

# Two-loop QCD corrections to gluon-gluon scattering\*

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**ABSTRACT:** We present the  $\mathcal{O}(\alpha_s^4)$  virtual QCD corrections to gluon-gluon scattering due to the interference of tree and two-loop amplitudes. We work in conventional dimensional regularisation and give analytic expressions renormalised in the  $\overline{\text{MS}}$  scheme. The structure of the infrared divergences agrees with that predicted by Catani while formulae for the finite remainder are given in terms of logarithms and polylogarithms that are real in the physical region. These results, together with those previously obtained for quark-quark and quark-gluon scattering, complete the two-loop matrix elements needed for the next-to-next-to-leading order contribution to inclusive jet production at hadron colliders.

**KEYWORDS:** QCD, Jets, LEP HERA and SLC Physics, NLO and NNLO Computations.

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# 1. Introduction

Accurate perturbative calculations beyond leading order in quantum chromodynamics (QCD) are an important ingredient in improving our understanding of jet production in current and future high energy collider experiments at the Tevatron and LHC. At present, next-to-leading order calculations have become standard and are used to make comparisons with experimental data. For example, the next-to-leading order  $\mathcal{O}(\alpha_s^3)$  predictions for jet production in  $p\bar{p}$  collisions [1, 2] based on the one-loop matrix elements computed by Ellis and Sexton [3] have been successfully compared with a wide variety of experimental observables using data from the Tevatron and the CERN  $Spp\bar{p}S$ . To date these comparisons have been limited by both experimental and theoretical uncertainties at the 10% level. However, improvements in detector technology, as well as the expected large increases in the luminosity of the colliding particles, should significantly improve the quality of the experimental data and will require more accurate theoretical calculations either to claim new physics or to refine our understanding of QCD.

The theoretical prediction may be improved by including the next-to-next-to-leading order perturbative predictions. This has the effect of (a) reducing the renormalisation scale dependence and (b) improving the matching of the parton level theoretical jet algorithm with the hadron level experimental jet algorithm, because the jet structure can be modeled by the presence of a third parton. The improvement in accuracy expected at next-to-next-to-leading order can be estimated using the renormalisation group equations together with the existing leading and next-to-leading order calculations and is at the 1-2% level for centrally produced jets with a transverse energy,  $E_T$ , of around 100 GeV.

The full next-to-next-to-leading order prediction requires the knowledge of the two-loop  $2 \rightarrow 2$  matrix elements as well as the contributions from the one-loop  $2 \rightarrow 3$  and tree-level  $2 \rightarrow 4$  processes. At large transverse energies,  $E_T \gg m_{\text{quark}}$ , the quark masses may be safely neglected and we therefore focus on the scattering of massless partons. Techniques for computing multiparticle tree amplitudes for  $2 \rightarrow 4$  processes, and the associated crossed processes, are well understood. For example, the helicity amplitudes for the six gluon  $gg \rightarrow gggg$ , four gluon-two quark  $\bar{q}q \rightarrow gggg$ , two gluon-four quark  $\bar{q}q \rightarrow \bar{q}'q'gg$  and six quark  $\bar{q}q \rightarrow \bar{q}'q'\bar{q}''q''$  have been computed in Refs. [4, 5, 6, 7]. Similarly, amplitudes for the one-loop  $2 \rightarrow 3$  parton sub-processes  $gg \rightarrow ggg$ ,  $\bar{q}q \rightarrow ggg$ ,  $\bar{q}q \rightarrow \bar{q}'q'g$ , and processes related to these by crossing symmetry, are also known and are available in [8, 9, 10] respectively.

Although the two-loop contribution for gluon-gluon scattering in  $N = 4$  supersymmetric models has been known for some time [11], the evaluation of the two-loop  $2 \rightarrow 2$  contributions for QCD processes has been a challenge for the past few years. This was mainly due to a lack of knowledge about planar and crossed double box integrals that arise at two-loops. In the massless parton limit and in dimensional

regularisation, analytic expressions for these basic scalar integrals have now been provided by Smirnov [12] and Tausk [13] as series in  $\epsilon = (4 - D)/2$ , where  $D$  is the space-time dimension, together with algorithms for reducing tensor integral to a basis set of known scalar (master) integrals [14, 15]. This makes the calculation of the two-loop amplitudes for  $2 \rightarrow 2$  QCD scattering processes possible.

Following on from the pioneering work of Bern, Dixon and Ghinculov [16] who completed the two-loop calculation of physical  $2 \rightarrow 2$  scattering amplitudes for the QED processes  $e^+e^- \rightarrow \mu^+\mu^-$  and  $e^+e^- \rightarrow e^-e^+$ , we have studied the  $\mathcal{O}(\alpha_s^4)$  contributions arising from the interference of two-loop and tree-level graphs for the QCD processes of quark-quark [17, 18, 19] and quark-gluon [20] scattering. In these papers we presented analytic expressions for the infrared pole structure (that ultimately cancels against contributions from the  $2 \rightarrow 3$  and  $2 \rightarrow 4$  processes), which agrees with that anticipated by Catani [21], as well as the finite remainder.

To complete the set of two-loop contributions to parton-parton scattering requires the study of (non-supersymmetric) gluon-gluon scattering. Bern, Dixon and Kosower [22] were the first to address this process and provided analytic expressions for the maximal-helicity-violating two-loop amplitude. Unfortunately, this amplitude vanishes at tree level and does not contribute to  $2 \rightarrow 2$  scattering at next-to-next-to-leading order  $\mathcal{O}(\alpha_s^4)$ . It is therefore the goal of this paper to provide analytic expressions for the  $\mathcal{O}(\alpha_s^4)$  two-loop corrections to gluon-gluon scattering

$$g + g \rightarrow g + g. \tag{1.1}$$

As is in Refs. [17, 18, 19, 20], we use the  $\overline{\text{MS}}$  renormalisation scheme to remove the ultraviolet singularities and conventional dimensional regularisation, where all external particles are treated in  $D$  dimensions. We provide expressions for the interference of tree-level and two-loop graphs. The infrared-pole structure agrees with that obtained using Catani's general factorisation formulae [21]. The finite remainders are the main new results presented in this paper and we give explicit analytic expressions valid for the gluon-gluon scattering process in terms of logarithms and polylogarithms that are real in the physical domain. For simplicity, we decompose our results according to the powers of the number of colours  $N$  and the number of light-quark flavours  $N_F$ .

Our paper is organised as follows. We first establish our notation in Sec. 2. Analytic expressions for the interference of the two-loop and tree-level amplitudes are given in Sec. 3. In Sec. 3.1 we adopt the notation used in Ref. [21], to isolate the infrared singularity structure of the two-loop amplitudes in the  $\overline{\text{MS}}$  scheme in terms of the one-loop bubble integral in  $D = 4 - 2\epsilon$  and the one-loop box integral in  $D = 6 - 2\epsilon$ . Analytic formulae connecting these integrals in the various kinematic regions are given in Appendix A. We demonstrate that the anticipated singularity structure agrees with our explicit calculation. The finite  $\mathcal{O}(\epsilon^0)$  is given in Sec. 3.2

in terms of logarithms and polylogarithms that have no imaginary parts. Finally we conclude with a brief summary of the results in Sec. 4.

## 2. Notation

For calculational purposes, the process we consider is

$$g(p_1) + g(p_2) + g(p_3) + g(p_4) \rightarrow 0, \quad (2.1)$$

where the gluons are all incoming with light-like momenta, satisfying

$$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0, \quad p_i^2 = 0.$$

The associated Mandelstam variables are given by

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_1 + p_3)^2, \quad s + t + u = 0. \quad (2.2)$$

The gluons also carry colour indexes,  $a_i$ , in the adjoint representation.

We work in conventional dimensional regularisation treating all external quark and gluon states in  $D$  dimensions and renormalise the ultraviolet divergences in the  $\overline{\text{MS}}$  scheme. The renormalised four point amplitude in the  $\overline{\text{MS}}$  scheme can be written

$$|\mathcal{M}\rangle = 4\pi\alpha_s \left[ |\mathcal{M}^{(0)}\rangle + \left(\frac{\alpha_s}{2\pi}\right) |\mathcal{M}^{(1)}\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2 |\mathcal{M}^{(2)}\rangle + \mathcal{O}(\alpha_s^3) \right], \quad (2.3)$$

where  $\alpha_s \equiv \alpha_s(\mu^2)$  is the running coupling at renormalisation scale  $\mu$  and the  $|\mathcal{M}^{(i)}\rangle$  represents the colour-space vector describing the renormalised  $i$ -loop amplitude. The dependence on both renormalisation scale  $\mu$  and renormalisation scheme is implicit.

We denote the squared amplitude summed over spins and colours by

$$\langle \mathcal{M} | \mathcal{M} \rangle = \sum |\mathcal{M}(g + g \rightarrow g + g)|^2 = \mathcal{D}(s, t, u). \quad (2.4)$$

which is symmetric under the exchange of  $s$ ,  $t$  and  $u$ . The function  $\mathcal{D}$  can be expanded perturbatively to yield

$$\mathcal{D}(s, t, u) = 16\pi^2\alpha_s^2 \left[ \mathcal{D}^4(s, t, u) + \left(\frac{\alpha_s}{2\pi}\right) \mathcal{D}^6(s, t, u) + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{D}^8(s, t, u) + \mathcal{O}(\alpha_s^3) \right], \quad (2.5)$$

where

$$\begin{aligned} \mathcal{D}^4(s, t, u) &= \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle \\ &= 16VN^2(1 - \epsilon)^2 \left( 3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{st}{u^2} \right), \end{aligned} \quad (2.6)$$

$$\mathcal{D}^6(s, t, u) = \left( \langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(1)} | \mathcal{M}^{(0)} \rangle \right), \quad (2.7)$$

$$\mathcal{D}^8(s, t, u) = \left( \langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle \right), \quad (2.8)$$

where  $N$  is the number of colours and  $V = N^2 - 1$ . Expressions for  $\mathcal{D}^6$  are given in Ref. [3] using dimensional regularisation to isolate the infrared and ultraviolet singularities.

In the following sections, we present expressions for the infrared singular and finite two-loop contributions to  $\mathcal{D}^8$

$$\mathcal{D}^{8(2\times 0)}(s, t, u) = \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle. \quad (2.9)$$

We defer the self-interference of the one-loop amplitudes

$$\mathcal{D}^{8(1\times 1)}(s, t, u) = \langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle, \quad (2.10)$$

to a later paper.

As in Refs. [17, 18, 19, 20], we use QGRAF [23] to produce the two-loop Feynman diagrams to construct  $|\mathcal{M}^{(2)}\rangle$ . We then project by  $\langle \mathcal{M}^{(0)} |$  and perform the summation over colours and spins. It should be noted that when summing over the gluon polarisations, we ensure that the polarisations states are transversal (i.e. physical) by the use of an axial gauge

$$\sum_{\text{spins}} \epsilon_i^\mu \epsilon_i^{\nu*} = -g^{\mu\nu} + \frac{n_i^\mu p_i^\nu + n_i^\nu p_i^\mu}{n_i \cdot p_i} \quad (2.11)$$

where  $p_i$  is the momentum,  $\epsilon_i$  is the polarisation vector and  $n_i$  is an arbitrary light-like 4-vector for gluon  $i$ . For simplicity, we choose  $n_1^\mu = p_2^\mu$ ,  $n_2^\mu = p_1^\mu$ ,  $n_3^\mu = p_4^\mu$  and  $n_4^\mu = p_3^\mu$ . Finally, the trace over the Dirac matrices is carried out in  $D$  dimensions using conventional dimensional regularisation. It is then straightforward to identify the scalar and tensor integrals present and replace them with combinations of the basis set of master integrals using the tensor reduction of two-loop integrals described in [14, 15, 24], based on integration-by-parts [25] and Lorentz invariance [26] identities. The final result is a combination of master integrals in  $D = 4 - 2\epsilon$  for which the expansions around  $\epsilon = 0$  are given in [12, 13, 14, 15, 24, 27, 28, 29, 30].

### 3. Two-loop contribution

We further decompose the two-loop contributions as a sum of two terms

$$\mathcal{D}^{8(2\times 0)}(s, t, u) = \mathcal{Poles}(s, t, u) + \mathcal{Finite}(s, t, u). \quad (3.1)$$

$\mathcal{Poles}$  contains infrared singularities that will be analytically canceled by those occurring in radiative processes of the same order (ultraviolet divergences are removed by renormalisation).  $\mathcal{Finite}$  is the remainder which is finite as  $\epsilon \rightarrow 0$ .

### 3.1 Infrared Pole Structure

Following the procedure outlined in Ref. [21], we can write the infrared pole structure of the two loop contributions renormalised in the  $\overline{\text{MS}}$  scheme in terms of the tree and unrenormalised one-loop amplitudes,  $|\mathcal{M}^{(0)}\rangle$  and  $|\mathcal{M}^{(1,un)}\rangle$  respectively, as

$$\begin{aligned} \mathcal{Poles} = 2 \text{Re} \left[ & -\frac{1}{2} \langle \mathcal{M}^{(0)} | \mathbf{I}^{(1)}(\epsilon) \mathbf{I}^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle - \frac{2\beta_0}{\epsilon} \langle \mathcal{M}^{(0)} | \mathbf{I}^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle \right. \\ & + \langle \mathcal{M}^{(0)} | \mathbf{I}^{(1)}(\epsilon) | \mathcal{M}^{(1,un)} \rangle \\ & + e^{-\epsilon\gamma} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left( \frac{\beta_0}{\epsilon} + K \right) \langle \mathcal{M}^{(0)} | \mathbf{I}^{(1)}(2\epsilon) | \mathcal{M}^{(0)} \rangle \\ & \left. + \langle \mathcal{M}^{(0)} | \mathbf{H}^{(2)}(\epsilon) | \mathcal{M}^{(0)} \rangle \right] \end{aligned} \quad (3.2)$$

where the Euler constant  $\gamma = 0.5772\dots$ . The first coefficient of the QCD beta function,  $\beta_0$ , for  $N_F$  (massless) quark flavours is

$$\beta_0 = \frac{11C_A - 4T_R N_F}{6}, \quad C_A = N, \quad T_R = \frac{1}{2}, \quad (3.3)$$

and the constant  $K$  is

$$K = \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R N_F. \quad (3.4)$$

Note that the unrenormalised one-loop amplitude  $|\mathcal{M}^{(1,un)}\rangle$  is what is obtained by direct Feynman diagram evaluation of the one-loop graphs.

It is convenient to decompose  $|\mathcal{M}^{(0)}\rangle$  and  $|\mathcal{M}^{(1,un)}\rangle$  in terms of  $SU(N)$  matrices in the fundamental representation,  $T^a$ , so that the tree amplitude may be written as [31, 32, 33]

$$|\mathcal{M}^{(0)}\rangle = \sum_{P(2,3,4)} \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \mathcal{A}_4^{\text{tree}}(1, 2, 3, 4), \quad (3.5)$$

while the one-loop amplitude has the form [34, 35]

$$\begin{aligned} |\mathcal{M}^{(1,un)}\rangle = & N \sum_{P(2,3,4)} \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \mathcal{A}_{4;1}^{[1]}(1, 2, 3, 4) \\ & + \sum_{Q(2,3,4)} \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}) \mathcal{A}_{4;3}^{[1]}(1, 2, 3, 4) \\ & + N_F \sum_{P(2,3,4)} \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \mathcal{A}_{4;1}^{[1/2]}(1, 2, 3, 4). \end{aligned} \quad (3.6)$$

In these expressions  $\sum_{P(2,3,4)}$  runs over the 6 permutations of indices of gluons 2, 3 and 4 while  $\sum_{Q(2,3,4)}$  includes the three choices of pairs of indices, as it is further detailed in Eq. (3.9). We note that the tree subamplitudes are further related by

cyclic and reflection properties as well as by the dual Ward identity [32, 36] and more general identities [37], while the subleading-colour loop amplitudes  $\mathcal{A}_{4;3}^{[1]}$  are related to the leading-colour amplitudes  $\mathcal{A}_{4;1}^{[1]}$  [34, 35]. Some of these relationships are made explicit using an alternative basis in terms of  $SU(N)$  matrices in the adjoint representation [38].

To evaluate Eq. (3.2) we find it convenient to express  $|\mathcal{M}^{(0)}\rangle$  and  $|\mathcal{M}^{(1,un)}\rangle$  as nine-dimensional vectors in colour space

$$|\mathcal{M}^{(0)}\rangle = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, 0, 0, 0)^T, \quad (3.7)$$

$$|\mathcal{M}^{(1,un)}\rangle = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6, \mathcal{L}_7, \mathcal{L}_8, \mathcal{L}_9)^T, \quad (3.8)$$

where  $()^T$  indicates the transpose vector. Here the  $\mathcal{T}_i$  and  $\mathcal{L}_i$  are the components of  $|\mathcal{M}^{(0)}\rangle$  and  $|\mathcal{M}^{(1,un)}\rangle$  in the colour space spanned by the (non-orthogonal) basis

$$\begin{aligned} \mathcal{C}_1 &= \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}), \\ \mathcal{C}_2 &= \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}), \\ \mathcal{C}_3 &= \text{Tr}(T^{a_1} T^{a_4} T^{a_2} T^{a_3}), \\ \mathcal{C}_4 &= \text{Tr}(T^{a_1} T^{a_3} T^{a_2} T^{a_4}), \\ \mathcal{C}_5 &= \text{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}), \\ \mathcal{C}_6 &= \text{Tr}(T^{a_1} T^{a_4} T^{a_3} T^{a_2}), \\ \mathcal{C}_7 &= \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}), \\ \mathcal{C}_8 &= \text{Tr}(T^{a_1} T^{a_3}) \text{Tr}(T^{a_2} T^{a_4}), \\ \mathcal{C}_9 &= \text{Tr}(T^{a_1} T^{a_4}) \text{Tr}(T^{a_2} T^{a_3}). \end{aligned} \quad (3.9)$$

The tree and loop amplitudes  $\mathcal{T}_i$  and  $\mathcal{L}_i$  are directly obtained in terms of  $\mathcal{A}_4^{\text{tree}}$ ,  $\mathcal{A}_{4;1}^{[1]}$ ,  $\mathcal{A}_{4;3}^{[1]}$  and  $\mathcal{A}_{4;1}^{[1/2]}$  by reading off from Eqs. (3.5) and (3.6). As we will see, the amplitudes themselves are not required since we compute the interference of tree and loop amplitudes directly.

In the same colour basis, the infrared-singularity operator  $\mathbf{I}^{(1)}(\epsilon)$  introduced by Catani [21] has the form

$$\mathbf{I}^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \left( \frac{1}{\epsilon^2} + \frac{\beta_0}{N\epsilon} \right) \times \begin{pmatrix} N(\mathbf{S}+\mathbf{T}) & 0 & 0 & 0 & 0 & 0 & (\mathbf{T}-\mathbf{U}) & 0 & (\mathbf{S}-\mathbf{U}) \\ 0 & N(\mathbf{S}+\mathbf{U}) & 0 & 0 & 0 & 0 & (\mathbf{U}-\mathbf{T}) & (\mathbf{S}-\mathbf{T}) & 0 \\ 0 & 0 & N(\mathbf{T}+\mathbf{U}) & 0 & 0 & 0 & 0 & (\mathbf{T}-\mathbf{S}) & (\mathbf{U}-\mathbf{S}) \\ 0 & 0 & 0 & N(\mathbf{T}+\mathbf{U}) & 0 & 0 & 0 & (\mathbf{T}-\mathbf{S}) & (\mathbf{U}-\mathbf{S}) \\ 0 & 0 & 0 & 0 & N(\mathbf{S}+\mathbf{U}) & 0 & (\mathbf{U}-\mathbf{T}) & (\mathbf{S}-\mathbf{T}) & 0 \\ 0 & 0 & 0 & 0 & 0 & N(\mathbf{S}+\mathbf{T}) & (\mathbf{T}-\mathbf{U}) & 0 & (\mathbf{S}-\mathbf{U}) \\ (\mathbf{S}-\mathbf{U}) & (\mathbf{S}-\mathbf{T}) & 0 & 0 & (\mathbf{S}-\mathbf{T}) & (\mathbf{S}-\mathbf{U}) & 2N\mathbf{S} & 0 & 0 \\ 0 & (\mathbf{U}-\mathbf{T}) & (\mathbf{U}-\mathbf{S}) & (\mathbf{U}-\mathbf{S}) & (\mathbf{U}-\mathbf{T}) & 0 & 0 & 2N\mathbf{U} & 0 \\ (\mathbf{T}-\mathbf{U}) & 0 & (\mathbf{T}-\mathbf{S}) & (\mathbf{T}-\mathbf{S}) & 0 & (\mathbf{T}-\mathbf{U}) & 0 & 0 & 2N\mathbf{T} \end{pmatrix} \quad (3.10)$$

where

$$\mathbf{S} = \left( -\frac{\mu^2}{s} \right)^\epsilon, \quad \mathbf{T} = \left( -\frac{\mu^2}{t} \right)^\epsilon, \quad \mathbf{U} = \left( -\frac{\mu^2}{u} \right)^\epsilon. \quad (3.11)$$

The matrix  $\mathbf{I}^{(1)}(\epsilon)$  acts directly as a rotation matrix on  $|\mathcal{M}^{(0)}\rangle$  and  $|\mathcal{M}^{(1,un)}\rangle$  in colour space, to give a new colour vector  $|X\rangle$ , equal to  $\mathbf{I}^{(1)}(\epsilon)|\mathcal{M}^{(0)}\rangle$ ,  $\mathbf{I}^{(1)}(\epsilon)\mathbf{I}^{(1)}(\epsilon)|\mathcal{M}^{(0)}\rangle$  or  $\mathbf{I}^{(1)}(\epsilon)|\mathcal{M}^{(1,un)}\rangle$ .

The contraction of the colour vector  $|X\rangle$  with the conjugate tree amplitude obeys the rule

$$\langle \mathcal{M}^{(0)} | X \rangle = \sum_{\text{spins}} \sum_{\text{colours}} \sum_{i,j=1}^9 \mathcal{T}_i^* X_j \mathcal{C}_i^* \mathcal{C}_j. \quad (3.12)$$

In evaluating these contractions, we typically encounter  $\sum_{\text{colours}} \mathcal{C}_i^* \mathcal{C}_j$  which is given by the  $ij$  component of the symmetric matrix  $\mathcal{C}$

$$\mathcal{C} = \frac{V}{16N^2} \begin{pmatrix} C_1 & C_2 & C_2 & C_2 & C_2 & C_3 & NV & -N & NV \\ C_2 & C_1 & C_2 & C_2 & C_3 & C_2 & NV & NV & -N \\ C_2 & C_2 & C_1 & C_3 & C_2 & C_2 & -N & NV & NV \\ C_2 & C_2 & C_3 & C_1 & C_2 & C_2 & -N & NV & NV \\ C_2 & C_3 & C_2 & C_2 & C_1 & C_2 & NV & NV & -N \\ C_3 & C_2 & C_2 & C_2 & C_2 & C_1 & NV & -N & NV \\ NV & NV & -N & -N & NV & NV & N^2V & N^2 & N^2 \\ -N & NV & NV & NV & NV & -N & N^2 & N^2V & N^2 \\ NV & -N & NV & NV & -N & NV & N^2 & N^2 & N^2V \end{pmatrix}, \quad (3.13)$$

with

$$C_1 = N^4 - 3N^2 + 3, \quad C_2 = 3 - N^2, \quad C_3 = 3 + N^2. \quad (3.14)$$

Similarly, we find that the interference of the tree-level amplitudes  $\sum_{\text{spins}} \mathcal{T}_i^* \mathcal{T}_j$  is given by  $\mathcal{T}\mathcal{T}_{ij}$ , where

$$\mathcal{T}\mathcal{T} = \frac{64(1-\epsilon)^2(t^2 + ut + u^2)^2}{s^2t^2u^2} \mathcal{V}^T \mathcal{V}, \quad (3.15)$$

and the vector  $\mathcal{V}$  is

$$\mathcal{V} = (u, t, s, s, t, u, 0, 0, 0), \quad (3.16)$$

while the interference of the tree-level amplitudes with one-loop amplitudes  $\sum_{\text{spins}} \mathcal{T}_i^* \mathcal{L}_j$  is given by  $\mathcal{T}\mathcal{L}_{ij}$ , where

$$\mathcal{T}\mathcal{L} = \mathcal{V}^T \mathcal{W}, \quad (3.17)$$

and the vector  $\mathcal{W}$  is

$$\mathcal{W} = (\mathcal{F}(s, t), \mathcal{F}(s, u), \mathcal{F}(u, t), \mathcal{F}(u, t), \mathcal{F}(s, u), \mathcal{F}(s, t), \mathcal{G}, \mathcal{G}, \mathcal{G}). \quad (3.18)$$



Here the function  $\mathcal{F}(s, t)$  is symmetric under the exchange of  $s$  and  $t$ , while  $\mathcal{G}$  is symmetric under the exchange of any two Mandelstam invariants, so that

$$\mathcal{F}(s, t) = f_1(s, t, u) + f_1(t, s, u), \quad (3.19)$$

$$\mathcal{G} = f_2(s, t, u) + f_2(s, u, t) + f_2(t, s, u) + f_2(t, u, s) + f_2(u, s, t) + f_2(u, t, s). \quad (3.20)$$

Here  $f_1$  and  $f_2$  are given in terms of the one-loop box integral in  $D = 6 - 2\epsilon$  dimensions and the one-loop bubble graph in  $D = 4 - 2\epsilon$ ,

$$\begin{aligned} f_1(s, t, u) = & \frac{16N(1-2\epsilon)}{s^2t^2} \left[ 2(1-\epsilon)^2 (s^4 + s^3t + st^3 + t^4) + 3(1-5\epsilon)s^2t^2 \right] \text{Box}^6(s, t) \\ & + \frac{8N_F(1-2\epsilon)}{st} \left[ (1-\epsilon)^2 (s^2 + t^2) + \epsilon(1+3\epsilon)st \right] \text{Box}^6(s, t) \\ & - \frac{16N(1-\epsilon)}{s^2t^2u\epsilon(3-2\epsilon)} \left[ (12-22\epsilon+12\epsilon^2+2\epsilon^3)s^4 + (24-58\epsilon+50\epsilon^2-6\epsilon^3-2\epsilon^4)s^3t \right. \\ & \quad \left. + (36-99\epsilon+93\epsilon^2-24\epsilon^3-2\epsilon^4)s^2t^2 + (1-\epsilon)(24-50\epsilon+23\epsilon^2)st^3 \right. \\ & \quad \left. + 4(1-\epsilon)(1-2\epsilon)(3-2\epsilon)t^4 \right] \text{Bub}(t) \\ & + \frac{16N_F}{st^2u(3-2\epsilon)} \left[ (4-12\epsilon+16\epsilon^2-4\epsilon^3)s^3 + (3-10\epsilon+23\epsilon^2-8\epsilon^3)s^2t \right. \\ & \quad \left. + (6-15\epsilon+21\epsilon^2-8\epsilon^3)st^2 + (1-\epsilon)(5-6\epsilon+2\epsilon^2)t^3 \right] \text{Bub}(t), \end{aligned} \quad (3.21)$$

$$\begin{aligned} f_2(s, t, u) = & \frac{32(1-2\epsilon)}{u^2} \left[ -4(1-\epsilon)^2st + 3(1-5\epsilon)u^2 \right] \text{Box}^6(u, t) \\ & + \frac{32(1-\epsilon)}{\epsilon su^2} \left[ 4(1-2\epsilon)(1-\epsilon)t^2 + (8-17\epsilon)(1-\epsilon)ut \right. \\ & \quad \left. + (6-20\epsilon+15\epsilon^2+\epsilon^3)u^2 \right] \text{Bub}(s). \end{aligned} \quad (3.22)$$

Series expansions around  $\epsilon = 0$  for the one-loop integrals are given in Appendix A.

Finally, the last term of Eq. (3.2) that involves  $\mathbf{H}^{(2)}(\epsilon)$  produces only a single pole in  $\epsilon$  and is given by

$$\langle \mathcal{M}^{(0)} | \mathbf{H}^{(2)}(\epsilon) | \mathcal{M}^{(0)} \rangle = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)} H^{(2)} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle \quad (3.23)$$

where the constant  $H^{(2)}$  is

$$H^{(2)} = \left( 2\zeta_3 + \frac{5}{3} + \frac{11}{36}\pi^2 \right) N^2 + \frac{20}{27} N_F^2 + \left( -\frac{\pi^2}{18} - \frac{89}{27} \right) NN_F - \frac{N_F}{N}, \quad (3.24)$$

and  $\zeta_n$  is the Riemann Zeta function with  $\zeta_2 = \pi^2/6$  and  $\zeta_3 = 1.202056\dots$ . We note that  $H^{(2)}$  is renormalisation-scheme dependent and Eq. (3.24) is valid in the  $\overline{\text{MS}}$  scheme. We also note that Eq. (3.24) differs from the corresponding expressions found in the singularity structure of two-loop quark-quark and quark-gluon scattering. This

is due to double emissions from the gluons. In fact, we note that  $H^{(2)}$  for quark-gluon scattering is the average of the  $H^{(2)}$  for gluon-gluon scattering and quark-quark scattering, as may be expected by counting the number of different types of radiating partons.

It can be easily noted that the leading infrared singularity in Eq. (3.2) is  $\mathcal{O}(1/\epsilon^4)$ . It is a very stringent check on the reliability of our calculation that the pole structure obtained by computing the Feynman diagrams directly and introducing series expansions in  $\epsilon$  for the scalar master integrals agrees with Eq. (3.2) through to  $\mathcal{O}(1/\epsilon)$ . We therefore construct the finite remainder by subtracting Eq. (3.2) from the full result.

### 3.2 Finite contributions

The finite two-loop contribution to  $\mathcal{D}^8(s, t, u)$  is defined as

$$\mathcal{F}inite(s, t, u) = \mathcal{D}^{8(2\times 0)}(s, t, u) - \mathcal{P}oles(s, t, u), \quad (3.25)$$

where we subtract the series expansions of both  $\mathcal{D}^{8(2\times 0)}(s, t, u)$  and  $\mathcal{P}oles(s, t, u)$  and set  $\epsilon \rightarrow 0$ . As usual, the polylogarithms  $\text{Li}_n(w)$  are defined by

$$\begin{aligned} \text{Li}_n(w) &= \int_0^w \frac{dt}{t} \text{Li}_{n-1}(t) \quad \text{for } n = 2, 3, 4 \\ \text{Li}_2(w) &= - \int_0^w \frac{dt}{t} \log(1-t). \end{aligned} \quad (3.26)$$

Using the standard polylogarithm identities [39], we retain the polylogarithms with arguments  $x$ ,  $1-x$  and  $(x-1)/x$ , where

$$x = -\frac{t}{s}, \quad y = -\frac{u}{s} = 1-x, \quad z = -\frac{u}{t} = \frac{x-1}{x}. \quad (3.27)$$

For convenience, we also introduce the following logarithms

$$X = \log\left(\frac{-t}{s}\right), \quad Y = \log\left(\frac{-u}{s}\right), \quad S = \log\left(\frac{s}{\mu^2}\right), \quad (3.28)$$

where  $\mu$  is the renormalisation scale.

We choose to present our results by grouping terms according to the power of the number of colours  $N$  and the number of light quarks  $N_F$ , so that

$$\mathcal{F}inite(s, t, u) = V \left( N^4 A + N^2 B + N^3 N_F C + N N_F D + N^2 N_F^2 E + N_F^2 F \right), \quad (3.29)$$

where

$$A = \left\{ \left( 48 \text{Li}_4(x) - 48 \text{Li}_4(y) - 128 \text{Li}_4(z) + 40 \text{Li}_3(x) X - 64 \text{Li}_3(x) Y - \frac{98}{3} \text{Li}_3(x) \right. \right.$$

$$\begin{aligned}
& +64 \operatorname{Li}_3(y) X - 40 \operatorname{Li}_3(y) Y + 18 \operatorname{Li}_3(y) + \frac{98}{3} \operatorname{Li}_2(x) X - \frac{16}{3} \operatorname{Li}_2(x) \pi^2 - 18 \operatorname{Li}_2(y) Y \\
& - \frac{37}{6} X^4 + 28 X^3 Y - \frac{23}{3} X^3 - 16 X^2 Y^2 + \frac{49}{3} X^2 Y - \frac{35}{3} X^2 \pi^2 - \frac{38}{3} X^2 - \frac{22}{3} S X^2 \\
& - \frac{20}{3} X Y^3 - 9 X Y^2 + 8 X Y \pi^2 + 10 X Y - \frac{31}{12} X \pi^2 - 22 \zeta_3 X + \frac{22}{3} S X + \frac{37}{27} X \\
& + \frac{11}{6} Y^4 - \frac{41}{9} Y^3 - \frac{11}{3} Y^2 \pi^2 - \frac{22}{3} S Y^2 + \frac{266}{9} Y^2 - \frac{35}{12} Y \pi^2 + \frac{418}{9} S Y + \frac{257}{9} Y \\
& + 18 \zeta_3 Y - \frac{31}{30} \pi^4 - \frac{11}{9} S \pi^2 + \frac{31}{9} \pi^2 + \frac{242}{9} S^2 + \frac{418}{9} \zeta_3 + \frac{2156}{27} S \\
& - \frac{11093}{81} - 8 S \zeta_3 \Big) \frac{t^2}{s^2} \\
& + \left( -256 \operatorname{Li}_4(x) - 96 \operatorname{Li}_4(y) + 96 \operatorname{Li}_4(z) + 80 \operatorname{Li}_3(x) X + 48 \operatorname{Li}_3(x) Y - \frac{64}{3} \operatorname{Li}_3(x) \right. \\
& - 48 \operatorname{Li}_3(y) X + 96 \operatorname{Li}_3(y) Y - \frac{304}{3} \operatorname{Li}_3(y) + \frac{64}{3} \operatorname{Li}_2(x) X - \frac{32}{3} \operatorname{Li}_2(x) \pi^2 + \frac{304}{3} \operatorname{Li}_2(y) Y \\
& + \frac{26}{3} X^4 - \frac{64}{3} X^3 Y - \frac{64}{3} X^3 + 20 X^2 Y^2 + \frac{136}{3} X^2 Y + 24 X^2 \pi^2 + 76 X^2 - \frac{88}{3} S X^2 \\
& + \frac{8}{3} X Y^3 + \frac{104}{3} X Y^2 - \frac{16}{3} X Y \pi^2 + \frac{176}{3} S X Y - \frac{136}{3} X Y - \frac{50}{3} X \pi^2 - 48 \zeta_3 X \\
& + \frac{2350}{27} X + \frac{440}{3} S X + 4 Y^4 - \frac{176}{9} Y^3 + \frac{4}{3} Y^2 \pi^2 - \frac{176}{3} S Y^2 - \frac{494}{9} Y \pi^2 + \frac{5392}{27} Y \\
& - 64 \zeta_3 Y + \frac{496}{45} \pi^4 - \frac{308}{9} S \pi^2 + \frac{200}{9} \pi^2 + \frac{968}{9} S^2 + \frac{8624}{27} S - \frac{44372}{81} \\
& \left. + \frac{1864}{9} \zeta_3 - 32 S \zeta_3 \right) \frac{t}{u} \\
& + \left( \frac{88}{3} \operatorname{Li}_3(x) - \frac{88}{3} \operatorname{Li}_2(x) X + 2 X^4 - 8 X^3 Y - \frac{220}{9} X^3 + 12 X^2 Y^2 + \frac{88}{3} X^2 Y + \frac{8}{3} X^2 \pi^2 \right. \\
& - \frac{88}{3} S X^2 + \frac{304}{9} X^2 - 8 X Y^3 - \frac{16}{3} X Y \pi^2 + \frac{176}{3} S X Y - \frac{77}{3} X \pi^2 + \frac{1616}{27} X \\
& + \frac{968}{9} S X - 8 \zeta_3 X + 4 Y^4 - \frac{176}{9} Y^3 - \frac{20}{3} Y^2 \pi^2 - \frac{176}{3} S Y^2 - \frac{638}{9} Y \pi^2 - 16 \zeta_3 Y \\
& + \frac{5392}{27} Y - \frac{4}{15} \pi^4 - \frac{308}{9} S \pi^2 - 20 \pi^2 - 32 S \zeta_3 + \frac{1408}{9} \zeta_3 + \frac{968}{9} S^2 - \frac{44372}{81} \\
& \left. + \frac{8624}{27} S \right) \frac{t^2}{u^2} \\
& + \left( \frac{44}{3} \operatorname{Li}_3(x) - \frac{44}{3} \operatorname{Li}_2(x) X - X^4 + \frac{110}{9} X^3 - \frac{22}{3} X^2 Y + \frac{14}{3} X^2 \pi^2 + \frac{44}{3} S X^2 \right. \\
& - \frac{152}{9} X^2 - 10 X Y + \frac{11}{2} X \pi^2 + 4 \zeta_3 X - \frac{484}{9} S X - \frac{808}{27} X + \frac{7}{30} \pi^4 - \frac{31}{9} \pi^2 \\
& \left. + \frac{11}{9} S \pi^2 - \frac{418}{9} \zeta_3 - \frac{242}{9} S^2 - \frac{2156}{27} S + 8 S \zeta_3 + \frac{11093}{81} \right) \frac{ut}{s^2}
\end{aligned}$$

$$\begin{aligned}
& + \left( -176 \text{Li}_4(x) + 88 \text{Li}_3(x) X - 168 \text{Li}_3(x) Y - \frac{206}{3} \text{Li}_3(x) + \frac{206}{3} \text{Li}_2(x) X \right. \\
& + \frac{65}{6} X^4 - \frac{40}{3} X^3 Y - \frac{295}{9} X^3 - 15 X^2 Y^2 + \frac{115}{3} X^2 Y + \frac{29}{3} X^2 \pi^2 - \frac{670}{9} X^2 \\
& - \frac{242}{3} S X^2 + \frac{64}{3} X Y \pi^2 + \frac{209}{3} X Y + 44 S X Y - \frac{1811}{36} X \pi^2 + \frac{8983}{27} X \\
& + \frac{1870}{9} S X - 18 \zeta_3 X + \frac{31}{20} \pi^4 - \frac{361}{18} \pi^2 - \frac{517}{18} S \pi^2 + \frac{1331}{9} S^2 + \frac{12452}{27} S \\
& \left. + \frac{1543}{9} \zeta_3 - \frac{129475}{162} - 44 S \zeta_3 \right) \} + \left\{ u \leftrightarrow t \right\}, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
B = & \left\{ \left( -288 \text{Li}_4(x) + 480 \text{Li}_4(y) - 288 \text{Li}_4(z) + 240 \text{Li}_3(x) X - 144 \text{Li}_3(x) Y \right. \right. \\
& + 224 \text{Li}_3(x) + 144 \text{Li}_3(y) X - 432 \text{Li}_3(y) Y - 224 \text{Li}_3(y) + 48 \text{Li}_2(x) X^2 \\
& - 224 \text{Li}_2(x) X - 176 \text{Li}_2(x) \pi^2 + 48 \text{Li}_2(y) Y^2 + 224 \text{Li}_2(y) Y - 16 X^4 + 112 X^3 Y \\
& - \frac{556}{3} X^3 - 48 X^2 Y^2 + 180 X^2 Y - 40 X^2 \pi^2 + 220 X^2 - 32 X Y^3 - 92 X Y^2 \\
& - 16 X Y \pi^2 - \frac{376}{3} X Y - 16 X \pi^2 - 80 X + 96 \zeta_3 X + 8 Y^4 + \frac{292}{3} Y^3 - 32 Y^2 \pi^2 \\
& \left. - \frac{284}{3} Y^2 + 16 Y \pi^2 + 80 Y - 96 \zeta_3 Y + \frac{38}{5} \pi^4 - 18 \pi^2 \right) \frac{t^2}{s^2} \\
& + \left( -576 \text{Li}_4(x) + 384 \text{Li}_4(y) - 1152 \text{Li}_4(z) + 1056 \text{Li}_3(x) X - 768 \text{Li}_3(x) Y \right. \\
& + 448 \text{Li}_3(x) + 768 \text{Li}_3(y) X - 768 \text{Li}_3(y) Y + 896 \text{Li}_3(y) - 192 \text{Li}_2(x) X^2 \\
& - 448 \text{Li}_2(x) X - 544 \text{Li}_2(x) \pi^2 - 384 \text{Li}_2(y) X Y - 896 \text{Li}_2(y) Y - 28 X^4 + 144 X^3 Y \\
& + \frac{320}{3} X^3 - 336 X^2 Y^2 - 224 X^2 Y - 40 X^2 \pi^2 - 64 X^2 - 32 X Y^3 + 128 X Y^2 \\
& - 64 X Y \pi^2 + \frac{1888}{3} X Y - 288 X \pi^2 + 160 X - 1248 \zeta_3 X - 240 Y^2 \pi^2 - 928 Y \pi^2 \\
& \left. + 768 \zeta_3 Y + \frac{1216}{15} \pi^4 - \frac{1912}{3} \pi^2 - 448 \zeta_3 \right) \frac{t}{u} \\
& + \left( -384 \text{Li}_4(y) - 384 \text{Li}_4(z) + 384 \text{Li}_3(x) X - 384 \text{Li}_3(x) Y + 384 \text{Li}_3(y) X \right. \\
& - 192 \text{Li}_2(x) X^2 - 192 \text{Li}_2(x) \pi^2 - 384 \text{Li}_2(y) X Y - 8 X^4 - 32 X^3 Y - 176 X^3 \\
& - 192 X^2 Y^2 + 352 X^2 Y - 80 X^2 \pi^2 + \frac{752}{3} X^2 - 32 X Y \pi^2 - 176 X \pi^2 - 384 \zeta_3 X \\
& \left. - 96 Y^2 \pi^2 - 352 Y \pi^2 + 384 \zeta_3 Y + 56 \pi^4 - \frac{968}{3} \pi^2 \right) \frac{t^2}{u^2} \\
& + \left( -192 \text{Li}_4(x) + 192 \text{Li}_3(x) X - 96 \text{Li}_2(x) X^2 - 4 X^4 - 32 X^3 Y + 88 X^3 \right. \\
& \left. + 12 X^2 Y^2 - 88 X^2 Y + 48 X^2 \pi^2 - \frac{376}{3} X^2 - 48 X Y \pi^2 + \frac{376}{3} X Y \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{15} \pi^4 + 18 \pi^2 \Big) \frac{ut}{s^2} \\
& + \left( 48 \operatorname{Li}_3(x) X + 144 \operatorname{Li}_3(x) Y + 672 \operatorname{Li}_3(x) - 48 \operatorname{Li}_2(x) X^2 - 672 \operatorname{Li}_2(x) X + 16 X^4 \right. \\
& - 32 X^3 Y - \frac{4}{3} X^3 + 24 X^2 Y^2 + 12 X^2 Y - 192 X^2 \pi^2 + \frac{1444}{3} X^2 + 72 X Y \pi^2 \\
& \left. + \frac{80}{3} X Y - 624 X \pi^2 + 80 X - 288 \zeta_3 X + \frac{509}{15} \pi^4 - 707 \pi^2 - 36 - 2800 \zeta_3 \right) \Big\} \\
& + \left\{ u \leftrightarrow t \right\}, \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
C = & \left\{ \left( -24 \operatorname{Li}_4(x) + 24 \operatorname{Li}_4(y) + 112 \operatorname{Li}_4(z) - 44 \operatorname{Li}_3(x) X + 56 \operatorname{Li}_3(x) Y + \frac{74}{3} \operatorname{Li}_3(x) \right. \right. \\
& - 56 \operatorname{Li}_3(y) X + 44 \operatorname{Li}_3(y) Y - 22 \operatorname{Li}_3(y) - \frac{74}{3} \operatorname{Li}_2(x) X + \frac{32}{3} \operatorname{Li}_2(x) \pi^2 + 22 \operatorname{Li}_2(y) Y \\
& + \frac{25}{4} X^4 - 26 X^3 Y + 4 X^3 + 14 X^2 Y^2 - \frac{37}{3} X^2 Y + 7 X^2 \pi^2 + \frac{27}{2} X^2 + 5 S X^2 \\
& + \frac{22}{3} X Y^3 + 11 X Y^2 - 4 X Y \pi^2 - 11 X Y + \frac{31}{6} X \pi^2 + 12 \zeta_3 X - \frac{637}{27} X - \frac{26}{3} S X \\
& - \frac{19}{12} Y^4 - \frac{16}{9} Y^3 + \frac{7}{3} Y^2 \pi^2 - \frac{221}{18} Y^2 - \frac{7}{3} S Y^2 - \frac{25}{6} Y \pi^2 + \frac{175}{9} Y - 12 \zeta_3 Y \\
& \left. - \frac{98}{9} S Y + \frac{1}{5} \pi^4 + \frac{2}{9} S \pi^2 + \frac{203}{54} \pi^2 - \frac{4}{9} \zeta_3 - \frac{88}{9} S^2 + \frac{4849}{162} - \frac{386}{27} S \right) \frac{t^2}{s^2} \\
& + \left( 224 \operatorname{Li}_4(x) + 48 \operatorname{Li}_4(y) - 48 \operatorname{Li}_4(z) - 88 \operatorname{Li}_3(x) X - 24 \operatorname{Li}_3(x) Y + \frac{124}{3} \operatorname{Li}_3(x) \right. \\
& + 24 \operatorname{Li}_3(y) X - 48 \operatorname{Li}_3(y) Y + \frac{280}{3} \operatorname{Li}_3(y) - \frac{124}{3} \operatorname{Li}_2(x) X + \frac{64}{3} \operatorname{Li}_2(x) \pi^2 \\
& - \frac{280}{3} \operatorname{Li}_2(y) Y - \frac{31}{6} X^4 + 6 X^3 Y - \frac{4}{3} X^3 - 3 X^2 Y^2 - \frac{56}{3} X^2 Y - \frac{55}{3} X^2 \pi^2 - 2 S X^2 \\
& - \frac{70}{3} X^2 - 6 X Y^3 - 26 X Y^2 - \frac{2}{3} X Y \pi^2 + 4 S X Y + \frac{148}{3} X Y - \frac{22}{3} X \pi^2 \\
& - \frac{124}{3} S X + \frac{938}{27} X + 64 \zeta_3 X + \frac{32}{9} Y^3 - 3 Y^2 \pi^2 + \frac{32}{3} S Y^2 - \frac{4}{9} Y \pi^2 - \frac{1096}{27} Y \\
& \left. + 24 \zeta_3 Y - \frac{829}{90} \pi^4 - \frac{10}{9} S \pi^2 - \frac{356}{27} \pi^2 - \frac{352}{9} S^2 - \frac{1544}{27} S - \frac{388}{9} \zeta_3 + \frac{9698}{81} \right) \frac{t}{u} \\
& + \left( -\frac{16}{3} \operatorname{Li}_3(x) + \frac{16}{3} \operatorname{Li}_2(x) X + \frac{40}{9} X^3 - \frac{16}{3} X^2 Y + \frac{22}{9} X^2 + \frac{16}{3} S X^2 - \frac{32}{3} S X Y \right. \\
& + \frac{14}{3} X \pi^2 - \frac{224}{27} X - \frac{352}{9} S X + \frac{32}{9} Y^3 + \frac{32}{3} S Y^2 + \frac{116}{9} Y \pi^2 - \frac{1096}{27} Y + \frac{56}{9} S \pi^2 \\
& \left. + \frac{340}{27} \pi^2 - \frac{1544}{27} S + \frac{9698}{81} + \frac{32}{9} \zeta_3 - \frac{352}{9} S^2 \right) \frac{t^2}{u^2}
\end{aligned}$$

$$\begin{aligned}
& + \left( -\frac{8}{3} \text{Li}_3(x) + \frac{8}{3} \text{Li}_2(x) X - \frac{20}{9} X^3 + \frac{4}{3} X^2 Y - \frac{11}{9} X^2 - \frac{8}{3} S X^2 + 11 X Y - X \pi^2 \right. \\
& + \frac{112}{27} X + \frac{176}{9} S X - \frac{2}{9} S \pi^2 - \frac{203}{54} \pi^2 + \frac{88}{9} S^2 - \frac{4849}{162} + \frac{386}{27} S + \frac{4}{9} \zeta_3 \left. \right) \frac{ut}{s^2} \\
& + \left( 136 \text{Li}_4(x) - 68 \text{Li}_3(x) X + 120 \text{Li}_3(x) Y + \frac{206}{3} \text{Li}_3(x) - \frac{206}{3} \text{Li}_2(x) X - \frac{71}{12} X^4 \right. \\
& + \frac{14}{3} X^3 Y - \frac{68}{9} X^3 + 15 X^2 Y^2 + \frac{5}{3} X^2 Y - \frac{29}{3} X^2 \pi^2 + \frac{973}{18} X^2 + \frac{77}{3} S X^2 \\
& - \frac{62}{3} X Y \pi^2 - \frac{139}{6} X Y - 8 S X Y - \frac{317}{18} X \pi^2 - \frac{1375}{27} X - \frac{626}{9} S X + 4 \zeta_3 X \\
& \left. - \frac{47}{30} \pi^4 + \frac{3799}{108} \pi^2 + \frac{47}{9} S \pi^2 - \frac{484}{9} S^2 - \frac{2825}{27} S + \frac{932}{9} \zeta_3 + \frac{70025}{324} \right) \left. \right\} \\
& + \left\{ u \leftrightarrow t \right\}, \tag{3.32}
\end{aligned}$$

$$\begin{aligned}
D = & \left\{ \left( 24 \text{Li}_4(x) - 24 \text{Li}_4(y) + 88 \text{Li}_4(z) - 52 \text{Li}_3(x) X + 36 \text{Li}_3(x) Y - \frac{46}{3} \text{Li}_3(x) \right. \right. \\
& - 36 \text{Li}_3(y) X + 52 \text{Li}_3(y) Y + \frac{46}{3} \text{Li}_3(y) - 4 \text{Li}_2(x) X^2 + \frac{46}{3} \text{Li}_2(x) X + \frac{44}{3} \text{Li}_2(x) \pi^2 \\
& - 16 \text{Li}_2(y) X Y + 4 \text{Li}_2(y) Y^2 - \frac{46}{3} \text{Li}_2(y) Y + \frac{79}{12} X^4 - \frac{82}{3} X^3 Y + \frac{817}{18} X^3 + 3 X^2 Y^2 \\
& - \frac{184}{3} X^2 Y + \frac{13}{3} X^2 \pi^2 - \frac{545}{6} X^2 + \frac{38}{3} X Y^3 + \frac{136}{3} X Y^2 + \frac{4}{3} X Y \pi^2 + \frac{155}{3} X Y \\
& - 10 X \pi^2 - 32 \zeta_3 X + \frac{76}{3} X - \frac{35}{12} Y^4 - \frac{529}{18} Y^3 + 3 Y^2 \pi^2 + \frac{235}{6} Y^2 + 10 Y \pi^2 - \frac{76}{3} Y \\
& \left. + 32 \zeta_3 Y - \frac{11}{30} \pi^4 + \frac{7}{2} \pi^2 + 8 \zeta_3 + 2 S - \frac{55}{6} \right) \frac{t^2}{s^2} \\
& + \left( 176 \text{Li}_4(x) - 48 \text{Li}_4(y) + 48 \text{Li}_4(z) - 104 \text{Li}_3(x) X + 32 \text{Li}_3(x) Y - \frac{92}{3} \text{Li}_3(x) \right. \\
& - 32 \text{Li}_3(y) X + 64 \text{Li}_3(y) Y - \frac{184}{3} \text{Li}_3(y) - 8 \text{Li}_2(x) X^2 + \frac{92}{3} \text{Li}_2(x) X + \frac{160}{3} \text{Li}_2(x) \pi^2 \\
& + 16 \text{Li}_2(y) X Y - 16 \text{Li}_2(y) Y^2 + \frac{184}{3} \text{Li}_2(y) Y - \frac{23}{6} X^4 - 10 X^3 Y - \frac{385}{9} X^3 + 19 X^2 Y^2 \\
& + \frac{161}{3} X^2 Y - 17 X^2 \pi^2 + \frac{80}{3} X^2 - \frac{14}{3} X Y^3 - 87 X Y^2 - \frac{26}{3} X Y \pi^2 - 260 X Y \\
& + \frac{215}{3} X \pi^2 - \frac{152}{3} X + 168 \zeta_3 X + 7 Y^2 \pi^2 + \frac{545}{3} Y \pi^2 + 8 Y - 32 \zeta_3 Y - \frac{571}{90} \pi^4 \\
& \left. + \frac{742}{3} \pi^2 + \frac{188}{3} \zeta_3 - \frac{110}{3} + 8 S \right) \frac{t}{u} \\
& + \left( 32 X^3 - 64 X^2 Y - \frac{310}{3} X^2 + 32 X \pi^2 + 64 Y \pi^2 + 8 Y + \frac{352}{3} \pi^2 + 8 S \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{110}{3} + 32\zeta_3 \Big) \frac{t^2}{u^2} \\
& + \left( -16X^3 + 16X^2Y + \frac{155}{3}X^2 - \frac{155}{3}XY - \frac{7}{2}\pi^2 - 8\zeta_3 - 2S + \frac{55}{6} \right) \frac{ut}{s^2} \\
& + \left( 64\text{Li}_4(x) - 20\text{Li}_3(x)X - 108\text{Li}_3(x)Y - 46\text{Li}_3(x) - 12\text{Li}_2(x)X^2 \right. \\
& + 46\text{Li}_2(x)X + \frac{5}{12}X^4 - 10X^3Y - \frac{401}{18}X^3 - \frac{21}{2}X^2Y^2 - \frac{34}{3}X^2Y - \frac{1}{3}X^2\pi^2 \\
& - \frac{1303}{6}X^2 - \frac{16}{3}XY\pi^2 - \frac{11}{6}XY + \frac{340}{3}X\pi^2 + 104\zeta_3X - \frac{52}{3}X - \frac{67}{20}\pi^4 \\
& \left. + \frac{2981}{12}\pi^2 + 11S + \frac{1166}{3}\zeta_3 - \frac{461}{12} \right) \Big\} + \left\{ u \leftrightarrow t \right\}, \tag{3.33}
\end{aligned}$$

$$\begin{aligned}
E = & \left\{ \left( -\frac{1}{3}X^3 - \frac{2}{3}SX^2 + \frac{2}{3}X^2 - \frac{2}{3}X\pi^2 + \frac{4}{3}SX - \frac{2}{3}X + \frac{1}{3}Y^3 + \frac{2}{9}Y^2 + \frac{2}{3}SY^2 \right. \right. \\
& + \left. \frac{2}{3}Y\pi^2 + \frac{4}{9}SY + \frac{2}{3}Y + \frac{2}{27}\pi^2 + \frac{8}{9}S^2 \right) \frac{t^2}{s^2} \\
& + \left( \frac{2}{3}X^3 - \frac{2}{3}X^2Y + \frac{4}{3}X^2 + \frac{4}{3}SX^2 - \frac{2}{3}XY^2 - \frac{8}{3}SXY + \frac{2}{3}X\pi^2 + \frac{8}{3}SX \right. \\
& + \left. \frac{4}{3}X - \frac{2}{3}Y\pi^2 - \frac{52}{27}\pi^2 + \frac{4}{3}S\pi^2 + \frac{32}{9}S^2 \right) \frac{t}{u} \\
& + \left( \frac{16}{9}X^2 + \frac{32}{9}SX - \frac{40}{27}\pi^2 + \frac{32}{9}S^2 \right) \frac{t^2}{u^2} \\
& + \left( -\frac{8}{9}X^2 - \frac{16}{9}SX - \frac{2}{27}\pi^2 - \frac{8}{9}S^2 \right) \frac{ut}{s^2} \\
& + \left( -X^3 - 2SX^2 + \frac{26}{9}X^2 - 2X\pi^2 + \frac{10}{3}X + \frac{52}{9}SX - \frac{43}{27}\pi^2 + \frac{44}{9}S^2 \right. \\
& \left. + \frac{1}{2} + 4S \right) \Big\} + \left\{ u \leftrightarrow t \right\}, \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
F = & \left\{ \frac{2}{3} \left( -X + Y \right) \left( 3X^2 - 4XY - 14X + 3Y^2 - 6Y + 2\pi^2 + 4 \right) \frac{t^2}{s^2} \right. \\
& + \left( 4X^3 - \frac{8}{3}X^2Y - \frac{8}{3}X^2 + \frac{8}{3}XY^2 + \frac{80}{3}XY - 4X\pi^2 + \frac{16}{3}X - \frac{8}{3}Y\pi^2 - 24\pi^2 \right) \frac{t}{u} \\
& - \frac{32}{3} \left( -X^2 + \pi^2 \right) \frac{t^2}{u^2} + \left( -\frac{16}{3}X^2 + \frac{16}{3}XY \right) \frac{ut}{s^2} \\
& \left. + \left( \frac{2}{3}X^3 + 2X^2Y + 20X^2 + \frac{4}{3}XY - \frac{16}{3}X\pi^2 + \frac{8}{3}X - \frac{64}{3}\pi^2 \right) \right\} + \left\{ u \leftrightarrow t \right\}. \tag{3.35}
\end{aligned}$$

## 4. Summary

In this paper we presented analytic expressions for the  $\mathcal{O}(\alpha_s^4)$  QCD corrections to the  $2 \rightarrow 2$  gluon-gluon scattering process due to the interference of the tree-level diagrams with the two-loop graphs in the  $\overline{\text{MS}}$  scheme. Throughout we employed conventional dimensional regularisation.

The renormalised matrix elements are infrared divergent and contain poles down to  $\mathcal{O}(1/\epsilon^4)$ . The singularity structure of one- and two-loop diagrams has been thoroughly studied by Catani [21] who provided a procedure for predicting the infrared behaviour of renormalised amplitudes. The anticipated pole structure agrees exactly with that obtained by direct Feynman diagram evaluation. In fact Catani's method does not determine the  $1/\epsilon$  poles exactly, but expects that the remaining unpredicted  $1/\epsilon$  poles are non-logarithmic and proportional to constants (colour factors,  $\pi^2$  and  $\zeta_3$ ). We find that this is indeed the case, and the constant  $H^{(2)}$  is given in Eq. (3.24). Its origin is in double emissions from the final state partons. It is related to that found for quark-gluon and quark-quark scattering in a straightforward way and therefore provides a very strong check on the reliability of our results.

The pole structure of the two-loop contribution is described by Eq. (3.2) while analytic formulae for the finite part according to the colour decomposition of Eq. (3.29) are given in Eqs. (3.30) to (3.35). The one-loop contributions to the two-loop pole structure are expressed in terms of the one-loop bubble graph in  $D = 4 - 2\epsilon$  dimensions and the one-loop box graph in  $D = 6 - 2\epsilon$  dimensions for which series expansions around  $\epsilon = 0$  are provided in Appendix A.

The results presented here, together with those previously computed for quark-quark scattering [17, 18, 19] and quark-gluon scattering [20] form a complete set of two-loop hard scattering matrix elements for parton-parton scattering at  $\mathcal{O}(\alpha_s^4)$ . They are vital ingredients for the next-to-next-to-leading order predictions for jet cross sections in hadron-hadron collisions. However, they are insufficient to make physical predictions and much work remains to be done. A major task is to establish a systematic procedure for analytically cancelling the infrared divergences between the tree-level  $2 \rightarrow 4$ , the one-loop  $2 \rightarrow 3$  and the  $2 \rightarrow 2$  processes for semi-inclusive jet cross sections. Recent progress in determining the singular limits of tree-level matrix elements when two particles are unresolved [40, 41] and the soft and collinear limits of one-loop amplitudes [42, 35, 43], together with the analytic cancellation of the infrared singularities in the somewhat simpler case of  $e^+e^- \rightarrow \text{photon} + \text{jet}$  at next-to-leading order [44], suggest that the technical problems will soon be solved for generic  $2 \rightarrow 2$  scattering processes.

A further complication is due to initial state radiation. Factorization of the collinear singularities from the incoming partons requires the evolution of the parton density functions to be known to an accuracy matching the hard scattering matrix element. This entails knowledge of the three-loop splitting functions. At three-



loop order, the even moments of the splitting functions are known for the flavour singlet and non-singlet structure functions  $F_2$  and  $F_L$  up to  $N = 12$  while the odd moments up to  $N = 13$  are known for  $F_3$  [45, 46]. The numerically small  $N_F^2$  non-singlet contribution is also known [47]. Van Neerven and Vogt have provided accurate parameterisations of the splitting functions in  $x$ -space [48, 49] which are now starting to be implemented in the global analyses [50].

Finally, and most importantly for phenomenological applications, a numerical implementation of the various contributions must be developed. The next-to-leading order programs for three jet production that have already been written provide a first step in this direction [51, 52]. We are confident that the problem of the numerical cancellation of residual infrared divergences will soon be addressed thereby enabling the construction of numerical programs to provide next-to-next-to-leading order QCD estimates of jet production in hadron collisions.

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## A. One-loop master integrals

In this appendix, we list the expansions for the one-loop box integrals in  $D = 6 - 2\epsilon$ . We remain in the physical region  $s > 0$ ,  $u, t < 0$ , and write coefficients in terms of logarithms and polylogarithms that are real in this domain. More precisely, we use the notation of Eqs. (3.27) and (3.28) to define the arguments of the logarithms and polylogarithms. The polylogarithms are defined as in Eq. (3.26).

We find that the box integrals have the expansion

$$\begin{aligned} \text{Box}^6(u, t) = & \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{2s\Gamma(1-2\epsilon)(1-2\epsilon)} \left(\frac{\mu^2}{s}\right)^\epsilon \left\{ \frac{1}{2} [(X-Y)^2 + \pi^2] \right. \\ & + 2\epsilon \left[ \text{Li}_3(x) - X\text{Li}_2(x) - \frac{1}{3}X^3 - \frac{\pi^2}{2}X \right] \\ & \left. - 2\epsilon^2 \left[ \text{Li}_4(x) + Y\text{Li}_3(x) - \frac{1}{2}X^2\text{Li}_2(x) - \frac{1}{8}X^4 - \frac{1}{6}X^3Y + \frac{1}{4}X^2Y^2 \right] \right\} \end{aligned}$$

$$-\frac{\pi^2}{4}X^2 - \frac{\pi^2}{3}XY - \frac{\pi^4}{45} + (u \leftrightarrow t) \Big\} + \mathcal{O}(\epsilon^3), \quad (\text{A.1})$$

and

$$\begin{aligned} \text{Box}^6(s, t) = & \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{2u\Gamma(1-2\epsilon)(1-2\epsilon)} \left(-\frac{\mu^2}{u}\right)^\epsilon \left\{ (X^2 + 2i\pi X) \right. \\ & + \epsilon \left[ \left( -2\text{Li}_3(x) + 2X\text{Li}_2(x) - \frac{2}{3}X^3 + 2YX^2 - \pi^2X + 2\zeta_3 \right) \right. \\ & \quad \left. \left. + i\pi \left( 2\text{Li}_2(x) + 4YX - X^2 - \frac{\pi^2}{3} \right) \right] \right. \\ & + \epsilon^2 \left[ \left( 2\text{Li}_4(z) + 2\text{Li}_4(y) - 2Y\text{Li}_3(x) - 2X\text{Li}_3(y) + (2XY - X^2 - \pi^2)\text{Li}_2(x) \right. \right. \\ & \quad \left. \left. + \frac{1}{3}X^4 - \frac{5}{3}X^3Y + \frac{3}{2}X^2Y^2 + \frac{2}{3}\pi^2X^2 - 2\pi^2XY + 2Y\zeta_3 + \frac{1}{6}\pi^4 \right) \right. \\ & \quad \left. \left. + i\pi \left( -2\text{Li}_3(x) - 2\text{Li}_3(y) + 2Y\text{Li}_2(x) + \frac{1}{3}X^3 - 2X^2Y + 3XY^2 \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\pi^2}{3}Y + 2\zeta_3 \right) \right] \right\} + \mathcal{O}(\epsilon^3). \quad (\text{A.2}) \end{aligned}$$

$\text{Box}^6(s, u)$  is obtained from Eq. (A.2) by exchanging  $u$  and  $t$ .

Finally, the one-loop bubble integral in  $D = 4 - 2\epsilon$  dimensions is given by

$$\text{Bub}(s) = \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)\epsilon} \left(-\frac{\mu^2}{s}\right)^\epsilon. \quad (\text{A.3})$$

## References

- [1] S.D. Ellis, Z. Kunzst and D.E. Soper, Phys. Rev. Lett. **69** (1992) 3615 [hep-ph/9208249].
- [2] W.T. Giele, E.W.N. Glover and D.A. Kosower, Phys. Rev. **D52** (1995) 1486 [hep-ph/9412338].
- [3] R.K. Ellis and J.C. Sexton, Nucl. Phys. **B269** (1986) 445.
- [4] J.F. Gunion and J. Kalinowski, Phys. Rev. **D34** (1986) 2119;  
S.J. Parke and T.R. Taylor, Nucl. Phys. **B269** (1986) 410;  
F.A. Berends and W.T. Giele, Nucl. Phys. **B294** (1987) 700;  
M. Mangano, S.J. Parke and Z. Xu, Nucl. Phys. **B298** (1988) 653.
- [5] Z. Kunszt, Nucl. Phys. **B271** (1986) 333;  
S.J. Parke and T.J. Taylor, Phys. Rev. **D35** (1987) 313.
- [6] J.F. Gunion and Z. Kunszt, Phys. Lett. **159B** (1985) 167.

- [7] J.F. Gunion and Z. Kunszt, Phys. Lett. **176B** (1986) 163.
- [8] Z. Bern, L. Dixon, D.A. Kosower, Phys. Rev. Lett **70** (1993) 2677 [hep-ph/9302280].
- [9] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. **B437** (1995) 259 [hep-ph/9409393].
- [10] Z. Kunszt, A. Signer and Z. Trócsányi, Phys. Lett. **B336** (1994) 529 [hep-ph/9405386].
- [11] Z. Bern, J.S. Rozowsky and B. Yan, Phys. Lett. **B401** (1997) 273, [hep-ph/9702424].
- [12] V.A. Smirnov, Phys. Lett. **B460** (1999) 397 [hep-ph/9905323].
- [13] J.B. Tausk, Phys. Lett. **B469** (1999) 225 [hep-ph/9909506].
- [14] V.A. Smirnov and O.L. Veretin, Nucl. Phys. **B566** (2000) 469 [hep-ph/9907385].
- [15] C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, Nucl. Phys. **B580** (2000) 577 [hep-ph/0003261].
- [16] Z. Bern, L. Dixon and A. Ghinculov, Phys. Rev. **D63** (2001) 053007 hep-ph/0010075.
- [17] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0010212.
- [18] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0011094.
- [19] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0012007.
- [20] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, hep-ph/0101304.
- [21] S. Catani, Phys. Lett. **B427** (1998) 161 [hep-ph/9802439].
- [22] Z. Bern, L. Dixon and D.A. Kosower, JHEP **0001** (2000) 027 [hep-ph/0001001].
- [23] P. Nogueira, J. Comput. Phys. **105** (1993) 279.
- [24] C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. **B575** (2000) 416, Erratum-  
ibid. **B585** (2000) 763 [hep-ph/9912251].
- [25] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, Nucl. Phys. **B174** (1980) 345 ;  
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159.
- [26] T. Gehrmann and E. Remiddi, Nucl. Phys. **B580** (2000) 485 [hep-ph/9912329].
- [27] C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. **B572** (2000) 307 [hep-  
ph/9907494].
- [28] R.J. Gonsalves, Phys. Rev. **D28** (1983) 1542;  
G. Kramer and B. Lampe, J. Math. Phys. **28** (1987) 945.
- [29] T. Gehrmann and E. Remiddi, Nucl. Phys. Proc. Suppl. **89** (2000) 251 [hep-  
ph/0005232].

- [30] C. Anastasiou, J.B. Tausk and M.E. Tejeda-Yeomans, Nucl. Phys. Proc. Suppl. **89** (2000) 262 [hep-ph/0005328].
- [31] P. Cvitanovic, P.G. Lauwers and P.N. Scharbach, Nucl. Phys. **B186** (1981) 165.
- [32] F.A. Berends and W.T. Giele, Nucl. Phys. **B294** (1987) 700;  
M. Mangano, S.J. Parke and Z. Xu, Nucl. Phys. **B298** (1988) 653.
- [33] D.A. Kosower, B.-H. Lee and V.P. Nair, Phys. Letts. **B201** (1988) 85;  
M. Mangano, Nucl. Phys. **B309** (1988) 461;  
D. Zeppenfeld, Int. J. Mod. Phys. **A3** (1988) 2175;  
D.A. Kosower, Nucl. Phys. **B315** (1989) 391.
- [34] Z. Bern and D.A. Kosower, Nucl. Phys. **B362** (1991) 389;  
Z. Bern, L. Dixon and D.A. Kosower, Ann. Rev. Nucl. Part. Sci. **46** (1996) 109 [hep-ph/9602280].
- [35] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. **B425** (1994) 217 [hep-ph/9403226].
- [36] F.A. Berends and W.T. Giele, Nucl. Phys. **B306** (1988) 759.
- [37] F.A. Berends and W.T. Giele, Nucl. Phys. **B313** (1989) 595;  
R. Kleiss and H. Kuijf, Nucl. Phys. **B312** (1989) 616.
- [38] V. Del Duca, L. Dixon and F. Maltoni, Nucl. Phys. **B571** (2000) 51 [hep-ph/9910563].
- [39] K.S. Kölbig, J.A. Mignaco and E. Remiddi, B.I.T. **10** (1970) 38.
- [40] J.M. Campbell and E.W.N. Glover, Nucl. Phys. **B527** (1998) 264 [hep-ph/9710255];  
S. Catani and M. Grazzini, Phys. Lett. **446B** (1999) 143 [hep-ph/9810389];  
S. Catani and M. Grazzini, Nucl. Phys. **B570** (2000) 287 [hep-ph/9908523];  
V. Del Duca, A. Frizzo and F. Maltoni, Nucl. Phys. **B568** (2000) 211 [hep-ph/9909464].
- [41] F.A. Berends and W.T. Giele, Nucl. Phys. **B313** (1989) 595;  
S. Catani, in Proceedings of the workshop on *New Techniques for Calculating Higher Order QCD Corrections*, report ETH-TH/93-01, Zurich (1992).
- [42] Z. Bern, V. Del Duca and C.R. Schmidt, Phys. Lett. **445B** (1998) 168 [hep-ph/9810409];  
Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, Phys. Rev. **D60** (1999) 116001 [hep-ph/9903516];  
S. Catani and M. Grazzini, Nucl. Phys. **B591** (2000) 435 [hep-ph/0007142].
- [43] D.A. Kosower, Nucl. Phys. **B552** (1999) 319;  
D.A. Kosower and P. Uwer, Nucl. Phys. **B563** (1999) 477 [hep-ph/9903515].
- [44] A. Gehrmann-De Ridder and E.W.N. Glover, Nucl. Phys. **B517** (1998) 269 [hep-ph/9707224].

- [45] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. **B427** (1994) 41; S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. **B492** (1997) 338 [hep-ph/9605317].
- [46] A. Retey and J.A.M. Vermaseren, hep-ph/0007294.
- [47] J.A. Gracey, Phys. Lett. **B322** (1994) 141 [hep-ph/9401214].
- [48] W.L. van Neerven and A. Vogt, Nucl. Phys. **B568** (2000) 263 [hep-ph/9907472]; Nucl. Phys. **588** (2000) 345 [hep-ph/0006154].
- [49] W.L. van Neerven and A. Vogt, Phys. Lett. **B490** (2000) 111 [hep-ph/0007362].
- [50] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, Eur. Phys. J. **C18** (2000) 117 [hep-ph/0007099].
- [51] Z. Trocsanyi, Phys. Rev. Lett. **77** (1996) 2182 [hep-ph/9610499].
- [52] W.B. Kilgore and W.T. Giele, Phys. Rev. **D55** (1997) 7183 [hep-ph/9610433]; hep-ph/0009193.