TOPOLOGICAL INVARIANTS FOR TILINGS

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This talk gave a brief – and personal – overview of some of the main themes in the recent and current study of aperiodic tilings by methods from topology. It was clearly not possible to cover everything, and similarly it is not possible to give a comprehensive bibliography in the space available here, even for the subjects touched upon. The interested reader should explore the topics further through the selected papers mentioned below and the further work they cite.

We restrict ourselves mainly to tilings of d-dimensional Euclidean space which are repetitive, aperiodic and of translationally finite local complexity (FLC). For such a tiling $T \subset \mathbb{R}^d$, the key to the topological approach is the space $\Omega = \Omega_T$, variously known as the *tiling space*, or *continuous hull* of T, the completion of the set of translates of T under the *tiling metric*. Under the assumptions above Ω naturally caries a minimal action of the translation group $\mathbb{R}^d$, and in many of the most popular classes of tilings, a unique ergodic probability measure.

The structure of Ω is fundamental to this work. Most lines of approach start from one or other of the observations that Ω can be **(a)** described (up to shape equivalence – see later) as an inverse limit of convenient finite CW complexes (*approximants*), or **(b)** given (up to homeomorphism) the structure of a fibre bundle over a d-torus with fibre a Cantor set [24]. The space may also be described as the classifying space of the holonomy groupoid associated with Ω.

For description **(a)**, there are a number of useful models. For primitive substitution tilings, the first constructions were those of [1, 14]. The desire to produce smaller models for the approximants led to a number of developments, including [2, 3] which implicitly involved working in the *shape category*, a notion formally explored in [8]. Recent work has explored further the use of minimal homotopy models for the approximants. For general tilings, inverse limit descriptions exist via various models [1, 3, 7], but without specific structure these are principally of theoretical use. Similarly, the Cantor bundle structure is computationally practical only in the case of a tangible description of the holonomy action of $\mathbb{Z}^d$ on the Cantor fibre; this can be given explicitly in the case of cut and project tilings [6].

Various results have been established exploring the relationships between the spaces Ω_T and Ω_S and the possible relationships of the underlying tilings T and S. Notable work in this thread includes [10] on deformations of tilings, and most recently [12] characterising homeomorphisms of tiling spaces.

Date: 2 October 2017. Lecture given at the Oberwolfach Workshop *Spectral Structures and Topological Methods in Mathematical Quasicrystals*, 1-7 October 2017.

Key words and phrases. aperiodic patterns, topological invariants, cohomology, K-theory.

Topological invariants for tilings typically study Ω_T, with or without additional structure, through the application of methods from algebraic topology. Typical applications to date have included characterisation results, identification of geometric properties of T, issues related to questions about pure point diffraction (for example work related to the Pisot conjecture, see [16] Chapter 2 for an overview), labelling of gaps in the spectrum of the Schrödingier operator associated to T [4, 5, 13], and results on the complexity of T, [11].

What algebro-topological tools should be employed? Homotopy groups are rich but hard to compute. For tiling spaces, the relevant variant of these are the *shape groups* $\underline{\pi}_*^{\mathrm{sh}}(-)$. The case of $d = 1$ was studied in [8] where the fundamental shape group $\underline{\pi}_1^{\mathrm{sh}}(\Omega)$ was shown to collect information relevant to embedding 1 dimensional tiling spaces in surfaces: the non-abelian nature of π_1 registered aspects not picked up by commutative invariants such as cohomology or K-theory. This is taken further in recent work of Gähler who uses the representation variety of $\underline{\pi}_1^{\mathrm{sh}}(\Omega)$ (more readily computable than $\underline{\pi}_1^{\mathrm{sh}}(\Omega)$ itself) in his classification of certain classes of 1 dimensional substitutions.

Cohomology is a long standing tool used for tiling spaces, but there are several variants in common use; we mention just three. Cech cohomology was the first, and perhaps most natural choice from its behaviour on inverse limits (in which it differs from singular or simplicial cohomology). The models [1, 2] for substitutions mentioned above make this is a computable and well understood invariant for such tilings, at least in low dimensions [23]. Recent work has explored more general situations, such as mixed substitutions [19, 21]. Cohomology gives some clear characterisations: for example, $H^*(\Omega; \mathbb{Q})$ is finite rank for an FLC substitution, but infinite for a generic cut and project tilling; the first cohomology $H^1(\Omega_T, \mathbb{R}^d)$ counts degrees of freedom for deformations of T, and so on.

Pattern Equivariant cohomology [15, 22] has proved a useful alternative approach, yielding the same algebraic invariant as the Cech theory, but in a way that elements can be realised in terms of geometric patches of T. A homological variant [25] shows that tiling spaces satisfy a Poincaré duality property analogous to that of manifolds, and has offered computational advantage, for example in the study of spaces remembering the symmetries of T [26].

The third variant can be thought of as the cohomology of the tiling groupoid, but in the case of an explicit Cantor bundle structure over a d-torus $\mathbb{T}^d$, this is equivalent to the group cohomology of $\mathbb{Z}^d = \pi_1(\mathbb{T}^d)$ with coefficients the continuous $\mathbb{Z}$-valued functions on the fibre. This too has its strengths, especially in the case of an explicit description of the bundle, such as for many of the cut and project tilings. See [16] Chapter 4 for a general introduction. Similar methods become natural to apply when studying tilings with rotations, as explored in recent work of the author with Walton.

Cohomology may be enriched with various additional structures, producing finer invariants. Included here are the Ruelle-Sullivan map of [18], the ordered cohomology of [20] and the homology core of [9]. The reader should consult those papers for statements of the advantages gained.

Aperiodic tilings are a fruitful source of examples for noncommutative geometry. Several C^*-algebras A_T have been constructed to model Ω_T and its paraphenalia, and their K-groups reflect the space and $\mathbb{R}^d$ action; in the case of a unique ergodic measure, there is also a trace map $K_*(A_T) \to \mathbb{R}$. See [17] for a discussion. Connes' Thom isomorphism identifies $K_*(A_T)$ with the topological K-theory $K^*(\Omega)$, and an Atiyah-Hirzebruch spectral sequence gives a method of calculating $K^*(\Omega)$ from the Cech cohomology $H^*(\Omega)$. Through these the noncommutative invariants can frequently be computed. A key object of study here has been the image of the tracial state, which is related to Bellissard's Gap labelling [4, 5, 13].

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