List k-Colouring P_t-Free Graphs: a Mim-width Perspective

Nick Brettell
School of Mathematics and Statistics, Victoria University of Wellington, New Zealand
nick.brettell@vuw.ac.nz

Jake Horsfield
School of Computing, University of Leeds, Leeds, UK
sc15jh@leeds.ac.uk

Andrea Munaro
School of Mathematics and Physics, Queen's University Belfast, UK
a.munaro@qub.ac.uk

Daniël Paulusma
Department of Computer Science, Durham University, UK
daniel.paulusma@durham.ac.uk

Abstract

A colouring of a graph $G = (V, E)$ is a mapping $c\colon V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ for every two adjacent vertices u and v of G. The List k-Colouring problem is to decide whether a graph $G = (V, E)$ with a list $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$ has a colouring c such that $c(u) \in L(u)$ for every $u \in V$. Let P_t be the path on t vertices and let $K_{1,s}^1$ be the graph obtained from the $(s+1)$-vertex star $K_{1,s}$ by subdividing each of its edges exactly once.

Recently, Chudnovsky, Spirkl and Zhong (DM 2020) proved that List 3-Colouring is polynomial-time solvable for $(K_{1,s}^1, P_t)$-free graphs for every $t \geq 1$ and $s \geq 1$. We generalize their result to List k-Colouring for every $k \geq 1$. Our result also generalizes the known result that for every $k \geq 1$ and $s \geq 0$, List k-Colouring is polynomial-time solvable for $(sP_1 + P_5)$-free graphs, which was proven for $s = 0$ by Hoàng, Kamiński, Lozin, Sawada, and Shu (Algorithmica 2010) and for every $s \geq 1$ by Couturier, Golovach, Kratsch and Paulusma (Algorithmica 2015).

We show our result by proving boundedness of an underlying width parameter. Namely, we show that for every $k \geq 1$, $s \geq 1$, $t \geq 1$, the class of $(K_k, K_{1,s}^1, P_t)$-free graphs has bounded mim-width and that a corresponding branch decomposition is "quickly computable" for these graphs.

2012 ACM Subject Classification Theory of computation → Graph algorithms analysis

Keywords and phrases Hereditary graph class, mim-width, colouring

Funding The research in this paper received support from the Leverhulme Trust (RPG-2016-258). The first author was supported by a Rutherford Foundation Postdoctoral Fellowship from Royal Society Te Apārangi.

1 Introduction

Width parameters play an important role in algorithmic graph theory, as evidenced by various surveys [12, 18, 19, 32, 33]. A graph class $\mathcal{G}$ has *bounded* width, for some width parameter, if there exists a constant c such that every graph in $\mathcal{G}$ has width at most c. Mim-width is a relatively young width parameter that was introduced by Vatshelle [37]. It is defined as follows. A *branch decomposition* for a graph G is a pair (T, δ), where T is a subcubic tree and δ is a bijection from $V(G)$ to the leaves of T. Every edge $e \in E(T)$ partitions the leaves of T into two classes, L_e and $\overline{L_e}$, depending on which component of $T - e$ they belong to. Hence, e induces a partition $(A_e, \overline{A_e})$ of $V(G)$, where $\delta(A_e) = L_e$ and $\delta(\overline{A_e}) = \overline{L_e}$. We let $G[A_e, \overline{A_e}]$ be the bipartite subgraph of G induced by the edges with one end-vertex in

A_e and the other in $\overline{A_e}$. A matching $F \subseteq E(G)$ of G is *induced* if there is no edge in G between vertices of different edges of F. We let $\text{cutmim}_G(A_e, \overline{A_e})$ be the size of a maximum induced matching in $G[A_e, \overline{A_e}]$. The *mim-width* $\text{mimw}_G(T, \delta)$ of (T, δ) is the maximum value of $\text{cutmim}_G(A_e, \overline{A_e})$ over all edges $e \in E(T)$. The *mim-width* $\text{mimw}(G)$ of G is the minimum value of $\text{mimw}_G(T, \delta)$ over all branch decompositions (T, δ) for G. See Figure 1 for an example.

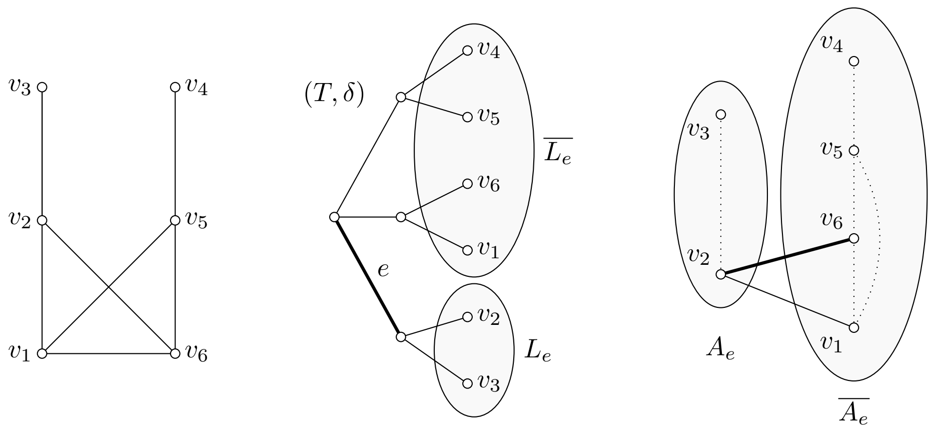

Figure 1 An example of a graph G with a branch decomposition (T, δ). The partition $(A_e, \overline{A_e})$ of $V(G)$ in the rightmost figure witnesses that $\text{mimw}_G(T, \delta) \geq 1$. It can be easily seen that $\text{mimw}_G(T, \delta) \leq 1$ and so $\text{mimw}(G) = 1$.

Vatshelle [37] proved that every class of bounded clique-width, or equivalently, bounded boolean-width, module-width, NLC-width or rank-width, has bounded mim-width, and that the converse is not true. That is, he proved that there exist graph classes of bounded mim-width that have unbounded clique-width. This means that proving that a problem is polynomial-time solvable for graph classes of bounded mim-width yields more tractable graph classes than doing this for clique-width. Hence, mim-width has greater *modeling power* than clique-width.

However, the *trade-off* is that fewer problems admit such an algorithm, as we explain below by means of a relevant example, namely the classical Colouring problem. Moreover, computing mim-width is NP-hard [36] and it is not possible to approximate in polynomial time the mim-width of a graph within a constant factor unless NP = ZPP [36]. It remains a challenging open problem to develop a polynomial-time algorithm for computing a branch decomposition with mim-width $f(k)$ for a graph with mim-width k. However, the latter has been shown possible for special graph classes $\mathcal{G}$. In such a case, we say that the mim-width of $\mathcal{G}$ is *quickly computable*. We can then develop a polynomial-time algorithm for the problem of interest via dynamic programming over the computed branch decomposition. We refer to [1, 2, 3, 5, 6, 7, 13, 22, 23, 24, 25] for a wide range of examples of graph classes and problems for which such dynamic programming algorithms have been obtained.

As mentioned, in this paper we focus on Graph Colouring, a central problem in Discrete Mathematics, Theoretical Computer Science and beyond. A *colouring* of a graph $G = (V, E)$ is a mapping $c \colon V \to \{1, 2, \ldots\}$ that gives each vertex $u \in V$ a *colour* $c(u)$ in such a way that, for every two adjacent vertices u and v, we have that $c(u) \neq c(v)$. If for every $u \in V$ we have $c(u) \in \{1, \ldots, k\}$, then we say that c is a *k-colouring* of G. The Colouring problem is to decide whether a given graph G has a k-colouring for some given integer $k \geq 1$. If k is *fixed*, that is, not part of the input, we call this the k-Colouring problem. A classical result of Lovász [30] states that k-Colouring is NP-complete even if $k = 3$.

The Colouring problem is an example of a problem that distinguishes between classes of bounded mim-width and bounded clique-width: it is polynomial-time solvable for every graph class of bounded clique-width [27] but NP-complete for circular-arc graphs [14], a class of graphs of mim-width at most 2 and for which mim-width is quickly computable [1]. When we fix k, we no longer have this distinction, as k-Colouring, for every fixed integer $k \geq 1$, is polynomial-time solvable for a graph class whose mim-width is bounded and quickly computable [7].

We consider the following generalization of k-Colouring. For an integer $k \geq 1$, a *k-list assignment* of a graph $G = (V, E)$ is a function L that assigns each vertex $u \in V$ a *list* $L(u) \subseteq \{1, 2, \ldots, k\}$ of *admissible* colours for u. A colouring c of G *respects* L if $c(u) \in L(u)$ for every $u \in V$. For a fixed integer $k \geq 1$, the List k-Colouring problem is to decide whether a given graph G with a k-list assignment L admits a colouring that respects L. Note that for $k_1 \leq k_2$, List k_1-Colouring is a special case of List k_2-Colouring and that by setting $L(u) = \{1, \ldots, k\}$ for every $u \in V$, we obtain the k-Colouring problem.

Given an instance (G, L) of List k-Colouring, one can construct an equivalent instance G' of k-Colouring by adding a clique on new vertices $u_1, \ldots, u_k$ to G and adding an edge between u_i and $v \in V(G)$ if and only if $i \notin L(u)$ (see, for example, [31]). Kwon [29] observed that $\text{mimw}(G') \leq \text{mimw}(G) + k$ and thus, as k-Colouring is polynomial-time solvable for graph classes whose mim-width is bounded and quickly computable [7], for every fixed integer $k \geq 1$, this leads to the following:

▶ **Theorem 1** ([29]). *For every $k \geq 1$, List k-Colouring is polynomial-time solvable for a graph class whose mim-width is bounded and quickly computable.*

In this paper we show that a number of known polynomial-time results for List k-Colouring on special graph classes can be obtained, and strengthened, by applying Theorem 1.

The classes that we consider belong to the framework of hereditary graph classes. A graph class is *hereditary* if it is closed under vertex deletion. It is well known and not difficult to see that hereditary graph classes are exactly those classes characterized by a (unique) set $\mathcal{F}$ of minimal forbidden induced subgraphs. If $|\mathcal{F}| = 1$ or $|\mathcal{F}| = 2$, we say that the hereditary graph class is *monogenic* or *bigenic*, respectively. In a recent study [5], boundedness or unboundedness of mim-width has been determined for all monogenic classes and a large number of bigenic classes. These results imply that a monogenic graph class has bounded mim-width if and only if it has bounded clique-width [5] but that this equivalence does not always hold for bigenic graph classes. As we focus on hereditary graph classes, our work can be seen as a continuation of the research in [5].

Related Work

We first need to introduce some more terminology. A graph G is *H-free*, for some graph H, if it contains no *induced* subgraph isomorphic to H, that is, we cannot modify G into H by a sequence of vertex deletions. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph is *$(H_1, \ldots, H_p)$-free* if it is H_i-free for every $i \in \{1, \ldots, p\}$. We denote the *disjoint union* of two graphs G_1 and G_2 by $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We let P_r and K_r denote the path and complete graph on r vertices, respectively.

The complexity of Colouring for H-free graphs has been settled for every graph H [28], but there are still infinitely many open cases for k-Colouring restricted to H-free graphs when H is a *linear forest*, that is, a disjoint union of paths. We refer to [15] for a survey and to [8, 10, 26] for updated summaries. In particular, Hoàng et al. [20] proved that for every integer $k \geq 1$, k-Colouring is polynomial-time solvable for P_5-free graphs. Their proof is in fact a proof for List k-Colouring. The result of [20] was generalized by Couturier et al. [11] as follows:

▶ **Theorem 2** ([11]). *For every $k \geq 1$ and $s \geq 0$, List k-Colouring is polynomial-time solvable for $(sP_1 + P_5)$-free graphs.*

For $r \geq 1$ and $s \geq 1$, we let $K_{r,s}$ denote the complete bipartite graph with partition classes of size r and s. The graph $K_{1,s}$ is also known as the $(s+1)$-vertex star. The 1-*subdivision* of a graph G is the graph obtained from G by subdividing each edge of G exactly once. We denote the 1-subdivision of a star $K_{1,s}$ by $K^1_{1,s}$; in particular $K^1_{1,2} = P_5$. Very recently, Chudnovsky, Spirkl and Zhong proved the following result:

▶ **Theorem 3** ([10]). *For every $s \geq 1$ and $t \geq 1$, List 3-Colouring is polynomial-time solvable for $(K^1_{1,s}, P_t)$-free graphs.*

For every $s \geq 1$ and $t \geq 2s+5$, the class of $(K^1_{1,s+2}, P_t)$-free graphs contains the class of $(sP_1 + P_5)$-free graphs. Hence, Theorem 3 generalizes Theorem 2 in the case $k = 3$. As $K_{1,s}$ is an induced subgraph of $K^1_{1,s}$, Theorem 3 also generalizes the following result in the case $r = 1$:

▶ **Theorem 4** ([17]). *For every $k \geq 1$, $r \geq 1$, $s \geq 1$ and $t \geq 1$, List k-Colouring is polynomial-time solvable for $(K_{r,s}, P_t)$-free graphs.*

Our Results

We prove the following result:

▶ **Theorem 5.** *For every $r \geq 1$, $s \geq 1$ and $t \geq 1$, the mim-width of the class of $(K_r, K^1_{1,s}, P_t)$-free graphs is bounded and quickly computable.*

We may assume without loss of generality that an instance of List k-Colouring is K_{k+1}-free, for otherwise it is a no-instance. Hence, combining Theorem 5 with Theorem 1 enables us to generalize both Theorems 2 and 3:

▶ **Corollary 6.** *For every $k \geq 1$, $s \geq 1$ and $t \geq 1$, List k-Colouring is polynomial-time solvable for $(K^1_{1,s}, P_t)$-free graphs.*

Corollary 6 is tight in the following sense. Let $L_{1,s}$ denote the subgraph obtained from $K^1_{1,s}$ by subdividing one edge exactly once; in particular $L_{1,2} = P_6$. Then, as List 4-Colouring is NP-complete for P_6-free graphs [16], we cannot generalize Corollary 6 to $(L_{1,s}, P_t)$-free graphs for $k \geq 4$, $s \geq 2$ and $t \geq 6$. Moreover, the mim-width of (K_4, P_6)-free graphs is unbounded [5] and so we cannot extend Theorem 5 to $(K_r, L_{1,s}, P_t)$-free graphs, for $r \geq 4$, $s \geq 2$ and $t \geq 6$, either.

Theorem 5 has other applications as well. Firstly, as mentioned earlier, there are many problems known to be XP parameterized by mim-width, so Theorem 5 implies that these problems are polynomial-time solvable for this graph class; in particular, this is the case for the broad class of problems known as Locally Checkable Vertex Subset and Vertex Partitioning problems. For a graph G, let $\omega(G)$ denote the size of a maximum clique in G. Chudnovsky et al. [9] gave for the class of $(K^1_{1,3}, P_6)$-free graphs an $n^{O(\omega(G)^3)}$-time algorithm for Max Partial H-Colouring, a problem equivalent to Independent Set if $H = P_1$ and to Odd Cycle Transversal if $H = P_2$. In other words, Max Partial H-Colouring is polynomial-time solvable for $(K^1_{1,3}, P_6)$-free graphs with bounded clique number. Moreover, they observed that Max Partial H-Colouring is polynomial-time solvable for graph classes whose mim-width is bounded and quickly computable. Hence, Theorem 5 generalizes their result for Max Partial H-Colouring to $(K_{1,s}^1, P_t)$-free graphs with bounded clique number, for any $s \geq 1$ and $t \geq 1$. However, the running time of the corresponding algorithm is worse than $n^{O(\omega(G)^3)}$ (see [9] for details).

It remains to prove Theorem 5, which we do in the next section. In Section 3 we give some directions for future work.

2 The Proof of Theorem 5

We first state two lemmas. The first lemma shows that given a partition of the vertex set of a graph G, we can bound the mim-width of G in terms of the mim-width of the graphs induced by each part and the mim-width between any two of the parts.

▶ **Lemma 7.** *Let G be a graph, and let $(X_1, \ldots, X_p)$ be a partition of $V(G)$ such that $\operatorname{cutmim}_G(X_i, X_j) \leq c$ for all distinct $i, j \in \{1, \ldots, p\}$, and $p \geq 2$. Then*

$$\operatorname{mimw}(G) \leq \max\left\{c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor, \max_{i\in\{1,\ldots,p\}}\{\operatorname{mimw}(G[X_i])\} + c(p-1)\right\}.$$

Moreover, if (T_i, δ_i) is a branch decomposition for $G[X_i]$ for each i, then we can construct, in $O(p)$ time, a branch decomposition (T, δ) for G with

$$\operatorname{mimw}(T,\delta) \leq \max\left\{c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor, \max_{i\in\{1,\ldots,p\}}\{\operatorname{mimw}(T_i,\delta_i)\} + c(p-1)\right\}.$$

Proof. We construct a branch decomposition (T, δ) for G with the desired mim-width as follows. Let T_0 be an arbitrary subcubic tree having p leaves $\ell_1, \ldots, \ell_p$. Fix for each $i \in \{1, \ldots, p\}$ a branch decomposition (T_i, δ_i) for $G[X_i]$. For each $i \in \{1, \ldots, p\}$, we choose an arbitrary leaf vertex v_i of T_i, we identify v_i with ℓ_i calling the resulting vertex ℓ_i, and we create a new pendant edge incident to ℓ_i, where the new leaf vertex adjacent to ℓ_i is called v_i. Then T is a subcubic tree whose set of leaves is the disjoint union of the leaves of T_i for each $i \in \{1, \ldots, p\}$. See Figure 2, for example. For a leaf v of T, we set $\delta(v) = \delta_i(v)$, where v is a leaf of T_i. Now (T, δ) is a branch decomposition for G, and clearly this branch decomposition can be constructed in $O(p)$ time. It remains to prove the upper bound for $\operatorname{mimw}(T, \delta)$.

Consider $e \in E(T)$ and the partition $(A_e, \overline{A_e})$ of $V(G)$. If $e \in E(T_0)$, then $A_e = \bigcup_{j\in J} X_j$ for some $J \subseteq \{1, \ldots, p\}$. If $e \in E(T_i)$ for some $i \in \{1, \ldots, p\}$, then either A_e or $\overline{A_e}$ is properly contained in X_i. The only other possibility is that e is one of the newly created pendant edges, in which case either A_e or $\overline{A_e}$ has size 1.

First suppose $e \in E(T_0)$, so $A_e = \bigcup_{j\in J} X_j$ for some $J \subseteq \{1, \ldots, p\}$. We claim that $\operatorname{cutmim}_G(A_e, \overline{A_e}) \leq c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor$. Let M be a maximum-sized induced matching in $G[A_e, \overline{A_e}]$. Let $K = \{1, \ldots, p\} \setminus J$. For each $j \in J$ and $k \in K$, there are at most c edges of M with one end in X_j and the other end in X_k, since $\operatorname{cutmim}_G(X_j, X_k) \leq c$. Thus $\operatorname{cutmim}_G(A_e, \overline{A_e}) \leq c|J||K|$, where $|J| + |K| = p$. As $c|J||K| \leq c\left\lfloor\left(\frac{p}{2}\right)\right\rfloor\left\lceil\left(\frac{p}{2}\right)\right\rceil = c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor$, the claim follows.

Now suppose $e \in E(T_i)$ for some $i \in \{1, \ldots, p\}$, so, without loss of generality, A_e is properly contained in X_i. We claim that $\operatorname{cutmim}_G(A_e, \overline{A_e}) \leq \operatorname{mimw}(G[X_i]) + c(p-1)$. Consider a maximum-sized induced matching M in $G[A_e, \overline{A_e}]$. As $A_e \subseteq X_i$, all the edges of M have one end in X_i. For each $j \in \{1, \ldots, p\}$ with $j \neq i$, there are at most c edges of M with one end in X_j, since $\operatorname{cutmim}_G(X_i, X_j) \leq c$. Since there are at most $\operatorname{mimw}(G[X_i])$ edges of M with both ends in X_i, we deduce that $\operatorname{cutmim}_G(A_e, \overline{A_e}) \leq \operatorname{mimw}(G[X_i]) + c(p-1)$, as claimed. The lemma follows. ◀

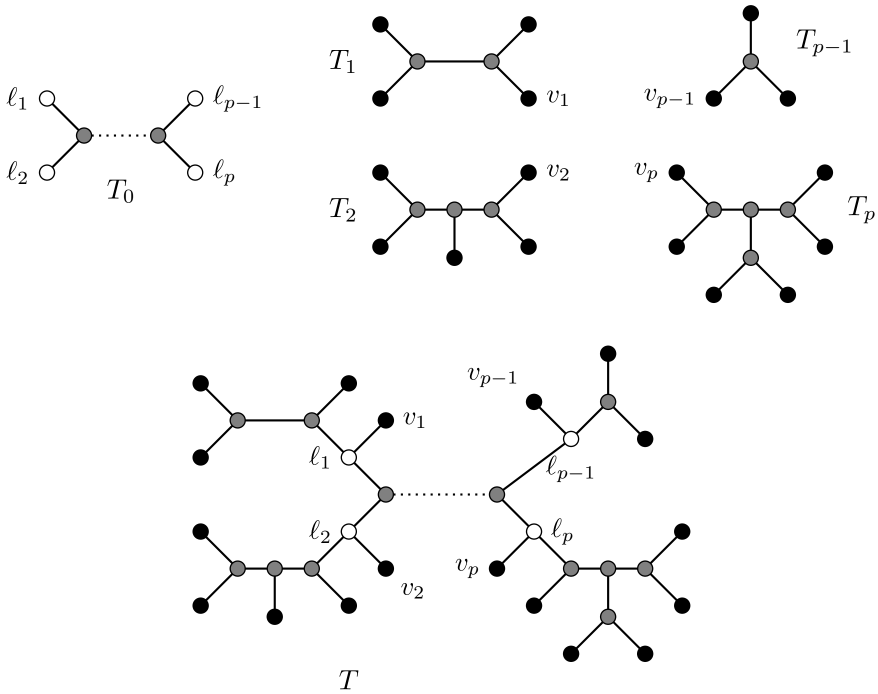

Figure 2 An example of the construction of T in the proof of Lemma 7.

A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* is a set of pairwise non-adjacent vertices. A *dominating set* is a set D of vertices such that every vertex not in D is adjacent to at least one vertex in D. Ramsey's Theorem states that for all positive integers k and ℓ, there exists an integer $R(k,\ell)$ such that every graph on at least $R(k,\ell)$ vertices contains a clique of size k or an independent set of size ℓ. A well-known, rough bound for $R(k,\ell)$ is $R(k,\ell) \leq \binom{k+\ell-2}{k-1} \leq (k+\ell-2)^{k-1}$.

For $r \geq 1$ and $s,t \geq 1$, let $M(r,s,t) = (1+R(r+1,R(r+1,s)))^{t-2}$. The next lemma has been proven by Chudnovsky, Spirkl and Zhong [10] for the case where $r=3$. The proof of the lemma is analogous to the proof in [10] for the case where $r=3$: replace each occurrence of "4" in the proofs of Lemmas 13 and 15 in [10] by "$r+1$".

▶ **Lemma 8** (cf. [10]). *For every $r \geq 1$, $s \geq 1$ and $t \geq 1$, a connected $(K_{r+1}, K^1_{1,s}, P_t)$-free graph contains a dominating set of size at most $M(r,s,t)$.*

We are now ready to prove Theorem 5. We in fact prove the following theorem, Theorem 9, which gives an explicit bound on the mim-width; Theorem 5 then follows from this.

▶ **Theorem 9.** *Let $r \geq 1$, $s \geq 1$ and $t \geq 1$, and let G be a $(K_r, K^1_{1,s}, P_t)$-free graph. Then $\text{mimw}(G) \leq g(r,s,t)$ where $g(r,s,t) = 2(r+s-1)^{2(r+1)^2(t+1)}$, and a branch decomposition (T,δ) of G with $\text{mimw}(T,\delta) \leq g(r,s,t)$ can be found in polynomial time.*

Proof. We may assume without loss of generality that G is connected. We use induction on r. If $r \leq 2$, then G is K_2-free, so $\text{mimw}(T,\delta) = 0$ for any branch decomposition (T,δ) of G, whereas $g(r,s,t)$ is positive for all $s,t \geq 1$; so the theorem holds trivially in this case.

Suppose that $r \geq 3$. By Lemma 8, we find that G has a dominating set D of size at most $M(r-1,s,t)$. Moreover, we can find D in polynomial time by brute force (or we can apply the $O(tn^2)$-time algorithm of [10]). We let $p = |D|$, so $p \leq M(r-1,s,t)$.

Let $f(r,s,t) = (r+s-1)^{2(r+1)^2(t+1)}$. We will show that there is a branch decomposition (T', δ') of $G - D$ with $\text{mimw}(T', \delta') \leq f(r,s,t)$. The theorem will then follow: to see this, observe that if (T', δ') is such a branch decomposition, then we can readily extend (T', δ') to a branch decomposition (T, δ) for G with mim-width at most $f(r,s,t) + p \leq f(r,s,t) + M(r-1,s,t) \leq g(r,s,t)$. Namely, we can obtain T in polynomial time from T' and an arbitrary subcubic tree T'' with $p+2$ leaves by identifying a leaf of T' with a leaf of T''. So it remains to prove that $\text{mimw}(G - D) \leq f(r,s,t)$, and that we can find a branch decomposition witnessing this bound, in polynomial time.

Let $V = V(G)$. We partition V as follows. We first fix an arbitrary ordering $d_1, \ldots, d_p$ on the vertices of D. Let X_1 be the set of vertices in $V \setminus D$ adjacent to d_1. For $i \in \{2, \ldots, p\}$, let X_i be the set of vertices in $V \setminus D$ adjacent to d_i, but non-adjacent to any d_h with $h \leq i-1$. Then $\{D, X_1, \ldots, X_p\}$ is a partition of V (where some of the sets X_i might be empty). Moreover, we found this partition in polynomial time.

By construction, d_i is adjacent to every vertex of X_i for each $i \in \{1, \ldots, p\}$. As G is K_r-free, this implies that each X_i induces a $(K_{r-1}, K_{1,s}^1, P_t)$-free subgraph of G. By the induction hypothesis, $\text{mimw}(G[X_i]) \leq f(r-1,s,t) + M(r-2,s,t)$, and a branch decomposition witnessing this mim-width bound can be computed in polynomial time, for every $i \in \{1, \ldots, p\}$.

Consider two sets X_i and X_j with $i < j$. We claim that $\text{cutmim}_G(X_i, X_j) < c = R(r-1, R(r-1,s))$. Towards a contradiction, suppose that $\text{cutmim}_G(X_i, X_j) \geq c$. Then, by definition, there exist two sets $A = \{a_1, a_2, \ldots, a_c\} \subseteq X_i$ and $B = \{b_1, b_2, \ldots, b_c\} \subseteq X_j$, each of size c, such that $\{a_1b_1, \ldots, a_cb_c\}$ is a set of c edges with the property that G does not contain any edges a_ib_j for $i \neq j$ (note that edges a_ia_j and b_ib_j may exist in G).

As $G[X_i]$ is K_{r-1}-free, and $|A| = c = R(r-1, R(r-1,s))$, Ramsey's Theorem tells us that $G[A]$ contains an independent set A' of size $c' = R(r-1,s)$. Assume without loss of generality that $A' = \{a_1, \ldots, a_{c'}\}$. Let $B' = \{b_1, \ldots, b_{c'}\}$. As $G[X_j]$ is K_{r-1}-free, $G[B']$ contains an independent set B'' of size s. Assume without loss of generality that $B'' = \{b_1, \ldots, b_s\}$. By construction, d_i is adjacent to every vertex of $\{a_1, \ldots, a_s\} \subseteq X_i$ and non-adjacent to every vertex of $\{b_1, \ldots, b_s\} \subseteq X_j$. Hence, $\{a_1, \ldots, a_s, b_1, \ldots, b_s, d_i\}$ induces a $K_{1,s}^1$ in G, a contradiction. We conclude that $\text{cutmim}_G(X_i, X_j) < c$.

Now, by Lemma 7, we have

$$\begin{aligned}
\text{mimw}(G-D) &\leq \max\left\{c\left\lfloor\left(\tfrac{p}{2}\right)^2\right\rfloor, \max_{i\in\{1,\ldots,p\}}\{\text{mimw}(G[X_i])\} + c(p-1)\right\} \\
&\leq \max\left\{cp^2,\ f(r-1,s,t) + M(r-2,s,t) + cp\right\}.
\end{aligned}$$

Recall that $R(k,\ell) \leq (k+\ell-2)^{k-1}$. We observe that $R(k, R(k,\ell)) \leq (k+\ell-2)^{k(k-1)}$. Hence, $c = R(r-1, R(r-1,s)) \leq (r+s-3)^{(r-1)(r-2)}$ and $p \leq M(r-1,s,t) = (1+R(r,R(r,s)))^{t-2} \leq \left(1 + (r+s-1)^{r(r+1)}\right)^{t-2} \leq \left((r+s-1)^{r(r+1)+1}\right)^{t-2}$. Thus

$$\begin{aligned}
cp^2 &\leq (r+s-3)^{(r-1)(r-2)}\left((r+s-1)^{r(r+1)+1}\right)^{2(t-2)} \\
&\leq (r+s-1)^{(r+1)^2}(r+s-1)^{2(r+1)^2(t-2)} \leq (r+s-1)^{2(r+1)^2 t} \leq f(r,s,t), \text{ and}
\end{aligned}$$

$$\begin{aligned}
& f(r-1,s,t) + M(r-2,s,t) + cp \\
\leq\ & (r+s-2)^{2r^2(t+1)} + \left((r+s-2)^{(r-1)r+1}\right)^{t-2} + (r+s-3)^{(r-1)(r-2)}\left((r+s-1)^{r(r+1)+1}\right)^{t-2} \\
\leq\ & (r+s-1)^{r^2(t+1)}\left((r+s-1)^{r^2(t+1)} + 1 + (r+s-1)^{(r+1)(t-1)}\right) \\
\leq\ & (r+s-1)^{r^2(t+1)}\left((r+s-1)^{r^2(t+1)+1}\right) \\
=\ & (r+s-1)^{2r^2(t+1)+1} \\
\leq\ & f(r,s,t).
\end{aligned}$$

So $\text{mimw}(G-D) \leq f(r,s,t)$ and the theorem follows by induction. ◀

3 Conclusions

We proved in Corollary 6 that for every $k \geq 1$, $s \geq 1$ and $t \geq 1$, List k-Colouring is polynomial-time solvable for $(K^1_{1,s}, P_t)$-free graphs by showing that the mim-width of these graphs is bounded and quickly computable. Huang [21] proved that 4-Colouring is NP-complete for P_7-free graphs and that 5-Colouring is NP-complete for P_6-free graphs. It is also known that List 4-Colouring is NP-complete for P_6-free graphs [16]. However, the List 3-Colouring problem is polynomial-time solvable for P_7-free graphs [4] and the computational complexities of 3-Colouring and List 3-Colouring are open for P_t-free graphs if $t \geq 8$. In particular, we do not know any integer t such that 3-Colouring or List 3-Colouring are NP-complete for P_t-free graphs. Recently, Pilipczuk, Pilipczuk and Rzążewski [35] gave, for every $t \geq 3$, a quasi-polynomial-time algorithm for 3-Colouring on the class of $\{C_{t+1}, C_{t+2}, \ldots\}$-free graphs; note that this class contains, for $t \geq 2$, the class of P_t-free graphs as a subclass. Hence, an extension of Corollary 6, which will require more research into the structure of P_t-free graphs, might still be possible for $k = 3$. We leave this for future work.

References

1 Rémy Belmonte and Martin Vatshelle. Graph classes with structured neighborhoods and algorithmic applications. *Theoretical Computer Science*, 511:54–65, 2013.

2 Benjamin Bergougnoux and Mamadou Moustapha Kanté. More applications of the d-neighbor equivalence: Connectivity and acyclicity constraints. *Proc. ESA 2019*, LIPIcs, 144:17:1–17:14, 2019.

3 Benjamin Bergougnoux, Charis Papadopoulos, and Jan Arne Telle. Node multiway cut and subset feedback vertex set on graphs of bounded mim-width. *Proc. WG 2020*, LNCS, 12301:388–400, 2020.

4 Flavia Bonomo, Maria Chudnovsky, Peter Maceli, Oliver Schaudt, Maya Stein, Mingxian Zhong. Three-coloring and List Three-Coloring of graphs without induced paths on seven vertices. *Combinatorica*, 38:779–801, 2018.

5 Nick Brettell, Jake Horsfield, Andrea Munaro, Giacomo Paesani, and Daniël Paulusma. Bounding the mim-width of hereditary graph classes. *Proc. IPEC 2020*, LIPIcs, 180:6:1–6:18, 2020.

6 Nick Brettell, Andrea Munaro and Daniël Paulusma. Solving problems on generalized convex graphs via mim-width. *CoRR*, arXiv:2008.09004, 2020.

7 Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theoretical Computer Science*, 511:66–76, 2013.

8 Maria Chudnovsky, Shenwei Huang, Sophie Spirkl, and Mingxian Zhong. List-three-coloring graphs with no induced $P_6 + rP_3$. *CoRR*, arXiv:1806.11196, 2018.

9 Maria Chudnovsky, Jason King, Michał Pilipczuk, Paweł Rzążewski, and Sophie Spirkl. Finding large H-colorable subgraphs in hereditary graph classes, *Proc. ESA 2020*, LIPIcs, 173:35:1–35:17, 2020.

10 Maria Chudnovsky, Sophie Spirkl, and Mingxian Zhong. List 3-coloring P_t-free graphs with no induced 1-subdivision of $K_{1,s}$. *Discrete Mathematics*, 343(11):112086, 2020.

11 Jean-François Couturier, Petr A. Golovach, Dieter Kratsch, and Daniël Paulusma. List coloring in the absence of a linear forest. *Algorithmica*, 71:21–35, 2015.

12 Konrad K. Dabrowski, Matthew Johnson and Daniël Paulusma. Clique-width for hereditary graph classes. *London Mathematical Society Lecture Note Series.* 456:1-56, 2019.

13 Esther Galby, Andrea Munaro, and Bernard Ries. Semitotal Domination: New hardness results and a polynomial-time algorithm for graphs of bounded mim-width. *Theoretical Computer Science*, 814:28–48, 2020.

14 M. R. Garey, David S. Johnson, G. L. Miller, and Christos H. Papadimitriou. The complexity of coloring circular arcs and chords. *SIAM Journal on Matrix Analysis and Applications*, 1(2):216–227, 1980.

15 Petr A. Golovach, Matthew Johnson, Daniël Paulusma, and Jian Song. A survey on the computational complexity of colouring graphs with forbidden subgraphs. *Journal of Graph Theory*, 84:331–363, 2017.

16 Petr A. Golovach, Daniël Paulusma, and Jian Song. Closing complexity gaps for coloring problems on H-free graphs. *Information and Computation* 237:204–214, 2014.

17 Petr A. Golovach, Daniël Paulusma and Jian Song. Coloring graphs without short cycles and long induced paths. *Discrete Applied Mathematics* 167:107–120, 2014.

18 Frank Gurski. The behavior of clique-width under graph operations and graph transformations. *Theory of Computing Systems*, 60:346–376, 2017.

19 Petr Hliněný, Sang-il Oum, Detlef Seese, and Georg Gottlob. Width parameters beyond tree-width and their applications. *The Computer Journal*, 51:326–362, 2008.

20 Chính T. Hoàng, Marcin Kamiński, Vadim V. Lozin, Joe Sawada, and Xiao Shu. Deciding k-Colorability of P_5-free graphs in polynomial time. *Algorithmica*, 57:74–81, 2010.

21 Shenwei Huang. Improved complexity results on k-coloring P_t-free graphs. *European Journal of Combinatorics*, 51 336-346, 2016.

22 Lars Jaffke, O-joung Kwon, Torstein J. F. Strømme, and Jan Arne Telle. Mim-width III. Graph powers and generalized distance domination problems. *Theoretical Computer Science*, 796:216–236, 2019.

23 Lars Jaffke, O-joung Kwon, and Jan Arne Telle. Mim-width I. Induced path problems. *Discrete Applied Mathematics*, 278:153–168, 2020.

24 Lars Jaffke, O-joung Kwon, and Jan Arne Telle. Mim-width II. The feedback vertex set problem. *Algorithmica*, 82:118–145, 2020.

25 Dong Yeap Kang, O-joung Kwon, Torstein J. F. Strømme, and Jan Arne Telle. A width parameter useful for chordal and co-comparability graphs. *Theoretical Computer Science*, 704:1–17, 2017.

26 Tereza Klimošová, Josef Malík, Tomáš Masařík, Jana Novotná, Daniël Paulusma, and Veronika Slívová. Colouring $(P_r + P_s)$-free graphs. *Algorithmica*, 82:1833–1858, 2020.

27 Daniel Kobler and Udi Rotics. Edge dominating set and colorings on graphs with fixed clique-width. *Discrete Applied Mathematics*, 126:197–221, 2003.

28 Daniel Král', Jan Kratochvíl, Zsolt Tuza, and Gerhard J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. *Proc. WG 2001*, LNCS, 2204:254–262, 2001.

29 O. Kwon. Personal communication, 2020.

30 László Lovász. Coverings and coloring of hypergraphs. *Congressus Numerantium*, VIII:3–12, 1973.

31 Daniel Lokshtanov, Dániel Marx, and Saket Saurabh. Known algorithms on graphs of bounded treewidth are probably optimal. *ACM Transactions on Algorithms* 14:13:1–13:30, 2018.

32 Marcin Kamiński, Vadim V. Lozin, and Martin Milanič. Recent developments on graphs of bounded clique-width. *Discrete Applied Mathematics* 157:2747–2761, 2009.

33 Sang-il Oum. Rank-width: Algorithmic and structural results. *Discrete Applied Mathematics* 231:15–24, 2017

34 Sang-il Oum and Paul Seymour, Approximating clique-width and branch-width. *Journal of Combinatorial Theory, Series B*, 96:514–528, 2006.

35 Marcin Pilipczuk, Michał Pilipczuk and Paweł Rzążewski. Quasi-polynomial-time algorithm for independent set in P_t-free and $C_{\geq t}$-free graphs via shrinking the space of connecting subgraphs. *CoRR*, arXiv:2009.13494, 2020.

36 Sigve Hortemo Sæther and Martin Vatshelle. Hardness of computing width parameters based on branch decompositions over the vertex set. *Theoretical Computer Science*, 615:120–125, 2016.

37 Martin Vatshelle. *New Width Parameters of Graphs.* PhD thesis, University of Bergen, 2012.