorem 2.7. The claim for $\lambda = \lambda_0$ is an immediate consequence of Proposition 2.13 applied to $A = (T - \lambda_0)^{-1}$, $A_n = (T_n - \lambda_0)^{-1}$, $n \in \mathbb{N}$. The generalized norm resolvent convergence for every $\lambda \in \varrho(T)$ follows from Proposition 2.16 ii). \square

3. GENERALIZED STRONG RESOLVENT (GSR) CONVERGENCE

In this section we establish sufficient conditions for gsr-convergence of a sequence of linear operators in varying Banach spaces. First, we find conditions that directly yield gsr-convergence, but then we also give sufficient conditions for gsr-convergence of a sequence of operators $A_n = T_n + S_n \in C(E_n)$, $n \in \mathbb{N}$, if we know it for the operators T_n , $n \in \mathbb{N}$ (see Subsection 3.1). Afterwards, in Subsection 3.2, we derive sufficient conditions on a sequence of 2×2 block operator matrices that imply gsr-convergence. Then we consider gsr-convergence of infinite operator matrices (see Subsections 3.3, 3.4).

3.1. Direct criteria and perturbation results. As in Section 2, let E_0 be a Banach space and $E, E_n \subset E_0$, $n \in \mathbb{N}$, be closed complemented subspaces, with $P : E_0 \rightarrow E$, $P_n : E_0 \rightarrow E_n$, $n \in \mathbb{N}$, denoting the projections on the respective subspaces (along the respective complements) that converge strongly, $P_n \xrightarrow{s} P$.

The next proposition is a generalization of Weidmann's result [28, Satz 9.29 a)] who considers selfadjoint operators.

Theorem 3.1. *Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$. Let $\Phi \subset \mathcal{D}(T)$ be a core of T such that for all $x \in \Phi$ there exists $n_0(x) \in \mathbb{N}$ with*

$$\forall n \geq n_0(x) : \quad P_n x \in \mathcal{D}(T_n), \quad \|T_n P_n x - Tx\|_{E_0} \rightarrow 0, \quad n \rightarrow \infty.$$

Suppose that $\Delta_b((T_n)_{n \in \mathbb{N}}) \cap \varrho(T) \neq \emptyset$. Then $T_n \xrightarrow{gsr} T$.

Proof. The proof of Weidmann's result [28, Satz 9.29 a)] remains valid in the non-selfadjoint case. The only place in the latter result where the selfadjointness of $T \in C(E)$, $T_n \in C(E_n)$, $n \in \mathbb{N}$, is used is the consequence

$$\Delta_b((T_n)_{n \in \mathbb{N}}) \cap \varrho(T) \neq \emptyset \tag{14}$$

(since $\mathbb{C} \setminus \mathbb{R}$ belongs to the intersection). Since here $T \in C(E)$, $T_n \in C(E_n)$, $n \in \mathbb{N}$, are not assumed to be selfadjoint, we require (14) to be satisfied by assumption. \square

For bounded operators strong convergence implies gsr-convergence.

Lemma 3.2. *Let $B \in L(E)$ and $B_n \in L(E_n)$, $n \in \mathbb{N}$, satisfy $B_n P_n \xrightarrow{s} BP$. Then*

$$\left\{ \lambda \in \mathbb{C} : |\lambda| > \limsup_{n \rightarrow \infty} \|B_n\| \right\} \subset \Delta_b((B_n)_{n \in \mathbb{N}}), \tag{15}$$

and

$$\forall \lambda \in \Delta_b((B_n)_{n \in \mathbb{N}}) \cap \varrho(B) : (B_n - \lambda)^{-1} P_n \xrightarrow{s} (B - \lambda)^{-1} P, \quad n \rightarrow \infty.$$

Proof. Let $\lambda \in \mathbb{C}$ satisfy $|\lambda| > \limsup_{n \rightarrow \infty} \|B_n\| =: M$. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ with

$$|\lambda| > \|B_n\| + \varepsilon, \quad n \geq n_0.$$

Now a Neumann series argument yields that, for $n \geq n_0$, the operator $(B_n - \lambda)$ is invertible, with

$$(B_n - \lambda)^{-1} = -\lambda^{-1}(I - \lambda^{-1}B_n)^{-1}, \quad \|(B_n - \lambda)^{-1}\| \leq \frac{|\lambda^{-1}|}{1 - |\lambda^{-1}|\|B_n\|} \leq \frac{1}{\varepsilon}.$$

This proves the inclusion (15).

Let $\lambda \in \Delta_b((B_n)_{n \in \mathbb{N}}) \cap \varrho(B)$. Then there exists $n_0 \in \mathbb{N}$ such that $\lambda \in \varrho(B_n)$, $n \geq n_0$, and $((B_n - \lambda)^{-1})_{n \geq n_0}$ is a bounded sequence. Let $y \in E_0$ and define $x := (B - \lambda)^{-1} P y$. It is easy to verify that, for every $n \geq n_0$,

$$\begin{aligned} & ((B_n - \lambda)^{-1} P_n - (B - \lambda)^{-1} P) y \\ &= (B_n - \lambda)^{-1} P_n (B P - B_n P_n) x - (B_n - \lambda)^{-1} P_n (P - P_n) y - (P - P_n) x. \end{aligned} \quad (16)$$

Since the sequence $(\|(B_n - \lambda)^{-1}\|)_{n \geq n_0}$ is bounded, the assumptions $B_n P_n \xrightarrow{s} B P$ and $P_n \xrightarrow{s} P$ imply that the right-hand side of (16) converges to 0. \square

Now we consider sums $A = T + S$ and $A_n = T_n + S_n$, $n \in \mathbb{N}$. We study perturbation results for generalized strong resolvent convergence, i.e. we establish sufficient conditions that $T_n \xrightarrow{gsr} T$ implies $A_n \xrightarrow{gsr} A$.

Theorem 3.3. *Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$. Let S and S_n , $n \in \mathbb{N}$, be linear operators in E and E_n , $n \in \mathbb{N}$, with $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $\mathcal{D}(T_n) \subset \mathcal{D}(S_n)$, $n \in \mathbb{N}$, respectively. Define*

$$A := T + S, \quad A_n := T_n + S_n, \quad n \in \mathbb{N}.$$

Suppose that there exist $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(T_n) \cap \varrho(T)$ and $\gamma_\lambda < 1$ with

$$\|S(T - \lambda)^{-1}\| < 1, \quad \|S_n(T_n - \lambda)^{-1}\| \leq \gamma_\lambda, \quad n \in \mathbb{N}. \quad (17)$$

If

$$\begin{aligned} & (T_n - \lambda)^{-1} P_n \xrightarrow{s} (T - \lambda)^{-1} P, \\ & S_n(T_n - \lambda)^{-1} P_n \xrightarrow{s} S(T - \lambda)^{-1} P, \end{aligned} \quad n \rightarrow \infty, \quad (18)$$

then $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(A_n) \cap \varrho(A)$ and $(A_n - \lambda)^{-1} P_n \xrightarrow{s} (A - \lambda)^{-1} P$ as $n \rightarrow \infty$.

Remark 3.4. The inequalities (17) imply that S is T -bounded with T -bound < 1 and S_n is T_n -bounded with T_n -bound $\leq \gamma_\lambda < 1$.

Proof of Theorem 3.3. Let λ and $\gamma_\lambda < 1$ satisfy the assumptions. For $n \in \mathbb{N}$, by a Neumann series argument, $(A_n - \lambda)$ is boundedly invertible and

$$\begin{aligned} & (A_n - \lambda)^{-1} = (T_n - \lambda)^{-1} (I + S_n(T_n - \lambda)^{-1})^{-1}, \\ & \|(I + S_n(T_n - \lambda)^{-1})^{-1}\| \leq \frac{1}{1 - \gamma_\lambda}. \end{aligned} \quad (19)$$

Analogously we obtain

$$(A - \lambda)^{-1} = (T - \lambda)^{-1} (I + S(T - \lambda)^{-1})^{-1}. \quad (20)$$

Thus $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(A_n) \cap \varrho(A)$. Let $x \in E_0$ and define $y := (I + S(T - \lambda)^{-1})^{-1} Px \in E$. Since $((T_n - \lambda)^{-1})_{n \in \mathbb{N}}$ is strongly convergent, $C := \sup_{n \in \mathbb{N}} \|(T_n - \lambda)^{-1}\| < \infty$. Then

$$\begin{aligned} & \|(A_n - \lambda)^{-1} P_n x - (A - \lambda)^{-1} Px\|_{E_0} \\ & \leq \left\| (T_n - \lambda)^{-1} P_n \left((I + S_n(T_n - \lambda)^{-1})^{-1} P_n x - (I + S(T - \lambda)^{-1})^{-1} Px \right) \right\|_{E_0} \\ & \quad + \left\| ((T_n - \lambda)^{-1} P_n - (T - \lambda)^{-1} P) (I + S(T - \lambda)^{-1})^{-1} Px \right\|_{E_0} \\ & \leq C \left\| (I + S_n(T_n - \lambda)^{-1})^{-1} P_n x - (I + S(T - \lambda)^{-1})^{-1} Px \right\|_{E_0} \\ & \quad + \left\| ((T_n - \lambda)^{-1} P_n - (T - \lambda)^{-1} P) y \right\|_{E_0}. \end{aligned}$$

The first convergence in (18) yields

$$\left\| ((T_n - \lambda)^{-1} P_n - (T - \lambda)^{-1} P) y \right\|_{E_0} \longrightarrow 0, \quad n \rightarrow \infty.$$

By (19) and (20), we have $-1 \in \Delta_b((S_n(T_n - \lambda)^{-1})_{n \in \mathbb{N}}) \cap \varrho(S(T - \lambda)^{-1})$. Hence, the second convergence in (18) implies that, by Lemma 3.2,

$$\left\| (I + S_n(T_n - \lambda)^{-1})^{-1} P_n x - (I + S(T - \lambda)^{-1})^{-1} Px \right\|_{E_0} \longrightarrow 0, \quad n \rightarrow \infty.$$

Altogether, we have $(A_n - \lambda)^{-1} P_n \xrightarrow{s} (A - \lambda)^{-1} P$, $n \rightarrow \infty$. \square

The following result is an immediate consequence of Theorem 3.3 for the case that the perturbations are bounded operators.

Corollary 3.5. *Let $T \in C(E)$ and $T_n \in C(E_n)$, $n \in \mathbb{N}$. Let $S \in L(E)$ and $S_n \in L(E_n)$, $n \in \mathbb{N}$. Define*

$$A := T + S, \quad A_n := T_n + S_n, \quad n \in \mathbb{N}.$$

Suppose that, for some $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(T_n) \cap \varrho(T)$ and some $\gamma_\lambda < 1$, we have

$$\|S\| < \|(T - \lambda)^{-1}\|^{-1}, \quad \|S_n\| \leq \gamma_\lambda \|(T_n - \lambda)^{-1}\|^{-1}, \quad n \in \mathbb{N},$$

and

$$(T_n - \lambda)^{-1} P_n \xrightarrow{s} (T - \lambda)^{-1} P, \quad S_n P_n \xrightarrow{s} S P, \quad n \rightarrow \infty.$$

Then $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(A_n) \cap \varrho(A)$ and $(A_n - \lambda)^{-1} P_n \xrightarrow{s} (A - \lambda)^{-1} P$ as $n \rightarrow \infty$.

3.2. Results for 2×2 block operator matrices. In this subsection we consider diagonally dominant 2×2 operator matrices. To this end, all spaces are assumed to consist of two components. For $i = 1, 2$, let $E_i^{(0)}$ be a Banach space and let $E_i, E_i^{(n)} \subset E_i^{(0)}$, $n \in \mathbb{N}$, be closed complemented subspaces; denote by $P_i : E_i^{(0)} \rightarrow E_i, P_i^{(n)} : E_i^{(0)} \rightarrow E_i^{(n)}$, $n \in \mathbb{N}$, projections on the respective subspaces (along the respective complements) that converge strongly, $P_i^{(n)} \xrightarrow{s} P_i, n \rightarrow \infty$. In the product space $\mathcal{E} := E_1 \oplus E_2$ we consider a 2×2 block operator matrix

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) := (\mathcal{D}(A) \cap \mathcal{D}(C)) \oplus (\mathcal{D}(B) \cap \mathcal{D}(D))$$

where $A : E_1 \rightarrow E_1, B : E_2 \rightarrow E_1, C : E_1 \rightarrow E_2, D : E_2 \rightarrow E_2$ are closable operators with dense domains. We assume that \mathcal{A} is densely defined.

Definition 3.6. [27, Definition 2.2.3] Let $\delta \geq 0$. The block operator matrix \mathcal{A} is called

- i) *diagonally dominant (of order δ)* if C is A -bounded with A -bound δ_C, B is D -bounded with D -bound δ_B , and $\delta = \max\{\delta_C, \delta_B\}$,

- ii) *off-diagonally dominant (of order δ)* if A is C -bounded with C -bound δ_A , D is B -bounded with B -bound δ_D , and $\delta = \max\{\delta_A, \delta_D\}$.

In the product space $\mathcal{E}^{(n)} := E_1^{(n)} \oplus E_2^{(n)}$ we consider a 2×2 block operator matrix

$$\mathcal{A}^{(n)} := \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}^{(n)}) := (\mathcal{D}(A^{(n)}) \cap \mathcal{D}(C^{(n)})) \oplus (\mathcal{D}(B^{(n)}) \cap \mathcal{D}(D^{(n)}))$$

with operators $A^{(n)} : E_1^{(n)} \rightarrow E_1^{(n)}$, $B^{(n)} : E_2^{(n)} \rightarrow E_1^{(n)}$, $C^{(n)} : E_1^{(n)} \rightarrow E_2^{(n)}$, $D^{(n)} : E_2^{(n)} \rightarrow E_2^{(n)}$ that are assumed to be closable with dense domains. We assume that $\mathcal{A}^{(n)}$ is densely defined as well.

Theorem 3.7. *Suppose that the following holds:*

- i) *the block operator matrices $\mathcal{A}, \mathcal{A}^{(n)}$, $n \in \mathbb{N}$, are diagonally dominant;*
ii) *there exists a core $\Phi_1 \subset \mathcal{D}(A)$ of A such that for all $x_1 \in \Phi_1$ there exists $n_1(x_1) \in \mathbb{N}$ with the property that*

$$P_1^{(n)}x_1 \in \mathcal{D}(A^{(n)}), \quad n \geq n_1(x_1), \\ \|A^{(n)}P_1^{(n)}x_1 - Ax_1\|_{E_1^{(0)}} + \|C^{(n)}P_1^{(n)}x_1 - Cx_1\|_{E_1^{(0)}} \longrightarrow 0, \quad n \rightarrow \infty;$$

- iii) *there exists a core $\Phi_2 \subset \mathcal{D}(D)$ of D such that for all $x_2 \in \Phi_2$ there exists $n_2(x_2) \in \mathbb{N}$ with the property that*

$$P_2^{(n)}x_2 \in \mathcal{D}(D^{(n)}), \quad n \geq n_2(x_2), \\ \|D^{(n)}P_2^{(n)}x_2 - Dx_2\|_{E_2^{(0)}} + \|B^{(n)}P_2^{(n)}x_2 - Bx_2\|_{E_2^{(0)}} \longrightarrow 0, \quad n \rightarrow \infty;$$

- iv) $\Delta_b((\mathcal{A}^{(n)})_{n \in \mathbb{N}}) \cap \varrho(\mathcal{A}) \neq \emptyset$.

Then $\mathcal{A}^{(n)} \xrightarrow{gsr} \mathcal{A}$.

Proof. First note that \mathcal{A} is closed by the assumption $\varrho(\mathcal{A}) \neq \emptyset$. Define the subspace $\Phi := \Phi_1 \oplus \Phi_2 \subset \mathcal{D}(\mathcal{A})$. We show that Φ is a core of \mathcal{A} .

It suffices for each $x := (x_1, x_2) \in \mathcal{D}(A) \oplus \mathcal{D}(D) = \mathcal{D}(\mathcal{A})$ to find a sequence $(x^{(m)})_{m \in \mathbb{N}} \subset \Phi$ such that $x^{(m)} \rightarrow x$, $\mathcal{A}x^{(m)} \rightarrow \mathcal{A}x$ as $m \rightarrow \infty$. Since Φ_1 is a core of A and Φ_2 is a core of D , there exist sequences $(x_1^{(m)})_{m \in \mathbb{N}} \subset \Phi_1$, $(x_2^{(m)})_{m \in \mathbb{N}} \subset \Phi_2$ with

$$x_1^{(m)} \rightarrow x_1, \quad Ax_1^{(m)} \rightarrow Ax_1, \quad m \rightarrow \infty, \\ x_2^{(m)} \rightarrow x_2, \quad Dx_2^{(m)} \rightarrow Dx_2,$$

Since C is A -bounded and B is D -bounded, there exist $a_1, b_1, a_2, b_2 \geq 0$ such that

$$\|C(x_1^{(m)} - x_1)\|_{E_1} \leq a_1\|x_1^{(m)} - x_1\|_{E_1} + b_1\|A(x_1^{(m)} - x_1)\|_{E_1} \longrightarrow 0, \\ \|B(x_2^{(m)} - x_2)\|_{E_2} \leq a_2\|x_2^{(m)} - x_2\|_{E_2} + b_2\|D(x_2^{(m)} - x_2)\|_{E_2} \longrightarrow 0, \quad m \rightarrow \infty.$$

Define $x^{(m)} := (x_1^{(m)}, x_2^{(m)}) \in \Phi$, $m \in \mathbb{N}$. The above convergences imply $x^{(m)} \rightarrow x$, $\mathcal{A}x^{(m)} \rightarrow \mathcal{A}x$ as $m \rightarrow \infty$; hence Φ is a core of \mathcal{A} .

Let $x := (x_1, x_2) \in \Phi$. Then, for $n \geq n_0(x) := \max\{n_1(x_1), n_2(x_2)\}$,

$$(P_1^{(n)} \oplus P_2^{(n)})x = (P_1^{(n)}x_1, P_2^{(n)}x_2) \in \mathcal{D}(A^{(n)}) \oplus \mathcal{D}(D^{(n)}) = \mathcal{D}(\mathcal{A}^{(n)}), \\ \mathcal{A}^{(n)}(P_1^{(n)} \oplus P_2^{(n)})x - \mathcal{A}x = \begin{pmatrix} (A^{(n)}P_1^{(n)}x_1 - Ax_1) + (B^{(n)}P_2^{(n)}x_2 - Bx_2) \\ (C^{(n)}P_1^{(n)}x_1 - Cx_1) + (D^{(n)}P_2^{(n)}x_2 - Dx_2) \end{pmatrix} \longrightarrow 0$$

as $n \rightarrow \infty$. Now the claim follows from Theorem 3.1. \square

3.3. Results for bounded infinite operator matrices. In this subsection we prove some useful lemmas about convergence of bounded infinite operator matrices; the unbounded case is considered in the next subsection.

Definition 3.8. Let $E_j^{(0)}$, $j \in \mathbb{N}$, be Banach spaces. Define

$$\mathcal{E}^{(0)} := l^2(E_j^{(0)} : j \in \mathbb{N}) := \left\{ (x_j)_{j \in \mathbb{N}} : x_j \in E_j^{(0)}, \sum_{j \in \mathbb{N}} \|x_j\|_{E_j^{(0)}}^2 < \infty \right\},$$

$$\|x\|_{\mathcal{E}^{(0)}} := \left(\sum_{j \in \mathbb{N}} \|x_j\|_{E_j^{(0)}}^2 \right)^{\frac{1}{2}}, \quad x = (x_j)_{j \in \mathbb{N}} \in \mathcal{E}^{(0)}.$$

For each $j \in \mathbb{N}$ let $E_j, E_j^{(n)} \subset E_j^{(0)}$, $n \in \mathbb{N}$, be closed complemented subspaces; let $P_j : E_j^{(0)} \rightarrow E_j$ and $P_j^{(n)} : E_j^{(0)} \rightarrow E_j^{(n)}$, $n \in \mathbb{N}$, be projections on the respective subspaces (along the respective complements). Then

$$\mathcal{E} := l^2(E_j : j \in \mathbb{N}), \quad \mathcal{E}^{(n)} := l^2(E_j^{(n)} : j \in \mathbb{N}) \subset \mathcal{E}^{(0)}, \quad n \in \mathbb{N},$$

are Banach spaces. Denote the projections of $\mathcal{E}^{(0)}$ onto the respective subspaces by the diagonal block operator matrices

$$\mathcal{P} := \text{diag}(P_j : j \in \mathbb{N}), \quad \mathcal{P}^{(n)} := \text{diag}(P_j^{(n)} : j \in \mathbb{N}), \quad n \in \mathbb{N}.$$

The following lemma is very useful for applications. It may be viewed as a Lebesgue's dominated convergence theorem with respect to a counting measure.

Lemma 3.9. Let $x^{(n)} := (x_j^{(n)})_{j \in \mathbb{N}} \in \mathcal{E}^{(0)}$, $n \in \mathbb{N}$, and $x := (x_j)_{j \in \mathbb{N}} \in \mathcal{E}^{(0)}$ with

$$\forall j \in \mathbb{N} : \|x_j^{(n)}\|_{E_j^{(0)}} \leq \|x_j\|_{E_j^{(0)}}, \quad n \in \mathbb{N}, \quad \|x_j^{(n)}\|_{E_j^{(0)}} \rightarrow 0, \quad n \rightarrow \infty. \quad (21)$$

Then $\|x^{(n)}\|_{\mathcal{E}^{(0)}} \rightarrow 0$, $n \rightarrow \infty$.

Notation 3.10. For $i, j \in \mathbb{N}$, let $B_{ij} \in L(E_j, E_i)$ and $B_{ij}^{(n)} \in L(E_j^{(n)}, E_i^{(n)})$, $n \in \mathbb{N}$. Consider the infinite block operator matrices $\mathcal{B}, \mathcal{B}^{(n)}$, $n \in \mathbb{N}$, in $\mathcal{E}, \mathcal{E}^{(n)}$, $n \in \mathbb{N}$, respectively, defined by

$$\mathcal{B} := (B_{ij})_{i,j=1}^{\infty}, \quad \mathcal{B}^{(n)} := (B_{ij}^{(n)})_{i,j=1}^{\infty}, \quad n \in \mathbb{N}.$$

- i) For each $i \in \mathbb{N}$ denote by $N_i \subset \mathbb{N}$ the set of indices j such that $B_{ij} \neq 0$ or $B_{ij}^{(n)} \neq 0$ for some $n \in \mathbb{N}$.
- ii) For each $j \in \mathbb{N}$ denote by $M_j \subset \mathbb{N}$ the set of indices i such that $B_{ij} \neq 0$ or $B_{ij}^{(n)} \neq 0$ for some $n \in \mathbb{N}$.

For $\mathcal{B} := (B_{ij})_{i,j=1}^{\infty} \in L(\mathcal{E})$ and $\mathcal{B}^{(n)} := (B_{ij}^{(n)})_{i,j=1}^{\infty} \in L(\mathcal{E}^{(n)})$, $n \in \mathbb{N}$, we consider the following cases:

- (a) We have $N := \sup_{i \in \mathbb{N}} \#N_i < \infty$, $M := \sup_{j \in \mathbb{N}} \#M_j < \infty$, and there exists $C \geq 0$ such that

$$\forall i, j \in \mathbb{N} : \|B_{ij}\| \leq C, \quad \|B_{ij}^{(n)}\| \leq C, \quad n \in \mathbb{N}.$$

- (b) There exist $C_i \geq 0$, $i \in \mathbb{N}$, such that $\sum_{i=1}^{\infty} C_i^2 \#N_i < \infty$ and

$$\forall i, j \in \mathbb{N} : \|B_{ij}\| \leq C_i, \quad \|B_{ij}^{(n)}\| \leq C_i, \quad n \in \mathbb{N}.$$

- (c) There exist $D_j \geq 0$, $j \in \mathbb{N}$, such that $\sum_{j=1}^{\infty} D_j^2 \#M_j < \infty$ and

$$\forall i, j \in \mathbb{N} : \|B_{ij}\| \leq D_j, \quad \|B_{ij}^{(n)}\| \leq D_j, \quad n \in \mathbb{N}.$$

Remark 3.11. i) Typical examples for case (a) are (uniformly) *banded operator matrices* for which there exists $L \in \mathbb{N}$ with

$$B_{ij} = B_{ij}^{(n)} = 0, \quad |i - j| > L, \quad n \in \mathbb{N};$$

in this case we have $N \leq 2L + 1$, $M \leq 2L + 1$.

ii) Typical examples for cases (b) and (c) are respectively lower and upper (uniformly) *semibanded operator matrices* for which there exists $L \in \mathbb{N}$ with

$$B_{ij} = B_{ij}^{(n)} = 0 \quad \text{for} \quad \begin{cases} j - i > L & \text{(case (b))}, \\ i - j > L & \text{(case (c))}; \end{cases}$$

we then have $\#N_i \leq i + L$ and $\#M_j \leq j + L$, respectively.

iii) Note that, in general, neither (a) is a special case of (b) or (c) nor vice versa. In fact, for (b) and (c), the norms of the entries need to decrease as $i \rightarrow \infty$ (in (b)) or $j \rightarrow \infty$ (in (c)), whereas in case (a) the norms just need to be uniformly bounded; vice versa, for (a), $(\#N_i)_{i \in \mathbb{N}}$ and $(\#M_j)_{j \in \mathbb{N}}$ need to be bounded sequences, whereas for (b) and (c) they may be unbounded.

Proposition 3.12. *Let $\mathcal{B} := (B_{ij})_{i,j=1}^\infty \in L(\mathcal{E})$ and $\mathcal{B}^{(n)} := (B_{ij}^{(n)})_{i,j=1}^\infty \in L(\mathcal{E}^{(n)})$, $n \in \mathbb{N}$, satisfy one of the cases (a), (b), (c). If*

$$K := \sup_{j \in \mathbb{N}} \max \left\{ \sup_{n \in \mathbb{N}} \|P_j^{(n)}\|, \|P_j\| \right\} < \infty$$

and

$$\forall i, j \in \mathbb{N}: \quad B_{ij}^{(n)} P_j^{(n)} \xrightarrow{s} B_{ij} P_j, \quad n \rightarrow \infty, \quad (22)$$

then we have $\mathcal{B}^{(n)} \mathcal{P}^{(n)} \xrightarrow{s} \mathcal{B} \mathcal{P}$ as $n \rightarrow \infty$.

Proof. (a) Let $y := (y_j)_{j \in \mathbb{N}} \in \mathcal{E}^{(0)}$. For each $i \in \mathbb{N}$ choose $e_i \in E_i^{(0)}$ such that $\|e_i\|_{E_i^{(0)}} = 1$ and define, with the constant C from (a),

$$x_i := \left(\sum_{j \in N_i} \|y_j\|_{E_j^{(0)}} \right) 2CK e_i, \quad x_i^{(n)} := \sum_{j \in N_i} \left(B_{ij}^{(n)} P_j^{(n)} - B_{ij} P_j \right) y_j \in E_i^{(0)}, \quad n \in \mathbb{N}.$$

The element $x := (x_i)_{i \in \mathbb{N}}$ belongs to $\mathcal{E}^{(0)}$ since

$$\begin{aligned} \|x\|_{\mathcal{E}^{(0)}}^2 &= \sum_{i=1}^\infty \|x_i\|_{E_i^{(0)}}^2 = 4C^2 K^2 \sum_{i=1}^\infty \left(\sum_{j \in N_i} \|y_j\|_{E_j^{(0)}} \right)^2 \leq 4C^2 K^2 \sum_{i=1}^\infty \#N_i \sum_{j \in N_i} \|y_j\|_{E_j^{(0)}}^2 \\ &\leq 4C^2 K^2 N \sum_{j=1}^\infty M_j \|y_j\|_{E_j^{(0)}}^2 \leq 4C^2 K^2 N M \|y\|_{\mathcal{E}^{(0)}}^2 < \infty. \end{aligned}$$

We fix an $i \in \mathbb{N}$. Since N_i is a finite set, the convergences in (22) imply

$$\|x_i^{(n)}\|_{E_i^{(0)}} \leq \|x_i\|_{E_i^{(0)}}, \quad n \in \mathbb{N}, \quad \|x_i^{(n)}\|_{E_i^{(0)}} \rightarrow 0, \quad n \rightarrow \infty.$$

Now Lemma 3.9 applied to $x^{(n)} := (x_i^{(n)})_{i \in \mathbb{N}} \in \mathcal{E}^{(0)}$, $n \in \mathbb{N}$, yields

$$\|(\mathcal{B}^{(n)} \mathcal{P}^{(n)} - \mathcal{B} \mathcal{P})y\|_{\mathcal{E}^{(0)}} = \|x^{(n)}\|_{\mathcal{E}^{(0)}} \rightarrow 0, \quad n \rightarrow \infty.$$

(b) First note that N_i is a finite set for every $i \in \mathbb{N}$. Proceed as in (a) with C replaced by C_i in the definition of x_i . Then

$$\sum_{i=1}^\infty \|x_i\|_{E_i^{(0)}}^2 \leq 4K^2 \sum_{i=1}^\infty C_i^2 \left(\sum_{j \in N_i} \|y_j\|_{E_j^{(0)}} \right)^2 \leq 4K^2 \sum_{i=1}^\infty C_i^2 \#N_i \sum_{j=1}^\infty \|y_j\|_{E_j^{(0)}}^2 < \infty.$$

The rest of the proof is analogous to (a).

(c) For $i \in \mathbb{N}$ define $x_i^{(n)}$, $n \in \mathbb{N}$, as in (a), and set

$$x_i := \left(\sum_{j \in N_i} D_j \|y_j\|_{E_j^{(0)}} \right) 2K e_i \in E_i^{(0)}, \quad i \in \mathbb{N}.$$

Then $x := (x_i)_{i \in \mathbb{N}}$ belongs to $\mathcal{E}^{(0)}$ since

$$\sum_{i=1}^{\infty} \|x_i\|_{E_i^{(0)}}^2 \leq 4K^2 \sum_{i=1}^{\infty} \sum_{j \in N_i} D_j^2 \sum_{k \in N_i} \|y_k\|_{E_k^{(0)}}^2 \leq 4K^2 \sum_{j=1}^{\infty} D_j^2 (\#M_j) \sum_{k=1}^{\infty} \|y_k\|_{E_k^{(0)}}^2 < \infty;$$

in particular, x_i is well-defined. It is left to show

$$\forall i \in \mathbb{N} : \|x_i^{(n)}\|_{E_i^{(0)}} \longrightarrow 0, \quad n \rightarrow \infty; \quad (23)$$

then the rest of the proof is completely analogous to (a). We fix an $i \in \mathbb{N}$ and let $\varepsilon > 0$ be arbitrary. First note that, for every $n \in \mathbb{N}$,

$$\|x_i^{(n)}\|_{E_i^{(0)}} \leq \sum_{j \in N_i} \|(B_{ij}^{(n)} P_j^{(n)} - B_{ij} P_j) y_j\|_{E_j^{(0)}} \leq \sum_{j \in N_i} 2D_j K \|y_j\|_{E_j^{(0)}} = \|x_i\|_{E_i^{(0)}} < \infty.$$

There exists $j_\varepsilon \in \mathbb{N}$ such that

$$\sum_{\substack{j \in N_i \\ j \geq j_\varepsilon}} \|(B_{ij}^{(n)} P_j^{(n)} - B_{ij} P_j) y_j\|_{E_j^{(0)}} \leq \sum_{\substack{j \in N_i \\ j \geq j_\varepsilon}} 2D_j K \|y_j\|_{E_j^{(0)}} < \frac{\varepsilon}{2}, \quad n \in \mathbb{N}.$$

The convergences in (22) imply the existence of $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{\substack{j \in N_i \\ j < j_\varepsilon}} \|(B_{ij}^{(n)} P_j^{(n)} - B_{ij} P_j) y_j\|_{E_j^{(0)}} < \frac{\varepsilon}{2}, \quad n \geq N_\varepsilon.$$

Altogether we have $\|x_i^{(n)}\|_{E_i^{(0)}} < \varepsilon$ for all $n \geq N_\varepsilon$; hence (23) is satisfied. \square

Corollary 3.13. *If $\sup_{j,n \in \mathbb{N}} \|P_j^{(n)}\| < \infty$ and $P_j^{(n)} \xrightarrow{s} P_j$, $n \rightarrow \infty$, for all $j \in \mathbb{N}$, then $\mathcal{P}^{(n)} \xrightarrow{s} \mathcal{P}$, $n \rightarrow \infty$.*

Proof. The claim follows immediately from Proposition 3.12, case (a); note that $\|P_j\| \leq \liminf_{n \rightarrow \infty} \|P_j^{(n)}\|$ by [17, Equation III.(3.2)]. \square

3.4. Results for unbounded infinite operator matrices. In this subsection we analyze whether a sequence of diagonally dominant infinite operator matrices converges in gsr-sense if the sequences for all diagonal elements do so.

Denote by $\mathcal{E}^{(0)}, \mathcal{E}, \mathcal{E}^{(n)}$, $n \in \mathbb{N}$, the same spaces as in the previous subsection. For $i, j \in \mathbb{N}$, let $A_{ij} : E_j \rightarrow E_i$ and $A_{ij}^{(n)} : E_j^{(n)} \rightarrow E_i^{(n)}$, $n \in \mathbb{N}$, be closable and densely defined, and let

$$\begin{aligned} \mathcal{A} &:= (A_{ij})_{i,j=1}^{\infty}, & \mathcal{D}(\mathcal{A}) &:= l^2 \left(\bigcap_{i \in \mathbb{N}} \mathcal{D}(A_{ij}) : j \in \mathbb{N} \right), \\ \mathcal{A}^{(n)} &:= (A_{ij}^{(n)})_{i,j=1}^{\infty}, & \mathcal{D}(\mathcal{A}^{(n)}) &:= l^2 \left(\bigcap_{i \in \mathbb{N}} \mathcal{D}(A_{ij}^{(n)}) : j \in \mathbb{N} \right), \quad n \in \mathbb{N}. \end{aligned}$$

We assume that \mathcal{A} and $\mathcal{A}^{(n)}$, $n \in \mathbb{N}$, are densely defined in \mathcal{E} and $\mathcal{E}^{(n)}$, $n \in \mathbb{N}$, respectively.

Definition 3.14. The block operator matrix \mathcal{A} is called *diagonally dominant* if, for every $j \in \mathbb{N}$, the operators A_{ij} , $i \in \mathbb{N}$, are A_{jj} -bounded. Diagonal dominance of $\mathcal{A}^{(n)}$, $n \in \mathbb{N}$, is defined analogously.

Theorem 3.15. *Assume that $\mathcal{A}, \mathcal{A}^{(n)}, n \in \mathbb{N}$, are diagonally dominant. Let*

$$\begin{aligned} \mathcal{T} &:= \text{diag}(A_{jj} : j \in \mathbb{N}), & \mathcal{D}(\mathcal{T}) &:= l^2(\mathcal{D}(A_{jj}) : j \in \mathbb{N}), \\ \mathcal{T}^{(n)} &:= \text{diag}(A_{jj}^{(n)} : j \in \mathbb{N}), & \mathcal{D}(\mathcal{T}^{(n)}) &:= l^2(\mathcal{D}(A_{jj}^{(n)}) : j \in \mathbb{N}), \quad n \in \mathbb{N}, \end{aligned} \quad (24)$$

and

$$\mathcal{S} := \mathcal{A} - \mathcal{T}, \quad \mathcal{S}^{(n)} := \mathcal{A}^{(n)} - \mathcal{T}^{(n)}, \quad n \in \mathbb{N}.$$

Suppose that there exist $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{T}^{(n)}) \cap \varrho(\mathcal{T})$ and $C_\lambda > 0, \gamma_\lambda < 1$ such that

$$\begin{aligned} \|(\mathcal{T}^{(n)} - \lambda)^{-1}\| &\leq C_\lambda, \quad n \in \mathbb{N}, \\ \|\mathcal{S}(\mathcal{T} - \lambda)^{-1}\| &< 1, \quad \|\mathcal{S}^{(n)}(\mathcal{T}^{(n)} - \lambda)^{-1}\| \leq \gamma_\lambda, \quad n \in \mathbb{N}. \end{aligned} \quad (25)$$

Further assume that the bounded operator matrices

$$\mathcal{B} := \mathcal{S}(\mathcal{T} - \lambda)^{-1}, \quad \mathcal{B}^{(n)} := \mathcal{S}^{(n)}(\mathcal{T}^{(n)} - \lambda)^{-1}, \quad n \in \mathbb{N},$$

satisfy one of the cases (a), (b), (c) after Notation 3.10. If

$$\begin{aligned} \forall j \in \mathbb{N} : & \quad (A_{jj}^{(n)} - \lambda)^{-1} P_j^{(n)} \xrightarrow{s} (A_{jj} - \lambda)^{-1} P_j, \\ \forall i, j \in \mathbb{N} : & \quad A_{ij}^{(n)} (A_{jj}^{(n)} - \lambda)^{-1} P_j^{(n)} \xrightarrow{s} A_{ij} (A_{jj} - \lambda)^{-1} P_j, \quad n \rightarrow \infty, \\ \forall j \in \mathbb{N} : & \quad P_j^{(n)} \xrightarrow{s} P_j, \end{aligned}$$

then $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{A}^{(n)}) \cap \varrho(\mathcal{A})$ and $(\mathcal{A}^{(n)} - \lambda)^{-1} \mathcal{P}^{(n)} \xrightarrow{s} (\mathcal{A} - \lambda)^{-1} \mathcal{P}$ as $n \rightarrow \infty$.

Remark 3.16. The inequalities (25) imply that \mathcal{S} is \mathcal{T} -bounded with \mathcal{T} -bound < 1 and $\mathcal{S}^{(n)}$ is $\mathcal{T}^{(n)}$ -bounded with $\mathcal{T}^{(n)}$ -bound $\leq \gamma_\lambda < 1$.

Proof of Theorem 3.15. Let $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{T}^{(n)}) \cap \varrho(\mathcal{T})$, $C_\lambda > 0$ and $\gamma_\lambda < 1$ satisfy the assumptions, and set $C := \max\{\|(\mathcal{T} - \lambda)^{-1}\|, C_\lambda\}$. Then

$$\begin{aligned} \|(A_{jj} - \lambda)^{-1}\| &\leq \|(\mathcal{T} - \lambda)^{-1}\| \leq C, \\ \|(A_{jj}^{(n)} - \lambda)^{-1}\| &\leq \|(\mathcal{T}^{(n)} - \lambda)^{-1}\| \leq C, \quad n \in \mathbb{N}. \end{aligned}$$

Proposition 3.12, case (a) applied to $(\mathcal{T} - \lambda)^{-1}$ and $(\mathcal{T}^{(n)} - \lambda)^{-1}$, $n \in \mathbb{N}$, yields

$$(\mathcal{T}^{(n)} - \lambda)^{-1} \mathcal{P}^{(n)} \xrightarrow{s} (\mathcal{T} - \lambda)^{-1} \mathcal{P}, \quad n \rightarrow \infty.$$

By applying Proposition 3.12 to $\mathcal{B}, \mathcal{B}^{(n)}, n \in \mathbb{N}$, we obtain, for all of the cases (a), (b), (c),

$$\mathcal{S}^{(n)}(\mathcal{T}^{(n)} - \lambda)^{-1} \mathcal{P}^{(n)} \xrightarrow{s} \mathcal{S}(\mathcal{T} - \lambda)^{-1} \mathcal{P}, \quad n \rightarrow \infty.$$

Corollary 3.13 yields $\mathcal{P}^{(n)} \xrightarrow{s} \mathcal{P}, n \rightarrow \infty$. Now the assertion follows from Theorem 3.3. \square

The following example illustrates Theorem 3.15, case (c).

Example 3.17. Let $\mathcal{E}^{(0)} = \mathcal{E} = \mathcal{E}^{(n)} := l^2(\mathbb{N}), n \in \mathbb{N}$. We define upper triangular matrices $\mathcal{A} := (A_{ij})_{i,j=1}^\infty, \mathcal{A}^{(n)} := (A_{ij}^{(n)})_{i,j=1}^\infty \in C(l^2(\mathbb{N}))$ by

$$A_{ij} := \begin{cases} j, & i < j, \\ j^3, & i = j, \\ 0, & \text{otherwise,} \end{cases} \quad A_{ij}^{(n)} := \begin{cases} j, & i < j \leq n, \\ j^3, & i = j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}, \mathcal{T}^{(n)}, n \in \mathbb{N}$, be defined as in (24). Since the latter are selfadjoint operators, it is obvious that $\mathbb{C} \setminus \mathbb{R} \subset \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{T}^{(n)}) \cap \varrho(\mathcal{T})$ and, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\forall j \in \mathbb{N}: \quad (A_{jj}^{(n)} - \lambda)^{-1} \xrightarrow{s} (A_{jj} - \lambda)^{-1}, \quad n \rightarrow \infty.$$

Moreover, we have, for $i \neq j$,

$$A_{ij}(A_{jj} - \lambda)^{-1} = \begin{cases} \frac{j}{j^3 - \lambda}, & i < j, \\ 0, & \text{otherwise,} \end{cases} \quad A_{ij}^{(n)}(A_{jj}^{(n)} - \lambda)^{-1} = \begin{cases} \frac{j}{j^3 - \lambda}, & i < j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, for $|\operatorname{Im}(\lambda)|$ sufficiently large, the matrices satisfy the assumptions of Theorem 3.15, case (c), with

$$D_j := \frac{j}{j^3 - \lambda} = \frac{1}{j^2 - \frac{\lambda}{j}}, \quad \#M_j = j - 1, \quad j \in \mathbb{N}.$$

So we conclude $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{A}^{(n)}) \cap \varrho(\mathcal{A})$ and $(\mathcal{A}^{(n)} - \lambda)^{-1} \xrightarrow{s} (\mathcal{A} - \lambda)^{-1}$ as $n \rightarrow \infty$.

4. DISCRETELY COMPACT RESOLVENTS

In this section we establish sufficient conditions for a sequence of operators $T_n, n \in \mathbb{N}$, in varying Banach spaces to have *discretely compact resolvents*, i.e.

$$\exists \lambda_0 \in \bigcap_{n \in \mathbb{N}} \varrho(T_n): \quad ((T_n - \lambda_0)^{-1})_{n \in \mathbb{N}} \text{ discretely compact;}$$

for the definition of discrete compactness see Definition 2.5.

As in Subsection 3.1, we first prove direct criteria and perturbation results (see Subsection 4.1). Secondly, we establish results for operator matrices, both 2×2 and infinite matrices (see Subsection 4.2). Finally, we study the case that the operator domains are contained in (varying) Sobolev spaces and derive discretely compact Sobolev embeddings (see Subsection 4.3).

Consider a Banach space E_0 and closed complemented subspaces $E, E_n \subset E_0, n \in \mathbb{N}$; as usual, we assume that the corresponding projections converge strongly, $P_n \xrightarrow{s} P$.

4.1. Direct criteria and perturbation results. The following result relates (discrete) resolvent compactness to (discrete) compactness of embeddings.

Proposition 4.1. *Let $T \in C(E)$ and $T_n \in C(E_n), n \in \mathbb{N}$. Define the Banach spaces $D := \mathcal{D}(T), D_n := \mathcal{D}(T_n), n \in \mathbb{N}$, equipped with the graph norms $\|\cdot\|_D := \|\cdot\|_T, \|\cdot\|_{D_n} := \|\cdot\|_{T_n}, n \in \mathbb{N}$, respectively. Let $J: D \rightarrow E, J_n: D_n \rightarrow E_n, n \in \mathbb{N}$, be the natural embeddings.*

- i) *Let $\varrho(T) \neq \emptyset$. The operator J is compact if and only if T has compact resolvent. The analogous result holds for J_n and T_n .*
- ii) *Let $\Delta_b((T_n)_{n \in \mathbb{N}}) \neq \emptyset$. The sequence $(J_n)_{n \in \mathbb{N}}$ is discretely compact if and only if for some (and hence for all) $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}})$ there exists $n_0 \in \mathbb{N}$ such that the sequence $((T_n - \lambda)^{-1})_{n \geq n_0}$ is discretely compact.*

Proof. We prove claim ii); claim i) is similar and well-known.

Let $\lambda \in \Delta_b((T_n)_{n \in \mathbb{N}})$ and $n_0 \in \mathbb{N}$ such that $\lambda \in \varrho(T_n)$ for $n \geq n_0$. Consider an infinite subset $I \subset \mathbb{N}$ and $y_n \in E_n, n \in I$, and define $x_n := (T - \lambda)^{-1}y_n \in D_n, n \in I$. First note that we have

$$\begin{aligned} \|x_n\|_{T_n} &= \|x_n\|_{E_n} + \|T_n x_n\|_{E_n} \leq (1 + |\lambda|)\|x_n\|_{E_n} + \|(T_n - \lambda)x_n\|_{E_n} \\ &\leq ((1 + |\lambda|)\|(T_n - \lambda)^{-1}\| + 1)\|y_n\|_{E_n}, \\ \|y_n\|_{E_n} &= \|(T_n - \lambda)x_n\|_{E_n} \leq \|T_n x_n\|_{E_n} + |\lambda|\|x_n\|_{E_n} \leq (1 + |\lambda|)\|x_n\|_{T_n}. \end{aligned}$$

Hence $(\|x_n\|_{T_n})_{n \in \mathbb{N}}$ is bounded if and only if $(\|y_n\|_{E_n})_{n \in I}$ is bounded.

Assume that $(J_n)_{n \in \mathbb{N}}$ is discretely compact and $(\|x_n\|_{T_n})_{n \in I}$ is bounded. Then the sequence of elements $(T_n - \lambda)^{-1}y_n = x_n \in D_n$, $n \in I$, has a subsequence that converges in E_0 with limit in E ; hence $((T_n - \lambda)^{-1})_{n \geq n_0}$ is discretely compact. Vice versa, the discrete compactness of $((T_n - \lambda)^{-1})_{n \geq n_0}$ implies discrete compactness of $(J_n)_{n \in \mathbb{N}}$. \square

We prove a perturbation result for discrete compactness of the resolvents. Note that the assumptions are similar to the ones used in the perturbation result for gsr-convergence (compare Theorem 3.3).

Theorem 4.2. *Let $T_n \in C(E_n)$, $n \in \mathbb{N}$. Let S_n , $n \in \mathbb{N}$, be linear operators in E_n , $n \in \mathbb{N}$, with $\mathcal{D}(T_n) \subset \mathcal{D}(S_n)$, $n \in \mathbb{N}$, respectively. Define*

$$A_n := T_n + S_n, \quad n \in \mathbb{N}.$$

Suppose that there exist $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(T_n)$ and $\gamma_\lambda < 1$ with

$$\|S_n(T_n - \lambda)^{-1}\| \leq \gamma_\lambda, \quad n \in \mathbb{N}. \quad (26)$$

Then $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(A_n)$. If the sequence $((T_n - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact, then so is $((A_n - \lambda)^{-1})_{n \in \mathbb{N}}$.

Remark 4.3. The inequalities (26) imply that, for every $n \in \mathbb{N}$, S_n is T_n -bounded with T_n -bound $\leq \gamma_\lambda < 1$.

Proof of Theorem 4.2. Let λ satisfy the assumptions. Then, using (19) and Lemma 2.8 i), the discrete compactness of $((T_n - \lambda)^{-1})_{n \in \mathbb{N}}$ implies that the sequence $((A_n - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact. \square

The following result is an immediate consequence of Theorem 4.2 for the case that the perturbations are bounded operators.

Corollary 4.4. *Let $T_n \in C(E_n)$, $n \in \mathbb{N}$, and $S_n \in L(E_n)$, $n \in \mathbb{N}$. Define*

$$A_n := T_n + S_n, \quad n \in \mathbb{N}.$$

Suppose that there exist $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(T_n)$ and $\gamma_\lambda < 1$ with

$$\|S_n\| \leq \gamma_\lambda \|(T_n - \lambda)^{-1}\|^{-1}, \quad n \in \mathbb{N}.$$

Then $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(A_n)$. If the sequence $((T_n - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact, then so is $((A_n - \lambda)^{-1})_{n \in \mathbb{N}}$.

4.2. Results for block operator matrices. In this subsection we consider (finite and infinite) operator matrices. We study whether a sequence of diagonally dominant operator matrices

$$\mathcal{A}^{(n)} := (T_{ij}^{(n)})_{i,j=1}^N, \quad n \in \mathbb{N},$$

has discretely compact resolvents if the sequences $(T_{ii}^{(n)})_{n \in \mathbb{N}}$, $i \in \mathbb{N}$, of all diagonal entries have discretely compact resolvents.

First, we consider diagonally dominant 2×2 operator matrices, i.e. $N = 2$. We use the same notation as in Subsection 3.2.

Theorem 4.5. *Suppose that the block operator matrices satisfy the following:*

- i) *the operators $A^{(n)}$, $n \in \mathbb{N}$, and $D^{(n)}$, $n \in \mathbb{N}$, have discretely compact resolvents, respectively;*
- ii) *the matrices $\mathcal{A}^{(n)}$, $n \in \mathbb{N}$, are diagonally dominant;*

iii) there exist $\lambda \in \bigcap_{n \in \mathbb{N}} (\varrho(A^{(n)}) \cap \varrho(D^{(n)}))$ and constants $\gamma_\lambda^{AC}, \gamma_\lambda^{DB} \geq 0$ such that

$$((A^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}, \quad ((D^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$$

are bounded sequences,

$$\|C^{(n)}(A^{(n)} - \lambda)^{-1}\| \leq \gamma_\lambda^{AC}, \quad \|B^{(n)}(D^{(n)} - \lambda)^{-1}\| \leq \gamma_\lambda^{DB}, \quad n \in \mathbb{N}, \quad (27)$$

and

$$\gamma_\lambda^{AC} \gamma_\lambda^{DB} < 1. \quad (28)$$

Then $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{A}^{(n)})$ and the sequence $((\mathcal{A}^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact.

Remark 4.6. The inequalities (27) imply that $\mathcal{A}^{(n)}$ is diagonally dominant of order $\max\{\gamma_\lambda^{AC}, \gamma_\lambda^{DB}\}$ (see Definition 3.6 i)). Note that (28) is more general than $\max\{\gamma_\lambda^{AC}, \gamma_\lambda^{DB}\} < 1$ and it implies $\min\{\gamma_\lambda^{AC}, \gamma_\lambda^{DB}\} < 1$.

Proof of Theorem 4.5. Let λ satisfy the assumption iii). We show the claim in the case $\gamma_\lambda^{DB} = \min\{\gamma_\lambda^{AC}, \gamma_\lambda^{DB}\} < 1$; the other case is analogous. For each $n \in \mathbb{N}$ define

$$\mathcal{T}^{(n)} := \begin{pmatrix} A^{(n)} & 0 \\ C^{(n)} & D^{(n)} \end{pmatrix}, \quad \mathcal{S}^{(n)} := \mathcal{A}^{(n)} - \mathcal{T}^{(n)} = \begin{pmatrix} 0 & B^{(n)} \\ 0 & 0 \end{pmatrix}.$$

Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} (\mathcal{T}^{(n)} - \lambda)^{-1} &= \begin{pmatrix} (A^{(n)} - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (D^{(n)} - \lambda)^{-1} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ -(D^{(n)} - \lambda)^{-1} C^{(n)} (A^{(n)} - \lambda)^{-1} & 0 \end{pmatrix}; \end{aligned}$$

denote the matrices on the right-hand side by $\mathcal{U}^{(n)}$, $\mathcal{V}^{(n)}$, $\mathcal{W}^{(n)}$, respectively. The assumption i) implies that $(\mathcal{U}^{(n)})_{n \in \mathbb{N}}$, $(\mathcal{V}^{(n)})_{n \in \mathbb{N}}$ are discretely compact sequences. By the assumption iii), $(C^{(n)}(A^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$ is a bounded sequence. Now i) and Lemma 2.8 i) imply the discrete compactness of $(\mathcal{W}^{(n)})_{n \in \mathbb{N}}$. By Lemma 2.8 iii), the sequence $((\mathcal{T}^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact. Assumption iii) yields the estimate

$$\begin{aligned} &\|\mathcal{S}^{(n)}(\mathcal{T}^{(n)} - \lambda)^{-1}\| \\ &\leq \max \left\{ \|B^{(n)}(D^{(n)} - \lambda)^{-1} C^{(n)}(A^{(n)} - \lambda)^{-1}\|, \|B^{(n)}(D^{(n)} - \lambda)^{-1}\| \right\} \\ &\leq \max\{\gamma_\lambda^{AC} \gamma_\lambda^{DB}, \gamma_\lambda^{DB}\} < 1, \quad n \in \mathbb{N}. \end{aligned}$$

Now the claim follows from Theorem 4.2. \square

Example 4.7. Let the operators $A^{(n)}$, $n \in \mathbb{N}$, and $D^{(n)}$, $n \in \mathbb{N}$, be selfadjoint and have discretely compact resolvents, respectively. If the operators $B^{(n)}, C^{(n)}$, $n \in \mathbb{N}$, are uniformly bounded, then the sequence of operators

$$\mathcal{A}^{(n)} := \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix}, \quad n \in \mathbb{N},$$

has discretely compact resolvents. This follows from Theorem 4.5 since for every $\gamma < 1$ there exists $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\operatorname{Im}(\lambda)|$ sufficiently large such that the assumption iii) of Theorem 4.5 holds with

$$\gamma_\lambda^{AC} = \gamma_\lambda^{DB} := \frac{1}{|\operatorname{Im}(\lambda)|} \sup_{n \in \mathbb{N}} \max \{ \|B^{(n)}\|, \|C^{(n)}\| \}.$$

Now we study sequences of diagonal operator matrices; first for finitely many diagonal elements, then for infinitely many. For block operator matrices that are not diagonal, one may combine Theorem 4.8/4.9 with Theorem 4.2 or its corollary (as in Theorem 4.5 for the 2×2 case).

Let $E_j^{(0)}$, $j \in \mathbb{N}$, be Banach spaces and, for every $j \in \mathbb{N}$, let $E_j, E_j^{(n)} \subset E_j^{(0)}$, $n \in \mathbb{N}$, be closed subspaces.

Theorem 4.8. *Let $k \in \mathbb{N}$ and consider Banach spaces*

$$\mathcal{E} := \bigoplus_{j=1, \dots, k} E_j, \quad \mathcal{E}^{(n)} := \bigoplus_{j=1, \dots, k} E_j^{(n)} \subset \mathcal{E}^{(0)} := \bigoplus_{j=1, \dots, k} E_j^{(0)}, \quad n \in \mathbb{N}.$$

Define

$$\mathcal{T}^{(n)} := \text{diag}(T_j^{(n)} : j = 1, \dots, k) \in C(\mathcal{E}^{(n)}), \quad n \in \mathbb{N}.$$

Suppose that there exists $\lambda \in \bigcap_{j=1, \dots, k} \bigcap_{n \in \mathbb{N}} \varrho(T_j^{(n)})$ such that

$$((T_j^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}, \quad j = 1, \dots, k,$$

are discretely compact sequences. Then we have $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{T}^{(n)})$ and the sequence $((\mathcal{T}^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact.

Proof. For $k = 2$ the claim is an immediate consequence of Lemma 2.8 iii) applied to

$$A_n^{(1)} = \begin{pmatrix} (T_1^{(n)} - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_n^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & (T_2^{(n)} - \lambda)^{-1} \end{pmatrix}, \quad n \in \mathbb{N}.$$

For $k \in \mathbb{N}$ with $k > 2$ the claim follows by induction. \square

Theorem 4.9. *Assume that $E_j^{(0)} = E_j$, $j \in \mathbb{N}$. Consider Banach spaces*

$$\mathcal{E}^{(n)} := l^2(E_j^{(n)} : j \in \mathbb{N}) \subset \mathcal{E} := l^2(E_j : j \in \mathbb{N}), \quad n \in \mathbb{N}.$$

Define

$$\mathcal{T}^{(n)} := \text{diag}(T_j^{(n)} : j \in \mathbb{N}) \in C(\mathcal{E}^{(n)}), \quad n \in \mathbb{N}.$$

Suppose that there exists $\lambda \in \bigcap_{j \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \varrho(T_j^{(n)})$ such that

$$((T_j^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}, \quad j \in \mathbb{N},$$

are discretely compact sequences. We further assume that

$$\sup_{n \in \mathbb{N}} \|(T_j^{(n)} - \lambda)^{-1}\| \longrightarrow 0, \quad j \rightarrow \infty. \quad (29)$$

Then we have $\lambda \in \bigcap_{n \in \mathbb{N}} \varrho(\mathcal{T}^{(n)})$ and the sequence $((\mathcal{T}^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact.

Proof. Let λ satisfy the assumptions. Fix an $n \in \mathbb{N}$. Since, by the assumption (29), the sequence $((T_j^{(n)} - \lambda)^{-1})_{j \in \mathbb{N}}$ is bounded, λ belongs to $\varrho(\mathcal{T}^{(n)})$. We define

$$\begin{aligned} \mathcal{A}^{(n)} &:= (\mathcal{T}^{(n)} - \lambda)^{-1} = \text{diag}((T_j^{(n)} - \lambda)^{-1} : j \in \mathbb{N}), \\ \mathcal{A}^{(n;k)} &:= \text{diag}((T_j^{(n)} - \lambda)^{-1} : j = 1, \dots, k) \oplus 0 \in L(\mathcal{E}^{(n)}), \quad k \in \mathbb{N}. \end{aligned}$$

From the assumption (29), it follows that

$$\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \|\mathcal{A}^{(n)} - \mathcal{A}^{(n;k)}\| = \lim_{k \rightarrow \infty} \sup_{j > k} \sup_{n \in \mathbb{N}} \|(T_j^{(n)} - \lambda)^{-1}\| = 0.$$

Since, by the assumptions, $((T_j^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$, $j \in \mathbb{N}$, are discretely compact sequences, Theorem 4.8 implies that $(\mathcal{A}^{(n;k)})_{n \in \mathbb{N}}$ is discretely compact for each $k \in \mathbb{N}$. Altogether, Proposition 2.9 yields that $(\mathcal{A}^{(n)})_{n \in \mathbb{N}} = ((\mathcal{T}^{(n)} - \lambda)^{-1})_{n \in \mathbb{N}}$ is discretely compact. \square

4.3. Discretely compact Sobolev embeddings. In this subsection we assume that $E_0 := L^p(\mathbb{R}^d)$ and $E := L^p(\Omega)$, where $p \in (1, \infty)$, $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ is an open subset. The space E_n is assumed to be $L^p(\Omega_n)$ for some open subset $\Omega_n \subset \mathbb{R}^d$ that may vary in $n \in \mathbb{N}$. Denote by m_d the Lebesgue measure on \mathbb{R}^d .

We consider operators $T_n \in C(E_n)$, $n \in \mathbb{N}$, whose domains are assumed to be subspaces of Sobolev spaces $W^{m,p}(\Omega_n)$, $n \in \mathbb{N}$, for some $m \in \mathbb{N}$. In this case it is sufficient to study discrete compactness of the Sobolev embeddings

$$\tilde{J}_n : W^{m,p}(\Omega_n) \rightarrow L^p(\Omega_n), \quad n \in \mathbb{N},$$

in order to conclude that the sequence of embeddings $J_n : D_n \rightarrow E_n$, $n \in \mathbb{N}$, (as defined in Proposition 4.1) is discretely compact (see Theorem 4.13 below).

Definition 4.10. Let Ω , Ω_n , $n \in \mathbb{N}$, be bounded open subsets of \mathbb{R}^d .

- i) The set Ω is said to have the *segment property* if there exist a finite open covering $\{O_k : k = 1, \dots, r\}$ of $\partial\Omega$ and corresponding vectors $\alpha_k \in \mathbb{R}^d \setminus \{0\}$, $k = 1, \dots, r$, such that

$$\overline{\Omega \cap O_k} + t\alpha_k \subset \Omega, \quad t \in (0, 1), \quad k = 1, \dots, r.$$

- ii) The pair $\{\Omega, \{\Omega_n : n \in \mathbb{N}\}\}$ is said to have the *uniform segment property* if there exist an open covering $\{O_k : k = 1, \dots, r\}$ of $\partial\Omega$ and corresponding vectors $\alpha_k \in \mathbb{R}^d \setminus \{0\}$, $k = 1, \dots, r$, such that $\{O_k : k = 1, \dots, r\}$ is an open covering of $\partial\Omega_n$ for sufficiently large $n \in \mathbb{N}$, say $n \geq n_0$, and

$$\begin{aligned} \forall n \geq n_0 : \quad & \overline{\Omega_n \cap O_k} + t\alpha_k \subset \Omega_n, \quad t \in (0, 1), \quad k = 1, \dots, r, \\ \forall \varepsilon \in (0, 1) \exists n_\varepsilon \in \mathbb{N} : \quad & \bigcup_{n \geq n_\varepsilon} (\Omega_n \cap O_k) + \varepsilon\alpha_k \subset \Omega, \quad k = 1, \dots, r. \end{aligned}$$

Remark 4.11. i) It is easy to see that if the pair $\{\Omega, \{\Omega_n : n \in \mathbb{N}\}\}$ has the uniform segment property, then Ω_n , $n \geq n_0$, all have the segment property. ii) If each compact subset $S \subset \Omega$ satisfies $S \subset \Omega_n$ for all sufficiently large $n \in \mathbb{N}$, then $\Omega \subset \bigcup_{n \geq n_\varepsilon} \Omega_n$ for each $n_\varepsilon \in \mathbb{N}$. If, in addition, the pair $\{\Omega, \{\Omega_n : n \in \mathbb{N}\}\}$ has the uniform segment property, then Ω has the segment property since, for each $\varepsilon \in (0, 1)$,

$$\overline{\Omega \cap O_k} + \varepsilon\alpha_k \subset \overline{\bigcup_{n \geq n_\varepsilon} (\Omega_n \cap O_k)} + \varepsilon\alpha_k \subset \Omega, \quad k = 1, \dots, r.$$

Example 4.12. i) The motivation for defining the segment property is that the interior of the set Ω shall not lie on both sides of the boundary. For instance, in \mathbb{R}^2 the set $B_1(0) \setminus ((0, 1) \times \{0\})$ does not have the segment property.

- ii) In dimension $d = 1$, it is easy to see that an open bounded subset $\Omega \subset \mathbb{R}$ has the segment property if Ω is the finite union of open bounded intervals I_l , $l = 1, \dots, L$, where the distance between any two different intervals is positive. Then $r = 2L$ and each O_k contains exactly one endpoint of an interval I_l ; the number α_k is positive (negative) if it is the left (right) endpoint, with $|\alpha_k|$ less than the length of I_l .

- iii) An example in dimension $d = 1$ for bounded open sets $\Omega, \Omega_n \subset \mathbb{R}$, $n \in \mathbb{N}$, such that the pair $\{\Omega, \{\Omega_n : n \in \mathbb{N}\}\}$ has the uniform segment property is

$$\begin{aligned}\Omega &:= \bigcup_{l=1, \dots, L} (a_l, b_l), & a_l < b_l < a_{l+1}, \\ \Omega_n &:= \bigcup_{l=1, \dots, L} (a_l^{(n)}, b_l^{(n)}), & a_l^{(n)} < b_l^{(n)} < a_{l+1}^{(n)},\end{aligned}$$

with $a_l^{(n)} \rightarrow a_l, b_l^{(n)} \rightarrow b_l, n \rightarrow \infty$, for every $l = 1, \dots, L$.

In the following we establish discretely compact Sobolev embeddings using earlier results by Grigorieff [14]. As Grigorieff's results are confined to dimension $d \geq 2$, we prove the case $d = 1$ separately; to avoid unnecessarily technicalities, in $d = 1$ we prove the result directly for the case studied in Example 4.12 iii) where $\Omega_n, n \in \mathbb{N}$, are unions of $L < \infty$ intervals whose endpoints converge to the ones of Ω .

Theorem 4.13. *For $d \geq 2$, suppose that $\Omega, \Omega_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, are bounded open subsets that satisfy the following:*

- (i) *each compact subset $S \subset \Omega$ is also a subset of Ω_n for all sufficiently large $n \in \mathbb{N}$;*
- (ii) *the pair $\{\Omega, \{\Omega_n : n \in \mathbb{N}\}\}$ has the uniform segment property;*
- (iii) *we have $m_d(\Omega_n \setminus \Omega) \rightarrow 0, n \rightarrow \infty$.*

For $d = 1$, suppose that $\Omega, \Omega_n \subset \mathbb{R}$, $n \in \mathbb{N}$, are as in Example 4.12 iii).

Let $T_n \in C(E_n)$, $n \in \mathbb{N}$, with $\mathcal{D}(T_n) \subset W^{m,p}(\Omega_n)$, $n \in \mathbb{N}$, for some $m \in \mathbb{N}$. If the embeddings

$$B_n : (\mathcal{D}(T_n), \|\cdot\|_{T_n}) \rightarrow W^{m,p}(\Omega_n), \quad n \in \mathbb{N},$$

are uniformly bounded, then the sequence $(J_n)_{n \in \mathbb{N}}$ of embeddings

$$J_n : (\mathcal{D}(T_n), \|\cdot\|_{T_n}) \rightarrow L^p(\Omega_n), \quad n \in \mathbb{N},$$

is discretely compact.

Proof. For dimension $d \geq 2$, the sequence $(\tilde{J}_n)_{n \in \mathbb{N}}$ of embeddings

$$\tilde{J}_n : W^{m,p}(\Omega_n) \rightarrow L^p(\Omega_n), \quad n \in \mathbb{N},$$

is discretely compact by [14, Satz 4.(9)]. Since $(B_n)_{n \in \mathbb{N}}$ is a bounded sequence by the assumptions, the claim follows from Lemma 2.8 i).

For dimension $d = 1$, let $I \subset \mathbb{N}$ be an infinite subset and let $f_n \in \mathcal{D}(T_n)$, $n \in I$, satisfy that $(\|f_n\|_{T_n})_{n \in I}$ is bounded. Then $(\|f_n\|_{W^{m,p}(\Omega_n)})_{n \in I}$ is bounded since $(B_n)_{n \in \mathbb{N}}$ is a bounded sequence. Define

$$\Lambda := \Omega \times (0, 1), \quad \Lambda_n := \Omega_n \times (0, 1), \quad n \in \mathbb{N}.$$

These sets are bounded open subsets of \mathbb{R}^2 . The idea of the proof is to show that $\Lambda, \Lambda_n, n \in \mathbb{N}$, satisfy assumptions (i)–(iii) for $d = 2$; then the sequence of embeddings

$$\tilde{J}_n^{(2)} : W^{m,p}(\Lambda_n) \rightarrow L^p(\Lambda_n), \quad n \in \mathbb{N}, \quad (30)$$

is discretely compact by [14, Satz 4.(9)]. From this, at the end, we conclude that the sequence of elements $f_n \in L^p(\Omega_n)$, $n \in I$, has a convergent subsequence in $L^p(\mathbb{R})$ with limit function in $L^p(\Omega)$.

It is easy to see that properties (i) and (iii) are satisfied for $\Lambda, \Lambda_n, n \in \mathbb{N}$. It remains to check (ii), i.e. the uniform segment property. There exists $\delta > 0$ such that $b_l - a_l > 3\delta$ for all $l = 1, \dots, L$ and $a_{l+1} - b_l > 2\delta$ for all $l = 1, \dots, L-1$. Note

that the latter implies $(b_l - \delta, b_l + \delta) \cap (a_{l+1} - \delta, a_{l+1} + \delta) = \emptyset$. Since $a_l^{(n)} \rightarrow a_l$ and $b_l^{(n)} \rightarrow b_l$ as $n \rightarrow \infty$, there exists $n_0 \in \mathbb{N}$ with

$$\forall n \geq n_0 : |a_l^{(n)} - a_l| < \delta, \quad |b_l^{(n)} - b_l| < \delta, \quad l = 1, \dots, L.$$

Since $b_l - a_l > 3\delta$ we thus have, for all $t \in (0, 1)$,

$$\forall n \geq n_0 : [a_l^{(n)}, a_l + \delta] + t\delta \subset \Omega_n, \quad [b_l - \delta, b_l^{(n)}] - t\delta \subset \Omega_n, \quad l = 1, \dots, L. \quad (31)$$

For any $\varepsilon \in (0, 1)$ there exists $n_\varepsilon \geq n_0$ such that

$$\sup_{n \geq n_\varepsilon} |a_l^{(n)} - a_l| < \varepsilon\delta, \quad \sup_{n \geq n_\varepsilon} |b_l^{(n)} - b_l| < \varepsilon\delta, \quad l = 1, \dots, L.$$

Hence, again with $b_l - a_l > 3\delta$,

$$\begin{aligned} \overline{\bigcup_{n \geq n_\varepsilon} (a_l^{(n)}, a_l + \delta)} + \varepsilon\delta &\subset (a_l - \varepsilon\delta, a_l + \delta) + \varepsilon\delta \subset \Omega, \\ \overline{\bigcup_{n \geq n_\varepsilon} (b_l - \delta, b_l^{(n)})} - \varepsilon\delta &\subset [b_l - \delta, b_l + \varepsilon\delta] - \varepsilon\delta \subset \Omega, \end{aligned} \quad l = 1, \dots, L. \quad (32)$$

Now (31) and (32) imply that $\{\Lambda, \{\Lambda_n : n \in \mathbb{N}\}\}$ has the uniform segment property with finite open covering

$$\begin{aligned} &\left\{ (a_l - \delta, a_l + \delta) \times (-1/2, 2/3), (a_l - \delta, a_l + \delta) \times (1/3, 3/2), \right. \\ &\quad (b_l - \delta, b_l + \delta) \times (-1/2, 2/3), (b_l - \delta, b_l + \delta) \times (1/3, 3/2), \\ &\quad \left. (a_l + \delta/2, b_l - \delta/2) \times (-1/2, 1/3), (a_l + \delta/2, b_l - \delta/2) \times (2/3, 3/2) : l = 1, \dots, L \right\} \end{aligned}$$

and corresponding set of non-zero vectors in \mathbb{R}^2

$$\left\{ \begin{pmatrix} \delta \\ 1/3 \end{pmatrix}, \begin{pmatrix} \delta \\ -1/3 \end{pmatrix}, \begin{pmatrix} -\delta \\ 1/3 \end{pmatrix}, \begin{pmatrix} -\delta \\ -1/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} : l = 1, \dots, L \right\}.$$

Altogether, assumptions (i)–(iii) are satisfied and thus [14, Satz 4.(9)] yields that the sequence of embeddings in (30) is discretely compact. Note that $f_n \in W^{m,p}(\Lambda_n)$, $n \in I$, and $(\|f_n\|_{W^{m,p}(\Lambda_n)})_{n \in I}$ is bounded. Hence there exist $f \in L^p(\Lambda)$ and an infinite subset $I_2 \subset I$ such that $(\tilde{J}_n^{(2)} f_n)_{n \in I_2}$ converges to f in $L^p(\mathbb{R}^2)$. Since $f \in L^p(\Lambda)$, we have $f(\cdot, x_2) \in L^p(\Omega)$ for almost all $x_2 \in (0, 1)$; denote by $\Theta_1 \subset (0, 1)$ the set of such x_2 . The convergence $\|f_n - f\|_{L^p(\mathbb{R}^2)} \rightarrow 0$ as $n \in I_2$, $n \rightarrow \infty$, implies the existence of an infinite subset $I_3 \subset I_2$ so that, for $n \in I_3$,

$$\int_{\mathbb{R}} |f_n(x_1) - f(x_1, x_2)|^p dx_1 \rightarrow 0, \quad n \rightarrow \infty,$$

for almost all $x_2 \in (0, 1)$ (see e.g. [18, Theorem B.98 (iii)] with $u \equiv 0$, $p = 1$); denote by $\Theta_2 \subset (0, 1)$ the set of such x_2 . For $x_2 \in \Theta_1 \cap \Theta_2$ we hence obtain $f(\cdot, x_2) \in L^p(\Omega)$ and $\|f_n - f(\cdot, x_2)\|_{L^p(\mathbb{R})} \rightarrow 0$ as $n \in I_3$, $n \rightarrow \infty$. So we have shown that $(J_n f_n)_{n \in I}$ has a convergent subsequence in $L^p(\mathbb{R})$ with limit in $L^p(\Omega)$. \square

5. APPLICATIONS TO DOMAIN TRUNCATION METHOD AND GALERKIN METHOD

In this section we give examples of spectrally exact operator approximations. All underlying spaces are Hilbert spaces H and $H_n \subset H$, $n \in \mathbb{N}$, with corresponding orthogonal projections P_n with $P_n \xrightarrow{s} I$. For an operator $T \in C(H)$ and approximating operators $T_n \in C(H_n)$, $n \in \mathbb{N}$, we check whether there exists an element $\lambda_0 \in \bigcap_{n \in \mathbb{N}} \varrho(T_n) \cap \varrho(T)$ such that

- $(T - \lambda_0)^{-1}$, $(T_n - \lambda_0)^{-1}$, $n \in \mathbb{N}$, are compact operators;
- the sequence $((T_n - \lambda_0)^{-1})_{n \in \mathbb{N}}$ is discretely compact;
- the resolvents converge strongly, $(T_n - \lambda_0)^{-1} P_n \xrightarrow{s} (T - \lambda_0)^{-1}$;
- the adjoint resolvents converge strongly, $(T_n^* - \overline{\lambda_0})^{-1} P_n \xrightarrow{s} (T^* - \overline{\lambda_0})^{-1}$.

If (a)–(c) are satisfied, then Theorem 2.6 is applicable which yields, in particular, that $(T_n)_{n \in \mathbb{N}}$ is a spectrally exact approximation of T . If, in addition, (d) holds, then Theorem 2.7 yields generalized norm resolvent convergence, i.e. $(T_n - \lambda)^{-1}P_n \rightarrow (T - \lambda)^{-1}$ for every $\lambda \in \varrho(T)$.

5.1. Operator matrices with singular Sturm-Liouville operator entries.

We study 2×2 block operator matrices with singular Sturm-Liouville-type operator entries and prove that the regularization via interval truncation is a spectrally exact approximation.

Let $(a, b) \subset \mathbb{R}$ be a finite interval. We consider Sturm-Liouville differential expressions τ of the form

$$(\tau f)(x) := -(p(x)f'(x))' + q(x)f(x), \quad x \in (a, b),$$

where $p, q : (a, b) \rightarrow \mathbb{R}$ are measurable functions with $1/p, q \in L^1_{\text{loc}}(a, b)$. Suppose that there exist $p_{\min} > 0$ and $q_{\min} \in \mathbb{R}$ such that

$$p \geq p_{\min}, \quad q \geq q_{\min} \quad \text{almost everywhere.}$$

We assume that τ is regular at b and in limit point case at a . Let $(a_n)_{n \in \mathbb{N}} \subset (a, b)$ with $a_n \searrow a, n \rightarrow \infty$. For $n \in \mathbb{N}$ let $P_n : L^2(a, b) \rightarrow L^2(a_n, b)$ be the orthogonal projection given by multiplication with the characteristic function of $[a_n, b]$, i.e. $P_n f := \chi_{[a_n, b]} f, n \in \mathbb{N}$. For $\beta \in [0, \pi)$ let $T_\tau(\beta), T_{\tau, n}(\beta), n \in \mathbb{N}$, be the selfadjoint realizations of τ in the Hilbert spaces $L^2(a, b), L^2(a_n, b), n \in \mathbb{N}$, respectively, with domains

$$\mathcal{D}(T_\tau(\beta)) := \left\{ f \in L^2(a, b) : \begin{array}{l} f, pf' \in \text{AC}_{\text{loc}}(a, b), \tau f \in L^2(a, b), \\ f(b) \cos \beta - (pf')(b) \sin \beta = 0 \end{array} \right\},$$

$$\mathcal{D}(T_{\tau, n}(\beta)) := \left\{ f \in L^2(a_n, b) : \begin{array}{l} f, pf' \in \text{AC}_{\text{loc}}(a_n, b), \tau f \in L^2(a_n, b), \\ f(b) \cos \beta - (pf')(b) \sin \beta = 0, f(a_n) = 0 \end{array} \right\}.$$

Theorem 5.1. *For $i = 1, 2$ let τ_i be a differential expression of the above form, let $\beta_i \in [0, \pi)$ and $\gamma_i \in \mathbb{C} \setminus \{0\}$. Denote by $\vartheta \in [0, \frac{\pi}{2}]$ the angle between $\gamma_1 \mathbb{R}$ and $\gamma_2 \mathbb{R}$. Let $s, t, u, v \in L^\infty(a, b)$ satisfy*

$$\frac{\|u\|_\infty \|s\|_\infty}{|\gamma_1| |\gamma_2| (\cos \frac{\vartheta}{2})^2} < 1, \quad (33)$$

and set

$$s_n := s|_{[a_n, b]}, \quad t_n := t|_{[a_n, b]}, \quad u_n := u|_{[a_n, b]}, \quad v_n := v|_{[a_n, b]}, \quad n \in \mathbb{N}.$$

Define the orthogonal projections $\mathcal{P}^{(n)} := \text{diag}(P_n, P_n), n \in \mathbb{N}$, and the 2×2 block operator matrices $\mathcal{A}, \mathcal{A}^{(n)}, n \in \mathbb{N}$, by

$$\begin{aligned} \mathcal{A} &:= \begin{pmatrix} \gamma_1 T_{\tau_1}(\beta_1) & s T_{\tau_2}(\beta_2) + t \\ u T_{\tau_1}(\beta_1) + v & \gamma_2 T_{\tau_2}(\beta_2) \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) := \mathcal{D}(T_{\tau_1}(\beta_1)) \oplus \mathcal{D}(T_{\tau_2}(\beta_2)), \\ \mathcal{A}^{(n)} &:= \begin{pmatrix} \gamma_1 T_{\tau_1, n}(\beta_1) & s_n T_{\tau_2, n}(\beta_2) + t_n \\ u_n T_{\tau_1, n}(\beta_1) + v_n & \gamma_2 T_{\tau_2, n}(\beta_2) \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}^{(n)}) &:= \mathcal{D}(T_{\tau_1, n}(\beta_1)) \oplus \mathcal{D}(T_{\tau_2, n}(\beta_2)). \end{aligned}$$

Then there exists $\lambda_0 \in \mathbb{C} \setminus (\gamma_1 \mathbb{R} \cup \gamma_2 \mathbb{R})$ such that $\mathcal{A}, \mathcal{A}^{(n)}, n \in \mathbb{N}$, satisfy the claims (a)–(c). If, in addition, $\mathcal{D}(T_{\tau_1}(\beta_1)) = \mathcal{D}(T_{\tau_2}(\beta_2)), \mathcal{D}(T_{\tau_1, n}(\beta_1)) = \mathcal{D}(T_{\tau_2, n}(\beta_2)), n \in \mathbb{N}$, and (33) is replaced by the stronger assumption

$$\max \left\{ \frac{\|u\|_\infty}{|\gamma_1| \cos \frac{\vartheta}{2}}, \frac{\|s\|_\infty}{|\gamma_2| \cos \frac{\vartheta}{2}} \right\} < 1, \quad (34)$$

then there exists $\lambda_0 \in \mathbb{C} \setminus (\gamma_1 \mathbb{R} \cup \gamma_2 \mathbb{R})$ so that all claims (a)–(d) are satisfied.

For the proof we need the following lemma.

Lemma 5.2. *Let $\beta \in [0, \pi)$. Then, for every $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, the operators $T_\tau(\beta)$, $T_{\tau,n}(\beta)$, $n \in \mathbb{N}$, satisfy the claims (a)–(d).*

Proof. Since the differential expression τ is in limit point case at the singular endpoint $x = a$, the fact that

$$\Phi_\tau(\beta) := \{f \in \mathcal{D}(T_\tau(\beta)) : f = 0 \text{ near } x = a\} \subset L^2(a, b)$$

is a core of $T_\tau(\beta)$ is a well-known result from Sturm-Liouville theory (see e.g. the proof of [29, Satz 14.12]). For $f \in \Phi_\tau(\beta)$ let $n_0(f) \in \mathbb{N}$ be such that $f(x) = 0$ for $x \in [a, a_{n_0(f)}]$. This implies $P_n f \in \mathcal{D}(T_{\tau,n}(\beta))$ for $n \geq n_0(f)$. The strong convergence $P_n \xrightarrow{s} I$, $n \rightarrow \infty$, implies

$$\begin{aligned} P_n f &\in \mathcal{D}(T_{\tau,n}(\beta)), \quad n \geq n_0(f), \\ \|T_{\tau,n}(\beta)P_n f - T_\tau(\beta)f\| &= \|P_n T_\tau(\beta)f - T_\tau(\beta)f\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (35)$$

The selfadjointness of $T_\tau(\beta)$, $T_{\tau,n}(\beta)$, $n \in \mathbb{N}$, implies

$$\mathbb{C} \setminus \mathbb{R} \subset \Delta_b((T_{\tau,n}(\beta))_{n \in \mathbb{N}}) \cap \varrho(T_\tau(\beta)). \quad (36)$$

Thus $T_{\tau,n}(\beta) \xrightarrow{gst} T_\tau(\beta)$ by Theorem 3.1. Therefore, (c) and (d) are satisfied for every $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.

Now we prove that (b) is satisfied for every $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$; the proof of (a) is analogous. To this end, we show that the embeddings

$$B_n : (\mathcal{D}(T_{\tau,n}(\beta)), \|\cdot\|_{T_{\tau,n}(\beta)}) \rightarrow W^{1,2}(a_n, b), \quad n \in \mathbb{N},$$

are uniformly bounded. Then Theorem 4.13 implies that the sequence $(J_n)_{n \in \mathbb{N}}$ of embeddings $J_n : (\mathcal{D}(T_{\tau,n}(\beta)), \|\cdot\|_{T_{\tau,n}(\beta)}) \rightarrow L^2(a_n, b)$, $n \in \mathbb{N}$, is discretely compact, and hence (b) follows from Proposition 4.1 ii) and (36). We fix an $n \in \mathbb{N}$ and denote by $\|\cdot\|_n$, $\langle \cdot, \cdot \rangle_n$ the norm and scalar product of $L^2(a_n, b)$. Let $f_n \in \mathcal{D}(T_{\tau,n}(\beta))$ satisfy $\|f_n\|_{T_{\tau,n}(\beta)} = \|f_n\|_n + \|T_{\tau,n}(\beta)f_n\|_n \leq 1$. We estimate

$$\begin{aligned} 1 &\geq \|T_{\tau,n}(\beta)f_n\|_n \|f_n\|_n \geq \langle T_{\tau,n}(\beta)f_n, f_n \rangle_n = \int_{a_n}^b (- (pf'_n)' \overline{f_n} + q|f_n|^2)(x) dx \\ &= (-pf'_n \overline{f_n})(x) \Big|_{x=a_n}^{x=b} + \int_{a_n}^b (p|f'_n|^2 + q|f_n|^2)(x) dx. \end{aligned}$$

If $\beta = 0$, then $f_n(b) = 0$ and $(-pf'_n \overline{f_n})(x) \Big|_{x=a_n}^{x=b} = 0$. If $\beta \in (0, \pi)$, then $(pf'_n)(b) = \cot \beta f_n(b)$ and $(-pf'_n \overline{f_n})(x) \Big|_{x=a_n}^{x=b} = -\cot \beta |f_n(b)|^2$. If $\beta \in [\pi/2, \pi)$, then the latter is non-negative. If $\beta \in (0, \pi/2)$, then

$$\begin{aligned} (-pf'_n \overline{f_n})(x) \Big|_{x=a_n}^{x=b} &= (-\cot \beta |f_n|^2)(x) \Big|_{x=a_n}^{x=b} = -\cot \beta \int_{a_n}^b \frac{d}{dx} (|f_n(x)|^2) dx \\ &\geq -\cot \beta 2 \|f'_n\|_n \|f_n\|_n \geq -\cot \beta \left(\varepsilon \|f'_n\|_n^2 + \frac{1}{\varepsilon} \|f_n\|_n^2 \right), \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. We also use the estimate

$$\int_{a_n}^b (p|f'_n|^2)(x) dx \geq p_{\min} \|f'_n\|_n^2.$$

Then, if we set $\varepsilon := p_{\min}/(2 \cot \beta)$, we obtain altogether

$$1 \geq \frac{1}{2} p_{\min} \|f'_n\|_n^2 + c_\beta \|f_n\|_n^2, \quad c_\beta := \begin{cases} q_{\min}, & \beta \in [\pi/2, \pi), \\ q_{\min} - \frac{2(\cot \beta)^2}{p_{\min}}, & \beta \in [0, \pi/2). \end{cases}$$

From this it is easy to see that the embeddings B_n , $n \in \mathbb{N}$, are uniformly bounded, and thus the claim follows. \square

Proof of Theorem 5.1. Let $\lambda_0 \in \mathbb{C} \setminus (\gamma_1 \mathbb{R} \cup \gamma_2 \mathbb{R})$ be on the unique line $\gamma \mathbb{R}$ (for some $\gamma \in \mathbb{C} \setminus \{0\}$) so that for $i = 1, 2$ the angle ϑ_i between $\gamma_i \mathbb{R}$ and $\gamma \mathbb{R}$ is $\frac{\pi - \vartheta}{2}$. Then

$$\sup_{\xi \in \gamma_i \mathbb{R}} \frac{|\xi|}{|\xi - \lambda_0|} = \frac{1}{\sin \vartheta_i} = \frac{1}{\cos \frac{\vartheta}{2}}, \quad i = 1, 2.$$

Then, using the selfadjointness of $T_{\tau,n}(\beta_i)$, $i = 1, 2$, and [17, Equation V.(3.17)],

$$\begin{aligned} \|(u_n T_{\tau,n}(\beta_1) + v_n)(\gamma_1 T_{\tau,n}(\beta_1) - \lambda_0)^{-1}\| &\leq \frac{\|u\|_\infty}{|\gamma_1|} \sup_{\xi \in \gamma_1 \mathbb{R}} \frac{|\xi|}{|\xi - \lambda_0|} + \frac{\|v\|_\infty}{\text{dist}(\lambda_0, \gamma_1 \mathbb{R})} \\ &= \frac{\|u\|_\infty}{|\gamma_1| \cos \frac{\vartheta}{2}} + \frac{\|v\|_\infty}{\text{dist}(\lambda_0, \gamma_1 \mathbb{R})} =: \gamma_{\lambda_0}^{(1)}, \\ \|(s_n T_{\tau,n}(\beta_1) + t_n)(\gamma_2 T_{\tau,n}(\beta_2) - \lambda_0)^{-1}\| &\leq \frac{\|s\|_\infty}{|\gamma_2|} \sup_{\xi \in \gamma_2 \mathbb{R}} \frac{|\xi|}{|\xi - \lambda_0|} + \frac{\|t\|_\infty}{\text{dist}(\lambda_0, \gamma_2 \mathbb{R})} \\ &= \frac{\|s\|_\infty}{|\gamma_2| \cos \frac{\vartheta}{2}} + \frac{\|t\|_\infty}{\text{dist}(\lambda_0, \gamma_2 \mathbb{R})} =: \gamma_{\lambda_0}^{(2)}. \end{aligned}$$

Note that, by the assumption (33), every $\lambda_0 \in \gamma \mathbb{R}$ with $|\lambda_0|$ sufficiently large satisfies $\gamma_{\lambda_0}^{(1)} \gamma_{\lambda_0}^{(2)} < 1$. Then Lemma 5.2 and Theorem 4.5 imply that $((\mathcal{A}^{(n)} - \lambda_0)^{-1})_{n \in \mathbb{N}}$ is discretely compact, i.e. (b) is satisfied. Claim (a) is shown analogously.

The generalized strong resolvent convergence in (c) follows from Lemma 5.2, (35) in its proof, and from Theorem 3.7 using that $\lambda_0 \in \Delta_b((\mathcal{A}^{(n)})_{n \in \mathbb{N}}) \cap \varrho(\mathcal{A}) \neq \emptyset$.

To prove (d), we first note that if (34) holds, then $\max\{\gamma_{\lambda_0}^{(1)}, \gamma_{\lambda_0}^{(2)}\} < 1$ for all $\lambda_0 \in \gamma \mathbb{R}$ with $|\lambda_0|$ sufficiently large. Therefore, the operator matrices \mathcal{A} , $\mathcal{A}^{(n)}$, $n \in \mathbb{N}$, are relatively bounded perturbations of diagonal operators with relative bound < 1 , and the same holds for the adjoint matrices if we assume that, in addition, $\mathcal{D}(T_{\tau_1}(\beta_1)) = \mathcal{D}(T_{\tau_2}(\beta_2))$ and $\mathcal{D}(T_{\tau_1,n}(\beta_1)) = \mathcal{D}(T_{\tau_2,n}(\beta_2))$ for $n \in \mathbb{N}$. Then [15, Corollary 1] yields

$$\begin{aligned} \mathcal{A}^* &= \begin{pmatrix} \overline{\gamma_1} T_{\tau_1}(\beta_1) & \overline{u} T_{\tau_1}(\beta_1) + \overline{v} \\ \overline{s} T_{\tau_2}(\beta_2) + \overline{t} & \overline{\gamma_2} T_{\tau_2}(\beta_2) \end{pmatrix}, & \mathcal{D}(\mathcal{A}^*) &= \mathcal{D}(\mathcal{A}), \\ (\mathcal{A}^{(n)})^* &= \begin{pmatrix} \overline{\gamma_1} T_{\tau_1,n}(\beta_1) & \overline{u_n} T_{\tau_1,n}(\beta_1) + \overline{v_n} \\ \overline{s_n} T_{\tau_2,n}(\beta_2) + \overline{t_n} & \overline{\gamma_2} T_{\tau_2,n}(\beta_2) \end{pmatrix}, & \mathcal{D}((\mathcal{A}^{(n)})^*) &= \mathcal{D}(\mathcal{A}^{(n)}). \end{aligned}$$

Now the strong convergence $((\mathcal{A}^{(n)})^* - \overline{\lambda_0})^{-1} \mathcal{P}^{(n)} \xrightarrow{s} (\mathcal{A}^* - \overline{\lambda_0})^{-1}$ is shown analogously as (c). \square

5.2. Domain truncation of magnetic Schrödinger operators on \mathbb{R}^d . Let $\Omega_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, be nested open sets that exhaust \mathbb{R}^d eventually. Denote by $\|\cdot\|$, $\langle \cdot, \cdot \rangle$, $\|\cdot\|_n$, $\langle \cdot, \cdot \rangle_n$, $n \in \mathbb{N}$, the norm and scalar product of $L^2(\mathbb{R}^d)$ and $L^2(\Omega_n)$, $n \in \mathbb{N}$, respectively. Let P_n , $n \in \mathbb{N}$, be the orthogonal projections of $L^2(\mathbb{R}^d)$ onto the respective subspaces, given by multiplication with the characteristic function χ_{Ω_n} . Then $P_n \xrightarrow{s} I$ as $n \rightarrow \infty$.

Consider the differential expression $\tau := -\Delta + p \cdot \nabla + v$ with a vector potential $p : \mathbb{R}^d \rightarrow \mathbb{C}^d$ and a scalar potential $v : \mathbb{R}^d \rightarrow \mathbb{C}$. An important application is given by the *magnetic Schrödinger operator* $\tau = (-i\nabla + A)^2 + V = -\Delta + p \cdot \nabla + v$ with $p = -iA$ and $v = -i\nabla \cdot A + A^2 + V$. The case $p \equiv 0$ was already studied in [5].

We assume that p and $v = q + r$ satisfy

- (i) $p \in L^\infty(\mathbb{R}^d)$;

- (ii) $q \in \text{AC}_{\text{loc}}(\mathbb{R}^d)$, $\text{Re } q \geq 0$, $|q(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$, and there are $a_{\nabla}, b_{\nabla} \geq 0$ with

$$|\nabla q(x)|^2 \leq a_{\nabla} + b_{\nabla} |q(x)|^2 \quad \text{for almost all } x \in \mathbb{R}^d; \quad (37)$$

- (iii) $r \in L^2_{\text{loc}}(\mathbb{R}^d)$, and there exist $a_r, b_r \geq 0$ with $b_r < 1$ such that

$$|r(x)|^2 \leq a_r + b_r |q(x)|^2 \quad \text{for almost all } x \in \mathbb{R}^d. \quad (38)$$

Theorem 5.3. *Let A and A_n , $n \in \mathbb{N}$, be the Dirichlet realizations of τ in the respective spaces,*

$$\begin{aligned} Af &:= \tau f, & \mathcal{D}(A) &:= \{f \in W^{2,2}(\mathbb{R}^d) : qf \in L^2(\mathbb{R}^d)\}, \\ A_n f &:= \tau f, & \mathcal{D}(A_n) &:= W^{2,2}(\Omega_n) \cap W_0^{1,2}(\Omega_n), \quad n \in \mathbb{N}. \end{aligned}$$

Then, for every real $\lambda_0 < 0$ with sufficiently large $|\lambda_0|$, the operators A, A_n , $n \in \mathbb{N}$, satisfy the claims (a)–(c). If, in addition, $p \in W^{1,\infty}(\mathbb{R}^d)$, then (d) is satisfied as well.

For the proof we use the following result.

Lemma 5.4. *Let T and T_n , $n \in \mathbb{N}$, be the Dirichlet realizations of the Schrödinger differential expression $\tau_0 := -\Delta + q$ in the respective spaces,*

$$\begin{aligned} Tf &:= \tau_0 f, & \mathcal{D}(T) &:= \{f \in W^{2,2}(\mathbb{R}^d) : qf \in L^2(\mathbb{R}^d)\}, \\ T_n f &:= \tau_0 f, & \mathcal{D}(T_n) &:= W^{2,2}(\Omega_n) \cap W_0^{1,2}(\Omega_n), \quad n \in \mathbb{N}. \end{aligned}$$

Then, for every real $\lambda_0 < 0$, the operators T, T_n , $n \in \mathbb{N}$, satisfy the claims (a)–(d).

Proof. To prove claim (b), we fix an $n \in \mathbb{N}$. First note that $q_n := q|_{\Omega_n}$ is bounded and thus T_n is a bounded perturbation of the Dirichlet Laplacian on Ω_n . By [13, Theorem VI.1.4], the operator T_n is m -accretive with compact resolvent. Therefore, every $\lambda_0 < 0$ satisfies $\lambda_0 \in \varrho(T_n)$ and, using [17, Problem V.3.31],

$$\|(T_n - \lambda_0)^{-1}\| \leq \frac{1}{|\lambda_0|}, \quad \|T_n(T_n - \lambda_0)^{-1}\| \leq 1. \quad (39)$$

So we have, in particular,

$$\forall \lambda_0 < 0 : \quad \lambda_0 \in \Delta_b((T_n)_{n \in \mathbb{N}}). \quad (40)$$

For any $f \in \mathcal{D}(T_n)$ we obtain, using integration by parts and $\text{Re } q_n \geq 0$ by assumption (ii),

$$\|T_n f\|_n^2 = \|\Delta f\|_n^2 + \|q_n f\|_n^2 + 2\text{Re}\langle -\Delta f, q_n f \rangle_n \geq \|\Delta f\|_n^2 + \|q_n f\|_n^2 + 2\text{Re}\langle \nabla f, (\nabla q_n) f \rangle_n.$$

Now, again using integration by parts and with (37), for any $\varepsilon, \delta > 0$,

$$\begin{aligned} & |2\text{Re}\langle \nabla f, (\nabla q_n) f \rangle_n| \\ & \leq \frac{1}{\varepsilon} \|\nabla f\|_n^2 + \varepsilon \|(\nabla q_n) f\|_n^2 \leq \frac{1}{\varepsilon} \langle -\Delta f, f \rangle_n + \varepsilon a_{\nabla} \|f\|_n^2 + \varepsilon b_{\nabla} \|q_n f\|_n^2 \\ & \leq \left(\varepsilon a_{\nabla} + \frac{1}{4\varepsilon\delta} \right) \|f\|_n^2 + \frac{\delta}{\varepsilon} \|\Delta f\|_n^2 + \varepsilon b_{\nabla} \|q_n f\|_n^2. \end{aligned}$$

Let $\alpha \in (0, 1)$ be arbitrary. We choose $\varepsilon > 0$ and then $\delta > 0$ both so small that $\max\{\delta/\varepsilon, \varepsilon b_{\nabla}\} \leq \alpha$. With these ε, δ , we set $C_{\alpha} := \varepsilon a_{\nabla} + 1/(4\varepsilon\delta)$ and arrive at

$$\|T_n f\|_n^2 + C_{\alpha} \|f\|_n^2 \geq (1 - \alpha) (\|\Delta f\|_n^2 + \|q_n f\|_n^2). \quad (41)$$

Now let $I \subset \mathbb{N}$ be an infinite subset and let $f_n \in \mathcal{D}(T_n)$, $n \in I$, be such that the sequence of graph norms $(\|f_n\|_{T_n})_{n \in I}$ is bounded. Then (41) implies that $(\|\Delta f_n\|_n)_{n \in I}$ and $(\|q_n f_n\|_n)_{n \in I}$ are bounded, and hence so is $(\|\nabla f_n\|_n)_{n \in I}$ since $2\|\nabla f_n\|_n^2 \leq \|f_n\|_n^2 + \|\Delta f_n\|_n^2$, $n \in I$. By extending every f_n by zero outside its domain Ω_n , we obtain $(f_n)_{n \in I} \subset W^{1,2}(\mathbb{R}^d)$, and the sequences $(\|\nabla f_n\|)_{n \in I}$ and

$(\|qf\|)_{n \in I}$ are both bounded. Since $|q(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$ by assumption (ii), *Rellich's criterion* (see [23, Theorem XIII.65]) implies that $(f_n)_{n \in I}$ has a subsequence that is convergent in $L^2(\mathbb{R}^d)$. So we have shown that the sequence $(J_n)_{n \in \mathbb{N}}$ of embeddings $J_n : (\mathcal{D}(T_n), \|\cdot\|_{T_n}) \rightarrow L^2(\Omega_n)$, $n \in \mathbb{N}$, is discretely compact, and hence, by Proposition 4.1 ii) and (40), claim (b) follows for every $\lambda_0 < 0$.

Let T_{\min}, T_{\max} be the realizations of $\tau_0 = -\Delta + q$ with

$$\mathcal{D}(T_{\min}) = \Phi := C_0^\infty(\mathbb{R}^d), \quad \mathcal{D}(T_{\max}) := \{f \in L^2(\mathbb{R}^d) : \tau f \in L^2(\mathbb{R}^d)\}.$$

By [13, Theorem VII.2.6, Corollary VII.2.7], T_{\max} is m -accretive and the closure of T_{\min} . Be proceeding as before, we arrive at the inequality (41) with T_n replaced by T_{\min} ; since $T_{\max} = \overline{T_{\min}}$, the inequality also holds for T_{\max} . Therefore, it is easy to see that $T_{\max} = T$. So T is m -accretive and Φ is a core of T . Claim (a) is shown analogously as (b).

For every $f \in \Phi$ there exists $n_0(f) \in \mathbb{N}$ such that $f \in \mathcal{D}(T_n)$ and $T_n f = P_n T f$ for $n \geq n_0(f)$. Hence Theorem 3.1 and (40) imply that (c) is satisfied for every $\lambda_0 < 0$. Since, by [13, Theorem VII.2.5], the adjoint operators T^*, T_n^* , $n \in \mathbb{N}$, are the Dirichlet realizations of $\tau^* = -\Delta + \bar{q}$ and \bar{q} satisfies the assumption (ii), we obtain analogously that (d) is satisfied for every $\lambda_0 < 0$. \square

Proof of Theorem 5.3. Let $\lambda_0 < 0$. The assumption (i) yields, for any $\beta > 0$,

$$\|p \cdot \nabla f\|^2 \leq \|p\|_\infty^2 \|\nabla f\|^2 \leq \frac{\|p\|_\infty^4}{4\beta} \|f\|^2 + \beta \|\Delta f\|^2. \quad (42)$$

Define $p_n := p|_{\Omega_n}$ and $r_n := r|_{\Omega_n}$ for $n \in \mathbb{N}$, and $S := A - T$, $S_n := A_n - T_n$, $n \in \mathbb{N}$. Then the estimates (42) and (38) imply, for any $\nu > 0$,

$$\begin{aligned} \|S_n(T_n - \lambda_0)^{-1}\|^2 &\leq \left(1 + \frac{1}{4\nu}\right) \|(p \cdot \nabla)(T_n - \lambda_0)^{-1}\|^2 + (1 + \nu) \|r_n(T_n - \lambda_0)^{-1}\|^2 \\ &\leq \left(\frac{\|p\|_\infty^4}{4\beta} \left(1 + \frac{1}{4\nu}\right) + a_r(1 + \nu)\right) \|(T_n - \lambda_0)^{-1}\|^2 \\ &\quad + \beta \left(1 + \frac{1}{4\nu}\right) \|\Delta(T_n - \lambda_0)^{-1}\|^2 + b_r(1 + \nu) \|q_n(T_n - \lambda_0)^{-1}\|^2. \end{aligned}$$

We choose ν and then β so small that $b := \max\{\beta(1 + 1/(4\nu)), b_r(1 + \nu)\} < 1$. Set

$$a := \frac{\|p\|_\infty^4}{4\beta} \left(1 + \frac{1}{4\nu}\right) + a_r(1 + \nu).$$

Then, by (41) and (39),

$$\begin{aligned} \|S_n(T_n - \lambda_0)^{-1}\|^2 &\leq \left(a + \frac{bC_\alpha}{1 - \alpha}\right) \|(T_n - \lambda_0)^{-1}\|^2 + \frac{b}{1 - \alpha} \|T_n(T_n - \lambda_0)^{-1}\|^2 \\ &\leq \left(a + \frac{bC_\alpha}{1 - \alpha}\right) \frac{1}{|\lambda_0|^2} + \frac{b}{1 - \alpha} =: (\gamma_{\lambda_0, \alpha})^2. \end{aligned}$$

Now we choose $\alpha \in (0, 1)$ so small that $b/(1 - \alpha) < 1$ and then $|\lambda_0|$ so large that $\gamma_{\lambda_0, \alpha} < 1$. Then Theorem 4.2 yields that $\lambda_0 \in \varrho(A_n)$, $n \in \mathbb{N}$, and $((A_n - \lambda_0)^{-1})_{n \in \mathbb{N}}$ is discretely compact, i.e. claim (b) holds for this λ_0 . Claim (a) is shown analogously.

Analogously as $\|S_n(T_n - \lambda_0)^{-1}\| \leq \gamma_{\lambda_0, \alpha} < 1$, $n \in \mathbb{N}$, if $|\lambda_0|$ is sufficiently large, one may show that $\|S(T - \lambda_0)^{-1}\| < 1$ if $|\lambda_0|$ is sufficiently large. Now claim (c) follows from Theorem 3.3 provided that $S_n(T_n - \lambda_0)^{-1} P_n \xrightarrow{s} S(T - \lambda_0)^{-1}$. To show the latter, we take $f \in \Phi = C_0^\infty(\mathbb{R}^d)$; the latter is a core of T (see the proof of Lemma 5.4). Define $g := (T - \lambda_0)f$. Then there exists $n_0(f) \in \mathbb{N}$ such that $\text{supp } g \subset \text{supp } f \subset \Omega_n$, $n \geq n_0(f)$. Hence, for $n \geq n_0(f)$,

$$S_n(T_n - \lambda_0)^{-1}g = S_n(T_n - \lambda_0)^{-1}(T - \lambda_0)f = S_n f = S f = S(T - \lambda_0)^{-1}g.$$

Now $S_n(T_n - \lambda_0)^{-1}P_n \xrightarrow{s} S(T - \lambda_0)^{-1}$ follows since $\{(T - \lambda_0)f : f \in \Phi\} \subset L^2(\mathbb{R}^d)$ is a dense subset and $\|S_n(T_n - \lambda_0)^{-1}\|$, $n \in \mathbb{N}$, are uniformly bounded.

Now assume that, in addition, $p \in W^{1,\infty}(\mathbb{R}^d)$. Then A^* and A_n^* , $n \in \mathbb{N}$, are Dirichlet realizations of the adjoint differential expression

$$\tau^* = -\Delta - \bar{p} \cdot \nabla + (-\nabla \cdot \bar{p} + \bar{v}).$$

Since the vector potential $\tilde{p} := \bar{p}$ and the scalar potential $\tilde{v} := \bar{q} + \bar{r}$ with $\bar{q} := \bar{q}$ and $\bar{r} := \bar{r} - \nabla \cdot \bar{p}$ satisfy assumptions (i)–(iii), the above arguments imply that $(A_n^* - \lambda_0)^{-1}P_n \xrightarrow{s} (A^* - \lambda_0)^{-1}$ for every $\lambda_0 < 0$ with $|\lambda_0|$ large enough; thus (d) is satisfied. \square

5.3. Galerkin approximation of block-diagonally dominant matrices. Let $\{e_k : k \in \mathbb{N}\}$ be the standard orthonormal basis of $H := l^2(\mathbb{N})$. Define the k -dimensional subspace $H_k := \text{span}\{e_i : i = 1, \dots, k\} \subset H$. Denote by $P_k : H \rightarrow H_k$, $k \in \mathbb{N}$, the corresponding orthogonal projections. Obviously, $P_k \xrightarrow{s} I$ as $k \rightarrow \infty$.

We study the Galerkin approximation of a closed operator $A \in C(H)$. To this end, we identify A with its matrix representation with respect to $\{e_k : k \in \mathbb{N}\}$,

$$A = (A_{ij})_{i,j=1}^{\infty}, \quad A_{ij} = \langle Ae_j, e_i \rangle, \quad i, j \in \mathbb{N}.$$

With $k_0 := 0$ and a strictly increasing sequence $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ define the diagonal blocks

$$B_n := (A_{ij})_{i,j=k_{n-1}+1}^{k_n}, \quad n \in \mathbb{N},$$

and split A as $A = T + S$ with $T := \text{diag}(B_n : n \in \mathbb{N})$. Define the Galerkin approximations $A_n := P_{k_n}A|_{H_{k_n}}$, $n \in \mathbb{N}$.

Theorem 5.5. *Assume that there exists $\lambda_0 \in \bigcap_{n \in \mathbb{N}} \varrho(B_n)$ with $\|(B_n - \lambda_0)^{-1}\| \rightarrow 0$ as $n \rightarrow \infty$. Then $\lambda_0 \in \varrho(T)$. If $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $\|S(T - \lambda_0)^{-1}\| < 1$, then A , A_n , $n \in \mathbb{N}$, satisfy the claims (a)–(c). If, in addition, $\mathcal{D}(T^*) \subset \mathcal{D}(S^*)$ and $\|S^*(T^* - \bar{\lambda}_0)^{-1}\| < 1$, then (d) is satisfied as well.*

The proof relies on the following lemma.

Lemma 5.6. *If $\|(B_n - \lambda_0)^{-1}\| \rightarrow 0$ as $n \rightarrow \infty$, then the block-diagonal operators T and $T_n := \text{diag}(B_k : k = 1, \dots, n)$, $n \in \mathbb{N}$, satisfy the claims (a)–(d).*

Proof. The assumption $\|(B_n - \lambda_0)^{-1}\| \rightarrow 0$ implies that $(T - \lambda_0)^{-1}$ is the norm limit of the finite-rank (and thus compact) operators $(T_n - \lambda_0)^{-1}P_{k_n}$ and hence compact by [17, Theorem III.4.7]. In addition, $(T_n - \lambda_0)^{-1} = P_{k_n}(T - \lambda_0)^{-1}|_{H_{k_n}}$ and therefore $(T_n - \lambda_0)^{-1}$, $n \in \mathbb{N}$, are compact and form a discretely compact sequence. Thus (a) and (b) are satisfied.

Again using $\|(B_n - \lambda_0)^{-1}\| \rightarrow 0$, we see that $(T_n - \lambda_0)^{-1}P_{k_n} \rightarrow (T - \lambda_0)^{-1}$ which implies, in particular, that (c) and (d) hold. \square

Proof of Theorem 5.5. Define the Galerkin approximations $S_n := P_{k_n}S|_{H_{k_n}}$, $n \in \mathbb{N}$. Note that

$$S_n(T_n - \lambda_0)^{-1} = P_{k_n}S(T - \lambda_0)^{-1}|_{H_{k_n}}, \quad n \in \mathbb{N}.$$

So we readily conclude that $\|S_n(T_n - \lambda_0)^{-1}\| \leq \|S(T - \lambda_0)^{-1}\| =: \gamma_{\lambda_0} < 1$ and $S_n(T_n - \lambda_0)^{-1}P_{k_n} \xrightarrow{s} S(T - \lambda_0)^{-1}$. Using Lemma 5.6 and Theorems 3.3, 4.2, we obtain claims (b) and (c) (and (a) analogously).

If the additional assumptions $\mathcal{D}(T^*) \subset \mathcal{D}(S^*)$ and $\|S^*(T^* - \bar{\lambda}_0)^{-1}\| < 1$ hold, then claim (d) is proved analogously; note that $(T + S)^* = T^* + S^*$ by [15, Corollary 1]. \square

Remark 5.7. If the assumptions of Theorem 5.5 are satisfied, then the Galerkin approximation $(A_n)_{n \in \mathbb{N}}$ with $A_n := P_{k_n} A|_{H_{k_n}}$, $n \in \mathbb{N}$, is spectrally exact. However, if we consider all $P_k A|_{H_k}$, $k \in \mathbb{N}$, then spurious eigenvalues may occur. As an example, let A be the selfadjoint Jacobi operator

$$A = \begin{pmatrix} 0 & q_1 & 0 & \dots \\ q_1 & 0 & q_2 & \ddots \\ 0 & q_2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad q_k := \begin{cases} k+1, & k \text{ odd,} \\ k/2, & k \text{ even.} \end{cases}$$

One may check that the assumptions of Theorem 5.5 are satisfied for $\lambda_0 = 0$ and

$$k_n = 2n, \quad B_n = \begin{pmatrix} 0 & q_{2n-1} \\ q_{2n-1} & 0 \end{pmatrix}, \quad n \in \mathbb{N}.$$

So the operators $P_{2n} A|_{H_{2n}}$, $n \in \mathbb{N}$, form a spectrally exact approximation of A . However, by induction over $n \in \mathbb{N}$ one may check that $\det(P_{2n-1} A|_{H_{2n-1}}) = 0$; hence $\lambda_0 = 0 \in \varrho(A)$ is an eigenvalue of every $P_{2n-1} A|_{H_{2n-1}}$, $n \in \mathbb{N}$, and thus a point of spectral pollution.

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