Filling the Complexity Gaps for Colouring Planar and Bounded Degree Graphs

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Abstract. We consider a natural restriction of the List Colouring problem, k-Regular List Colouring, which corresponds to the List Colouring problem where every list has size exactly k. We give a complete classification of the complexity of k-Regular List Colouring restricted to planar graphs, planar bipartite graphs, planar triangle-free graphs and to planar graphs with no 4-cycles and no 5-cycles. We also give a complete classification of the complexity of this problem and a number of related colouring problems for graphs with bounded maximum degree.

Keywords: list colouring, choosability, planar graphs, maximum degree

1 Introduction

A *colouring* of a graph is a labelling of the vertices so that adjacent vertices do not have the same label. We call these labels *colours*. Graph colouring problems are central to the study of combinatorial algorithms and they have many theoretical and practical applications. A typical problem asks whether a colouring exists under certain constraints, or how difficult it is to find such a colouring. For example, in the List Colouring problem, a graph is given where each vertex has a list of colours and one wants to know if the vertices can be coloured using only colours in their lists. The Choosability problem asks whether such list colourings are guaranteed to exist whenever all the lists have a certain size. In fact, an enormous variety of colouring problems can be defined and there is now a vast literature on this subject. For longer introductions to the type of problems we consider we refer to two recent surveys [9,15].

In this paper, we are concerned with the computational complexity of colouring problems. For many such problems, the complexity is well understood in the case where we allow every graph as input, so it is natural to consider problems

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with restricted inputs. We consider a variant of the LIST COLOURING problem, closely related to CHOOSABILITY, and give a complete classification of its complexity for planar graphs and a number of subclasses of planar graphs by combining known results with new results. Some of the known results are for (planar) graphs with bounded degree. We use these results to fill some more complexity gaps by giving a complete complexity classification of a number of colouring problems for graphs with bounded maximum degree.

1.1 Terminology

A *colouring* of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We say that $c(u)$ is the *colour* of u. For a positive integer k, if $1 \leq c(u) \leq k$ for all $u \in V$, then c is a *k-colouring* of G. We say that G is *k-colourable* if a k-colouring of G exists. The COLOURING problem is to decide whether a graph G is k-colourable for some given integer k. If k is fixed, that is, not part of the input, we obtain the k-COLOURING problem.

A *list assignment* of a graph $G = (V, E)$ is a function L with domain V such that for each vertex $u \in V$, $L(u)$ is a subset of $\{1, 2, \ldots\}$. This set is called the *list* of *admissible* colours for u. If $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$, then L is a *k-list assignment*. The *size* of a list assignment L is the maximum list size $|L(u)|$ over all vertices $u \in V$. A colouring c *respects* L if $c(u) \in L(u)$ for all $u \in V$. Given a graph G with a k-list assignment L, the LIST COLOURING problem is to decide whether G has a colouring that respects L. If k is fixed, then we have the LIST k-COLOURING problem. Fixing the size of L to be at most ℓ gives the ℓ-LIST COLOURING problem. We say that a list assignment L of a graph $G = (V, E)$ is *ℓ-regular* if, for all $u \in V$, $L(u)$ contains exactly ℓ colours. This gives us the following problem, which is one focus of this paper. It is defined for each integer $\ell \geq 1$ (note that ℓ is *fixed*; that is, ℓ is not part of the input).

ℓ-REGULAR LIST COLOURING
Instance: a graph G with an ℓ-regular list assignment L.
Question: does G have a colouring that respects L?

A *k-precolouring* of a graph $G = (V, E)$ is a function $c_W : W \to \{1, 2, \ldots, k\}$ for some subset $W \subseteq V$. A k-colouring c of G is an *extension* of a k-precolouring c_W of G if $c(v) = c_W(v)$ for each $v \in W$. Given a graph G with a precolouring c_W, the PRECOLOURING EXTENSION problem is to decide whether G has a k-colouring that extends c_W. If k is fixed, we obtain the k-PRECOLOURING EXTENSION problem.

The relationships amongst the problems introduced are shown in Fig. 1.

For an integer $\ell \geq 1$, a graph $G = (V, E)$ is *ℓ-choosable* if, for every ℓ-regular list assignment L of G, there exists a colouring that respects L. The corresponding decision problem is the CHOOSABILITY problem. If ℓ is fixed, we obtain the ℓ-CHOOSABILITY problem.

We emphasize that ℓ-REGULAR LIST COLOURING and ℓ-CHOOSABILITY are two fundamentally different problems. For the former we must decide whether there exists a colouring that respects a *particular* ℓ-regular list assignment. For

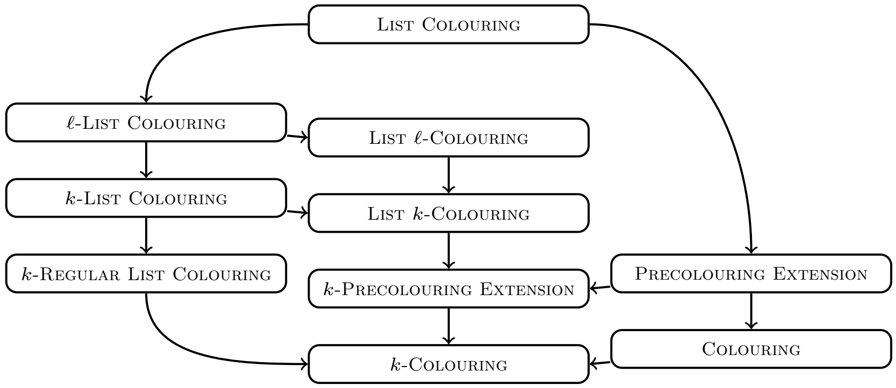

Fig. 1. Relationships between Colouring and its variants. An arrow from one problem to another indicates that the latter is (equivalent to) a special case of the former; k and ℓ are any two arbitrary integers for which $\ell \geq k$. For instance, k-Colouring is a special case of k-Regular List Colouring. This can be seen by giving the list $L(u) = \{1, \ldots, k\}$ to each vertex u in an instance graph of Colouring. We also observe that ℓ-Regular List Colouring and k-Regular List Colouring are not comparable for any $k \neq \ell$.

the latter we must decide whether or not *every* ℓ-regular list assignment has a colouring that respects it. As we will see later, this difference also becomes clear from a complexity point of view: for some graph classes ℓ-Regular List Colouring is computationally easier than ℓ-Choosability, whereas, perhaps more surprisingly, for other graph classes, the reverse holds.

For two vertex-disjoint graphs G and H and positive integer k, we let $G+H$ denote the disjoint union $(V(G) \cup V(H), E(G) \cup E(H))$, and kG denote the disjoint union of k copies of G. If G is a graph containing an edge e or a vertex v then $G-e$ and $G-v$ denote the graphs obtained from G by deleting e or v, respectively. If G' is a subgraph of G then $G-G'$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(G')$. We let C_n, K_n and P_n denote the cycle, complete graph and path on n vertices, respectively. A wheel is a cycle with an extra vertex added that is adjacent to all other vertices. The wheel on n vertices is denoted W_n; note that $W_4 = K_4$. A graph on at least three vertices is 2-*connected* if it is connected and there is no vertex whose removal disconnects it. A *block* of a graph is a maximal subgraph that is connected and cannot be disconnected by the removal of one vertex (so a block is either 2-connected, a K_2 or an isolated vertex). A *block graph* is a connected graph in which every block is a complete graph. A *Gallai tree* is a connected graph in which every block is a complete graph or a cycle. We say that B is a *leaf-block* of a connected graph G if B contains exactly one cut vertex u of G and $B \setminus u$ is a component of $G-u$. For a set of graphs $\mathcal{H}$, a graph G is $\mathcal{H}$-*free* if G contains no induced subgraph isomorphic to a graph in $\mathcal{H}$, whereas G is $\mathcal{H}$-*subgraph-free* if it contains no subgraph isomorphic to a graph in $\mathcal{H}$. The *girth* of a graph is the length of its shortest cycle.

1.2 Known Results for Planar Graphs

We start with a classical result observed by Erdős et al. [12] and Vizing [26].

Theorem 1 ([12,26]). 2-List Colouring *is polynomial-time solvable.*

Garey et al. proved the following result, which is in contrast to the fact that every planar graph is 4-colourable by the Four Colour Theorem [2].

Theorem 2 ([13]). 3-Colouring *is* NP*-complete for planar graphs of maximum degree* 4*.*

Next we present results found by several authors on the existence of k-choosable graphs for various graph classes.

Theorem 3. *The following statements hold for k-choosability:*

(i) *Every planar graph is* 5*-choosable* [24]*.*
(ii) *There exists a planar graph that is not* 4*-choosable* [28]*.*
(iii) *Every planar triangle-free graph is* 4*-choosable* [19]*.*
(iv) *Every planar graph with no* 4*-cycles is* 4*-choosable* [20]*.*
(v) *There exists a planar triangle-free graph that is not* 3*-choosable* [29]*.*
(vi) *There exists a planar graph with no* 4*-cycles, no* 5*-cycles and no intersecting triangles that is not* 3*-choosable* [23]*.*
(vii) *Every planar bipartite graph is* 3*-choosable* [1]*.*

We note that smaller examples of graphs than were used in the original proofs have been found for Theorems 3.(ii) [17], 3.(v) [22] and 3.(vi) [33] and that Theorem 3.(vi) strengthens a result of Voigt [30]. We recall that Thomassen [25] first showed that every planar graph of girth at least 5 is 3-choosable, and that a number of results have since been obtained on 3-choosability of planar graphs in which certain cycles are forbidden; see, for example, [7,10,31,32].

We will also use the following result of Chlebík and Chlebíková.

Theorem 4 ([8]). List Colouring *is* NP*-complete for* 3*-regular planar bipartite graphs that have a list assignment in which each list is one of* $\{1, 2\}$*,* $\{1, 3\}$*,* $\{2, 3\}$*,* $\{1, 2, 3\}$ *and all the neighbours of each vertex with three colours in its list have two colours in their lists.*

1.3 New Results for Planar Graphs

Theorems 1–3 have a number of immediate consequences for the complexity of ℓ-Regular List Colouring when restricted to planar graphs. For instance, Theorem 2 implies that 3-Regular List Colouring is NP-complete for planar graphs, whereas Theorem 3.(i) shows that 5-Regular List Colouring is polynomial-time solvable on this graph class. As such, it is a natural question to determine the complexity for the missing case $\ell = 4$. In this section we settle this missing case and also present a number of new hardness results for ℓ-Regular List Colouring restricted to various subclasses of planar graphs. At the end of this section we show how to combine the known results with our new ones to obtain a number of dichotomies (Corollaries 3–6). We deduce some of our new results from two more general theorems, namely Theorems 5 and 6, which we state below; see Section 2 for a proof of Theorem 5 (we omitted the proof of Theorem 6).

Theorem 5. *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then 4-*Regular List Colouring *is* NP*-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 4-choosable.*

Note that the class of $\mathcal{H}$-subgraph-free graphs is contained in the class of $\mathcal{H}$-free graphs. Hence, whenever a problem is NP-complete for $\mathcal{H}$-subgraph-free graphs, it is also NP-complete for $\mathcal{H}$-free graphs.

Combining Theorem 5 with Theorem 3.(ii) yields the following result which, as we will see later, was the only case for which the complexity of k-Regular List Colouring for planar graphs was not settled.

Corollary 1. 4-Regular List Colouring *is* NP*-complete for planar graphs.*

Theorem 5 has more applications. For instance, consider the non-4-choosable planar graph H from the proof of Theorem 1.7 in [17]. It can be observed that H is W_p-subgraph-free for all $p \geq 8$. Wheels are 2-connected and planar. Hence if $\mathcal{H}$ is any finite set of wheels on at least eight vertices then 4-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs.

Our basic idea for proving Theorem 5 is to pick a minimal counterexample H with list assignment L (which must exist due to Theorem 3.(ii)). We select an "appropriate" edge $e = uv$ and consider the graph $F' = F - e$. We reduce from an appropriate colouring problem restricted to planar graphs and use copies of F' as a gadget to ensure that we can enforce a regular list assignment. The proof of the next theorem also uses this idea.

Theorem 6. *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then 3-*Regular List Colouring *is* NP*-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 3-choosable.*

Theorem 6 has a number of applications. For instance, if we let $\mathcal{H} = \{K_3\}$ then Theorem 6, combined with Theorem 3.(v), leads to the following result.

Corollary 2. 3-Regular List Colouring *is* NP*-complete for planar triangle-free graphs.*

Theorem 6 can also be used for other classes of graphs. For example, let $\mathcal{H}$ be a finite set of graphs, each of which includes a 2-connected graph on at least five vertices as a subgraph. Let $\mathcal{I}$ be the set of these 2-connected graphs. The graph K_4 is a planar $\mathcal{I}$-subgraph-free graph that is not 3-choosable (since it is not 3-colourable). Therefore, Theorem 6 implies that 3-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs. We can obtain more hardness results by taking some other planar graph that is not 3-choosable, such as a wheel on an even number of vertices. Also, if we let $\mathcal{H} = \{C_4, C_5\}$ we can use Theorem 6 by combining it with Theorem 3.(vi) to find that 3-Regular List Colouring is NP-complete for planar graphs with no 4-cycles and no 5-cycles. We strengthen this result as follows (proof omitted).

Theorem 7. 3-Regular List Colouring *is* NP*-complete for planar graphs with no* 4*-cycles, no* 5*-cycles and no intersecting triangles.*

Corollaries 1 and 2 and Theorem 7 can be seen as strengthenings of Theorems 3.(ii), 3.(v) and 3.(vi), respectively. Moreover, they complement Theorem 2, which implies that 3-List Colouring is NP-complete for planar graphs, and a result of Kratochvíl [18] that, for planar bipartite graphs, 3-Precolouring Extension is NP-complete. Corollaries 1 and 2 also complement results of Gutner [17] who showed that 3-Choosability and 4-Choosability are Π_2^p-complete for planar triangle-free graphs and planar graphs, respectively. However, we emphasize that, for special graph classes, it is not necessarily the case that ℓ-Choosability is computationally harder than ℓ-Regular List Colouring. For instance, contrast the fact that Choosability is polynomial-time solvable on $3P_1$-free graphs [14] with our next result (proof omitted).

Theorem 8. 3-Regular List Colouring *is* NP*-complete for* $(3P_1, P_1 + P_2)$*-free graphs.*

Our new results, combined with known results, close a number of complexity gaps for the ℓ-Regular List Colouring problem. Combining Corollary 1 with Theorems 1, 2 and 3.(i) gives us Corollary 3. Combining Theorem 7 with Theorems 1 and 3.(iv) gives us Corollary 4. Combining Corollary 2 with Theorems 1 and 3.(iii) gives us Corollary 5, whereas Theorems 1 and 3.(vii) imply Corollary 6.

Corollary 3. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar graphs, is* NP*-complete if* $\ell \in \{3, 4\}$ *and polynomial-time solvable otherwise.*

Corollary 4. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar graphs with no* 4*-cycles and no* 5*-cycles and no intersecting triangles, is* NP*-complete if* $\ell = 3$ *and polynomial-time solvable otherwise (even if we allow intersecting triangles and* 5*-cycles).*

Corollary 5. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar triangle-free graphs, is* NP*-complete if* $\ell = 3$ *and polynomial-time solvable otherwise.*

Corollary 6. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar bipartite graphs, is polynomial-time solvable.*

1.4 Known Results for Bounded Degree Graphs

First we present a result of Kratochvíl and Tuza [19].

Theorem 9 ([19]). List Colouring *is polynomial-time solvable on graphs of maximum degree at most* 2.

Brooks' Theorem [6] states that every graph G with maximum degree d has a d-colouring unless G is a complete graph or a cycle with an odd number of vertices. The next result of Vizing [27] generalizes Brooks' Theorem to list colourings.

Theorem 10 ([27]). *Let d be a positive integer. Let $G = (V, E)$ be a connected graph of maximum degree at most d and let L be a d-regular list assignment for G. If G is not a cycle or a complete graph then G has a colouring that respects L.*

And we need another result of Chlebík and Chlebíková [8].

Theorem 11 ([8]). Precolouring Extension *is polynomial-time solvable on graphs of maximum degree* 3.

1.5 New Results for Bounded Degree Graphs

The following result is obtained by making a connection to Gallai trees (proof omitted).

Theorem 12. *Let k be a positive integer. Then k-*Precolouring Extension *is polynomial-time solvable for graphs of maximum degree at most k.*

We have the following two classifications. The first one is an observation obtained by combining only previously known results, whereas the second one also makes use of our new result.

Corollary 7. *Let d be a positive integer. The following two statements hold for graphs of maximum degree at most d.*

(i) List Colouring *is* NP*-complete if $d \geq 3$ and polynomial-time solvable if $d \leq 2$.*
(ii) Precolouring Extension *and* Colouring *are* NP*-complete if $d \geq 4$ and polynomial-time solvable if $d \leq 3$.*

Proof. We first consider (i). If $d \geq 3$, we use Theorem 4. If $d \leq 2$, we use Theorem 9. We now consider (ii). If $d \geq 4$, we use Theorem 2. If $d \leq 3$, we use Theorem 11. □

Corollary 8. *Let d and k be two positive integers. The following two statements hold for graphs of maximum degree at most d.*

(i) k-List Colouring *and* List k-Colouring *are* NP*-complete if* $k \geq 3$ *and* $d \geq 3$ *and polynomial-time solvable otherwise.*

(ii) k-Regular List Colouring *and* k-Precolouring Extension *are* NP*-complete if* $k \geq 3$ *and* $d \geq k+1$ *and polynomial-time solvable otherwise.*

Proof. We first consider (i). If $k \geq 3$ and $d \geq 3$, we use Theorem 4. If $k \leq 2$ or $d \leq 2$, we use Theorems 1 or 9, respectively.

We now consider (ii). We start with the hardness cases and so let $k \geq 3$ and $d \geq k+1$.

First consider k-Precolouring Extension. Theorem 2 implies that 3-Colouring is NP-complete for graphs of maximum degree at most d for all $d \geq 4$. The $k = 3$ case follows immediately from this result. Suppose $k \geq 4$ and $d \geq k+1$. Consider a graph G of maximum degree 4. For each vertex v, we add $k-3$ new vertices $x_1^v, \ldots, x_{k-3}^v$ and edges $vx_1^v, \ldots, vx_{k-3}^v$. Let G' be the resulting graph. Note that G' has maximum degree at most $4 + k - 3 = k + 1 \leq d$. We define a precolouring c on the newly added vertices by assigning colour $i+3$ to each x_i^v. Then G' has a k-colouring extending c if and only if G has a 3-colouring.

Now consider k-Regular List Colouring. The $k = 3$ case follows immediately from Theorem 2. Suppose $k \geq 4$ and $d \geq k+1$. Consider a graph G of maximum degree 4. We define the list $L(v) = \{1, \ldots, k\}$ for each vertex $v \in V(G)$. For each vertex v, we add $k-3$ new vertices $x_1^v, \ldots, x_{k-3}^v$ and edges $vx_1^v, \ldots, vx_{k-3}^v$. We define the list $L(x_i^v) = \{i, k+1, k+2, \ldots, 2k-1\}$ for each x_i^v. For each vertex x_i^v, we also add k new vertices $w_1(x_i^v), \ldots, w_k(x_i^v)$ and edges such that $x_i^v, w_1(x_i^v), \ldots, w_k(x_i^v)$ form a clique (on $k+1$ vertices). We define the list $L(w_j(x_i^v)) = \{k+1, \ldots, 2k\}$ for each $w_j(x_i^v)$. Let G' be the resulting graph. Note that G' has maximum degree at most $k+1$ and that the resulting list assignment L is a k-regular list assignment of G'. Then G' has a k-colouring respecting L if and only if G has a 3-colouring.

We continue with the polynomial-time solvable cases. If $k \leq 2$, the result follows from Theorem 1. Suppose that $k \geq 3$ and $d \leq k$. Then the result for k-Regular List Colouring follows from Theorems 9 and 10 and the result for k-Precolouring Extension follows from Theorem 12. $\square$

Note that Corollary 8 does not contain a dichotomy for k-Colouring restricted to graphs of maximum degree at most d. A full classification of this problem is open, but a number of results are known. Molloy and Reed [21] classified the complexity for all pairs (k, d) for sufficiently large d. Emden-Weinert et al. [11] proved that k-Colouring is NP-complete for graphs of maximum degree at most $k + \lceil \sqrt{k} \rceil - 1$.

2 The Proof of Theorem 5

We need an additional result (proof omitted).

Theorem 13. *For every integer* $p \geq 3$*,* 3-List Colouring *is* NP*-complete for planar graphs of girth at least* p *that have a list assignment in which each list is one of* $\{1,2\}$*,* $\{1,3\}$*,* $\{2,3\}$*,* $\{1,2,3\}$*.*

We are now ready to prove Theorem 5, which we restate below.

Theorem 5 (restated). *Let $\mathcal{H}$ be a finite set of* 2*-connected planar graphs. Then* 4-Regular List Colouring *is* NP*-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not* 4*-choosable.*

Proof. The problem is readily seen to be in NP. Let F be a planar $\mathcal{H}$-subgraph-free graph with a 4-regular list assignment L such that F has no colouring respecting L. We may assume that F is minimal (with respect to the subgraph relation). In particular, this means that F is connected. Let r be the length of a longest cycle in any graph of $\mathcal{H}$. We reduce from the problem of 3-List Colouring restricted to planar graphs of girth at least $r+1$ in which each vertex has list $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ or $\{1,2,3\}$. This problem is NP-complete by Theorem 13. Let a graph G and list assignment L_G be an instance of this problem. We will construct a planar $\mathcal{H}$-subgraph-free graph G' with a 4-regular list assignment L' such that G has a colouring that respects L_G if and only if G' has a colouring that respects L'.

If every pair of adjacent vertices in F has the same list, then the problem of finding a colouring that respects L is just the problem of finding a 4-colouring which, by the Four Colour Theorem [2], we know is possible. Thus we may assume that, on the contrary, there is an edge $e = uv$ such that $|L(u) \cap L(v)| \leq 3$. Let $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, and moreover, for any colouring of F' that respects L, u and v are coloured alike (otherwise we would have a colouring of F that respects L). Let T be the set of possible colours that can be used on u and v in colourings of F' that respect L and let $t = |T|$. As $T \subseteq L(u) \cap L(v)$, we have $1 \leq t \leq 3$. Up to renaming the colours in L, we can build copies of F' with 4-regular list assignments such that

(i) the set T is any given list of colours of size t, and
(ii) the vertex corresponding to u has any given list of 4 colours containing T.

We will implicitly make use of this several times in the remainder of the proof.

We say that a vertex w in G is a *bivertex* or *trivertex* if $|L_G(w)|$ is 2 or 3, respectively. We construct a planar $\mathcal{H}$-subgraph-free graph G' and 4-regular list assignment L' as follows.

First suppose that $t = 1$. For each bivertex w in G, we do as follows. We add two copies of F' to G, which we label $F_1(w)$ and $F_2(w)$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w for $i \in \{1,2\}$ and we set $U^w = \{u_1^w, u_2^w\}$. We add the edges wu_1^w and wu_2^w. We give list assignments to the vertices of $F_1(w)$ and $F_2(w)$ such that $T = \{4\}$ for F_1 and $T = \{5\}$ for F_2. We let $L'(w) = L_G(w) \cup \{4,5\}$. For each trivertex w in G, we do as follows. We add one copy of F' to G, which we label $F_1(w)$. The vertex in $F_1(w)$ corresponding to u is labelled u_1^w and we set $U^w = \{u_1^w\}$. We add the edge wu_1^w. We give list assignments to vertices of $F_1(w)$ such that $T = \{4\}$ for F_1. We let $L'(w) = L_G(w) \cup \{4\}$. This completes the construction of G' and L' when $t = 1$.

Now suppose that $t = 2$. Let $s = r$ if r is even and $s = r + 1$ if r is odd (so s is even in both cases). For each bivertex w in G, we do as follows. We add a copy of F' to G, which we label $F_1(w)$, and identify the vertex in $F_1(w)$ corresponding to u with w. We give list assignments to vertices of $F_1(w)$ such that $T = L_G(w)$ and $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add s copies of F' to G which we label $F_i(w)$, $1 \leq i \leq s$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w. Let $U^w = \{u_i^w \mid 1 \leq i \leq s\}$. Add edges such that the union of w and U^w induces a cycle on $s+1$ vertices. For all $1 \leq i \leq s$, we give list assignments to vertices of $F_i(w)$ such that $T = \{4, 5\}$. We let $L'(w) = \{1, 2, 3, 4\}$. This completes the construction of G' and L' when $t = 2$.

Now suppose that $t = 3$. For each bivertex w in G, we do as follows. We add two copies of F' to G which we label $F_1(w)$ and $F_2(w)$, such that for $i \in \{1, 2\}$, the vertex in $F_i(w)$ corresponding to u is identified with w. We give list assignments to vertices of $F_1(w)$ and $F_2(w)$ such that $T = L_G(w) \cup \{4\}$ for $F_1(w)$, $T = L_G(w) \cup \{5\}$ for $F_2(w)$ and $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add a copy of F' to G which we label $F_1(w)$, such that the vertex in $F_1(w)$ corresponding to u is identified with w. We give list assignments to the vertices of $F_1(w)$ such that $T = \{1, 2, 3\}$ and $L'(w) = \{1, 2, 3, 4\}$. This completes the construction of G' and L' when $t = 3$.

Note that G' is planar. Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' is also $\mathcal{H}$-subgraph-free. Therefore for all w, H is not fully contained in any $F_i(w)$. Since H is 2-connected and since for all w only one vertex of any $F_i(w)$ has a neighbour outside of $F_i(w)$, we find that H has at most one vertex in each $F_i(w)$. In particular, H cannot contain any vertex of any $F_i(w)$ in which the vertex corresponding to u has been attached to w (as opposed to being identified with w); this includes the case when the union of w and U^w induces a cycle on $s + 1$ vertices. Hence we have found that H is a subgraph of G, which contradicts the fact that G has girth at least $r + 1$. Therefore G' is $\mathcal{H}$-subgraph-free.

Note that in any colouring of G' that respects L', each copy of F' must be coloured such that the vertices corresponding to u and v have the same colour, which must be one of the colours from the corresponding set T. If $t = 1$ and w is a trivertex, this means that the unique neighbour of w in U^w must be coloured with colour 4, so w cannot be coloured with colour 4. Similarly, if $t = 1$ and w is a bivertex or $t = 2$ and w is a trivertex then the two neighbours of w in U^w must be coloured with colours 4 and 5, so w cannot be coloured with colours 4 or 5. If $t = 2$ and w is a bivertex or $t = 3$ and w is a trivertex then w belongs to a copy of F' with $T = L_G(w)$, so w cannot have colour 4 or 5. If $t = 3$ and w is a bivertex then w belongs to two copies of F', one with $T = L_G(w) \cup \{4\}$ and one with $T = L_G(w) \cup \{5\}$. Therefore, w must be coloured with a colour from the intersection of these two sets, that is it must be coloured with a colour from $L_G(w)$. Therefore none of the vertices of G can be coloured 4 or 5. Thus the problem of finding a colouring of G' that respects L' is equivalent to the problem of finding a colouring of G that respects L_G. This completes the proof. $\square$

3 Conclusions

As well as filling the complexity gaps of a number of colouring problems for graphs with bounded maximum degree, we have given several dichotomies for the k-Regular List Colouring problem restricted to subclasses of planar graphs. In particular we showed NP-hardness of the cases $k = 3$ and $k = 4$ restricted to planar $\mathcal{H}$-subgraph-free graphs for several sets $\mathcal{H}$ of 2-connected planar graphs. Our method implies that for such sets $\mathcal{H}$ it suffices to find a counterexample to 3-choosability or to 4-choosability, respectively. It is natural to ask whether we can determine the complexity of 3-Regular List Colouring and 4-Regular List Colouring for any class of planar $\mathcal{H}$-subgraph-free graphs. However, we point out that even when restricting $\mathcal{H}$ to be a finite set of 2-connected planar graphs, this would be very hard (and beyond the scope of this paper) as it would require solving several long-standing conjectures in the literature. For example, when $\mathcal{H} = \{C_4, C_5, C_6\}$, Montassier [22] conjectured that every planar $\mathcal{H}$-subgraph-free graph is 3-choosable.

A drawback of our method is that we need the set of graphs $\mathcal{H}$ to be 2-connected. If we forbid a set $\mathcal{H}$ of graphs that are not 2-connected, the distinction between polynomial-time solvable and NP-complete cases is not clear, and both cases may occur even if we forbid only one graph.

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