Colouring H-free Graphs of Bounded Diameter

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Abstract

The Colouring problem is to decide if the vertices of a graph can be coloured with at most k colours for an integer k, such that no two adjacent vertices are coloured alike. A graph G is H-free if G does not contain H as an induced subgraph. It is known that Colouring is NP-complete for H-free graphs if H contains a cycle or claw, even for fixed $k \geq 3$. We examine to what extent the situation may change if in addition the input graph has bounded diameter.

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1 Introduction

Graph colouring is one of the best studied concepts in Computer Science and Mathematics. This is mainly due to its many practical and theoretical applications and its many natural variants and generalizations. Over the years, numerous surveys and books on graph colouring were published (see, for example, [1, 4, 18, 21, 26, 28, 31]).

A *(vertex) colouring* of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots\}$ that assigns each vertex $u \in V$ a *colour* $c(u)$ in such a way that $c(u) \neq c(v)$ whenever $uv \in E$. If $1 \leq c(u) \leq k$, then c is said to be a *k-colouring* of G and G is said to be *k-colourable*. The Colouring problem is to decide if a given graph G has a k-colouring for some given integer k. If k is *fixed*, that is, k is not part of the input, we denote the problem by k-Colouring. It is well known that even 3-Colouring is NP-complete [23].

In this paper we aim to increase our understanding of the computational hardness of Colouring. One way to do this is to consider inputs from families of graphs to learn more about the kind of graph structure that causes the hardness. This led to a highly extensive study of Colouring and k-Colouring for many special graph classes. The best-known result in this direction is due to Grötschel, Lovász, and Schrijver, who proved that Colouring is polynomial-time solvable for perfect graphs [13].

Perfect graphs form an example of a graph class that is closed under vertex deletion. Such graph classes are also called *hereditary*. Hereditary graph classes are ideally suited for a *systematic* study in the computational complexity of graph problems. Not only do they capture a very large collection of many well-studied graph classes, but they are also exactly the graph classes that can be characterized by a unique set $\mathcal{H}$ of minimal forbidden induced subgraphs. When solving an NP-hard problem under input restrictions, it is standard practice to consider, for example, first the case where $\mathcal{H}$ has small size, or where each $H \in \mathcal{H}$ has small size.

We note that the set $\mathcal{H}$ defined above may be infinite. If not, say $\mathcal{H} = \{H_1, \ldots, H_p\}$ for some positive integer p, then the corresponding hereditary graph class $\mathcal{G}$ is said to be *finitely defined*. Formally, a graph G is $(H_1, \ldots, H_p)$*-free* if for each $i \in \{1, \ldots, p\}$, G is H_i*-free*, where the latter means that G does not contain an induced subgraph isomorphic to H_i.

We emphasize that the borderline between NP-hardness and tractability is often far from clear beforehand and jumps in computational complexity can be extreme. In order to illustrate this behaviour of graph problems, we present the following example of a (somewhat artificial) graph problem related to vertex colouring.

Colouring-or-Subgraph	
Instance:	an n-vertex graph G
Question:	is G $\lceil\sqrt{\log n}\rceil$-colourable or H-free for some graph H with $\lvert V(H)\rvert \leq \lceil\sqrt{\log n}\rceil$?

▶ **Theorem 1.** *The* Colouring-or-Subgraph *problem is* NP*-hard, but constant-time solvable for every hereditary graph class not equal to the class of all graphs.*

Proof. We reduce from 3-Colouring, which we recall is NP-complete [23]. Let G be an n-vertex graph. Set $k = \lceil\sqrt{\log n}\rceil$. Add $k-3$ pairwise adjacent vertices to G. Make the new vertices also adjacent to every vertex of G. Add each possible graph on k vertices as a connected component to G. The resulting graph G' has $n+(k-3)+k\cdot 2^{\frac{k(k-1)}{2}} < 3n^2$ vertices. By construction, G' contains every graph on at most k vertices as an induced subgraph. Hence, G' is a yes-instance of Colouring-or-Subgraph if and only if G' is k-colourable, and the latter holds if and only if G is 3-colourable.

Now let $\mathcal{G}$ be a hereditary graph class for which there exist at least one graph H such that every graph $G \in \mathcal{G}$ is H-free. Let $\ell = \lvert V(H)\rvert$. We claim that Colouring-or-Subgraph is constant-time solvable for $\mathcal{G}$. Let $G \in \mathcal{G}$ be an n-vertex graph. If $n \leq 2^{|\ell|^2}$, then G has constant size and the problem is constant-time solvable. If $n > 2^{|\ell|^2}$, then $\ell = \lvert V(H)\rvert < \sqrt{\log n} \leq \lceil\sqrt{\log n}\rceil$. Hence G is a yes-instance of Colouring-or-Subgraph, as G is H-free and H has less than $\lceil\sqrt{\log n}\rceil$ vertices. ◀

In this paper, we consider the problems Colouring and k-Colouring. In order to describe known results and our new results we first give some terminology and notation.

1.1 Terminology and Notation

The *disjoint union* of two vertex-disjoint graphs F and G is the graph $G + F = (V(F) \cup V(G), E(F) \cup E(G))$. The disjoint union of s copies of a graph G is denoted sG. A *linear forest* is the disjoint union of paths. The *length* of a path or a cycle is the number of its edges. The *distance* $\text{dist}(u, v)$ between two vertices u, v in a graph G is the length of a shortest induced path between them. The *diameter* of a graph G is the maximum distance over all pairs of vertices in G. The *girth* of a graph G is the length of a shortest induced cycle of G. The graphs C_r, P_r and K_r denote the cycle, path and complete graph on r vertices, respectively.

A *polyad* is a tree where exactly one vertex has degree at least 3. We will use several special polyads in our paper. The graph $K_{1,r}$ denotes the $(r + 1)$-vertex *star*, that is, the graph with vertices $x, y_1, \ldots, y_r$ and edges xy_i for $i = 1, \ldots, r$. The graph $K_{1,3}$ is also called the *claw*. The *subdivision* of an edge uw in a graph removes uw and replaces it with a new vertex v and edges uv, vw. We let $K^\ell_{1,r}$ denote the ℓ*-subdivided star*, which is the graph obtained from a star $K_{1,r}$ by subdividing one edge of $K_{1,r}$ exactly ℓ times. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, which is the tree with one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. Note that $S_{1,1,1} = K_{1,3}$. The graph $S_{1,1,2} = K_{1,3}^1$ is also known as the *chair*.

1.2 Known Results

The computational complexity of Colouring has been fully classified for H-free graphs: if H is an induced subgraph of $P_1 + P_3$ or of P_4, then Colouring for H-free graphs is polynomial-time solvable, and otherwise it is NP-complete [20]. In contrast, the complexity classification for k-Colouring restricted to H-free graphs is still incomplete. It is known that for every $k \geq 3$, k-Colouring for H-free graphs is NP-complete if H contains a cycle [10] or an induced claw [16, 22]. However, the remaining case where H is a linear forest has not been settled yet even if H consists of a single path. For P_t-free graphs, the cases $k \leq 2$, $t \geq 1$ (trivial), $k \geq 3$, $t \leq 5$ [14], $k = 3$, $6 \leq t \leq 7$ [2] and $k = 4$, $t = 6$ [6] are polynomial-time solvable and the cases $k = 4$, $t \geq 7$ [17] and $k \geq 5$, $t \geq 6$ [17] are NP-complete. The cases where $k = 3$ and $t \geq 8$ are still open. For further details, including for linear forests H of more than one connected component, see the survey paper [11] or some recent papers [5, 12, 19].

1.3 Our Focus

We consider H-free graphs where H contains a cycle or claw. In this case, k-Colouring restricted to H-free graphs is NP-compete for every $k \geq 3$, as mentioned above. However, we re-examine the situation after adding a diameter constraint to our input graphs. If the diameter is 1, then G is a complete graph, and Colouring becomes trivial. As such, our research question is:

To what extent does bounding the diameter help making Colouring *and* k-Colouring *tractable on* H*-free graphs?*

We remark that H-free graphs of diameter at most d for some integer d are no longer hereditary, which requires some care in the proof of our results. We also note that by a straightforward reduction from 3-Colouring one can show that k-Colouring is NP-complete for graphs of diameter d for all pairs (k, d) with $k \geq 3$ and $d \geq 2$ except for two cases, namely $(k, d) \in \{(3, 2), (3, 3)\}$. Mertzios and Spirakis [24] settled the case $(k, d) = (3, 3)$ by proving that 3-Colouring is NP-complete even for C_3-free graphs of diameter 3. The case $(k, d) = (3, 2)$ is still open.

1.4 Our Results

We complement the bounded diameter results of Mertzios and Spirakis [24] by presenting a set of new results for Colouring and k-Colouring for H-free graphs of bounded diameter when H contains a claw or a cycle. Results for the case where H has a cycle usually follow from stronger results for graphs of girth at least g for some fixed integer g. In particular, Emden-Weinert, Hougardy and Kreuter [10] proved that for all integers $k \geq 3$ and $g \geq 3$, k-Colouring is NP-complete for graphs with girth at least g and with maximum degree at most $6k^{13}$ (for more results on Colouring for graphs of maximum degree, see [3, 7, 25]).

First, in Section 3 we research the effect on bounding the diameter of k-Colouring and Colouring restricted to polyad-free graphs for various polyads. Our first result, which formed together with the result of [24] the starting point of our investigation, is that k-Colouring is constant-time solvable for $K_{1,r}$-free graphs of diameter d for any fixed integers

Colours	Diameter	H-free	Complexity	Theorem
fixed k	d	$K_{1,r}$	P	9
input k	d	$K_{1,4}$	NP-c	10
3	d	$K^1_{1,3}$	P	12(1)
3	2	$K^2_{1,r}$	P	12(2)
3	4	$K^3_{1,4}$	NP-c	12(3)
4	2	$K^1_{1,3}$	NP-c	12(4)
3	2	$S_{1,2,2}$	P	13

Figure 1 Our polynomial-time (P) and NP-complete (NP-c) results for polyad-free graphs.

$d \geq 1$, $k \geq 1$ and $r \geq 1$. We also show that this does not hold for COLOURING (when k is part of the input). We then extend these results for larger polyads; see also Figure 1.

Second, in Section 4 we perform a similar study for graphs of bounded diameter and girth. We provide new polynomial-time and NP-hardness results for k-COLOURING, identifying and narrowing the gap between tractability and intractability, in particular for the case where $k = 3$ (see also Figure 2). Section 5 contains some open questions and directions for future work.

diameter \ girth	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	≥ 11	≥ 12
≤ 1	P	P	P	P	P	P	P	P	P	P
≤ 2	?	?	P	P	P	P	P	P	P	P
≤ 3	NP-c	NP-c	?	?	P	P	P	P	P	P
≤ 4	NP-c	NP-c	NP-c	NP-c	?	?	P	P	P	P
≤ 5	NP-c	NP-c	NP-c	NP-c	?	?	?	?	?	P

Figure 2 The complexity of 3-COLOURING for graphs of diameter at most d and girth at least g.

2 Preliminaries

In this section we complement Section 1.1 by giving some additional terminology and notation. We also recall some useful results from the literature.

Let $G = (V, E)$ be a graph. A vertex $u \in V$ is *dominating* if u is adjacent to every other vertex of G. For a set $S \subseteq V$, the graph $G[S]$ denotes the subgraph of G induced by S. The *neighbourhood* of a vertex $u \in V$ is the set $N(u) = \{v \mid uv \in E\}$ and the *degree* of u is the size of $N(u)$. For a set $U \subseteq V$, we write $N(U) = \bigcup_{u \in U} N(u) \setminus U$. For a set $U \subseteq V$ and a vertex $u \in U$, the *private neighbourhood* of u with respect to U is the set $N(u) \setminus (N(U \setminus \{u\}) \cup U)$ of *private neighbours* of u with respect to U, which is the set of neighbours of u outside U that are not a neighbour of any other vertex of U. If every vertex of G has degree p, then G is *(p)-regular*.

We will use the aforementioned results of Král' et al.; Holyer; Leven and Galil; Emden-Weinert, Hougardy and Kreuter; and Mertzios and Spirakis.

▶ **Theorem 2** ([20]). *Let H be a graph. If $H \subseteq_i P_4$ or $H \subseteq_i P_1 + P_3$, then* COLOURING *restricted to H-free graphs is polynomial-time solvable, otherwise it is* NP*-complete.*

▶ **Theorem 3** ([16, 22]). *For every integer $k \geq 3$, k-Colouring is NP-complete for claw-free graphs.*

▶ **Theorem 4** ([10]). *For all integers $k \geq 3$ and $g \geq 3$, k-Colouring is NP-complete for graphs with girth at least g (and with maximum degree at most $6k^{13}$).*

▶ **Theorem 5** ([24]). *3-Colouring is NP-complete for C_3-free graphs of diameter 3.*

A *list assignment* of a graph $G = (V, E)$ is a function L that prescribes a *list of admissible colours* $L(u) \subseteq \{1, 2, \ldots\}$ to each $u \in V$. A colouring c *respects* L if $c(u) \in L(u)$ for every $u \in V$. If $|L(u)| \leq 2$ for each $u \in V$, then L is also called a 2-*list assignment*. The 2-List Colouring problem is to decide if a graph G with a 2-list assignment L has a colouring that respects G. Our strategy for obtaining a polynomial-time algorithm for 3-Colouring is often to reduce the input to a polynomial number of instances of 2-List Colouring. The reason is that we can then apply the following well-known result of Edwards.

▶ **Theorem 6** ([9]). *The 2-List Colouring problem is linear-time solvable.*

We will also use the following result, which includes the Hoffman-Singleton Theorem, which provides a description of regular graphs of diameter 2 and girth 5.

▶ **Theorem 7** ([8, 15, 30]). *For every $d \geq 1$, every graph of diameter d and girth $2d + 1$ is p-regular for some integer p. Moreover, if $d = 2$, then there are only four such graphs (with $p = 2, 3, 7, 57$, respectively) and if $d \geq 3$, then such graphs are cycles (of length $2d + 1$).*

A *clique* in a graph is a set of pairwise adjacent vertices, and an *independent set* is a set of pairwise non-adjacent vertices. By Ramsey's Theorem [27], there exists a constant, which we denote by $R(k, r)$, such that any graph on at least $R(k, r)$ vertices contains either a clique of size k or an independent set of size r.

3 Polyad-Free Graphs of Bounded Diameter

In this section we prove, among other things, our results on Colouring and k-Colouring for polyad-free graphs of bounded diameter; see also Figure 1. We first make an observation.

▶ **Lemma 8.** *If G is a graph of diameter d that is not a tree, then G contains an induced cycle of length at most $2d + 1$.*

Proof. As G is not a tree and G is connected, G must contain a cycle C. Suppose that C has length at least $2d + 2$. Since G has diameter d, there exists a path of length at most d in G between any two vertices u and v at distance $d + 1$ in C. The vertices of this path, together with the vertices of the path of length $d + 1$ between u and v on C, induce a subgraph of G that contains an induced cycle C' of length at most $2d + 1$. ◀

We now state our first result, which forms the starting point of the research in this section.

▶ **Theorem 9.** *For all integers $d, k, r \geq 1$, k-Colouring is constant-time solvable for $K_{1,r}$-free graphs of diameter d.*

Proof. Let $G = (V, E)$ be a $K_{1,r}$-free graph of diameter d. We prove that if G has size larger than some constant $\beta(k, r)$, which we determine below, then G is not k-colourable. If $|V(G)| \leq \beta(k, r)$, we can solve k-Colouring in constant time.

As G is $K_{1,r}$-free, Ramsey's Theorem tells us that the neighbourhood of every vertex $u \in V$ with degree at least $R(k, r)$ contains a clique of size k. In that case $N(u) \cup \{u\}$ is a clique of size $k + 1$. Hence, to be k-colourable, every vertex of G must have degree less than $R(k, r)$, so G must have at most $\beta(k, r) = 1 + R(k, r) + R(k, r)^2 + \ldots + R(k, r)^d$ vertices. ◀

If k is not part of the input, Theorem 9 no longer holds. This is shown by the following more general theorem. In this theorem we assume that $H \not\subseteq_i P_1 + P_3$ and $H \not\subseteq_i P_4$, as in those cases Colouring is polynomial-time solvable for all H-free graphs due to Theorem 2. Note that Theorem 10 covers all remaining cases except the case where $H = K_{1,3}$.

▶ **Theorem 10.** *Let H be a graph with $H \not\subseteq_i P_1 + P_3$ and $H \not\subseteq_i P_4$ and d be an integer. Then* Colouring *for H-free graphs of diameter at most d is*

1. NP*-complete if H has no dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$ and $d \geq 2$;*
2. NP*-complete if $H \neq K_{1,3}$ and H has a dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$ and $d \geq 3$.*

Proof. **1.** Let H have no dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$. We define H' as $H - u$ if H has a dominating vertex u and as H itself otherwise. By construction, $H' \not\subseteq_i P_1 + P_3$ and $H' \not\subseteq_i P_4$. Hence, Colouring is NP-complete for H'-free graphs due to Theorem 2. Let G be an H'-free graph. Add a dominating vertex to G. The new graph G' has diameter 2 and is H-free. Moreover, G is k-colourable if and only if G' is $(k + 1)$-colourable.

2. Let $H \neq K_{1,3}$ have a dominating vertex u such that $H - u \subseteq_i P_1 + P_3$ or $H - u \subseteq_i P_4$. Then H cannot be a forest, as in that case H would be in $\{P_1, P_2, P_3, K_{1,3}\}$. Hence, H has an induced cycle C_r for some $r \geq 3$. If $r = 3$, then 3-Colouring is NP-complete for H-free graphs of diameter 3, as it is so for C_3-free graphs of diameter 3 due to Theorem 5. If $r \geq 4$, then Colouring is NP-complete even for H-free graphs of diameter 2, as it is so for C_r-free graphs of diameter 2 due to **1**. ◀

It is a natural question whether we can extend Theorem 9 to H-free graphs of diameter d, where H is a slightly larger tree than a star. The first interesting case is where H is an ℓ-subdivided star $K_{1,r}^{\ell}$ for some integer $\ell \geq 1$ and $r \geq 3$. We prove a number of results for various values of d,k,ℓ. For one of our proofs and also for the proof of our next result we need the following theorem.

▶ **Theorem 11.** 3-Colouring *can be solved in polynomial time for C_5-free graphs of diameter at most 2.*

Proof. If G is bipartite, then G is 3-colourable. If G contains a K_4, then G is not 3-colourable. We check these properties in polynomial time, and from now on we assume that G is K_4-free and non-bipartite. The latter implies that G must have an odd induced cycle C_r for some odd integer r. As G has diameter 2, we find that $r \leq 5$ due to Lemma 8. As G is C_5-free, it follows that $r = 3$.

Let C be a triangle in G. We write $N_0 = V(C) = \{x_1, x_2, x_3\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$. As G has diameter 2, for every $i \in \{1, 2, 3\}$, it holds that every vertex in N_2 has a neighbour in N_1 that is adjacent to x_i.

We let T consist of all vertices of N_2 that have a neighbour in N_1 that is adjacent to exactly two vertices of N_0. We claim that $N_2 = T$. In order to see this, let $u \in N_2$. If u has a neighbour $y \in N_1$ adjacent to every x_i, then G contains a K_4, a contradiction. Hence, u must have three distinct neighbour y_1, y_2, y_3, such that for $i \in \{1, 2, 3\}$, it holds that $N(y_i) \cap N_0 = \{x_i\}$. If $\{y_1, y_2, y_3\}$ is a clique, then G has a K_4 on vertices u, y_1, y_2, y_3, a contradiction. Hence, we may assume without loss of generality that y_1 and y_2 are non-adjacent. However, then $\{u, y_1, x_1, x_2, y_2\}$ induces a C_5 in G, another contradiction. We conclude that $T = N_2$.

If G has a 3-colouring c, then we may assume without loss of generality that $c(x_i) = i$ for $i \in \{1, 2, 3\}$. Hence, our algorithm assigns colours 1, 2, 3 to x_1, x_2, x_3, respectively. This reduces the list of admissible colours of the vertices of N_1 by at least one colour. In particular, vertices in N_1 that have two neighbours in N_0 can be coloured with only one colour. Our algorithm assigns this colour to such vertices. This means that any of their neighbours in $T = N_2$ can be coloured with at most two colours. So, after propagation, we have obtained either two adjacent vertices that are coloured alike, in which case G is not 3-colourable, or we have constructed an instance of 2-List Colouring. We can solve such an instance in linear time due to Theorem 6. ◀

We are now ready to state our results for $K^{\ell}_{1,r}$, where we exclude the cases that are tractable in general, namely where $d = 1$, or $k \leq 2$, or $r \leq 2$ (the latter case corresponds to the case where $H = K^{+}_{1,2} = P_4$, so we can use Theorem 2). Note that for $k \geq 4$ all interesting cases are NP-complete, whereas for $k = 3$ the situation is less clear.

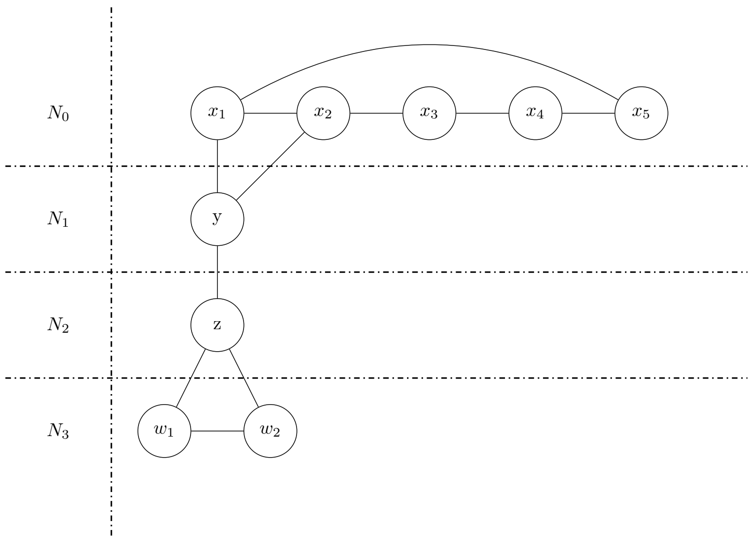

Figure 3 An example of a decomposition of a chair-free graph of diameter 3 into sets $N_0, \ldots, N_3$ where $p = 5$ and $y \in N_1$ has two "descendants" in N_3. To prevent an induced chair, y must be adjacent to exactly two (adjacent) vertices of N_0, and w_1 and w_2 must be adjacent to each other.

▶ **Theorem 12.** *Let d, k, ℓ, r be four integers with $d \geq 2$, $k \geq 3$, $\ell \geq 1$ and $r \geq 3$. Then k-Colouring for $K^{\ell}_{1,r}$-free graphs of diameter at most d is:*
1. *polynomial-time solvable if $d \geq 2$, $k = 3$, $\ell = 1$ and $r = 3$*
2. *polynomial-time solvable if $d = 2$, $k = 3$, $\ell = 2$ and $r \geq 3$*
3. NP-*complete if $d \geq 4$, $k = 3$, $\ell \geq 3$ and $r \geq 4$*
4. NP-*complete if $d \geq 2$, $k \geq 4$, $\ell \geq 1$ and $r \geq 3$.*

Proof. 1. Recall that $K^1_{1,3}$ is the chair $S_{1,1,2}$. Let G be a chair-free graph of diameter d. If G is a tree, then G is even 2-colourable. We check in $O(n^4)$ time if G has a K_4. If so, then G is not 3-colourable. From now on we assume that G is not a tree and that G is K_4-free. As G is not a tree and G is connected, G contains an induced cycle of length at most $2d + 1$ by Lemma 8. We can find a largest induced cycle C of length at most $2d + 1$ in $O(n^{2d+1})$ time. Let $|V(C)| = p$. We write $N_0 = V(C) = \{x_1, x_2, \ldots, x_p\}$ and for $i \geq 1$, $N_i = N(N_{i-1}) \setminus N_{i-2}$. So the sets N_i partition $V(G)$, and the distance of a vertex $u \in N_i$ to N_0 is i.

Case 1. $4 \leq p \leq 2d+1$.
This case is illustrated in Figure 3. We consider every possible 3-colouring of C. Let c be such a 3-colouring. Every vertex with two differently coloured neighbours can only be coloured with one remaining colour. We assign this unique colour to such a vertex and apply this rule as long as possible. This takes polynomial time. The remaining vertices have a list of admissible colours that either consists of two or three colours, and vertices in the latter case belong to $V(G) \setminus (N_0 \cup N_1)$ (as $N(N_0) = N_1$).

If $N_2 = \emptyset$, then $V(G) = N_0 \cup N_1$. Then, we obtained an instance of 2-List Colouring, which we can solve in linear time due to Theorem 6. Now assume that $N_2 \neq \emptyset$. Let $z \in N_2$. Then z has a neighbour $y \in N_1$, which in turn has a neighbour $x \in N_0$. If y is adjacent to neither neighbour of x on N_0, then z, y, x and these two neighbours induce a chair in G, a contradiction. Hence, y must be adjacent to at least one neighbour of x on N_0, meaning that y must have received a colour by our algorithm. Consequently, z must have a list of admissible colours of size at most 2.

From the above we deduce that every vertex in N_2 has only two available colours in its list. We now consider the vertices of N_3. Let $z' \in N_3$. Then z' has a neighbour $z \in N_2$, which in turn has a neighbour $y \in N_1$, which in turn has a neighbour $x \in N_0$, say $x = x_1$. If y has two non-adjacent neighbours in N_0, then z', z, y and these two non-adjacent neighbours of y induce a chair in G, a contradiction. Combined with the fact deduced above, we conclude that y must have exactly two neighbours in N_0 and these two neighbours must be adjacent, say x_2 is the other neighbour of y in N_0.

Suppose x_1 and x_2 are both adjacent to a vertex $y' \in N_1 \setminus \{y\}$ that is adjacent to a vertex in N_2 that has a neighbour in N_3. Then, just as in the case of vertex y, the two vertices x_1 and x_2 are the only two neighbours of y' in N_0. If y and y' are not adjacent, this means that x_2, x_3, x_4, y, y' induce a chair in G, a contradiction. Hence y and y' must be adjacent. However, then x_1, x_2, y, y' form a K_4, a contradiction. This means that every pair of adjacent vertices of N_0 can have at most one common neighbour in N_1 that is adjacent to a vertex in N_2 with a neighbour in N_3. We already deduced that every vertex of N_1 with a "descendant" in N_3 has exactly two neighbours in N_0, which are adjacent. Hence, we conclude that the number of such vertices of N_1 is at most p.

We now observe that for $i \geq 2$, every vertex in N_i has at most two neighbours in N_{i+1}. This can be seen as follows. If $v \in N_i$ has two non-adjacent neighbours w_1, w_2 in N_{i+1}, then we pick a neighbour u of v in N_{i-1}, which has a neighbour t in N_{i-2}. Then v, u, t, w_1, w_2 induce a chair in G, a contradiction. Hence , the neighbourhood of every vertex in N_i in N_{i+1} is a clique, which must have size at most 2 due to the K_4-freeness of G. As the number of vertices in N_1 with a "descendant" in N_3 is at most p, this means that there are at most $2^{i-1}p$ vertices in N_i with a neighbour in N_{i+1}. Therefore the total number of vertices not belonging to any of the sets N_0, N_1 or N_2 is at most $\sum_{i=3}^{d} 2^{i-1}p$.

This means the total number of vertices not belonging to N_1 or N_2 is at most $\beta(d) = \sum_{i=3}^{d} 2^{i-1}p + p \leq \sum_{i=3}^{d} 2^{i-1}(2d+1) + 2d + 1$. Let T_c be this set. We consider every possible 3-colouring of $G[T_c]$. As we already deduced that the vertices in $N_1 \cup N_2$ have a list of size at most 2, for each case we obtain an instance of 2-List Colouring, which we can solve in linear time due to Theorem 6. As the total number of instances we need to consider is at most $3^p \times 3^{\beta(d)} \leq 3^{2d+1} \times 3^{\beta(d)}$, our algorithm runs in polynomial time.

Case 2. $p = 3$.
As p was the size of a largest induced cycle of length at most $2d+1$ and $2d+1 \geq 5$, we find that G is C_4-free. As G is K_4-free, each vertex of N_1 is adjacent to at most two vertices of N_0. If a vertex $x \in N_0$ has two independent private neighbours u and v in N_1 with respect

to N_0, then every neighbour w of u in N_2 must also be a neighbour of v and vice versa, since G is chair-free. However, this is not possible, as x, u, w, v induce a C_4. We conclude that u and v must be adjacent. Therefore, as G is K_4-free, every vertex of N_0 has at most two private neighbours in N_1, with respect to N_0, that have a neighbour in N_2.

By the same arguments as above we deduce that every two vertices of N_0 have at most one common neighbour in N_1 that is adjacent to a vertex in N_2. Combined with the above, we find that there at most $6 + 3 = 9$ vertices in N_1 that have a neighbour in N_2. If a vertex in N_1 has two independent neighbours in N_2, then G contains an induced chair, which is not possible. Hence the neighbourhood of a vertex in N_1 in N_2 is a clique, which has size at most 2 due to the K_4-freeness of G. We conclude that $|N_2| \leq 9 \times 2 = 18$. Similarly, every vertex in N_i for $i \geq 3$ has at most two neighbours in N_{i+1}. Therefore the number of vertices in N_i for $i \geq 3$ is at most $18 \times 2^{i-2}$. This means that the total number of vertices outside $N_0 \cup N_1 \cup N_2$ is at most $\beta(d) = \sum_{i=3}^{d} 18 \times 2^{i-2}$. Let T be this set. We consider every possible 3-colouring of $G[T]$ and every possible 3-colouring of C. For each case we obtain an instance of 2-List Colouring, which we can solve in linear time due to Theorem 6. As the total number of instances we need to consider is at most $3^d \times 3^{\beta(d)}$, our algorithm runs in polynomial time.

2. Let G be a $K_{1,r}^2$-free graph of diameter at most 2. We first check in $O(n^4)$ time if G is K_4-free. If not, then G is not 3-colourable. We then check in $O(n^5)$ time if G has an induced C_5. If G is C_5-free, then we use Theorem 11. From now on, suppose that G is K_4-free and that G contains an induced cycle C of length 5, say on vertices $x_1, \ldots, x_5$ in that order. We write $N_0 = V(C) = \{x_1, \ldots, x_5\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$.

Let N_2' be the set of vertices in N_2 that are adjacent to some vertex in N_1 that is a private neighbour of some vertex in N_0 with respect to N_0. As G is K_4-free, the private neighbourhood $P(x_i)$ of each vertex $x_i \in N_0$ with respect to N_0 does not contain a clique of size 3. Moreover, if $P(x_i)$ contains an independent set I of size $r - 1$ for some $i \in \{1, \ldots, 5\}$, then $I \cup \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$ induces a $K_{1,r}^2$, which is not possible. Now let $v \in P(x_i)$ for some $i \in \{1, \ldots 5\}$, say $i = 1$. As G is K_4-free, the set $N(v) \cap N_2$ does not contain a clique of size 3. Moreover, if $N(v) \cap N_2$ contains an independent set I' of size $r - 1$, then $I' \cup \{v, x_1, x_2, x_3, \}$ induces a $K_{1,r}^2$, which is not possible. Hence, $|N(v) \cap N_2| \leq R(3, r - 1)$ by Ramsey's Theorem. We conclude that $|N_2'| \leq 5R(3, r - 1)^2$.

We now consider all possible 3-colourings of C. Let c be such a 3-colouring. We assume without loss of generality that $c(x_1) = c(x_3) = 1$, $c(x_2) = c(x_4) = 2$ and $c(x_5) = 3$. Moreover, every vertex that has two differently coloured neighbours can only be coloured with one remaining colour. We assign this unique colour to such a vertex and apply this rule as long as possible. This takes polynomial time. The remaining vertices have a list of admissible colours that either consists of two or three colours, and vertices in the latter case must belong to N_2 (as $N(N_0) = N_1$).

Let T_c be the set of vertices in N_2 that still have a list of size 3. We will prove that $T_c \subseteq N_2'$. Let $u \in T_c$. As G has diameter 2, we find that u has a neighbour v adjacent to x_5. Then v cannot be adjacent to any of $x_1, \ldots, x_4$, as otherwise v would have a unique colour and u would not be in T_c. Hence, v is a private neighbour of x_5 with respect to N_0. We conclude that all vertices in T_c belong to N_2', which implies that $|T_c| \leq |N_2'| \leq 5R(3, r - 1)^2$.

We now consider every possible 3-colouring of $G[T_c]$. Then all uncoloured vertices have a list of size at most 2. In other words, we created an instance of 2-List Colouring, which we solve in linear time using Theorem 6. As the number of 3-colourings of C is at most 3^5 and for each 3-colouring c of C the number of 3-colourings of $G[T_c]$ is at most $3^{5R(3,r-1)^2}$, the total running time of our algorithm is polynomial.

3. We consider the standard reduction from the NP-complete problem NAE 3-SAT [29], where each variable appears in at most three clauses and each literal appears in at most two. Given a CNF formula ϕ, we construct the graph G as follows:

- Add a vertex v_{x_i} for each literal x_i.
- Add an edge between each literal and its negation.
- Add a vertex z adjacent to every literal vertex.
- For each clause C_i add a triangle T_i with vertices $c_{i_1}, c_{i_2}, c_{i_3}$.
- Fix an arbitrary order of the literals of C_i, $x_{i_1}, x_{i_2}, x_{i_3}$ and add an edge $x_{i_j}c_{i_j}$.

Given a 3-colouring of G, assume z is assigned colour 1. Then each literal vertex is assigned either colour 2 or colour 3. If, for some clause C_i, the vertices x_{i_1}, x_{i_2} and $x_{i,3}$ are all assigned the same colour, then T_i cannot be coloured. Therefore, if we set literals whose vertices are coloured with colour 2 to be true and those coloured with colour 3 to be false, each clause must contain at least one true literal and at least one false literal.

If ϕ is satisfiable then we can colour vertex z with colour 1, each true literal with colour 2 and each false literal with colour 3. Then, since each clause has at least one true literal and at least one false literal, each triangle has neighbours in two different colours. This implies that each triangle is 3-colourable. Therefore G is 3-colourable if and only if ϕ is satisfiable.

We next show that G has diameter at most 4. First note that any literal vertex is adjacent to z and any clause vertex is adjacent to some literal vertex so any vertex is at distance at most 2 from z. Therefore any two vertices are at distance at most 4.

Finally we show that G is $K_{1,4}^3$-free. Any literal vertex has degree at most 4 since it appears in at most two clauses. However it has at most 3 independent neighbours since its negation is adjacent to z. Each clause vertex has at most 3 neighbours so the only vertex with four independent neighbours is d. The longest induced path including z has length at most 4 since any such path contains at most one literal and at most two vertices of any triangle. Therefore G is $K_{1,4}^3$-free.

4. This follows from Theorem 3. Let $k^* \geq 3$. We take a claw-free graph G and add a dominating vertex to it. The new graph G' has diameter at most 2 and is $K_{1,3}^1$-free. Let $k = k^* + 1 \geq 4$. Then G is k^*-colourable if and only if G' is k-colourable. ◀

Subdividing two edges of the claw yields another interesting case, namely where $H = S_{1,2,2}$. For $k \geq 4$, Theorem 12 tells us that k-Colouring is NP-complete for $S_{1,2,2}$-free graphs of diameter 2. For $k = 3$, we could only prove polynomial-time solvability if $d = 2$.

▶ **Theorem 13.** 3-Colouring *can be solved in polynomial time for $S_{1,2,2}$-free graphs of diameter at most* 2.

Proof. Let G be an $S_{1,2,2}$-free graph of diameter at most 2. We first check in $O(n^5)$ time if G has an induced C_5. If G is C_5-free, then we use Theorem 11. Suppose G contains an induced cycle C of length 5, say on vertices $x_1, \ldots, x_5$ in that order. We write $N_0 = V(C) = \{x_1, \ldots, x_5\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$. As G has diameter 2, for every $i \in \{1, 2, 3\}$, every vertex in N_2 has a neighbour in N_1 that is adjacent to x_i.

We let T consist of all vertices of N_2 that have a neighbour in N_1 that is adjacent to two adjacent vertices of N_0. So the colour of any vertex of T will be fixed in any 3-colouring after colouring the five vertices of N_0. We claim that $N_2 = T$. In order to see this, let $u \in N_2$. As G has diameter 2, we find that u must have a neighbour $v \in N_1$ adjacent to a vertex of N_0, say x_1. Then v is not adjacent to x_5 or x_2. If v is not adjacent to x_3 either, then the vertices x_1, x_5, x_2, x_3, v, u induce a $S_{1,2,2}$ with center x_1, a contradiction. So v must be adjacent to

x_3, meaning v is not adjacent to x_4. However, now x_3, x_2, x_4, x_5, v, u induce a $S_{1,2,2}$ with center x_3, another contradiction.

We now "guess" the 3-colouring of C by considering all 3^5 possibilities if necessary. We then proceed as in the proof of Theorem 11. That is, we observe that every vertex of N_1 can only be coloured with two possible colours and that after propagation, every uncoloured vertex of N_2 can only be coloured with two possible colours as well (as $T = N_2$). Then it remains to solve an instance of 2-List Colouring, which takes linear time by Theorem 6. As we need to do this at most 3^5 times, the total running time of our algorithm is polynomial. ◀

4 Graphs of Bounded Diameter and Girth

In this section we will examine the trade-offs for k-Colouring between diameter and girth. Recall that Mertzios and Spirakis [24] proved that 3-Colouring is NP-complete for graphs of diameter 3 and girth 4 (Theorem 5). We extend their result in our next theorem, partially displayed in Figure 2. This theorem shows that there is still a large gap for which we do not know the computational complexity of 3-Colouring for graphs of diameter d and girth g.

▶ **Theorem 14.** *Let d, g, k be three integers with $d \geq 2$, $g \geq 3$ and $k \geq 3$. Then k-Colouring for graphs of diameter at most d and girth at least g is*

1. *polynomial-time solvable if $g \geq 2d + 1$*
2. NP-*complete if $d = 3$ and $g \leq 4$ and $k = 3$*
3. NP-*complete if $4p \leq d \leq 4p + 3$ and $g \leq 4p + 2$ for some integer $p \geq 1$ and $k = 3$.*

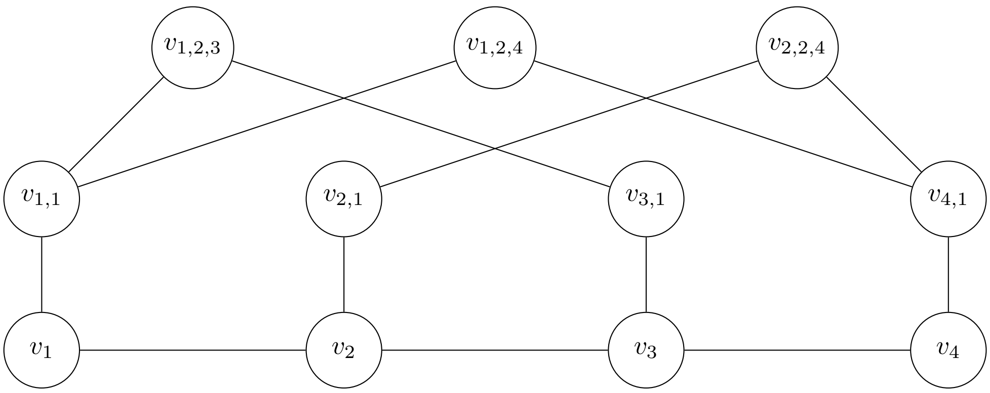

Figure 4 An example of a graph G', constructed in the proof of Theorem 14(3), for $p = 1$.

Proof. **1.** This case follows from Theorem 7. **2.** This case is Theorem 5 (proven in [24]).

3. We reduce 3-Colouring for graphs of girth at least $8p - 3$, which is NP-complete by Theorem 4, to 3-Colouring for graphs of diameter at most $4p$ and girth at least $4p + 2$. Construct the graph G' as follows (see Figure 4 for an example):

- label the vertices of G v_1 to v_n;
- for each vertex of G, add a new neighbour $v_{i,1}$;
- for every two vertices v_i and v_j such that $\text{dist}(v_i, v_j) > l = 2p - 1$ add new vertices to form the path $v_{i,1}v_{i,2,j}...v_{i,p+1,j}v_{j,p,i}...v_{j,1}$.

First we show that G' has diameter at most $4p$. For any two vertices v_i and v_j of G either $\text{dist}(v_i, v_j) \leq l$ or we have the path $v_{i,1}v_{i,2,j}...v_{i,p+1,j}v_{j,p,i}...v_{j,1}$ and $\text{dist}(v_i, v_j) \leq 2p+2$. Similarly, $\text{dist}(v_i, v_{j,1}) \leq 2p+1$ and $\text{dist}(v_{i,1}, v_{j,1}) \leq 2p+1$. Now consider two vertices $v_{a,r,b}$ and $v_{c,q,d}$ for $2 \leq r \leq p+1$, $2 \leq q \leq p+1$. If $\text{dist}(v_a, v_c) \leq l$ then $\text{dist}(v_{a,r,b}, v_{c,q,d}) \leq r + q + l \leq (p+1) + (p+1) + (2p-1) \leq 4p+1$. Otherwise we have the path $v_{a,r,b}..v_{a,1}v_{a,2,c}...v_{a,p+1,c}v_{c,p,a}...v_{c,1}v_{c,2,d}...v_{c,q,d}$. This gives $\text{dist}(v_{a,r,b}, v_{c,q,d}) \leq (r-1) + p + p + (q-1) \leq 4p$. In fact, if $\text{dist}(v_{a,r,b}, v_{c,q,d}) = 4p+1$, then we must have $r = q = p+1$ and $\text{dist}(v_a, v_c) = \text{dist}(v_a, v_d) = \text{dist}(v_b, v_c) = \text{dist}(v_b, v_d) = 2p-1$. In this case we have two paths of length at most $4p-2$ between v_a and v_b, one containing v_c and the other containing v_d. These paths must be distinct since the existence of the vertex $v_{c,p+1,d}$ implies that $\text{dist}(v_c, v_d) > 2p-1$. Therefore we have a cycle in G of length at most $8p-4$ which contradicts the assumption that G has girth at least $8p-3$. This implies that the diameter of G' is at most $4p$.

Since G has girth at least $8p-3$, every cycle in G' of length less than $4p+2$ must contain at least one vertex of $V(G') \setminus V(G)$. Since all the vertices of $V(G') \setminus V(G)$ except the vertices $v_{i,1}$ have degree 2, any such cycle C must contain the path $v_{i,1}..v_{i,p+1,j}...v_j$ for some v_i, v_j at distance greater than l. This path has length $2p+1$. If C contains $v_{i,2,m}$ for some m different from j then it contains the path $v_{i,2,m}...v_{m,1}$ and has length at least $4p+2$. Similarly, this is the case if C contains $v_{j,2,m}$ for m different from i. Otherwise C contains v_i and v_j which are at distance at least l and has length at least $(2p+1)+2+(2p-1) = 4p+2$.

Finally, we show that G is 3-colourable if and only if G' is 3-colourable. The latter holds if and only if the subgraph G'' of G' induced by $V(G) \cup \{v_{i,1} \mid 1 \leq i \leq n\}$ is 3-colourable, since every other vertex of G' has degree 2. The graph G is 3-colourable if and only if G'' is 3-colourable, since G is an induced subgraph of G'' and each vertex of $V(G'') \setminus V(G)$ has degree 1. Therefore, G is 3-colourable if and only if G' is 3-colourable. ◀

5 Conclusions

We proved a number of new results for Colouring and k-Colouring for polyad-free graphs of bounded diameter and for graphs of bounded diameter and girth. In particular we identified and narrowed a number of complexity gaps. This leads us to some natural open problems. Our first two open problems follow from Theorem 10. The third open problem comes from Theorem 12; note that $K_{1,3}^2 = S_{1,1,3}$. Our fourth open problem stems from Theorem 13. Recall that determining the complexity of 3-Colouring for graphs of diameter 2 is still wide open. This question is covered by the fifth open problem.

▷ Open Problem 1. Does there exist an integer d such that Colouring is NP-complete for $K_{1,3}$-free graphs of diameter d?

▷ Open Problem 2. What is the complexity of Colouring for C_3-free graphs of diameter 2, or equivalently, graphs of diameter 2 and girth 4?

▷ Open Problem 3. What are the complexities of 3-Colouring for $K_{1,4}^1$-free graphs of diameter 3 and for $K_{1,3}^2$-free graphs of diameter 3?

▷ Open Problem 4. Do there exist integers d, h, i, j such that 3-Colouring is NP-complete for $S_{h,i,j}$-free graphs of diameter d?

▷ Open Problem 5. What is the complexity of the open cases in Figure 2 and in particular of 3-Colouring for graphs of diameter 2 and for graphs of diameter 2 and girth 4?

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