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Calculus of Variations

BV functions and sets of finite perimeter on configuration spaces

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Abstract

In this paper, we aim to develop the foundations of a theory of BV functions in the configuration space over the Euclidean space $\mathbb{R}^n$ equipped with the Poisson measure π. We first construct the m-codimensional Poisson measure—formally written as "$(\infty - m)$-dimensional Poisson measure"—on the configuration space. We then show that our construction is consistent with potential theory induced by the infinitely many independent Brownian motions by establishing relations between the m-codimensional Poisson measure and Bessel capacities. Secondly, we introduce three different definitions of BV functions based on the variational, relaxation, and semigroup approaches, and prove the equivalence of them. In the process, we prove the p-Bakry–Émery inequality on the configuration space for any $1 < p < \infty$. Thirdly, we construct perimeter measures and introduce an appropriate notion of measure-theoretic boundary, called the reduced boundary. We then prove that the perimeter measure can be expressed by the 1-codimensional Poisson measure restricted to the reduced boundary, which is a generalisation of De Giorgi's identity to the configuration space. Finally, we construct the total variation measures for functions of bounded variation, and prove the Gauß–Green formula.

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Contents

1 Introduction

The purpose of this paper is to establish the foundations for functions of bounded variations (BV functions) in the space of all locally finite point measures (without multiplicity) in the Euclidean space $\mathbb{R}^n$, denoted by $\Upsilon(\mathbb{R}^n)$ and called the *configuration space*. The space $\Upsilon(\mathbb{R}^n)$ is endowed with the vague topology τ_v, the L^2-transportation (extended) distance d_Υ, which stems from the optimal transport problem, and the Poisson measure π whose intensity measure is the Lebesgue measure $\mathbf{L}^n$ on $\mathbb{R}^n$. The resulting *topological (extended) metric measure structure* $(\Upsilon(\mathbb{R}^n), \tau_v, \mathsf{d}_\Upsilon, \pi)$ plays a fundamental role to describe dynamical systems of infinite particles stemming from statistical physics, random point processes, random graphs and integral geometry, representation theory of diffeomorphism groups on manifolds, and many others. Instead of giving enormous numbers of related references here, we refer the reader to [23, Section1.6] for an overview of the aforementioned subjects.

The studies of BV functions and sets of finite perimeter beyond the standard Euclidean space have seen a thriving development in the last years, see [3–7, 9, 10, 14, 19, 36] and references therein. However, all of these results do not cover the configuration space $\Upsilon(\mathbb{R}^n)$. The space $(\Upsilon(\mathbb{R}^n), \tau_v, \mathsf{d}_\Upsilon, \pi)$ is known to possesses several pathological properties (see details in [23]):

- the extended distance d_Υ is not continuous with respect the topology τ_v;
- d_Υ-metric balls are negligible with respect to the Poisson measure π;
- d_Υ-Lipschitz functions are not necessarily π-measurable;
- the Riesz–Markov–Kakutani representation theorem does not hold.

For these reasons, the study of the configuration space $(\Upsilon(\mathbb{R}^n), \tau_v, \mathsf{d}_\Upsilon, \pi)$ does not fall into the standard framework of metric measure geometry. Furthermore, the lack of the Riesz–Markov–Kakutani's representation theorem causes further complexity to construct total variation measures supporting the Gauß–Green formula by means of standard functional-analytic technique.

In the setting of infinite-dimensional spaces, the study of geometric measure theory has been pioneered by Feyel and de la Pradelle [28], Fukushima [31], Fukushima and Hino [32] and Hino [34] in the Wiener space. In [28], they constructed the finite-codimensional Gauß–Hausdorff measure in the Wiener space and investigated its relation to capacities. In [31, 32], they developed the theory of functions of bounded variation and constructed perimeter measures, and prove the Gauß–Green formula. Based on these results, Hino introduced in [34] a notion of reduced boundary and investigated relations between the one-codimensional Hausdorff–Gauß measure and the perimeter measures. Further fine properties were investigated by Ambrosio and Figalli [11], Ambrosio et al. [12, 15–17]. The notion of functions of bounded variation has been studied also in a Gelfand triple by Röckner et al. [39–41]. All of the aforementioned results rely heavily on the linear structure of the Wiener space or the Hilbert space, which is used to perform finite-dimensional approximations. However, the configuration space does not have a linear structure and there is no chance to apply similar techniques.

1.1 Non-linear dimension reduction and overview of the main results

To overcome the difficulties explained above, we develop a *non-linear dimensional reduction* tailored to the configuration space $\Upsilon(\mathbb{R}^n)$. A key observation is that $\Upsilon(B_r)$, the configuration space over the Euclidean closed metric ball B_r centred at the origin o with radius $r > 0$, is essentially finite dimensional. More precisely, due to the compactness of B_r, $\Upsilon(B_r)$ can be written as the disjoint union $\sqcup_{k\in\mathbb{N}}\Upsilon^k(B_r)$ of the k-particle configuration spaces $\Upsilon^k(B_r)$, each of which is isomorphic to the quotient space of the k-product space $B_r^{\times k}$ by the k-th symmetric group. In light of this observation, the main task is to lift geometric measure theory on $\Upsilon(B_r)$ to the infinite-dimensional space $\Upsilon(\mathbb{R}^n)$ by finite-dimensional approximations.

In this paper, we first construct the m-codimensional Poisson measure on the configuration space (Theorem 3.7, Definition 3.8), and study its relation to $(1, p)$-Bessel capacities (Theorem 4.3). Secondly, we introduce three different definitions of functions of bounded variation based on the variational, relaxation and the semigroup approaches, and prove their equivalence (Theorem 5.18). In the process of showing the equivalence of these three definitions, we prove the p-Bakry–Émery inequality (Theorem 5.16) for the heat semigroup on $\Upsilon(\mathbb{R}^n)$

for $1 < p < \infty$, which was previously known only for $p = 2$ in Erbar–Huesmann [26]. Thirdly, we construct perimeter measures and introduce the notion of the reduced boundary in Sect. 6. We then prove that the perimeter measure can be expressed by the 1-codimensional Poisson measure restricted to the reduced boundary (Theorem 6.15). Fourthly, we construct the total variation measures for functions of bounded variation and prove the Gauß–Green formula (Theorem 7.7).

We now explain each result in details.

1.2 *m*-Codimensional Poisson measure

The first main result of this paper is the construction of the m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$. Since $\Upsilon(\mathbb{R}^n)$ is infinite-dimensional, it is formally written as

$$\text{“}(\infty - m)\text{-dimensional Poisson measure”}.$$

In the case of finite-dimensional spaces, usually the construction of finite-codimensional measures builds upon covering arguments, which heavily rely on the volume doubling property of the ambient measure. However, this property does not hold for the Poisson measure π on $\Upsilon(\mathbb{R}^n)$.

We construct the m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$ by passing to the limit of finite dimensional approximations obtained by using the m-codimensional Poisson measure on $\Upsilon(B_r)$. The key step in the construction is to prove the *monotonicity* of these finite dimensional approximations with respect to the radius r, allowing us to find a unique limit measure. More in details, based on the decomposition $\Upsilon(B_r) = \sqcup_{k\in\mathbb{N}}\Upsilon^k(B_r)$, we build $\rho^m_{\Upsilon(B_r)}$, the *spherical Hausdorff measure of codimension m* in $\Upsilon(B_r)$, by summing the m-codimensional spherical Hausdorff measure $\rho^{m,k}_{\Upsilon(B_r)}$ on the k-particle configuration space $\Upsilon^k(B_r)$, which is obtained by the quotient measure of the m-codimensional spherical Hausdorff measure on the k-product space $B_r^{\times k}$ with a suitable renormalisation corresponding to the Poisson measure. The *localised m-codimensional Poisson measure* ρ^m_r of a set $A \subset \Upsilon(\mathbb{R}^n)$ is then obtained by averaging the $\rho^m_{\Upsilon(B_r)}$-measure of sections of A with the Poisson measure $\pi_{B_r^c}$ on $\Upsilon(B_r^c)$, i.e.

$$\rho^m_r(A) := \int_{\Upsilon(B_r^c)} \rho^m_{\Upsilon(B_r)}(\{\gamma \in \Upsilon(B_r) : \gamma + \eta \in A\}) d\pi_{B_r^c}(\eta)\,.$$

We prove that ρ^m_r is well-defined on Borel sets (indeed, we prove it for all Suslin sets), and that it is monotone increasing with respect to r (Theorem 3.7, Definition 3.8). In particular, we can define *the m-codimensional Poisson measure* as

$$\rho^m := \lim_{r\to\infty} \rho^m_r\,.$$

We refer the readers to Sect. 3 for the detailed construction of ρ^m.

1.3 Bessel capacity

In Sect. 4, we compare the m-codimensional Poisson measure ρ^m and $\mathrm{Cap}_{\alpha,p}$, the *Bessel capacity* induced by the Dirichlet form associated with infinite independent Brownian motions constructed in Albeverio et al. [2]. We prove that zero capacity sets are ρ^m negligible provided $\alpha p > m$ (Theorem 4.3). This result, that is well-known in the case of finite-dimensional spaces, proves that our m-codimensional Poisson measure ρ^m behaves coherently with the potential-analytic structure of $\Upsilon(\mathbb{R}^n)$. To prove it, we introduce the (α, p)-Bessel capacity $\mathrm{Cap}_{\alpha,p}^{\Upsilon(B_r)}$ on $\Upsilon(B_r)$ and the localised (α, p)-Bessel capacity $\mathrm{Cap}_{\alpha,p}^r$ on $\Upsilon(\mathbb{R}^n)$ based on the localisation argument of the L^p-heat semigroup $\{T_t\}$ on $\Upsilon(\mathbb{R}^n)$. We prove that $\mathrm{Cap}_{\alpha,p}$ is approximated by $\mathrm{Cap}_{\alpha,p}^r$ as $r \to \infty$, hence we can obtain the proof by lifting the corresponding result for $\rho^m_{\Upsilon(B_r)}$ and $\mathrm{Cap}_{\alpha,p}^{\Upsilon(B_r)}$ in $\Upsilon(B_r)$ (see Proposition 4.14). We refer the readers to Sect. 4 for the detailed arguments.

As an application, we prove in Corollary 7.4 that, if $\mathrm{Cap}_{1,2}(A) = 0$ then $|\mathrm{D}F|(A) = 0$ for every $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n)) \cap L^2(\Upsilon(\mathbb{R}^n), \pi)$, where $|\mathrm{D}F|$ is the total variation measure (Definition 7.2) and $\mathrm{BV}(\Upsilon(\mathbb{R}^n))$ is the space of functions of bounded variation (Definition 5.19). The latter result will be fundamental for applications to stochastic analysis of infinite-particle diffusions, which will be the subject of a forthcoming paper.

1.4 Functions of bounded variations and Caccioppoli sets

In the second part of this paper we develop the theory of functions of bounded variation and sets of finite perimeter in $\Upsilon(\mathbb{R}^n)$. In Sect. 5 we propose three different notions of functions with bounded variation. The first one follows the classical *variational approach*, the second one is built upon the *relaxation approach*, while the third one relies on the regularisation properties of the *heat semigroup*. It turns out that they are all equivalent, as shown in Sect. 5.5, and the resulting class is denoted by $\mathrm{BV}(\Upsilon(\mathbb{R}^n))$. For $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$ we define a total variation measure $|\mathrm{D}F|$ and prove a *Gauß–Green formula* (see Theorem below). We remark that in our infinite-dimesional setting, Riesz–Markov–Kakutani's representation theorem is not available due to the lack of local compactness. In particular, the construction of the total variation measure is not straightforward. We follow an unusual path to show its existence: we first develop the theory of *sets with finite perimeter* relying on the non-linear dimension reduction. We then employ the *coarea formula* to build the total variation measure of a function of bounded variation as a superposition of perimeter measures.

Sets of finite perimeter are those Borel sets E such that $\chi_E \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, where χ_E denotes the indicator function of E. In Sect. 6, we study their structure by means of the non-linear reduction approach, a part of which uses a strategy inspired by Hino [34] for the study of Wiener spaces. The key result in this regard is Proposition 5.5 saying that if E has finite perimeter then the projection $E_{\eta,r} := \{\gamma \in \Upsilon(B_r) : \gamma + \eta \in E\}$ has finite *localised total variation* in B_r, for $\pi_{B_r^c}$-a.e. $\eta \in B_r^c$. Hence, we can reduce the problem to the study of sections that are sets with finite perimeter in $\Upsilon(B_r)$. As already remarked, the latter is essentially a finite dimensional space, so we can appeal to classical tools of geometric measure theory to attack the problem.

The *reduced boundary* $\partial^* E$ of a set of finite perimeter $E \subset \Upsilon(\mathbb{R}^n)$ is then defined in terms of the reduced boundary of the sections $E_{\eta,r}$, through a limit procedure. The resulting object allows us to represent the perimeter measure as

$$\|E\| = \rho^1|_{\partial^* E} ,$$

which is a generalisation of the identity proven in the Euclidean setting by De Giorgi [21, 22].

Our approach to the BV theory deviates from the standard one. We define the total variation measure $|\mathrm{D}F|$ of a function $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$ by imposing the validity of the coarea formula. More precisely, we show that dt-a.e. level set $\{F > t\}$ is of finite perimeter and we set

$$|\mathsf{D}F| := \int_{-\infty}^{\infty} \big\|\{F > t\}\big\| dt \,,$$

taking advantage of the perimeter measure $\|\{F > t\}\|$ that has been already defined using finite dimensional approximations. The reason for this non-standard treatment is that we are not able to build directly $|\mathsf{D}F|$ through a finite dimensional approximation, since the latter does not have a simple expression in terms of 1-codimensional Poisson measure ρ^1 restricted to a suitable subset. Our approach is, however, consistent with the standard one, as shown in Corollary 7.3 and in Theorem 7.7.

We summarise the main results in Sects. 6 and 7 concerning functions of bounded variations and a sets of finite perimeter. We denote by $\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ the space of cylinder vector fields on $\Upsilon(\mathbb{R}^n)$ and by $(T\Upsilon, \langle\cdot,\cdot\rangle_{T\Upsilon})$ the tangent bundle to $\Upsilon(\mathbb{R}^n)$ with the pointwise inner product $\langle\cdot,\cdot\rangle_{T\Upsilon}$ (see Sect. 2.5).

Theorem (Theorems 6.15, 7.7) *For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, there exists a unique positive finite measure $|\mathsf{D}F|$ on $\Upsilon(\mathbb{R}^n)$ and a π-a.e. unique $T\Upsilon$-valued measurable function σ on $\Upsilon(\mathbb{R}^n)$ so that $|\sigma|_{T\Upsilon} = 1$ $|\mathsf{D}F|$-a.e., and*

$$\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F \, d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma\rangle_{T\Upsilon} d|\mathsf{D}F| \,, \quad \forall V \in \mathrm{CylV}(\Upsilon) \,.$$

If, furthermore, $F = \chi_E$, then

$$|\mathsf{D}\chi_E| = \rho^1|_{\partial^* E} \,.$$

1.5 Potential applications

Our theory of functions of bounded variation has several potential applications to related fields such as singular boundary problems of infinite interacting diffusions. In the case of the Euclidean space $\mathbb{R}^n$—the the case of *one particle* Brownian motion—there is a connection between the theory of BV functions and stochastic analysis: the (modified) reflected Brownian motion on an open set $A \subset \mathbb{R}^n$ is *semi-martingale* if and only if A is Caccioppoli. Furthermore, the modified reflected Brownian motion satisfies the *generalised Skorokhod equation* and the *generalised Itô's formula*, where the reflection at the boundary is phrased by the local time in terms of the reduced boundary (see, [30, Theorem 7.1, 7.2]). As an infinite dimensional counterpart, one can expect that the main results in this paper would be useful to construct infinite particle diffusions with singular boundary conditions (cf. [32, Theorem 4.4.] in the case of the Wiener space).

1.6 Structure of the paper

In Sect. 2, we collect preliminary results regarding the the configuration space, Suslin sets and measurability of sections. In Sect. 3, we construct the m-codimensional measure. Relations with the Bessel capacity are studied in Sect. 4. Section 5 is devoted to the study of functions of bounded variation. We introduce three different notion and prove the equivalence. In Sect. 6, we introduce and study sets of finite perimeter. We build the notion of reduced boundary and the perimeter measure, and we show an integration by parts formula. In Sect. 7, we introduce the total variation measure of functions with bounded variations by employing the coarea formula, and prove a Gauß–Green type integration-by-parts formula.

2 Preliminaries

2.1 Notational convention

In this paper, the bold fonts $\mathbf{S}, \mathbf{L}, \ldots$ are mainly used for objects in product spaces or vector-valued objects, while the serif fonts $\mathsf{S}, \mathsf{D}, \ldots$ are used for objects in the quotient space of product spaces with respect to the k-symmetric group $\mathfrak{S}_k$ or for objects in the configuration space $\Upsilon(\mathbb{R}^n)$.

The lower-case fonts $f, g, h, v, w, \ldots$ are mainly used for functions on the base space $\mathbb{R}^n$, while the upper-case fonts $F, G, H, V, W, \ldots$ are used for functions on the configuration space $\Upsilon(\mathbb{R}^n)$.

We denote by χ_E the indicator function of E, i.e., $\chi_E = 1$ on E and $\chi_E = 0$ on E^c. Let $\Omega \subset \mathbb{R}^n$ be a closed domain. We denote by $C_*^\infty(\Omega)$ the space of smooth functions with compact support in $\Omega \backslash \partial\Omega$ (i.e., functions vanish at the boundary $\partial\Omega$), while $C_c^\infty(\Omega)$ denotes the space of compactly supported smooth functions on Ω (functions do not necessarily vanish at the boundary $\partial\Omega$). Note that $C_*^\infty(\Omega) \subset C_c^\infty(\Omega)$ in general, but these two function spaces coincide, i.e. $C_c^\infty(\mathbb{R}^n) = C_*^\infty(\mathbb{R}^n)$, when we take $\Omega = \mathbb{R}^n$.

2.2 Configuration spaces

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $B_r := B_r(0) \subset \mathbb{R}^n$ be the closed ball with radius $r > 0$ centred at the origin 0. Let δ_x denote the point measure at $x \in \mathbb{R}^n$, i.e. $\delta_x(A) = 1$ if and only if $x \in A$. We denote by $\Upsilon(\mathbb{R}^n)$ the *configuration space over* $\mathbb{R}^n$ *without multiplicity*, i.e. the set of all locally finite point measures γ on $\mathbb{R}^n$ so that $\gamma(\{x\}) \in \{0, 1\}$ for every $x \in \mathbb{R}^n$. Elements in $\Upsilon(\mathbb{R}^n)$ can be written as $\gamma = \sum_{i=1}^N \delta_{x_i}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$. Let $\Upsilon(A)$ denote the configuration space over a Polish subspace $A \subset \mathbb{R}^n$ defined analogously to $\Upsilon(\mathbb{R}^n)$, and $\Upsilon^k(A)$ denote the space of k-configurations on a subset A, i.e. $\Upsilon^k(A) = \{\gamma \in \Upsilon(A) : \gamma(A) = k\}$. We equip $\Upsilon(\mathbb{R}^n)$ with the vague topology τ_v, i.e., $\gamma_n \in \Upsilon(\mathbb{R}^n)$ converges to $\gamma \in \Upsilon(\mathbb{R}^n)$ in τ_v if and only if $\gamma_n(f) \to \gamma(f)$ for any $f \in C_c(\mathbb{R}^n)$. For a subset $A \subset \mathbb{R}^n$, we equip $\Upsilon(A)$ with the relative topology as a subset in $\Upsilon(\mathbb{R}^n)$. Let $\mathscr{B}(\Upsilon(A), \tau_v)$ denote the Borel σ-algebra associated with the vague topology τ_v. For a set $A \subset \mathbb{R}^n$, let $\mathrm{pr}_A : \Upsilon(\mathbb{R}^n) \to \Upsilon(A)$ be the projection defined by the restriction of configurations on A, i.e. $\mathrm{pr}_A(\gamma) = \gamma|_A$.

Given $A \subset \mathbb{R}^n$, an open or closed domain, we denote by π_A the *Poisson measure* on $\Upsilon(A)$ whose intensity measure is the Lebesgue measure restricted to A, namely, π_A is the unique Borel probability measure so that, for all $f \in C_c(A)$, the following holds

$$\int_{\Upsilon(A)} e^{f^*} d\pi_A = \exp\left\{\int_A (e^f - 1) d\mathbf{L}^n(x)\right\}. \tag{2.1}$$

Here $\mathbf{L}^n$ denotes the n-dimensional Lebesgue measure. See [33] for a reference for the expression (2.1). We write $\pi = \pi_{\mathbb{R}^n}$. Note that π_A coincides with the push-forward measure $\pi_A = (\mathrm{pr}_A)_\# \pi$. Let

$$\mathrm{diag}_k := \{(x)_{1\le i\le m} \in (\mathbb{R}^n)^{\times k} : \exists i, j \text{ s.t. } x_i = x_j\},$$

denote the set of all sub-diagonals in $(\mathbb{R}^n)^{\times k}$, and let $\mathfrak{S}_k$ denote the k-symmetric group. For any set $A \subset \mathbb{R}^n$, we identify

$$\Upsilon^k(A) \cong (A^{\times k} \setminus \mathrm{diag}_k)/\mathfrak{S}_k, \quad k \in \mathbb{N}.$$

Let $\mathsf{s}_k : A^{\times k} \setminus \mathrm{diag}_k \to \Upsilon^k(A)$ be the canonical projection with respect to the action of $\mathfrak{S}_k$, i.e. $\mathsf{s}_k : (x_i)_{1\le i\le k} \mapsto \sum_{i=1}^k \delta_{x_i}$. We say that a function $f : \sqcup_{k=1}^\infty (\mathbb{R}^n)^{\times k} \to \mathbb{R}$ is *symmetric* iff $f(\mathbf{x}_{\sigma_k}) = f(\mathbf{x}_k)$ with $\mathbf{x}_{\sigma_k} := (x_{\sigma_k(1)}, \dots, x_{\sigma_k(k)})$ for every permutation $\sigma_k \in \mathfrak{S}_k$ and every $k \in \mathbb{N}$.

For $\mathbf{x}_k, \mathbf{y}_k \in A^{\times k}$ with $\mathbf{s}_k(\mathbf{x}_k) = \gamma \in \Upsilon^k(A)$ and $\mathbf{s}_k(\mathbf{y}_k) = \eta \in \Upsilon^k(A)$, define *the L^2-transportation distance* $\mathsf{d}_{\Upsilon^k}(\gamma, \eta)$ on $\Upsilon^k(A)$ by the quotient metric w.r.t. $\mathfrak{S}_k$:

$$\mathsf{d}_{\Upsilon^k}(\gamma, \eta) = \inf_{\sigma_k \in \mathfrak{S}_k} |\mathbf{x}_{\sigma_k} - \mathbf{y}_k|_{\mathbb{R}^{nk}} . \tag{2.2}$$

Here $|\mathbf{x}_k - \mathbf{y}_k|_{\mathbb{R}^{nk}}$ denotes the standard Euclidean distance in $\mathbb{R}^{nk}$.

Remark 2.1 (Polishness/lack of completeness)

(a) The space $\Upsilon(\mathbb{R}^n)$ equipped with the vague topology is a Polish space. The subspace $\Upsilon^k(A) \subset \Upsilon(\mathbb{R}^n)$ is a Polish subspace for every $k \in \mathbb{N}$ if A is a Polish subspace in $\mathbb{R}^n$. This fact will play a role later in Sect. 3 to discuss Suslin sets.
(b) The metric space $(\Upsilon^k(A), \mathsf{d}_{\Upsilon^k})$ is not complete even if A is closed, due to the lack of multiple configurations in $\Upsilon^k(A)$. This fact is, however, irrelevant to the rest of arguments.

2.3 Spherical Hausdorff measure

Let (X, d) be a metric space and n be the Hausdorff dimension of X. For $m \le n$, the *m-dimensional spherical Hausdorff measure* $\mathbf{S}_X^m$ on X is defined as the restriction of the following outer measure $\mathbf{S}_X^m$ on $\mathbf{S}_X^m$-measurable sets (i.e., the Carathéodory measurable sets):

$$\mathbf{S}_X^m(A) := \lim_{\varepsilon\to 0} \mathbf{S}_{X,\varepsilon}^m(A) := \lim_{\varepsilon\to 0} \inf\Big\{\sum_{i\in\mathbb{N}} \mathsf{diam}(B_i)^m : \\ B_i \text{ open ball with } \mathsf{diam}(B_i) < \varepsilon,\ A \subset \sum_{i\in\mathbb{N}} B_i\Big\}. \tag{2.3}$$

Here $\mathsf{diam}(B_i) = \sup\{d(x, y) : x, y \in B_i\}$ denotes the diameter of B_i. We call $\mathbf{S}_{X,\varepsilon}^m$ the *m-dimensional ε-Hausdorff measure*. If $X = \mathbb{R}^n$, we simply write $\mathbf{S}^m$ and $\mathbf{S}_\varepsilon^m$ instead of $\mathbf{S}_{\mathbb{R}^n}^m$ and $\mathbf{S}_{\mathbb{R}^n,\varepsilon}^m$ respectively.

Remark 2.2 (Comparison with the standard Hausdorff measure) In the case of $m < n$, the spherical Hausdorff measure $\mathbf{S}_X^m$ does not coincide with the standard Hausdorff measure in general, the latter is smaller since it is defined allowing *all non-empty coverings* instead of *open balls*. In the case of $m = n$ and $X = \mathbb{R}^n$, however, $\mathbf{S}^m$ coincides with the standard n-dimensional Hausdorff measure and also with the n-dimensional Lebesgue measure ([27, 2.10.35]). Note that $\mathbf{S}^m$ is a Borel measure, but not σ-finite for $m < n$.

For a bounded set $A \subset \mathbb{R}^n$, let $\mathbf{S}^n|_A$ be the spherical Hausdorff measure restricted to A. The spherical Hausdorff measure $(\mathbf{S}^n|_A)^{\otimes k}$ on $A^{\times k}$ can be push-forward to the k-configuration space $\Upsilon^k(A)$ by the projection map s_k, i.e.

$$\mathsf{S}_A^k := \frac{1}{k!}(\mathsf{s}_k)_\#(\mathbf{S}^n|_A)^{\otimes k} .$$

It is immediate by construction to see that S_A^k is the spherical Hausdorff measure on $\Upsilon^k(A)$ induced by the L^2-transportation distance d_{Υ^k} up to constant multiplication. We introduce the

m-codimensional spherical Hausdorff measure and the m-codimensional ε-spherical Hausdorff measure on $\Upsilon^k(A)$ as follows

$$\mathsf{S}_A^{m,k} = \frac{1}{k!}(\mathsf{s}_k)_\#(\mathbf{S}^{nk-m}|_{A^{\times k}}), \quad \mathsf{S}_{A,\varepsilon}^{m,k} = \frac{1}{k!}(\mathsf{s}_k)_\#(\mathbf{S}_\varepsilon^{nk-m}|_{A^{\times k}}). \tag{2.4}$$

One can immediately see that $\mathsf{S}_{A,\varepsilon}^{m,k}$ is (up to constant multiplication) the m-codimensional ε-spherical Hausdorff measure on $\Upsilon^k(A)$ associated with the L^2-transportation distance d_{Υ^k}.

2.4 Regularity of the spherical Hausdorff measures

In this section, we prove the upper semi-continuity of the ε-spherical Hausdorff measure on sections of compact sets, which will be of use in Sect. 3.

Proposition 2.3 *Let (X, d_X), (Y, d_Y) be metric spaces, and $K \subset X \times Y$ be a compact set. Then, the map $Y \ni y \mapsto \mathbf{S}^m_{X,\varepsilon}(K^y)$ is upper semi-continuous. Here, $K^y := \{x \in X : (x, y) \in K\}$.*

Proof Let us fix $y \in Y$ and a sequence $y_n \to y$. The family of compact sets $(K^{y_n} \times \{y_n\})_{n\in\mathbb{N}} \subset K$ is precompact with respect to the Hausdorff topology in K endowed with the product metric (e.g., [20, Theorem 7.3.8]). In particular, we can take a (non-relabeled) subsequence so that $K^{y_n} \times \{y_n\} \to \bar{K} \times \{y\} \subset K$, as $n \to \infty$ in the Hausdorff topology, and $\bar{K} \subset K^y$ by the definition of K^y.

Let us fix $\delta > 0$ and a family of open balls $B_1, \dots, B_\ell \subset X$ with radius smaller than $\varepsilon(1-\delta) > 0$ so that

$$\bar{K} \subset \bigcup_{i=1}^{\ell} B_i,$$

and

$$\mathbf{S}^m_{X,\varepsilon}(\bar{K}) \geq c(m) \sum_{i=1}^{\ell} r_i^m - \delta. \tag{2.5}$$

Here $c(m)$ denotes the constant depending on m such that $\mathbf{L}^m(B_i) = c(m) r_i^m$. Note that we can always take $\ell = \ell(\delta)$ to be finite for any $\delta > 0$ by the compactness of $\bar{K}$. Let $\underline{r} = \underline{r}(\delta) := \min\{r_i : 1 \leq i \leq l(\delta)\} > 0$ be the minimum radius among $\{B_i\}_{1\leq i\leq l}$.

We claim that there exists $\bar{k} = \bar{k}(\delta) \in \mathbb{N}$ so that $K^{y_{n_k}} \subset \cup_{i=1}^{\ell} B(x_i, \frac{1}{1-\delta} r_i)$ for any $k \geq \bar{k}$. Here x_i and r_i are the centre and the radius of B_i.

Indeed, by the Hausdorff convergence of $K^{y_{n_k}}$ to $\bar{K}$, there exists $\bar{k} := \bar{k}(\delta) \in \mathbb{N}$ such that, for any $k > \bar{k}$, it holds that $K^{y_{n_k}} \subset B_{\underline{r}\delta}(\bar{K})$. Here, $B_{\underline{r}\delta}(\bar{K})$ denotes the $\underline{r}\delta$-neighbourhood of $\bar{K}$ in X, i.e., $B_{\underline{r}\delta}(\bar{K}) := \{x \in X : d(x, \bar{K}) < \underline{r}\delta\}$. Hence, for any $z \in K^{y_{n_k}}$, we can always find $x \in \bar{K}$ such that $d(x, z) < \underline{r}\delta$. Since $x \in B_i$ for some $i = 1, \dots, \ell$, we conclude $z \in B(x_i, \frac{1}{1-\delta} r_i)$ by noting that $\frac{1}{1-\delta} r_i - r_i = \frac{\delta}{1-\delta} r_i \geq \delta \underline{r}$.

By using the claim in the previous paragraph, the monotonicity $\mathbf{S}^m_{X,a} \geq \mathbf{S}^m_{X,b}$ whenever $a \leq b$, and (2.5), we obtain that

$$\mathbf{S}^m_{X,\varepsilon}(K^{y_{n_k}}) \leq c(m)(1+\delta)^m \sum_{i=1}^{\ell} r_i^m \leq (1+\delta)^m \mathbf{S}^m_{X,\varepsilon(1-\delta)}(\bar{K}) + \delta(1+\delta)^m c(m), \tag{2.6}$$

for any $k \geq \bar{k}(\delta)$. By taking $\delta \to 0$ after taking $k \to \infty$, we conclude that

$$\limsup_{k\to\infty} \mathbf{S}^m_{X,\varepsilon}(K^{y_{n_k}}) \leq \mathbf{S}^m_{X,\varepsilon}(\bar{K}) \leq \mathbf{S}^m_{X,\varepsilon}(K^y), \tag{2.7}$$

which is the sought conclusion. □

2.5 Differential structure on configuration spaces

In this section, $\Omega \subset \mathbb{R}^n$ will denote either a closed domain with smooth boundary or the whole Euclidean space $\mathbb{R}^n$. Below we review the natural differential structure of $\Upsilon(\Omega)$, obtained by lifting the Euclidean one on Ω. We follow closely the presentation in [2].
Cylinder functions, vector fields and divergence.

Definition 2.4 (*Cylinder functions*) We define the class of *cylinder functions* as

$$\mathrm{CylF}(\Upsilon(\Omega)) := \{\Phi(f_1^*, \dots, f_k^*) \,:\, \Phi \in C_b^\infty(\mathbb{R}^k),\ f_i \in C_c^\infty(\Omega),\ k \in \mathbb{N}\}, \tag{2.8}$$

where $f^*(\gamma) := \int_\Omega f d\gamma$ for every $\gamma \in \Upsilon(\Omega)$. We call f_i *inner function* and Φ *outer function*.

The tangent space $T_\gamma\Upsilon(\Omega)$ at $\gamma \in \Upsilon(\Omega)$ is identified with the Hilbert space of measurable γ-square-integrable vector fields $V : \Omega \to T(\mathbb{R}^n)$ equipped with the scalar product: for γ-measurable $V, W : \Omega \to T(\mathbb{R}^n)$,

$$\langle V, W\rangle_{T\Upsilon} = \int_\Omega \langle V(x), W(x)\rangle_{T\mathbb{R}^n} d\gamma(x)\,,$$
$$|V|^2_{T\Upsilon} = \int_\Omega \langle V(x), V(x)\rangle_{T\mathbb{R}^n} d\gamma(x)\,.$$

We define the tangent bundle of $\Upsilon(\Omega)$ by $T\Upsilon(\Omega) := \sqcup_{\gamma\in\Upsilon(\Omega)} T_\gamma\Upsilon(\Omega)$.

Definition 2.5 (*Cylinder vector fields*) We define two classes of *cylinder vector fields* as

$$\mathrm{CylV}(\Upsilon(\Omega)) := \left\{V(\gamma, x) = \sum_{i=1}^k F_i(\gamma)v_i(x) : F_i \in \mathrm{CylF}(\Upsilon(\Omega)),\ v_i \in C_c^\infty(\Omega; \mathbb{R}^n), k \in \mathbb{N}\right\},$$
$$\mathrm{CylV}_*(\Upsilon(\Omega)) := \left\{V(\gamma, x) = \sum_{i=1}^k F_i(\gamma)v_i(x) : F_i \in \mathrm{CylF}(\Upsilon(\Omega)),\ v_i \in C_*^\infty(\Omega; \mathbb{R}^n), k \in \mathbb{N}\right\}.$$

Notice that $\mathrm{CylV}_*(\Upsilon(\Omega)) \subset \mathrm{CylV}(\Upsilon(\Omega))$, and $\mathrm{CylV}_*(\Upsilon(\Omega)) = \mathrm{CylV}(\Upsilon(\Omega))$ when $\Omega = \mathbb{R}^n$. Using the tensorial notation, we can write

$$\begin{aligned}\mathrm{CylV}(\Upsilon(\Omega)) &= \mathrm{CylF}(\Upsilon(\Omega)) \otimes_\mathbb{R} C_c^\infty(\Omega; \mathbb{R}^n)\\ \mathrm{CylV}_*(\Upsilon(\Omega)) &= \mathrm{CylF}(\Upsilon(\Omega)) \otimes_\mathbb{R} C_*^\infty(\Omega; \mathbb{R}^n)\,.\end{aligned} \tag{2.9}$$

Let $p \in [1, \infty)$. For $V \in \mathrm{CylV}(\Upsilon(\Omega))$, we define

$$\|V\|^p_{L^p(T\Upsilon(\Omega))} := \|V\|^p_{L^p(\Upsilon(\Omega)\to T\Upsilon(\Omega),\pi_\Omega)} := \int_{\Upsilon(\Omega)} |V(\gamma)|^p_{T_\gamma\Upsilon} d\pi_\Omega(\gamma)\,, \tag{2.10}$$

and introduce the associated Banach space by

$$L^p(T\Upsilon(\Omega), \pi_\Omega) := \text{the completion of } \mathrm{CylV}(\Upsilon(\Omega)) \text{ with respect to } \|\cdot\|_{L^p(T\Upsilon(\Omega))}\,.$$

See [1, the fifth displayed formula on p. 23] in the case of $p = 2$.

Remark 2.6 When $p = 2$, the closure $L^p(T\Upsilon(\Omega), \pi_\Omega)$ coincides with the L^2-section of vector fields $L^2(\Upsilon(\Omega) \to T\Upsilon(\Omega), \pi_\Omega)$ defined as the direct integral of the Hilbert spaces $(T_\gamma\Upsilon(\Omega), \langle\cdot,\cdot\rangle_{T\Upsilon})$ with respect to π_Ω. See, for instance, the proof of [25, p. 165, 3rd bullet point].

Proposition 2.7 *Let* $1 \le p < \infty$. *Then,*

$$\|V\|_{L^p(T\Upsilon(\Omega))} < \infty\,, \quad V \in \mathrm{CylV}(\Upsilon(\Omega))\,. \tag{2.11}$$

Moreover, $\mathrm{CylV}_*(\Upsilon(\Omega))$ *is dense in* $L^p(T\Upsilon(\Omega), \pi_\Omega)$.

Proof Let $V(\gamma, x) = \sum_{i=1}^k F_i(\gamma)v_i(x)$. Then, we have that

$$\int_{\Upsilon(\Omega)} |V|^p_{T_\gamma\Upsilon(\Omega)} d\pi_\Omega(\gamma) \le \max_{1\le i\le k} \|F\|_{L^\infty} \sum_{i,j=1}^k \int_{\Upsilon(\Omega)} \left(\int_\Omega |v_i||v_j|d\gamma\right)^{p/2} d\pi_\Omega(\gamma).$$

By the exponential integrability implied by (2.1), we obtain that the function $\gamma \mapsto G(\gamma) := \int_\Omega |v_i||v_j|d\gamma$ is $L^p(\Upsilon(\Omega), \pi_\Omega)$ for any $1 \le p < \infty$, which concludes the first assertion.

The density of $\mathrm{CylV}_*(\Upsilon(\Omega))$ in $L^p(T\Upsilon(\Omega), \pi_\Omega)$ follows from the density of $C_*^\infty(\Omega; \mathbb{R}^n)$ in $L^p(\Omega; \mathbb{R}^n)$. More precisely, we check that for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ and $\varepsilon > 0$ there exists $W \in \mathrm{CylV}_*(\Upsilon(\Omega))$ such that $\int_{\Upsilon(\Omega)} |V - W|^p_{T_\gamma\Upsilon} d\pi_\Omega \le \varepsilon$. To this aim we write $V = \sum_{i=1}^k F_i v_i$ and pick $w_i \in C_*^\infty(\Omega)$ such that $\sum_{i=1}^k \|v_i - w_i\|_{L^p} < \varepsilon$ and set $W := \sum_{i=1}^k F_i w_i$. It is straightforward to see that W satisfies the needed estimate. By noting that $L^p(T\Upsilon(\Omega), \pi_\Omega)$ is defined as the completion of $\mathrm{CylV}(\Upsilon(\Omega))$ with respect to the norm $\|V\|_{L^p(T\Upsilon(\Omega))}$, the proof is complete. □

Definition 2.8 (*Directional derivatives*. [2, Def. 3.1]) Let $F = \Phi(f_1^*, \ldots, f_k^*) \in \mathrm{CylF}(\Upsilon(\Omega))$ and $v \in C_*^\infty(\Omega, \mathbb{R}^n)$. We denote by ϕ the flow associated to v, i.e.

$$\frac{d}{dt}\phi_t(x) = v(\phi_t(x)), \quad \phi_*(x) = x \in \Omega\,.$$

The *directional derivative* $\nabla_v F(\gamma) \in T_\gamma\Upsilon(\Omega)$ is defined as

$$\nabla_v F(\gamma) := \frac{d}{dt} F(\phi_t(\gamma))\Big|_{t=0}\,,$$

where $\phi_t(\gamma) := \sum_{x\in\gamma} \delta_{\phi_t(x)}$.

Definition 2.9 (*Gradient of cylinder functions*. [2, Def. 3.3]) The gradient $\nabla_{\Upsilon(\Omega)}F$ of $F \in \mathrm{CylF}(\Upsilon(\Omega))$ at $\gamma \in \Upsilon(\Omega)$ is defined as the unique vector field $\nabla_{\Upsilon(\Omega)}F$ so that

$$\nabla_v F(\gamma) = \langle \nabla_{\Upsilon(\Omega)}F, v\rangle_{T_\gamma\Upsilon(\Omega)}, \quad \gamma \in \Upsilon(\Omega),\ v \in C_*^\infty(\Omega, \mathbb{R}^n).$$

By the expression (2.8), the gradient $\nabla_{\Upsilon(\Omega)}F$ can be written as

$$\nabla_{\Upsilon(\Omega)}F(\gamma) = \sum_{i=1}^k \partial_i\Phi(f_1^*, \ldots, f_k^*)(\gamma)\nabla_{\mathbb{R}^n} f_i \in T_\gamma\Upsilon(\Omega)\,, \tag{2.12}$$

where $\nabla_{\mathbb{R}^n}$ is the gradient operator in $\mathbb{R}^n$. When $\Omega = \mathbb{R}^n$, we simply write $\nabla := \nabla_{\Upsilon(\mathbb{R}^n)}$ in the rest of the paper when no confusion occurs.

Notice that $\nabla_{\Upsilon(\Omega)}F \in \mathrm{CylV}(\Upsilon(\Omega))$ for any $F \in \mathrm{CylF}(\Upsilon(\Omega))$ by (2.12). In particular, for any $F \in \mathrm{CylF}(\Omega)$, it holds that $\nabla_{\Upsilon(\Omega)}F \in L^p(T\Upsilon(\Omega), \pi_\Omega)$ for any $1 \le p < \infty$ by Proposition 2.7.

Remark 2.10 (Ampleness of L^∞-vector fields) By Proposition 2.7, $\mathrm{CylV}_*(\Upsilon(\Omega)) \subset L^p(T\Upsilon(\Omega), \pi_\Omega)$ for any $p \in [1, \infty)$, while the inclusion is false for $p = \infty$. See [23, Example 4.35] for a counterexample. However, $\mathrm{CylV}_*(\Upsilon(\Omega))$ can be approximated by the subspace of bounded cylinder vector fields with respect to the pointwise convergence and the convergence in the $L^p(\Upsilon(\Omega) \to T\Upsilon(\Omega))$-norm for $1 \le p < \infty$. Indeed, given $\varepsilon > 0$ and $V = \sum_{i=1}^k F_i(\gamma)v_i(x) \in \mathrm{CylV}_*(\Upsilon(\Omega))$ it holds

$$|V|^2_{T_\gamma\Upsilon} = \sum_{i,j=1}^k F_i(\gamma)F_j(\gamma)\int_{\Upsilon(\Omega)} v_i(x)\cdot v_j(x)d\gamma(x)\,, \qquad \frac{1}{1+\varepsilon|V|^2_{T_\gamma\Upsilon}} \in \mathrm{CylF}(\Upsilon(\Omega))\,,$$

hence

$$V_\varepsilon := \frac{1}{1+\varepsilon|V|^2_{T_\gamma\Upsilon(\Omega)}} V \in \mathrm{CylV}(\Upsilon(\Omega))\,.$$

Finally, notice that for $\gamma \in \Upsilon(\Omega)$ it holds

$$|V - V_\varepsilon|_{T_\gamma\Upsilon(\Omega)} = \varepsilon\frac{|V|^3_{T_\gamma\Upsilon(\Omega)}}{1+\varepsilon|V|^2_{T_\gamma\Upsilon(\Omega)}} \le \varepsilon|V|^3_{T_\gamma\Upsilon(\Omega)} \to 0\,, \quad \text{as}\,\varepsilon \to 0\,.$$

Moreover, for every $1 \le p < \infty$ we have

$$\|V - V_\varepsilon\|_{L^p(T\Upsilon(\Omega))} \le \varepsilon\|V\|^3_{L^{3p}(T\Upsilon(\Omega))} \to 0\,, \quad \text{as}\,\varepsilon \to 0\,.$$

We now define the adjoint operator of the gradient $\nabla_{\Upsilon(\Omega)}$.

Definition 2.11 (*Divergence*. [2, Def. 3.5]) Let $1 < p < \infty$. We say that $V \in L^p(T\Upsilon(\Omega), \pi_\Omega)$ is in the domain $\mathcal{D}(\nabla^*_{\Upsilon(\Omega)})$ of the divergence if there exists a unique function $\nabla^*_{\Upsilon(\Omega)}V \in L^p(\Upsilon(\Omega), \pi_\Omega)$ such that

$$\int_{\Upsilon(\Omega)} \langle V, \nabla_{\Upsilon(\Omega)}F\rangle_{T_\gamma\Upsilon} d\pi_\Omega(\gamma) = -\int_{\Upsilon(\Omega)} F(\nabla^*_{\Upsilon(\Omega)}V)d\pi_\Omega\,, \quad F \in \mathrm{CylF}(\Upsilon(\Omega)) \tag{2.13}$$

When $\Omega = \mathbb{R}^n$, we simply write $\nabla^* := \nabla^*_{\Upsilon(\mathbb{R}^n)}$ in the rest of the paper when no confusion occurs.

Proposition 2.12 *The following inclusion holds:*

$$\mathrm{CylV}_*(\Upsilon(\Omega)) \subset \mathcal{D}(\nabla^*_{\Upsilon(\Omega)})\,.$$

Furthermore, for $V(\gamma, x) = \sum_{i=1}^m F_i(\gamma)v_i(x) \in \mathrm{CylV}_*(\Upsilon(\Omega))$,

$$\nabla^*_{\Upsilon(\Omega)}V(\gamma) = \sum_{i=1}^m \nabla_{v_i}F_i(\gamma) + \sum_{i=1}^m F_i(\gamma)(\nabla^*_{\mathbb{R}^n}v_i)^*\gamma\,, \tag{2.14}$$

where $\nabla^*_{\mathbb{R}^n}$ *is the divergence operator in* $\mathbb{R}^n$. *In particular,* $\nabla^*_{\Upsilon(\Omega)}V \in L^p(\Upsilon(\Omega), \pi_\Omega)$ *for every* $p \in [1, \infty)$.

Proof Let $r > 0$ be such that $\mathrm{supp}(v_i) \subset \Omega_r := \{x \in \Omega : d(x, \partial\Omega) > r\}$. For any $\varepsilon < r/2$ we define $\phi_\varepsilon \in C^\infty_*(\Omega)$ satisfying $\phi = 1$ on Ω_ε. For any $i = 1, \dots, m$ we write $F_i = \Phi_i(f^*_{1,i}, \dots, f^*_{k_i,i})$ and set $F^\varepsilon_i := \Phi_i((\phi_\varepsilon f_{1,i})^*, \dots, (\phi_\varepsilon f_{k_i,i})^*)$. Observe that $V_\varepsilon := \sum_{i=1}^m F^\varepsilon_i(\gamma)v_i \in \mathrm{CylV}_*(\Upsilon(\Omega))$ and also $V_\varepsilon \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ by construction. Furthermore, we note that $F \in \mathrm{CylF}(\Upsilon(\Omega))$ can be extended to $\widetilde{F} \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ with $F = \widetilde{F}$ on $\Upsilon(\Omega)$

by extending each inner function $f_i \in C_c^\infty(\Omega)$ to $\widetilde{f_i} \in C_c^\infty(\mathbb{R}^n)$ with $f_i = \widetilde{f_i}$ on Ω (e.g., by Whitney's extension theorem). Thus, $\nabla^*_{\Upsilon(\Omega)}$ and $\nabla^*_{\Upsilon(\mathbb{R}^n)}$ defined in (2.13) are consistent, so that $\nabla^*_{\Upsilon(\Omega)} V_\varepsilon(\gamma) = \nabla^*_{\Upsilon(\mathbb{R}^n)} V_\varepsilon(\gamma)$. By [2, Prop. 3.1], therefore, we have

$$\begin{aligned}\nabla^*_{\Upsilon(\Omega)} V_\varepsilon(\gamma) = \nabla^*_{\Upsilon(\mathbb{R}^n)} V_\varepsilon(\gamma) &= \sum_{i=1}^m \nabla_{v_i} F_i^\varepsilon(\gamma) + \sum_{i=1}^m F_i^\varepsilon(\gamma)(\nabla^*_{\mathbb{R}^n} v_i)^*\gamma \\ &= \sum_{i=1}^m \nabla_{v_i} F_i(\gamma) + \sum_{i=1}^m F_i^\varepsilon(\gamma)(\nabla^*_{\mathbb{R}^n} v_i)^*\gamma \,.\end{aligned}$$

Here we used the fact that $F_i(\gamma) = F_i^\varepsilon(\gamma)$ for any γ concentrated on the support of v_i. The sought conclusion (2.14) follows from the observation that $F_i^\varepsilon \to F_i$ in $L^p(\Upsilon(\Omega), \pi_\Omega)$ and $V_\varepsilon \to V$ in $L^p(T\Upsilon(\Omega), \pi_\Omega)$ combined with (2.13). The last assertion is then a direct consequence from Proposition 2.7 and (2.14). □

Sobolev spaces. We now introduce the $(1, p)$-Sobolev space. The operator

$$\nabla_{\Upsilon(\Omega)} : \mathrm{CylF}(\Upsilon(\Omega)) \subset L^p(\Upsilon(\Omega), \pi_\Omega) \to \mathrm{CylV}(\Upsilon(\Omega)) \tag{2.15}$$

is densely defined and closable. The latter fact is a direct consequence of the integration-by-parts formula (2.14). Indeed, we observe that, if $F_n \in \mathrm{CylF}(\Upsilon(\Omega))$, $F_n \to 0$ in $L^p(\Upsilon(\Omega), \pi_\Omega)$, and $\nabla_{\Upsilon(\Omega)} F_n \to W$ in $L^p(T\Upsilon(\Omega), \pi_\Omega)$, then for any $V \in \mathrm{CylV}_*(\Upsilon(\Omega))$, it holds

$$\begin{aligned}\int_{\Upsilon(\Omega)} \langle V, W\rangle_{T_\gamma\Upsilon} d\pi_\Omega(\gamma) &= \lim_{n\to\infty} \int_{\Upsilon(\Omega)} \langle V, \nabla_{\Upsilon(\Omega)} F_n\rangle_{T_\gamma\Upsilon} d\pi_\Omega(\gamma) \\ &= -\lim_{n\to\infty} \int_{\Upsilon(\Omega)} (\nabla^*_{\Upsilon(\Omega)} V) F_n d\pi_\Omega(\gamma) = 0\,,\end{aligned}$$

yielding $W = 0$ as a consequence of the density of $\mathrm{CylV}_*(\Upsilon(\Omega))$ in $L^p(T\Upsilon(\Omega), \pi_\Omega)$ by Proposition 2.7. The above argument justifies the following definition.

Definition 2.13 ($H^{1,p}$-*Sobolev spaces*) Let $1 < p < \infty$. We define $H^{1,p}(\Upsilon(\Omega), \pi_\Omega)$ as the closure of $\mathrm{CylF}(\Upsilon(\Omega))$ in $L^p(\Upsilon(\Omega), \pi_\Omega)$ with respect to the following $(1, p)$-Sobolev norm:

$$\|F\|^p_{H^{1,p}(\Upsilon(\Omega))} := \|F\|^p_{L^p(\Upsilon(\Omega))} + \|\nabla_{\Upsilon(\Omega)} F\|^p_{L^p(T\Upsilon(\Omega))}\,.$$

We set $\|F\|_{H^{1,p}} := \|F\|_{H^{1,p}(\Upsilon(\mathbb{R}^n))}$. When $p = 2$, we write the corresponding Dirichlet form (i.e., a closed form satisfying the unit contraction property [35, Def. 4.5]) by

$$\mathcal{E}_{\Upsilon(\Omega)}(F, G) := \int_{\Upsilon(\Omega)} \left\langle \nabla_{\Upsilon(\Omega)} F, \nabla_{\Upsilon(\Omega)} G\right\rangle_{T_\gamma\Upsilon(\Omega)} d\pi_\Omega(\gamma), \quad F, G \in H^{1,2}(\Upsilon(\Omega), \pi_\Omega)\,.$$

We set $\mathcal{E} := \mathcal{E}_{\Upsilon(\mathbb{R}^n)}$.

Remark 2.14 (The case of $p = 1$) As is indicated by (2.14), it is not true in general that $\nabla^*_{\Upsilon(\Omega)} V \in L^\infty(\Upsilon(\Omega), \pi_\Omega)$ since arbitrarily many finite particles can be concentrated on the supports of inner functions of $F \in \mathrm{CylF}(\Upsilon(\Omega))$ and vector fields v_i. See [23, Example 4.35] for more detail. Due to this fact, the standard integration by part argument for the closability of the operator $\nabla_{\Upsilon(\Omega)} : \mathrm{CylF}(\Upsilon(\Omega)) \to \mathrm{CylV}(\Upsilon(\Omega)) \subset L^p(T\Upsilon(\Omega), \pi_\Omega)$ does not work in the case of $p = 1$. For this reason, we restricted the definition of the $H^{1,p}$-Sobolev spaces to the case $1 < p < \infty$ in Definition 2.13.

Once the closed form $\mathcal{E}_{\Upsilon(\Omega)}$ on $L^2(\Upsilon(\Omega), \pi_\Omega)$ is constructed, one can define the infinitesimal generator on $L^2(\Upsilon(\Omega), \pi_\Omega)$ as the unique non-positive definite self-adjoint operator

Definition 2.15 (*Laplace operator* [2, Theorem 4.1]) *The* $L^2(\Upsilon(\Omega), \pi_\Omega)$*-Laplace operator* $\Delta_{\Upsilon(\Omega)}$ with domain $\mathcal{D}(\Delta_{\Upsilon(\Omega)})$ is defined as the unique non-positive definite self-adjoint operator $\Delta_{\Upsilon(\Omega)}$ so that

$$\mathcal{E}_{\Upsilon(\Omega)}(F, G) = -\int_{\Upsilon(\Omega)} (\Delta_{\Upsilon(\Omega)} F) G d\pi_\Omega, \quad F \in \mathcal{D}(\Delta_{\Upsilon(\Omega)}),\ G \in \mathcal{D}(\mathcal{E}_{\Upsilon(\Omega)}) .$$

In the case of $\Omega = \mathbb{R}^n$, employing (2.12) and (2.14), one can compute that

$$\Delta_{\Upsilon(\mathbb{R}^n)} F := \nabla^*_{\Upsilon(\mathbb{R}^n)} \nabla_{\Upsilon(\mathbb{R}^n)} F, \quad F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)) .$$

When $\Omega = \mathbb{R}^n$, we shortly write $\Delta = \Delta_{\Upsilon(\mathbb{R}^n)}$ in the rest of the paper when no confusion occurs.

Let $\{T_t^{\Upsilon(\Omega)}\}$ and $\{G_\alpha^{\Upsilon(\Omega)}\}$ be the strongly continuous Markovian L^2-semigroup and resolvent, respectively, corresponding to the energy $\mathcal{E}_{\Upsilon(\Omega)}$. We set $G_\alpha := G_\alpha^{\Upsilon(\mathbb{R}^n)}$ and $T_t := T_t^{\Upsilon(\mathbb{R}^n)}$. By the Riesz–Thorin Interpolation Theorem, $T_t^{\Upsilon(\Omega)}$ and $\{G_\alpha^{\Upsilon(\Omega)}\}$ can be uniquely extended to L^p strongly continuous Markovian semigroup and resolvent, respectively, for every $1 \le p < \infty$ (see e.g. [43, Section 2, p. 70]).

2.6 Product semigroups and exponential cylinder functions

In this section, we relate the finite-product semigroup on $\Omega^{\times k}$ and the semigroup on $\Upsilon^k(\Omega)$ when $\Omega \subset \mathbb{R}^n$ is a bounded closed domain with smooth boundary. To this aim we introduce a class of test functions, which is suitable to compute the semigroups.

Definition 2.16 (*Exponential cylinder functions*. [2, (4.12)]) Let $\Omega \subset \mathbb{R}^n$ be a bounded closed domain with smooth boundary, or $\Omega = \mathbb{R}^n$. The class $\mathrm{ECyl}(\Upsilon(\Omega))$ of *exponential cylinder functions* is defined as the vector space spanned by

$$\left\{\exp\left\{\log(1+f)^*\right\} \ :\ f \in \mathcal{D}(\Delta_\Omega)\,,\ \Delta_\Omega f \in L^1(\Omega),\ -\delta \le f \le 0 \text{ for some } \delta \in (0,1)\right\} .$$

Here $(\Delta_\Omega, \mathcal{D}(\Delta_\Omega))$ denotes the L^2-Neumann Laplacian on Ω when $\Omega \subsetneq \mathbb{R}^n$.

The space $\mathrm{ECyl}(\Upsilon(\Omega))$ is dense in $L^p(\Upsilon(\Omega), \pi_\Omega)$ for any $1 \le p < \infty$ (see [2, p. 479]). Noting that Δ_Ω is essentially self-adjoint on the core $C_c^\infty(\Omega) \cap \{\frac{\partial f}{\partial n} = 0 \text{ in } \partial\Omega\}$, where $\frac{\partial}{\partial n}$ is the normal derivative on $\partial\Omega$, and the corresponding L^2-semigroup $\{T_t^\Omega\}$ is conservative, we can apply the same argument in the proof of [2, Prop. 4.1] to obtain the following: $T_t^{\Upsilon(\Omega)} \mathrm{ECyl}(\Upsilon(\Omega)) \subset \mathrm{ECyl}(\Upsilon(\Omega))$ and

$$T_t^{\Upsilon(\Omega)} \exp\left\{\log(1+f)^*\right\} = \exp\left\{\log\left(1 + (T_t^\Omega f)\right)^*\right\} . \tag{2.16}$$

Let $T_t^{\Omega,\otimes k}$ be the k-tensor semigroup of T_t^Ω, i.e. the unique semigroup in $L^p(\Omega^{\times k})$ satisfying

$$T_t^{\Omega,\otimes k} f(x_1, \ldots, x_k) := T_t^\Omega f_1(x_1) \cdots T_t^\Omega f_k(x_k) , \quad \text{for every } k \in \mathbb{N} , \tag{2.17}$$

whenever $f(x_1, \ldots, x_k) = f_1(x_1) \cdots f_k(x_k)$ with $f_i \in L^\infty(\Omega)$ for $i = 1, \ldots, k$.

Proposition 2.17 *Let $\Omega \subset \mathbb{R}^n$ be a bounded closed domain with smooth boundary and $1 \le p < \infty$. For $F \in L^p(\Upsilon^k(\Omega), \pi_\Omega|_{\Upsilon^k(\Omega)})$, it holds*

$$T_t^{\Omega,\otimes k}(F \circ \mathsf{s}_k) = (T_t^{\Upsilon^k(\Omega)} F) \circ \mathsf{s}_k, \quad \mathbf{S}^{kn}_{\Omega^{\times k}}\text{-a.e.} \tag{2.18}$$

Proof Since $\mathrm{ECyl}(\Upsilon^k(\Omega))$ is dense in $L^p(\Upsilon^k(\Omega))$ for any $1 \le p < \infty$, it suffices to show (2.18) only for $F \in \mathrm{ECyl}(\Upsilon^k(\Omega))$. Furthermore, we can reduce the argument to the case $F = \exp\{\log(1+f)^*\}$ by using the linearity of semigroups. From (2.17) and (2.16) we get

$$\begin{aligned} T_t^{\Omega,\otimes k}(F \circ \mathsf{s}_k)(x_1,\ldots,x_k) &= T_t^{\Omega,\otimes k}(\exp\{\log(1+f)^*\} \circ \mathsf{s}_k)(x_1,\ldots,x_k) \\ &= T_t^{\Omega,\otimes k}\Big(\prod_{i=1}^k (1+f)(\cdot_i)\Big)(x_1,\ldots,x_k) \\ &= \prod_{i=1}^k \Big(1 + T_t^{\Omega} f(x_i)\Big) \\ &= \exp\Big\{\log\big(1 + (T_t^{\Omega} f)^*\big)\Big\} \circ \mathsf{s}_k(x_1,\ldots,x_k) \\ &= T_t^{\Upsilon(\Omega)} \exp\big\{\log(1+f)^*\big\} \circ \mathsf{s}_k(x_1,\ldots,x_k)\,. \square \end{aligned}$$

2.7 Suslin sets

Let X be a set. We denote by $\mathbb{N}^{\mathbb{N}}$ the space of all infinite sequences $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers. For $\phi \in \mathbb{N}^{\mathbb{N}}$, we write $\phi|_l \in \mathbb{N}^l$ for the restriction of ϕ to the first l elements, i.e., $\phi|_l := (\phi_i : 1 \le i \le l)$. Let $\mathcal{S} := \cup_{l\in\mathbb{N}}\mathbb{N}^l$, and for $\sigma \in \mathcal{S}$, we denote the length of the sequence σ by $\#\sigma := \#\{\sigma_i\}$. Let $\mathscr{E} \subset 2^X$ be a family of subsets in X. We write $\mathcal{S}(\mathscr{E})$ for the family of sets expressible in the following form:

$$\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{l\ge 1} E_{\phi|_l},$$

for some family $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ in $\mathscr{E}$. A family $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ is called *Suslin scheme*; the corresponding set $\cup_{\phi\in\mathbb{N}^{\mathbb{N}}} \cap_{l\ge 1} E_{\phi|_l}$ is its *kernel*; the operation

$$\{E_\sigma\}_{\sigma\in\mathcal{S}} \mapsto \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{l\ge 1} E_{\phi|_l},$$

is called *Suslin's operation*. We denote by $\mathcal{S}(\mathscr{E})$ the family of sets generated from sets in $\mathscr{E}$ by Suslin's operation, whose elements are called an $\mathscr{E}$-Suslin set (or simply Suslin set). It is known that $\mathcal{S}(\mathscr{E})$ is closed under Suslin's operation ([44], and e.g., [29, 421D Theorem]). If E_σ is compact for all $\sigma \in \mathcal{S}$, we call $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ a *compact* Suslin scheme. We say that $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ is regular if $E_\sigma \subset E_\tau$ whenever $\#\tau \le \#\sigma$ and $\sigma_i \le \tau_i$ for any $i < \#\sigma$ ([29, 421X (n) & 422 H Theorem (b)]).

In the following remark, we list basic properties of Suslin sets in a Polish space and relations to Choquet capacities and Borel measures. In the rest of this section, we assume that

$$\begin{aligned} &(X,\tau) \text{ is a Polish space, } \mathsf{c} \text{ is a Choquet capacity on } X, \quad \mu \text{ is a bounded Borel measure,} \\ &\mathscr{E} := \mathcal{C}(X) := \{C : \text{closed set in } X\}\,. \end{aligned} \tag{2.19}$$

We refer the readers to, e.g., [29, 432I Definition] for the definition of Choquet capacity.

Remark 2.18 Under the assumption (2.19), the following hold:

(i) Every Borel set is a Suslin set, i.e., $\mathcal{B}(\tau) \subset \mathcal{S}(\mathscr{E})$ (e.g., [29, 423B(a) and 423F(a)]);
(ii) Every Suslin set is μ-measurable, i.e., $\mathcal{S}(\mathscr{E}) \subset \overline{\mathcal{B}}^{\mu}(\tau)$ (e.g., [29, 431B Corollary]);
(iii) Let A be a Suslin set in X. Then, A is the kernel of a *compact regular* Suslin scheme $\{E_\sigma\}_{\sigma \in \mathcal{S}}$. Furthermore, it holds that

$$\mathsf{c}(A) = \sup_{\psi \in \mathbb{N}^{\mathbb{N}}} \mathsf{c}(A_\psi), \quad A_\psi = \bigcup_{\phi \leq \psi} \bigcap_{l \geq 1} E_{\phi|_l}, \tag{2.20}$$

whereby $\phi \leq \psi$ means that $\phi(l) \leq \psi(l)$ for all $l \in \mathbb{N}$ (e.g., [29, 423B Theorem & the proof of 432J Theorem]). By the regularity of $\{E_\sigma\}_{\sigma \in \mathcal{S}}$, (2.20) can be reduced to the following form:

$$\mathsf{c}(A) = \sup_{\psi \in \mathbb{N}^{\mathbb{N}}} \mathsf{c}(A_\psi), \quad A_\psi = \bigcap_{l \geq 1} E_{\psi|_l}, \quad \psi \in \mathbb{N}^{\mathbb{N}}; \tag{2.21}$$

(iv) A subset $A \subset X$ is Suslin iff A is analytic iff A is K-analytic ([29, 423E Theorem (b)]. See [29, 422F, 423A Definitions] for the definitions of *K-analyticity* and *analyticity* respectively). As every K-analytic set is capacitable (e.g., [29, 432J]), in particular, we have that $\mathsf{c}(A)$ is well-defined for every Suslin set A as

$$\mathsf{c}(A) = \sup\{\mathsf{c}(K) : K \subset A \text{ compact}\}. \tag{2.22}$$

3 Finite-codimensional Poisson measures

In this section, we construct finite-codimensional Poisson measures on $\Upsilon(\mathbb{R}^n)$. As a first step we prove measurability results for sections of *Suslin* subsets of the configuration space.

3.1 Measurability of sections of Suslin sets

Let $B \subset \mathbb{R}^n$. For $A \subset \Upsilon(\mathbb{R}^n)$ and $\eta \in \Upsilon(B)$, the section $A_{\eta,B} \subset \Upsilon(B^c)$ of A at η is defined as

$$A_{\eta,B} = \{\gamma \in \Upsilon(B^c) : \gamma + \eta \in A\}. \tag{3.1}$$

The subset of $A_{\eta,B}$ consisting of k-particle space $\Upsilon^k(B^c)$ is denoted by $A^k_{\eta,B} := A_{\eta,B} \cap \Upsilon^k(B^c)$. To shorten the notation we often write $A_{\eta,r}$ in place of A_{η,B_r^c}, where $\overline{B_r}$ is the closed ball centred at the origin.

Lemma 3.1 *Let $B \subset \mathbb{R}^n$ be a Borel set. If A is Suslin in $\Upsilon(\mathbb{R}^n)$ then $A^k_{\eta,B}$ is Suslin in $\Upsilon^k(B^c)$ for every $\eta \in \Upsilon(B)$, $k \in \mathbb{N}$ and $r > 0$.*

Proof We can express $A_{\eta,B} = \mathrm{pr}_{B^c}\big(\mathrm{pr}_B^{-1}(\eta) \cap A\big)$. The set $\mathrm{pr}_B^{-1}(\eta) \cap A$ is Suslin in $\Upsilon(\mathbb{R}^n)$ whenever A is Suslin. Set $\Upsilon_{\eta,B}(\mathbb{R}^n) = \mathrm{pr}_B^{-1}(\eta) \cap \Upsilon(\mathbb{R}^n)$, which is Suslin. The map $\mathrm{pr}_{B^c} : \Upsilon_{\eta,B}(\mathbb{R}^n) \to \Upsilon(B^c)$ is continuous. Thus, $A_{\eta,B}$ is the continuous image $\mathrm{pr}_{B^c}\big(\mathrm{pr}_B^{-1}(\eta) \cap A\big)$ of the Suslin set $\mathrm{pr}_B^{-1}(\eta) \cap A$ in the Suslin Hausdorff space $\Upsilon_{\eta,B}(\mathbb{R}^n)$. Hence, $A_{\eta,B}$ is Suslin ([29, 423B Proposition (b) & 423E Theorem (b)]). Since $A^k_{\eta,B} = A_{\eta,B} \cap \Upsilon^k(B^c)$ and $\Upsilon^k(B^c)$ is Borel in $\Upsilon(B^c)$, we conclude that $A^k_{\eta,B}$ is Suslin. □

Lemma 3.2 *Let $B \subset \mathbb{R}^n$ be an open set. Let $A \subset \Upsilon(\mathbb{R}^n)$ be the kernel of a compact Suslin's scheme $\{E_\sigma\}_{\sigma\in\mathcal{S}}$, i.e., $A = \cup_{\phi\in\mathbb{N}^\mathbb{N}} \cap_{l\geq 1} E_{\phi|_l}$ with E_σ compact for any $\sigma \in \mathcal{S}$. Then, $A_{\eta,B}$ is the kernel of the compact Suslin scheme $\{(E_\sigma)_{\eta,r}\}_{\sigma\in\mathcal{S}}$.*

Proof By expressing $(E_\sigma)_{\eta,B} = \mathrm{pr}_{B^c}\big(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap E_\sigma\big)$, where $\Upsilon_{\eta,B}(\mathbb{R}^n) = \mathrm{pr}_B^{-1}(\eta) \cap \Upsilon(\mathbb{R}^n)$, we see that $(E_\sigma)_{\eta,B}$ is compact since $\Upsilon_{\eta,B}(\mathbb{R}^n)$ is closed, E_σ is compact by the hypothesis, pr_{B^c} is continuous on $\Upsilon_{\eta,B}(\mathbb{R}^n)$ and every continuous image of a compact set is compact. To see that $A_{\eta,B}$ is the kernel of $\{(E_\sigma)_{\eta,r}\}_{\sigma\in\mathcal{S}}$,

$$
\begin{aligned}
A_{\eta,B} &= p_r\big(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap A\big) = p_r\Bigg(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} E_{\phi|_l}\Bigg) \\
&= p_r\Bigg(\bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} \Upsilon_{\eta,B}(\mathbb{R}^n) \cap E_{\phi|_l}\Bigg) \\
&= \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} p_r\big(\Upsilon_{\eta,B}(\mathbb{R}^n) \cap E_{\phi|_l}\big) = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{l\geq 1} (E_\sigma)_{\eta,B}\,. \quad \square
\end{aligned}
$$

3.2 Localised finite-codimensional Poisson measures

In this section, we construct *a localised* version of the m-codimensional Poisson measure ρ_r^m, which will be used to construct the m-condimensional Poisson measure by taking the limit for $r \to \infty$. We also show that Suslin sets are contained in the domain of the finite-codimensional Poisson measure.

Let $A \subset \Upsilon(\mathbb{R}^n)$ be a Suslin subset. By Lemma 3.1, the set $A^k_{\eta,r} = A^k_{\eta,B_r^c}$ is Suslin. Since $\mathsf{S}^{m,k}_{B_r}$ is a Choquet capacity, the expression $\mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r})$ is well-defined and satisfies (2.22), which in particular implies that $A^k_{\eta,r}$ is a $\mathsf{S}^{m,k}_{B_r}$-measurable set. We define the domain $\mathscr{D}^m$ of the m-codimensional measures by

$$
\mathscr{D}^m := \bigcap_{r>0} \mathscr{D}^m_r\,, \tag{3.2}
$$

where the localised domain $\mathscr{D}^m_r$ is defined by

$$
\mathscr{D}^m_r := \{A \subset \Upsilon(\mathbb{R}^n) \,:\, \text{the map } \Upsilon(B_r^c) \ni \eta \mapsto \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) \text{ is } \pi_{B_r^c}\text{-measurable for every } k\}\,.
$$

We first introduce the m-codimensional Poisson measure on the configuration space $\Upsilon(B_r)$ over the ball B_r.

Definition 3.3 The *m-codimensional Poisson measure $\rho^m_{\Upsilon(B_r)}$ on* $\Upsilon(B_r)$ is defined as

$$
\rho^m_{\Upsilon(B_r)}(A) := e^{-\mathbf{S}^n(B_r)} \sum_{k=1}^{\infty} \mathsf{S}^{m,k}_{B_r}(A^k) \quad \text{for every Suslin set } A \text{ in } \Upsilon(B_r)\,, \tag{3.3}
$$

where $A^k = A \cap \Upsilon^k(B_r)$.

Remark 3.4 Notice that $\rho^0_{\Upsilon(B_r)} = \pi_{B_r}$, in other words the 0-codimension Poisson measure $\rho^0_{\Upsilon(B_r)}$ on $\Upsilon(B_r)$ is the Poisson measure π_{B_r} on $\Upsilon(B_r)$. It can be shown by noting that the m-dimensional spherical Hausdorff measure $\mathbf{S}^m$ and the n-dimensional Lebesgue measure $\mathbf{L}^n$ coincide when $m = n$ (see Remark 2.2).

We introduce the localised m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$ by averaging the m-codimensional Poisson measure $\rho^m_{\Upsilon(B_r)}$ by means of $\pi_{B_r^c}$.

Definition 3.5 The *localised m-codimensional Poisson measure ρ^m_r on* $\Upsilon(\mathbb{R}^n)$ is defined by

$$\rho^m_r(A) = \int_{\Upsilon(B_r^c)} \rho^m_{\Upsilon(B_r)}(A_{\eta,r}) d\pi_{B_r^c}(\eta), \quad A \in \mathscr{D}^m. \tag{3.4}$$

Before investigating the main properties of ρ^m_r, we check that sufficiently many sets are contained in $\mathscr{D}^m$, i.e. we show that all Suslin sets are contained in the domain $\mathscr{D}^m$ for $m \le n$.

Proposition 3.6 *Any Suslin set in $\Upsilon(\mathbb{R}^n)$ is contained in $\mathscr{D}^m$ for $m \le n$.*

Proof Let $A \subset \Upsilon(\mathbb{R}^n)$ be a Suslin set. Let $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ be a Suslin scheme whose kernel is A. Noting that $\Upsilon(B_r^c)$ is Polish, by applying (i) of Remark 2.18 with $X = \Upsilon(B_r^c)$ and $\mu = \pi_{B_r^c}$, any Suslin set is $\pi_{B_r^c}$-measurable. Hence, it suffices to show that every super-level set $\{\eta : \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) > a\}$ is Suslin for any $a \in \mathbb{R}$, $r > 0$, $k \in \mathbb{N}$ and $m \le n$. Note that $A^k_{\eta,r}$ is Suslin by Lemma 3.1, whence the expression $\{\eta : \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) > a\}$ is well-defined as was discussed in the paragraph before (3.2).

Since $\Upsilon(\mathbb{R}^n)$ is Polish, by using (iii) in Remark 2.18, we may assume that $\{E_\sigma\}_{\sigma\in\mathcal{S}}$ is a compact regular Suslin scheme. By Lemma 3.2 and $\Upsilon(B_r) = \sqcup_{k\in\mathbb{N}}\Upsilon^k(B_r)$, we see that $A^k_{\eta,r} \subset \Upsilon^k(B_r)$ is the kernel of the compact regular Suslin scheme $\{(E_\sigma)^k_{\eta,r}\}_{\sigma\in\mathcal{S}}$, whereby $(E_\sigma)^k_{\eta,r} := (E_\sigma)_{\eta,B_r^c} \cap \Upsilon^k(B_r)$. Since $\mathsf{S}^{m,k}_{B_r}$ is an outer measure on $\Upsilon^k(B_r)$ by construction, $\mathsf{S}^{m,k}_{B_r}$ is a Choquet capacity on $\Upsilon^k(B_r)$. Hence, by applying (2.21) in (iii) of Remark 2.18 with $X = \Upsilon^k(B_r)$ and $\mathsf{c} = \mathsf{S}^{m,k}_{B_r}$, we obtain that

$$\mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) = \sup_{\psi\in\mathbb{N}^\mathbb{N}} \mathsf{S}^{m,k}_{B_r}((A^k_{\eta,r})_\psi), \quad (A^k_{\eta,r})_\psi = \bigcap_{l\ge 1}(E_{\psi|_l})^k_{\eta,r}, \quad \psi \in \mathbb{N}^\mathbb{N}.$$

Thus, noting the monotonicity $\mathsf{S}^{m,k}_{B_r,\varepsilon} \le \mathsf{S}^{m,k}_{B_r,\delta}$ $(\delta \le \varepsilon)$ of the ε-Hausdorff measure defined in (2.3), the super-level set $\{\eta : \mathsf{S}^{m,k}_r(A^k_{\eta,r}) > a\}$ can be expressed in the following way:

$$\{\eta : \mathsf{S}^{m,k}_{B_r}(A^k_{\eta,r}) > a\} = \bigcup_{\varepsilon>0}\bigcup_{\psi\in\mathbb{N}^\mathbb{N}} \{\eta : \mathsf{S}^{m,k}_{B_r,\varepsilon}\big((A^k_{\eta,r})_\psi\big) > a\}.$$

Since the space $\mathcal{S}(\mathscr{E})$ of Suslin sets is closed under Suslin's operation, it suffices to show that $\{\eta : \mathsf{S}^{m,k}_{B_r,\varepsilon}\big((A^k_{\eta,r})_\psi\big) > a\}$ is Suslin.

We equip $\Upsilon^k(B_r)$ with the L^2-transportation distance d_{Υ^k} as defined in (2.2), and equip $\Upsilon(B_r^c)$ with some distance d generating the vague topology. By Proposition 2.3 and noting that $(A^k_{\eta,r})_\psi$ is compact and that $\mathsf{S}^{m,k}_{B_r,\varepsilon}$ is (up to constant multiplication) the m-codimensional ε-spherical Hausdorff measure on $\Upsilon^k(B_r)$ associated with d_{Υ^k}, we conclude that $\{\eta : \mathsf{S}^{m,k}_{B_r,\varepsilon}\big((A^k_{\eta,r})_\psi\big) > a\}$ is open in $\Upsilon(B_r^c)$ for any $a \in \mathbb{R}$, $r > 0$, $k \in \mathbb{N}$ and $m \le n$. $\square$

3.3 Finite-codimensional Poisson measures

In this section, we construct the m-codimensional Poisson measure on $\Upsilon(\mathbb{R}^n)$, which is the first main result of this paper. By Proposition 3.6, the set function ρ^m_r given in (3.4) turned out to be well-defined in the sense that the space $\mathcal{S}(\mathscr{E})$ of all Suslin sets in $\Upsilon(\mathbb{R}^n)$ is contained in its domain $\mathscr{D}^m$. We show the following monotonicity result which allows us to pass to the limit of ρ_r^m as $r \to \infty$.

Theorem 3.7 *The map $r \mapsto \rho_r^m(A)$ is monotone non-decreasing for any $A \in \mathcal{S}(\mathscr{E})$.*

The proof of Theorem 3.7 is given at the end of this section. We can now introduce the *m-codimensional Poisson measure* on $\Upsilon(\mathbb{R}^n)$ as the monotone limit of ρ_r^m on the space $\mathcal{S}(\mathscr{E})$ of Suslin sets:

$$\rho^m(A) = \lim_{r\to\infty} \rho_r^m(A), \quad \forall A \in \mathcal{S}(\mathscr{E}). \tag{3.5}$$

Definition 3.8 (*m-codimensional Poisson Measure*) Let $\mathfrak{D}^m$ be the completion of $\mathcal{S}(\mathscr{E})$ with respect to ρ^m. The measure $(\rho^m, \mathfrak{D}^m)$ is called the *m-codimensional Poisson measure* on $\Upsilon(\mathbb{R}^n)$.

Remark 3.9 We give two remarks below:

(i) Note $\rho^0 = \pi$, i.e. 0-codimensional Poisson measure ρ^0 on $\Upsilon(\mathbb{R}^n)$ is the Poisson measure π on $\Upsilon(\mathbb{R}^n)$ by noting that the m-dimensional spherical Hausdorff measure $\mathbf{S}^m$ and the n-dimensional Lebesgue measure $\mathbf{L}^n$ coincide when $m = n$ (see Remark 2.2).
(ii) The construction of ρ^m, a priori, depends on the choice of the exhaustion $\{B_r\} \subset \mathbb{R}^n$. However, in Proposition 3.13, we will see that it is not the case.

The rest of this section is devoted to the proof of Theorem 3.7. Let us begin with a definition.

Definition 3.10 (*Section of functions, multi-section*) Let $M, N \subset \mathbb{R}^n$ be two disjoint sets and $L = M \sqcup N$. For every $F : \Upsilon(L) \to \mathbb{R}$ and $\xi \in \Upsilon(M)$, define $F_{\xi,M} : \Upsilon(N) \to \mathbb{R}$ as

$$F_{\xi,M}(\zeta) := F(\zeta + \xi), \quad \zeta \in \Upsilon(N). \tag{3.6}$$

For a set $A \subset \Upsilon(\mathbb{R}^n)$, let $A_{\xi,\eta,M,N}$ denote the *multi-section* both at $\xi \in \Upsilon(M)$ and $\zeta \in \Upsilon(N)$:

$$A_{\xi,\zeta,M,N} := \{\gamma \in \Upsilon(L^c) : \gamma + \xi + \zeta \in A\}, \quad \text{and} \quad A^k_{\xi,\zeta,M,N} = A_{\xi,\zeta,M,N} \cap \Upsilon^k(L^c). \tag{3.7}$$

Lemma 3.11 *Let A be a Suslin set in $\Upsilon(\mathbb{R}^n)$. Let $M, N \subset \mathbb{R}^n$ be two disjoint Borel sets. Set $L = M \sqcup N$. Let $F : \Upsilon(L) \to \mathbb{R}$ be defined by $\gamma \mapsto F(\gamma) := \mathsf{S}^{m,k}_{L^c}(A^k_{\gamma,L})$. Then,*

$$F_{\xi,M}(\zeta) = \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M}), \quad \forall \xi \in \Upsilon(M),\ \forall \zeta \in \Upsilon(N). \tag{3.8}$$

Proof The set $A^k_{\zeta,\xi,N,M}$ is Suslin by the same argument as in Lemma 3.1. Thus, $\mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M})$ is well-defined. By Definition 3.10, we have that

$$\begin{aligned} F_{\xi,M}(\zeta) &= F(\zeta + \xi) = \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta+\xi,L}) \\ &= \mathsf{S}^{m,k}_{L^c}\big(\{\gamma \in \Upsilon(L^c) : \gamma + \xi + \zeta \in A\}\big) = \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M}). \square \end{aligned}$$

The next lemma is straightforward since the Poisson measures π_M and π_N are mutually singular.

Lemma 3.12 *With the same notation M, N and L as in Lemma 3.11. For any bounded measurable function G on $\Upsilon(L)$,*

$$\int_{\Upsilon(L)} G(\eta) d\pi_L(\eta) = \int_{\Upsilon(N)} \int_{\Upsilon(M)} G_{\xi,M}(\zeta) d\pi_M(\xi) d\pi_N(\zeta). \tag{3.9}$$

Proof of Theorem 3.7 Let $\mathcal{A}_{r,\varepsilon} := B_{r+\varepsilon} \setminus B_r$ be the annulus of width ε and radius r. Fix $A \in \mathcal{S}(\mathscr{E})$, $r > 0$, $\varepsilon > 0$ and $\zeta \in \Upsilon(B^c_{r+\varepsilon})$. We claim that

$$\mathsf{S}^{m,k}_{B_{r+\varepsilon}}(A^k_{\zeta,B^c_{r+\varepsilon}}) \geq \sum_{j=0}^{k} \int_{\Upsilon(\mathcal{A}_{r,\varepsilon})} \mathsf{S}^{m,k}_{B_r}\left(A^{k-j}_{\zeta,\xi,B^c_{r+\varepsilon},\mathcal{A}_{r,\varepsilon}}\right) d\mathsf{S}^j_{\mathcal{A}_{r,\varepsilon}}(\xi). \tag{3.10}$$

Let us first show how (3.10) concludes the proof. For simplicity of notation, we set $M = \mathcal{A}_{r,\varepsilon}$, $N = B^c_{r+\varepsilon}$ and $L = M \sqcup N$. Then, (3.10) is reformulated as follows:

$$\mathsf{S}^{m,k}_{N^c}(A^k_{\zeta,N}) \geq \sum_{j=0}^{k} \int_{\Upsilon(M)} \mathsf{S}^{m,k}_{L^c}(A^{k-j}_{\zeta,\xi,N,M}) d\mathsf{S}^j_M(\xi).$$

Then, by using Lemmas 3.12 and 3.11 we deduce

$$\begin{aligned}
\rho^m_r(A) &= e^{-\mathbf{S}^n(L^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(L)} \mathsf{S}^{m,k}_{L^c}(A^k_{\eta,L}) d\pi_L(\eta) \\
&= e^{-\mathbf{S}^n(L^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(N)} \int_{\Upsilon(M)} \left(\mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,L})\right)_{\xi,M} d\pi_M(\xi) d\pi_N(\zeta) \\
&= e^{-\mathbf{S}^n(L^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(N)} \int_{\Upsilon(M)} \mathsf{S}^{m,k}_{L^c}(A^k_{\zeta,\xi,N,M}) d\pi_M(\xi) d\pi_N(\zeta) \\
&= e^{-\mathbf{S}^n(L^c)} e^{-\mathbf{S}^n(M)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \int_{\Upsilon(N)} \int_{\Upsilon(M)} \mathsf{S}^{m,k-j}_{L^c}(A^{k-j}_{\zeta,\xi,N,M}) d\mathsf{S}^j_M(\xi) d\pi_N(\zeta) \\
&\leq e^{-\mathbf{S}^n(N^c)} \sum_{k=0}^{\infty} \int_{\Upsilon(N)} \mathsf{S}^{m,k}_{N^c}(A^k_{\zeta,N}) d\pi_N(\zeta) \\
&= \rho^m_{r+\varepsilon}(A) .
\end{aligned}$$

To show (3.10), it is enough to verify that, for any bounded measurable function F on $\Upsilon(\mathbb{R}^n)$,

$$\int_{\Upsilon(N^c)} F_{\zeta,N}(\gamma) d\mathsf{S}^{m,k}_{N^c}(\gamma) \geq \sum_{j=0}^{k} \int_{\Upsilon(M)} \int_{\Upsilon(L^c)} (F_{\zeta,N})_{\xi,M}(\gamma) d\mathsf{S}^{m,k-j}_{L^c}(\gamma) d\mathsf{S}^j_M(\xi). \tag{3.11}$$

By the definition of $\mathsf{S}^{m,k}_{N^c}$, the L.H.S. of (3.11) can be deduced as follows:

$$\int_{\Upsilon(N^c)} F_{\zeta,N}(\gamma) d\mathsf{S}^{m,k}_{N^c}(\gamma) = \frac{1}{k!} \int_{(N^c)^{\otimes k}} (F_{\zeta,N} \circ \mathbf{s}_k)(\mathbf{x}_k) d\mathbf{S}^{nk-m}_{N^c}(\mathbf{x}_k),$$

whereby $\mathbf{x}_k := (x_*, \ldots, x_{k-1})$ and $\mathbf{x}_* = x_*$. Furthermore, by the definition of $(F_{\zeta,N})_{\xi,M}$, the R.H.S. of (3.11) can be deduced as follows:

$$\begin{aligned}
&\int_{\Upsilon(M)} \int_{\Upsilon(L^c)} (F_{\zeta,N})_{\xi,M}(\gamma) d\mathsf{S}^{m,k-j}_{L^c}(\gamma) d\mathsf{S}^j_M(\xi) \\
&= \int_{\Upsilon(M)} \int_{\Upsilon(L^c)} (F_{\zeta,N})(\gamma + \xi) d\mathsf{S}^{m,k-j}_{L^c}(\gamma) d\mathsf{S}^j_M(\xi) \\
&= \frac{1}{j!(k-j)!} \int_{M^{\times j}} \int_{(L^c)^{\times(k-j)}} (F_{\zeta,N} \circ \mathbf{s}_k)(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{n(k-j)-m}_{L^c}(\mathbf{x}_{k-j}) d\mathbf{S}^{nj}_M(\mathbf{y}_j),
\end{aligned}$$

whereby $(\mathbf{x}_{k-j}, \mathbf{y}_j) = (x_*, \dots, x_{k-j-1}, y_*, \dots, y_{j-1})$. Hence, in order to conclude (3.11), it suffices to show the following inequality: for any bounded measurable symmetric function f on $(\mathbb{R}^n)^{\times k}$,

$$\int_{B_{r+\varepsilon}^{\times k}} f(\mathbf{x}_k) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_k) \geq \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \int_{B_r^{\times(k-j)}} \int_{\mathcal{A}_{r,\varepsilon}^{\times j}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{n(k-j)-m}_{\mathcal{A}_{r,\varepsilon}}(\mathbf{x}_{k-j}) d\mathbf{S}^{nj}_{B_r}(\mathbf{y}_j).$$

By using the symmetry of f and a simple combinatorial argument, we obtain

$$\int_{B_{r+\varepsilon}^{\times k}} f(\mathbf{x}_k) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_k) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \int_{B_r^{\times(k-j)}} \int_{\mathcal{A}_{r,\varepsilon}^{\times j}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_{k-j}, \mathbf{y}_j),$$

while [27, 2.10.27, p. 190] implies

$$\int_{B_r^{\times(k-j)}} \int_{\mathcal{A}_{r,\varepsilon}^{\times j}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{nk-m}_{B_{r+\varepsilon}}(\mathbf{x}_{k-j}, \mathbf{y}_j)$$
$$\geq \int_{B_r^{\times(k-j)}} \int_{\mathcal{A}_{r,r+\varepsilon}^{\times j}} f(\mathbf{x}_{k-j}, \mathbf{y}_j) d\mathbf{S}^{n(k-j)-m}_{\mathcal{A}_{r,\varepsilon}}(\mathbf{x}_{k-j}) d\mathbf{S}^{nj}_{B_r}(\mathbf{y}_j). \qquad \square$$

3.4 Independence of ρ^m from the exhaustion

So far we have built the m-codimensional measure ρ^m by passing to the limit a sequence of finite dimensional measures ρ^m_r. The latter have been constructed by relying on the exhaustion $\{B_r\}_{r>0}$ of $\mathbb{R}^n$. Hence, a priori, ρ^m depends on the chosen exhaustion. In this subsection we make a remark that this is actually not the case.

Let $\Omega \subset \mathbb{R}^n$ be a compact set. Following closely the proof in Sect. 3.3 we can prove that

$$\rho^m_\Omega(A) := e^{-\mathbf{S}^n(\Omega)} \sum_{k=1}^{\infty} \int_{\Upsilon(\Omega^c)} \mathsf{S}^{m,k}_\Omega(A^k_{\eta,\Omega^c})\, d\pi_{\Omega^c}(\eta). \tag{3.12}$$

is well defined for any Suslin set A.

The next proposition can be proven by arguing as in Theorem 3.7. We omit the proof.

Proposition 3.13 (Independence from exhaustion) *Let $0 < r < R < \infty$ and $\Omega \subset \mathbb{R}^n$ be a compact subset satisfying $B_r \subset \Omega \subset B_R$. Then*

$$\rho^m_r(A) \leq \rho^m_\Omega(A) \leq \rho^m_R(A)\,, \quad \textit{for every Suslin set } A\,. \tag{3.13}$$

In particular ρ^m does not depend on the choice of the exhaustion.

4 Bessel capacity and finite-codimensional Poisson measure

In this section, we discuss a relation between *Bessel capacities* and finite-codimensional Poisson measures ρ^m. This will play a significant role to develop fundamental relations between potential analysis induced by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and theory of BV functions in Sects. 5 and 7.

Definition 4.1 (*Bessel operator*) Let $\alpha > 0$ and $1 \le p < \infty$. We set

$$B_{\alpha,p} := \frac{1}{\Gamma(\alpha/2)} \int_*^\infty e^{-t} t^{\alpha/2-1} T_t^{(p)} dt\,, \tag{4.1}$$

where $T_t^{(p)}$ is the L^p-heat semigroup, see Sect. 2.5.

Notice that $B_{\alpha,p}$ is well defined for $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ and satisfies

$$\|B_{\alpha,p} F\|_{L^p} \le \|F\|_{L^p}\,, \tag{4.2}$$

due to the contractivity of $T_t^{(p)}$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$.

Definition 4.2 (*Bessel capacity*) Let $\alpha > 0$ and $1 \le p < \infty$. The (α, p)-*Bessel capacity* is defined as

$$\mathrm{Cap}_{\alpha,p}(E) := \inf\{\|F\|_{L^p}^p : B_{\alpha,p} F \ge 1 \text{ on } E,\ F \ge 0\}\,, \tag{4.3}$$

for any $E \subset \Upsilon(\mathbb{R}^n)$.

We are now ready to state the main theorem of this section.

Theorem 4.3 *Let $\alpha p > m$. Then,* $\mathrm{Cap}_{\alpha,p}(E) = 0$ *implies $\rho^m(E) = 0$ for any $E \in \mathcal{S}(\mathscr{E})$.*

We briefly explain the heuristic idea of proof. In view of the identities

$$\rho^m(E) = \lim_{r\to\infty} \rho_r^m(E)\,,$$

$$\rho_r^m(E) = e^{-\mathsf{S}^n(B_r)} \sum_{k=1}^\infty \int_{\Upsilon(B_r^c)} \mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) d\pi_{B_r^c}(\eta)\,,$$

it is enough to prove that $\mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) = 0$ for $\pi_{B_r^c}$-a.e. η, all $k \in \mathbb{N}$ and $r > 0$. This, together with the implication

$$\mathrm{Cap}_{\alpha,p}(E) = 0 \implies \mathrm{Cap}_{\alpha,p}^{\eta,r}(E_{\eta,r}^k) = 0, \quad \text{for } \pi_{B_r^c}\text{-a.e. } \eta \text{ and all } k \in \mathbb{N} \text{ and } r > 0\,, \tag{4.4}$$

where $\mathrm{Cap}_{\alpha,p}^{\eta,r}$ is the Bessel (α, p)-capacity on $\Upsilon^k(B_r)$, reduces the problem to the corresponding problem in the finite dimensional setting. To be more precise, we will show that

$$\mathrm{Cap}_{\alpha,p}^{\eta,r}(E_{\eta,r}^k) = 0 \implies \mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) = 0\,.$$

In the rest of this section, we implement the aforementioned idea. The key point is to show (4.4), for which we introduce localisations of functional-analytic objects in Sects. 4.1 and 4.2. We then introduce *localised Bessel operators* and *localised Bessel capacities* in Sect. 4.3.

4.1 Localisation of sets and functions

Lemma 4.4 *Let $A \subset \Upsilon(\mathbb{R}^n)$ be a π-measurable set. Let $B \subset \mathbb{R}^n$ be a Borel set. Then, $A_{\eta,B}$ is π_{B^c}-measurable for π_B-a.e. $\eta \in \Upsilon(B)$. Moreover, if $\pi(A) = 0$, then $\pi_{B^c}(A_{\eta,B}) = 0$ for a.e. $\eta \in \Upsilon(B)$.*

Proof By hypothesis, there exist Borel sets $\underline{A} \subset A \subset \overline{A}$ so that $\pi(\overline{A}\setminus\underline{A}) = 0$. By (i) in Remark 2.18, $\underline{A}$ and $\overline{A}$ are Suslin. By Lemma 3.1, $\underline{A}_{\eta,B}$ and $\overline{A}_{\eta,B}$ are Suslin. By the standard disintegration argument as in Lemma 3.12, it holds that

$$0 = \pi(\overline{A} \setminus \underline{A}) = \int_{\Upsilon(B)} \pi_{B^c}((\overline{A} \setminus \underline{A})_{\eta,B}) d\pi_B(\eta). \tag{4.5}$$

Therefore, there exists a π_B-measurable set $\Omega \subset \Upsilon(B)$ so that $\pi_{B^c}((\overline{A}\setminus\underline{A})_{\eta,B}) = 0$ for any $\eta \in \Omega$. By noting that $\underline{A}_{\eta,B} \subset A_{\eta,B} \subset \overline{A}_{\eta,B}$, we conclude that $A_{\eta,B}$ is π_B-measurable since, up to π_B negligible sets, it coincides with a Suslin set and every Suslin set is π_B-measurable by (ii) in Remark 2.18. The proof of the first assertion is complete.

If $\pi(A) = 0$ the disintegration

$$0 = \pi(A) = \int_{\Upsilon(B)} \pi_{B^c}(A_{\eta,B}) d\pi_B(\eta)\,, \tag{4.6}$$

immediately gives the second assertion. □

Corollary 4.5 *Let $A \subset \Upsilon(\mathbb{R}^n)$ be a π-measurable set, $B \subset \mathbb{R}^n$ a Borel set, and let g be a π-measurable function on $\Upsilon(\mathbb{R}^n)$ with $g \geq 1$ π-a.e. on A. Then, for π_B-a.e. η it holds*

$$g_{\eta,B} \geq 1, \quad \pi_{B^c}\text{-a.e. on} A_{\eta,B}\,. \tag{4.7}$$

Proof Taking $\widetilde{A} = A \setminus \{g \geq 1\}$ and applying Lemma 4.4 with $\widetilde{A}$ in place of A, we obtain the conclusion. □

Lemma 4.6 *Let $1 \leq p < \infty$ and $r > 0$. Let $F^n, F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ such that $F^n \to F$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$ as $n \to \infty$. Then, there exists a subsequence (non-relabelled) of (F^n) and a measurable set $A_r \subset \Upsilon(\mathbb{R}^n)$ so that $\pi_{B_r^c}(A_r) = 1$ and*

$$F^n_{\eta,r} \to F_{\eta,r}, \quad \text{in } L^p(\pi_{B_r}), \text{for any } \eta \in A_r\,.$$

Note that $F_{\eta,r} := F_{\eta,B_r^c}$ was defined in Definition 3.10.

Proof By Lemma 3.12, we have that

$$\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} |F^n_{\eta,r} - F_{\eta,r}|^p d\pi_{B_r}\right) d\pi_{B_r^c}(\eta) = \int_{\Upsilon(\mathbb{R}^n)} |F^n - F|^p d\pi \to 0, \quad \text{as } n \to \infty\,. \tag{4.8}$$

In particular, up to subsequence $\int_{\Upsilon(B_r)} |F^n_{\eta,r} - F_{\eta,r}|^p d\pi_{B_r} \to 0$ for $\pi_{B_r^c}$-a.e. η, which completes the proof. □

4.2 Localisation of energies, resolvents and semigroups

In this section, we localise differential operators and related objects introduced in Sect. 2.5.

Let $r > 0$. The *localised energy* $(\mathcal{E}_r, \mathcal{D}(\mathcal{E}_r))$ is defined as the following direct integral

$$\mathcal{E}_r(F) = \int_{\Upsilon(B_r^c)} \mathcal{E}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta)\,, \quad \mathcal{D}(\mathcal{E}_r) := \{F \in L^2(\Upsilon(\mathbb{R}^n), \pi) : \mathcal{E}_r(F) < \infty\}\,. \tag{4.9}$$

The form is closed by [18, Proposition V.3.1.1]. For $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$,

$$\mathcal{E}_r(F) = \int_{\Upsilon(\mathbb{R}^n)} |\nabla_r F|^2_{T\Upsilon} d\pi \quad F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)), \tag{4.10}$$

where

$$\nabla_r F(\gamma, x) := \chi_{B_r}(x) \nabla F(\gamma, x)\,. \tag{4.11}$$

See e.g., [45, Prop. 3.9]. We denote by $\{G^r_\alpha\}_{\alpha>0}$ and $\{T^r_t\}_{t>0}$ the L^2-resolvent operator and the semigroup associated with $(\mathcal{E}_r, \mathcal{D}(\mathcal{E}_r))$, respectively. Recall that $\{G_\alpha^{\Upsilon(B_r)}\}_\alpha$ and $\{T_t^{\Upsilon(B_r)}\}$ denote the L^2-resolvent operator and the semigroup corresponding to $(\mathcal{E}_{\Upsilon(B_r)}, H^{1,2}(\Upsilon(B_r), \pi))$. The relation between $\{G^r_\alpha\}_{\alpha>0}, \{T^r_t\}_{t>0}$ and $\{G_\alpha^{\Upsilon(B_r)}\}_\alpha, \{T_t^{\Upsilon(B_r)}\}$ is given below.

Proposition 4.7 ([46, Corollary 4.11]) *Let $\alpha > 0$, $t > 0$, and $r > 0$ be fixed. Then, for any bounded measurable function F, it holds that*

$$G^r_\alpha F(\gamma) = G_\alpha^{\Upsilon(B_r)} F_{\gamma|_{B_r^c}, r}(\gamma|_{B_r}) , \tag{4.12}$$

$$T^r_t F(\gamma) = T_t^{\Upsilon(B_r)} F_{\gamma|_{B_r^c}, r}(\gamma|_{B_r}) , \tag{4.13}$$

for π-a.e. $\gamma \in \Upsilon(\mathbb{R}^n)$.

Remark 4.8 Although Proposition 4.7 provides the statement only for the L^2-semigroups and resolvents, it is straightforward to extend it to the L^p-semigroups and resolvents for any $1 \le p < \infty$.

Proposition 4.9 *The form $(\mathcal{E}_r, \mathcal{D}(\mathcal{E}_r))$ is monotone non-decreasing in r, i.e. for any $s \le r$,*

$$\mathcal{D}(\mathcal{E}_r) \subset \mathcal{D}(\mathcal{E}_s), \quad \mathcal{E}_s(F) \le \mathcal{E}_r(F), \quad F \in \mathcal{D}(\mathcal{E}_r).$$

Furthermore, the following two forms coincide: letting $\bar{\mathcal{E}}(F) := \lim_{r\to\infty} \mathcal{E}_r(F)$ and $\mathcal{D}(\bar{\mathcal{E}}) = \{F \in \cap_{r>0}\mathcal{D}(\mathcal{E}_r) : \lim_{r\to\infty} \mathcal{E}_r(F) < \infty\}$,

$$(\bar{\mathcal{E}}, \mathcal{D}(\bar{\mathcal{E}})) = (\mathcal{E}, H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)).$$

Proof The monotone increasing property is a direct application of [46, Proposition 4.13]. The second assertion follows from the fact that $(\Delta, \mathrm{CylF}(\Upsilon(\mathbb{R}^n)))$ is essentially self-adjoint by [2, Theorem 5.3] and that $\bar{\mathcal{E}}$ and $\mathcal{E}$ coincide on $\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$. □

Remark 4.10 In [46, Corollary 4.11, Proposition 4.13], the statements deal with the case where the reference measure is the law of the sine_β point process. The case of the Poisson point process corresponds to $\beta = 0$, and the same proofs there apply to the case of the Poisson point process in this paper.

The next proposition shows the monotonicity property for the resolvent operator G^r_α and the semigroup T^r_t.

Proposition 4.11 *The resolvent operator $\{G^r_\alpha\}_\alpha$ and the semigroup $\{T^r_t\}_t$ are monotone non-increasing on non-negative functions, i.e.,*

$$G^r_\alpha F \le G^s_\alpha F, \quad T^r_t F \le T^s_t F, \quad \textit{for every non-negative } F \in L^2(\Upsilon(\mathbb{R}^n), \pi), \quad s \le r. \tag{4.14}$$

Furthermore, $\lim_{r\to\infty} G^r_\alpha F = G_\alpha F$ and $\lim_{r\to\infty} T^r_t F = T_t F$ for $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$ and $\alpha, t > 0$.

Proof Thanks to the identity

$$G^r_\alpha = \int_*^\infty e^{-\alpha t} T^r_t \, dt ,$$

it suffices to show (4.14) only for T^r_t. By a direct application of [37, Theorem 3.3] and the monotonicity of the Dirichlet form in Proposition 4.9, we obtain the monotonicity of the semigroup. The second part of the statement follows from the monotone convergence $\mathcal{E}_r \uparrow \mathcal{E}$ combined with [38, S.14, p.372]. □

4.3 Localised Bessel operators

Let $B^r_{\alpha,p}$ and $B^{\Upsilon(B_r)}_{\alpha,p}$ be the (α, p)-Bessel operators corresponding to $\{T^r_t\}_{t>0}$ and $\{T^{\Upsilon(B_r)}_t\}_{t>0}$, respectively defined in the analogous way as in (4.1). The corresponding (α, p)-Bessel capacities are denoted by $\mathrm{Cap}^r_{\alpha,p}$ and $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}$ defined in the analogous way as in (4.3)

Lemma 4.12 $\mathrm{Cap}^r_{\alpha,p}(E) \le \mathrm{Cap}_{\alpha,p}(E)$ *for every* $E \subset \Upsilon(\mathbb{R}^n)$ *and* $r > 0$.

Proof It suffices to show that $B^r_{\alpha,p}F \le B_{\alpha,p}F$ for any $F \ge 0$ with $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$, which immediately follows from Proposition 4.11 and (4.1). □

Lemma 4.13 *If* $\mathrm{Cap}_{\alpha,p}(E) = 0$*, then* $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}(E_{\eta,r}) = 0$ *for* $\pi_{B^c_r}$*-a.e.* η *and every* $r > 0$.

Proof By Lemma 4.12 we may assume $\mathrm{Cap}^r_{\alpha,p}(E) = 0$ for any $r > 0$. Let $\{F_n\} \subset L^p(\Upsilon(\mathbb{R}^n), \pi)$ be a sequence so that $F_n \ge 0$, $B^r_{\alpha,p}F_n \ge 1$ on E, and $\|F_n\|^p_{L^p} \to 0$. By Lemma 4.5, $(F_n)_{\eta,r} \ge 0$ for $\pi_{B^c_r}$-a.e. η. Furthermore, by Lemma 4.6, there exists $A_r \subset \Upsilon(B^c_r)$ and a (non-relabelled) subsequence $(F_n)_{\eta,r}$ so that $\pi_{B^c_r}(A_r) = 1$, and for every $\eta \in A_r$,

$$(F_n)_{\eta,r} \to 0, \quad \text{in } L^p(\Upsilon(B_r), \pi_{B_r})\,. \tag{4.15}$$

By Proposition 4.7 and Remark 4.8, we have that

$$\begin{aligned}(B^r_{\alpha,p}F_n)_{\eta,r} &= \left(\frac{1}{\Gamma(\alpha/2)}\int_*^\infty e^{-t}t^{\alpha/2-1}T^r_t F_n dt\right)_{\eta,r} \\ &= \frac{1}{\Gamma(\alpha/2)}\int_*^\infty e^{-t}t^{\alpha/2-1}\left(T^r_t F_n\right)_{\eta,r} dt \\ &= \frac{1}{\Gamma(\alpha/2)}\int_*^\infty e^{-t}t^{\alpha/2-1}T^{\Upsilon(B_r)}_t (F_n)_{\eta,r} dt \\ &= B^{\Upsilon(B_r)}_{\alpha,p}(F_n)_{\eta,r}\,.\end{aligned} \tag{4.16}$$

Note that we dropped the specification of p in the semigroups for notational simplicity in (4.16).

Since $B^r_{\alpha,p}F_n \ge 1$ on E, by applying Corollary 4.5, we obtain that $(B^r_{\alpha,p}F_n)_{\eta,r} \ge 1$ on $E_{\eta,r}$ for $\pi_{B^c_r}$-a.e. η. Thus, by (4.16), $B^{\Upsilon(B_r)}_{\alpha,p}(F_n)_{\eta,r} \ge 1$ on $E_{\eta,r}$ for $\pi_{B^c_r}$-a.e. η. By (4.15), we conclude that $\mathrm{Cap}^{\Upsilon(B_r)}_{\alpha,p}(E_{\eta,r}) = 0$ for $\pi_{B^c_r}$-a.e. η and any $r > 0$. □

4.4 Finite-dimensional counterpart

In this section, we develop the finite-dimensional counterpart of Theorem 4.3. The goal is to prove the following proposition.

Proposition 4.14 *Let* $\alpha p > m$*. If* $\mathrm{Cap}^{\Upsilon^k(B_r)}_{\alpha,p}(E) = 0$*, then* $\mathsf{S}^{m,k}_{B_r}(E) = 0$ *for any* $k \in \mathbb{N}$.

Proof Recall that $T^{\Omega,\otimes k}_t$ is the k-tensor semigroup of T^Ω_t as defined in (2.17). Let $B^{B^{\times k}_r}_{\alpha,p}$ be the corresponding Bessel operator defined analogously as in (4.1), and $\mathrm{Cap}^{B^{\times k}_r}_{\alpha,p}$ be the corresponding (α, p)-capacity.

Let $\{F_m\} \subset L^p(\Upsilon(B_r), \pi_{B_r})$ be a sequence so that $F_m \geq 0$ and $B_{\alpha,p}^{\Upsilon^k(B_r)} F_m \geq 1$ on $E \subset \Upsilon^k(B_r)$, and $\|F_m\|_{L^p} \to 0$. By Proposition 2.17 and the definition of Bessel operator, we have

$$B_{\alpha,p}^{\Upsilon^k(B_r)} F_m \circ \mathsf{s}_k = B_{\alpha,p}^{B_r^{\times k}} (F_m \circ \mathsf{s}_k)\,,$$

hence $F_m \circ \mathsf{s}_k \geq 0$, $B_{\alpha,p}^{B_r^{\times k}} (F_m \circ \mathsf{s}_k) \geq 1$ on $\mathsf{s}_k^{-1}(E)$. Furthermore,

$$\|F_m \circ \mathsf{s}_k\|_{L^p(B_r^{\times k})} = C(k,n,r)\|F_m\|_{L^p(\Upsilon^k(B_r))} \to 0\,, \quad \text{as}\, m \to \infty\,,$$

where $C(k,n,r) > 0$ comes from the constant appearing in front of the Hausdorff measure in the definition of π_{B_r}. This implies that $\mathrm{Cap}_{\alpha,p}^{B_r^{\times k}}(\mathsf{s}_k^{-1}(E)) = 0$. We can now rely on standard capacity estimates in the Euclidean setting (see, e.g. [47, Theorem 2.6.16]) to conclude that $\mathbf{S}^{nk-m}(\mathsf{s}_k^{-1}(E)) = 0$. Recalling (2.4), we have that

$$\mathsf{S}_{B_r}^{m,k}(E) = \frac{1}{k!}(\mathsf{s}_k)_\#\mathbf{S}^{nk-m}(E) = \frac{1}{k!}\mathbf{S}^{nk-m}(\mathsf{s}_k^{-1}(E)) = 0\,.\square$$

4.5 Proof of Theorem 4.3

Let $E \in \mathcal{S}(\mathscr{E})$ such that $\mathrm{Cap}_{\alpha,p}(E) = 0$. Thanks to Lemma 4.13 we have $\mathrm{Cap}_{\alpha,p}^{\Upsilon(B_r)}(E) = 0$ for any $r > 0$, hence $\mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) = 0$ for any $k \in \mathbb{N}$ as a consequence of Proposition 4.14. It implies

$$\rho_r^m(E) = e^{-\mathbf{S}^n(B_r)} \sum_{k=1}^{\infty} \int_{\Upsilon(B_r^c)} \mathsf{S}_{B_r}^{m,k}(E_{\eta,r}^k) d\pi_{B_r^c}(\eta) = 0\,,$$

for any $r > 0$. Recalling that $\rho_r^m(E) \uparrow \rho^m(E)$ by (3.5), we obtain the sought conclusion.

5 Functions of bounded variation

In this section, we introduce functions of bounded variations (called *BV functions*) on $\Upsilon(\mathbb{R}^n)$ following three different approaches: the variational approach (Sect. 5.1), the relaxation approach (Sect. 5.2), and the semigroup approach (Sect. 5.3). In Sect. 5.5, we prove that they all coincide.

5.1 Variational approach

Let us begin by introducing a class of BV functions through integration by parts. We then discuss localisation properties.

Definition 5.1 (*BV functions I: variational approach*) Let $\Omega \subset \mathbb{R}^n$ be either a closed domain with smooth boundary or $\mathbb{R}^n$. For $F \in \cup_{p>1} L^p(\Upsilon(\Omega), \pi_\Omega)$, we define the total variation as

$$\mathcal{V}_{\Upsilon(\Omega)}(F) := \sup\left\{\int_{\Upsilon(\Omega)} (\nabla_{\Upsilon(\Omega)}^* V) F d\pi_\Omega : V \in \mathrm{CylV}_*(\Upsilon(\Omega)),\ |V|_{T\Upsilon(\Omega)} \leq 1\right\}. \quad (5.1)$$

When $\Omega = \mathbb{R}^n$, we simply write $\mathcal{V}(F) := \mathcal{V}_{\Upsilon(\mathbb{R}^n)}(F)$. We say that F *is BV in the variational sense* if $\mathcal{V}(F) < \infty$.

Remark 5.2 The assumption $F \in \cup_{p>1} L^p(\Upsilon(\Omega), \pi_\Omega)$ plays an important role in Definition 5.1, ensuring that $\int_{\Upsilon(\Omega)} (\nabla^*_{\Upsilon(\Omega)} V) F d\pi_\Omega$ is well defined for any $V \in \mathrm{CylV}(\Upsilon(\Omega))$. Indeed, one can easily prove that $\nabla^*_{\Upsilon(\Omega)} V \in \cup_{1\le p<\infty} L^p(\Upsilon(\Omega), \pi_\Omega)$ for any $V \in \mathrm{CylV}_*(\Upsilon(\Omega))$, but it is not $L^\infty(\Upsilon(\Omega), \pi_\Omega)$ in general.

Remark 5.3 As it was shown in Remark 2.10, the set of $V \in \mathrm{CylV}_*(\Upsilon(\Omega))$ with $|V|_{T\Upsilon} \le 1$ is dense in $\mathrm{CylV}_*(\Upsilon(\Omega))$ with respect to the topology of point-wise convergence and the $L^p(\Upsilon(\Omega) \to T\Upsilon(\Omega), \pi_\Omega)$ topology for $1 \le p < \infty$.

In order to localise the total variation we employ a family of cylinder vector fields concentrated on B_r, for some $r > 0$.

Definition 5.4 For $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$, we define the *localised total variation* as

$$\mathcal{V}_r(F) := \sup\left\{ \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi : V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon(\mathbb{R}^n)} \le 1 \right\}, \tag{5.2}$$

where

$$\mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n)) := \left\{ V(\gamma, x) = \sum_{i=1}^k F_i(\gamma) v_i(x) : F_i \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)),\ v_i \in C^\infty_*(B_r; \mathbb{R}^n), k \in \mathbb{N} \right\}.$$

The next result shows that $\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) < \infty$ for $\pi_{B_r^c}$-a.e. η whenever $\mathcal{V}_r(F) < \infty$. It is the key step to perform our nonlinear dimension reduction. Indeed it allows to reduce the study of BV functions on $\Upsilon(\mathbb{R}^n)$ to their sections, which live on the finite dimensional space $\Upsilon(B_r)$.

Proposition 5.5 *Let $r > 0$ and $p > 1$. For $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ with $\mathcal{V}_r(F) < \infty$, it holds*

$$\int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta) = \mathcal{V}_r(F)\,. \tag{5.3}$$

Let us begin with a simple technical lemma.

Lemma 5.6 *Let $r > 0$. For $V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$, and $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds*

$$\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r}(\gamma) \nabla^*_{\Upsilon(B_r)} V_{\eta,r}(\gamma) d\pi_{B_r}(\gamma) \right) d\pi_{B_r^c}(\eta) = \int_{\Upsilon(\mathbb{R}^n)} F \nabla^* V d\pi\,. \tag{5.4}$$

Proof of Lemma 5.6 Recall that for $r > 0$ and $\eta \in \Upsilon(B_r^c)$ we have $V_{\eta,r} \in \mathrm{CylV}_*(B_r)$. By the divergence formula (2.14) and the disintegration Lemma 3.12, we have that

$$\begin{aligned}
&\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r}(\gamma) \nabla^*_{\Upsilon(B_r)} V_{\eta,r}(\gamma) d\pi_{B_r}(\gamma) \right) d\pi_{B_r^c}(\eta) \\
&= -\int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r}(\gamma) \Big(\sum_{i=1}^k \nabla_{v_i}(F_i)_{\eta,r}(\gamma) \right. \\
&\quad \left. + \sum_{i=1}^k (F_i)_{\eta,r}(\gamma) (\nabla^*_{\mathbb{R}^n} v_i)^*(\gamma) \Big) d\pi_{B_r}(\gamma) \right) d\pi_{B_r^c}(\eta) \\
&= -\int_{\Upsilon(B_r^c)} \int_{\Upsilon(B_r)} \left(F \Big(\sum_{i=1}^k \nabla_{v_i} F_i + \sum_{i=1}^k F_i (\nabla^*_{\mathbb{R}^n} v_i)^* \Big) \right)_{\eta,r} (\gamma) d\pi_{B_r}(\gamma) d\pi_{B_r^c}(\eta)
\end{aligned}$$

$$= -\int_{\Upsilon(\mathbb{R}^n)} F\Big(\sum_{i=1}^{k} \nabla_{v_i} F_i + \sum_{i=1}^{k} F_i (\nabla^*_{\mathbb{R}^n} v_i)^*\Big) d\pi$$

$$= \int_{\Upsilon(\mathbb{R}^n)} F \nabla^* V d\pi \,. \square$$

Proof of Proposition 5.5 We first prove that

$$\int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta) \geq \mathcal{V}_r(F)\,. \tag{5.5}$$

Let $V_i \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$ with $|V_i|_{T\Upsilon} \leq 1$ so that

$$\mathcal{V}_r(F) = \lim_{i\to\infty} \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V_i) F d\pi.$$

Observe that $(V_i)_{\eta,r} \in \mathrm{CylV}_*(\Upsilon(B_r))$, then by definition of $\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r})$ we get

$$\int_{\Upsilon(B_r)} ((\nabla^* V_i)F)_{\eta,r} d\pi_{B_r} = \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} (V_i)_{\eta,r}) F_{\eta,r} d\pi_{B_r} \leq \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}), \quad i \in \mathbb{N}\,.$$

Therefore, by Lemma 5.6,

$$\begin{aligned}
\mathcal{V}_r(F) &= \lim_{i\to\infty} \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V_i) F d\pi \\
&= \lim_{i\to\infty} \int_{\Upsilon(B_r^c)} \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} (V_i)_{\eta,r}) F_{\eta,r} d\pi_{B_r} d\pi_{B_r^c}(\eta) \\
&\leq \int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta)\,,
\end{aligned}$$

which completes the proof of (5.5).

Let us now pass to the proof of the opposite inequality

$$\int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) d\pi_{B_r^c}(\eta) \leq \mathcal{V}_r(F)\,. \tag{5.6}$$

The idea of the proof is inspired by [34, Proposition 3.2] in the case of the Wiener space. We divide it into three steps.

Step 1. We show the existence of $\{V_i : i \in \mathbb{N}\} \subset \mathrm{CylV}_*(\Upsilon(B_r))$ such that $|V_i|_{T\Upsilon} \leq 1$ and

$$\mathcal{V}_{\Upsilon(B_r)}(G) = \sup_{i\in\mathbb{N}} \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} V_i) G d\pi_{B_r}\,, \tag{5.7}$$

for any $G \in \cup_{p>1} L^p(\Upsilon(B_r), \pi_{B_r})$.

First we observe that there exists $\mathcal{F}F := \{G_i : i \in \mathbb{N}\} \subset \mathrm{CylF}(\Upsilon(B_r))$ such that any cylinder function can be approximated strongly in $H^{1,q}(\Upsilon(B_r))$ for any $q < \infty$, by elements of $\mathcal{F}F$. Let $D \subset C^\infty_*(B_r; \mathbb{R}^n)$ be a countable dense subset, w.r.t. the C^1-norm: $\|v\|_{C^1(B_r)} := \|\nabla_{\mathbb{R}^n} v\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}$. We define the countable family

$$\begin{aligned}
\mathcal{F}V := \Big\{ &\beta V(\gamma, x) \phi_\alpha(|V|_{T_\gamma \Upsilon}) : V(\gamma, x) \\
&= \sum_{i=1}^{m} w_i(x) G_i(\gamma),\ \alpha, \beta \in \mathbb{Q}^+,\ m \in \mathbb{N},\ w_i \in D,\ G_i \in \mathcal{F}F \Big\}\,,
\end{aligned}$$

where $\phi_\alpha \in C^\infty([0,\infty))$ satisfies $0 \le \phi_\alpha \le 1$, $|\phi'_\alpha| \le 2/\alpha$ and $\phi_\alpha(t) = 1$ on $[0, 1+\alpha]$, $\phi(t) = 0$ on $[1+2\alpha, \infty)$.

Fix $\delta > 0$, $q \in [1,\infty)$ and $V \in \mathrm{CylV}_*(\Upsilon(B_r))$ with $|V|_{T_\gamma \Upsilon} \le 1$. To prove (5.7) it suffices to show that there exists $W \in \mathcal{F}V$ with $|W|_{T\Upsilon} \le 1$ such that $\|\nabla^*_{\Upsilon(B_r)}(V - W)\|_{L^q(\Upsilon(B_r))} \le \delta$.

Fix $t \in (q, 2q)$ and $\varepsilon \in (0, 1/9)$. Letting $V = \sum_{i=1}^m F_i v_i \in \mathrm{CylV}_*(\Upsilon(B_r))$, we pick $G_i \in \mathcal{F}F$ and $w_i \in D$ such that

$$\sum_{i=1}^m \Big(\|v_i - w_i\|_{C^1(B_r)} + \|F_i - G_i\|_{L^t(\Upsilon(B_r))} + \|\nabla_{\Upsilon(B_r)}(F_i - G_i)\|_{L^t(\Upsilon(B_r))} \Big) < \varepsilon, \quad (5.8)$$

and consider $\bar W := \sum_{i=1}^m w_i G_i$. By using the divergence formula (2.14), we can obtain that

$$\int_{\Upsilon(B_r)} |\nabla^*_{\Upsilon(B_r)}(\bar W - V)|^t d\pi_{B_r} + \int_{\Upsilon(B_r)} \big| |\bar W|_{T_\gamma\Upsilon} - |V|_{T_\gamma\Upsilon} \big|^t d\pi_{B_r} \le C\varepsilon^t , \qquad (5.9)$$

where $C = \max\{\|w_i\|_{C^1}, \|G_i\|_{L^t(\Upsilon(B_r))}, \|\nabla G_i\|_{L^t(\Upsilon(B_r))} : 1 \le i \le m\}$ does not depend on ε. We assume without loss of generality that $\varepsilon, \varepsilon^{\frac{1}{10t}} \in \mathbb{Q}$ and set

$$W := (1 - 2\varepsilon^{\frac{1}{10t}}) \phi_{\varepsilon^{\frac{1}{10t}}} \big(|\bar W|^2_{T_\gamma \Upsilon} \big) \bar W \in \mathcal{F}V , \qquad (5.10)$$

which satisfies

$$|W|_{T_\gamma\Upsilon} = (1 - 2\varepsilon^{\frac{1}{10t}}) \phi_{\varepsilon^{\frac{1}{10t}}} \big(|\bar W|^2_{T_\gamma \Upsilon} \big) |\bar W|_{T_\gamma\Upsilon} \le (1 - 2\varepsilon^{\frac{1}{10t}})(1 + 2\varepsilon^{\frac{1}{10t}}) \le 1 .$$

We now check that $\|\nabla^*_{\Upsilon(B_r)}(V - W)\|_{L^q(\Upsilon(B_r))} \le \delta$. From the identity

$$\begin{aligned}\nabla^*_{\Upsilon(B_r)} W = {} & (1 - 2\varepsilon^{\frac{1}{10t}}) \phi_{\varepsilon^{\frac{1}{10t}}} \big(|\bar W|^2_{T_\gamma \Upsilon} \big) (\nabla^*_{\Upsilon(B_r)} \bar W) \\ & - 2(1 - 2\varepsilon^{\frac{1}{10t}}) \phi'_{\varepsilon^{\frac{1}{10t}}} \big(|\bar W|^2_{T_\gamma \Upsilon} \big) |\bar W|^2_{T_\gamma\Upsilon} ,\end{aligned}$$

and the inequality

$$\begin{aligned} & \big| \phi'_{\varepsilon^{\frac{1}{10t}}} \big(|\bar W|^2_{T_\gamma \Upsilon} \big) \big| |\bar W|^2_{T_\gamma\Upsilon} \le 2\varepsilon^{-\frac{1}{10t}} \chi_{\{1+\varepsilon^{\frac{1}{10t}} \le |\bar W|^2_{T_\gamma\Upsilon} \le 1+2\varepsilon^{\frac{1}{10t}}\}} |\bar W|^2_{T_\gamma\Upsilon} \\ & \le 5\varepsilon^{-\frac{1}{10t}} \chi_{\{|\bar W|^2_{T_\gamma\Upsilon} \ge 1+\varepsilon^{\frac{1}{10t}}\}} ,\end{aligned}$$

we obtain

$$\begin{aligned} \|\nabla^*_{\Upsilon(B_r)}(W - \bar W)\|_{L^q} \le {} & \Big\| \Big((1 - 2\varepsilon^{\frac{1}{10t}}) \phi_{\varepsilon^{\frac{1}{10t}}} (|\bar W|^2_{T_\gamma \Upsilon}) - 1 \Big) (\nabla^*_{\Upsilon(B_r)} \bar W) \Big\|_{L^q(\Upsilon(B_r))} \\ & + 5\varepsilon^{-\frac{1}{10t}} \Big\| \chi_{\{|\bar W|^2_{T_\gamma\Upsilon} \ge 1+\varepsilon^{\frac{1}{10t}}\}} \Big\|_{L^q(\Upsilon(B_r))} \\ \le {} & 5\varepsilon^{\frac{1}{10t}} \|\nabla^*_{\Upsilon(B_r)} \bar W\|_{L^q(\Upsilon(B_r))} + \Big\| \chi_{\{|\bar W|^2_{T_\gamma\Upsilon} \ge 1+\varepsilon^{\frac{1}{10t}}\}} (\nabla^*_{\Upsilon(B_r)} \bar W) \Big\|_{L^q(\Upsilon(B_r))} \\ & + 5\varepsilon^{-\frac{1}{10t}} \Big\| \chi_{\{|\bar W|^2_{T_\gamma\Upsilon} \ge 1+\varepsilon^{\frac{1}{10t}}\}} \Big\|_{L^q(\Upsilon(B_r))} \\ \le {} & C\Big(\|\nabla^*_{\Upsilon(B_r)} \bar W\|_{L^t(\Upsilon(B_r))}, t, q \Big) \\ & \cdot \Big(\varepsilon^{\frac{1}{10t}} + \varepsilon^{-\frac{1}{10t}} \Big\| \chi_{\{|\bar W|^2_{T_\gamma\Upsilon} \ge 1+\varepsilon^{\frac{1}{10t}}\}} \Big\|_{L^t(\Upsilon(B_r))} \Big), \end{aligned} \qquad (5.11)$$

where we estimated $\|\chi_{\{|\bar{W}|^2_{T_\gamma\Upsilon}\geq 1+\varepsilon^{\frac{1}{10t}}\}}(\nabla^*_{\Upsilon(B_r)}\bar{W})\|_{L^q(\Upsilon(B_r))}$ by means of the Hölder inequality and using that $t < 2q$. The Chebyshev inequality and (5.9) give

$$\begin{aligned}\left\|\chi_{\{|\bar{W}|^2_{T_\gamma\Upsilon}\geq 1+\varepsilon^{\frac{1}{10t}}\}}\right\|_{L^t(\Upsilon(B_r))} &\leq \left\|\chi_{\{|\bar{W}|_{T_\gamma\Upsilon}\geq 1+\varepsilon^{\frac{1}{3t}}\}}\right\|_{L^t(\Upsilon(B_r))}\\ &\leq \left\|\chi_{\{||\bar{W}|_{T_\gamma\Upsilon}-|V|_{T_\gamma\Upsilon}|\geq \varepsilon^{\frac{1}{3t}}\}}\right\|_{L^t(\Upsilon(B_r))}\\ &\leq \left(\varepsilon^{-\frac{1}{3t}}\left\||\bar{W}|_{T_\gamma\Upsilon}-|V|_{T_\gamma\Upsilon}\right\|_{L^1(\Upsilon(B_r))}\right)^{1/t}\\ &\leq C\varepsilon^{\frac{1}{t}-\frac{1}{3t^2}} \leq C\varepsilon^{\frac{1}{2t}} \quad (\varepsilon<1),\end{aligned}$$

where $C = \max\{\|w_i\|_{C^1}, \|G_i\|_{L^1(\Upsilon(B_r))}, \|\nabla G_i\|_{L^1(\Upsilon(B_r))} : 1 \leq i \leq m\}$ is independent of ε. Therefore, we conclude

$$\begin{aligned}\|\nabla^*_{\Upsilon(B_r)}(W-V)\|_{L^q(\Upsilon(B_r))} &\leq \|\nabla^*_{\Upsilon(B_r)}(W-\bar{W})\|_{L^q(\Upsilon(B_r))} + \|\nabla^*_{\Upsilon(B_r)}(\bar{W}-V)\|_{L^q(\Upsilon(B_r))}\\ &\leq C(\varepsilon^{\frac{1}{10t}}+\varepsilon^{\frac{1}{5t}})+\varepsilon \leq \delta\,,\end{aligned}$$

provided ε is small enough. The proof of (5.7) is complete.

Step 2. We conclude the proof of (5.6).

Note that the map $\gamma \mapsto F(\gamma)\nabla^*_{\Upsilon(B_r)}V(\gamma|_{B_r})$ is π-measurable. Furthermore, by Lemma 4.4, F_{η,B_r^c} is π_{B_r}-measurable and the map

$$\Upsilon(B_r^c) \ni \eta \mapsto \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)}V)F_{\eta,B_r^c}d\pi_{B_r}$$

is $\pi_{B_r^c}$-measurable. Therefore, the map $\eta \mapsto \mathcal{V}_{\Upsilon(B_r)}(F_{\eta,B_r^c})$ is $\pi_{B_r^c}$-measurable.

Fix now $\varepsilon > 0$ and define a sequence $\{C_j : j \in \mathbb{N}\}$ of subsets in $\Upsilon(B_r^c)$ so that $C_* = \emptyset$, and

$$C_j := \Big\{\eta \in \Upsilon(B_r^c) : F_{\eta,r} \text{ is } \pi_{B_r}\text{-measurable and,}$$
$$\int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)}V_j)F_{\eta,r}d\pi_{B_r} \geq (1-\varepsilon)\mathcal{V}_{\Upsilon(B_r)}(F_{\eta,r}) \wedge \varepsilon^{-1}\Big\} \setminus \bigcup_{i=1}^{j-1} C_i\,,$$

where the family $\{V_i : i \in \mathbb{N}\}$ has been built in Step 1.

Then, C_j is $\pi_{B_r^c}$-measurable for any j and $\pi_{B_r^c}(\Upsilon(B_r^c) \setminus \cup_{j=1}^\infty C_j) = 0$. Set

$$W_n^\eta(\gamma) := W_n(\gamma+\eta) := \sum_{j=1}^n V_j(\gamma)\chi_{C_j}(\eta), \quad \gamma \in \Upsilon(B_r),\ \eta \in \Upsilon(B_r^c)\,.$$

We approximate χ_{C_j} by $\{F_j^i\}_{i\in\mathbb{N}} \subset \mathrm{CylF}(\Upsilon(B_r^c))$ with $|F_j^i| \leq 1$ in the strong $L^{p'}(\Upsilon(B_r^c), \pi_{B_r^c})$ topology, where $\frac{1}{p'}+\frac{1}{p}=1$. Thus, setting $W_n^i(\gamma+\eta) := \sum_{j=1}^n V_j(\gamma)F_j^i$ (η), we see that

$$\int_{\Upsilon(B_r^c)} \|\nabla^*_{\Upsilon(B_r)}(W_n - W_n^i)(\cdot+\eta)\|_{L^{p'}(\Upsilon(B_r))}d\pi_{B_r^c}(\eta) \to 0 \quad \text{as } i \to \infty\,.$$

Notice that $W_n^i \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$, hence

$$\lim_{i\to\infty}\int_{\Upsilon(B_r^c)}\left(\int_{\Upsilon(B_r)}(\nabla^*_{\Upsilon(B_r)}W_n^i(\cdot+\eta))f_{\eta,r}d\pi_{B_r}\right)d\pi_{B_r^c}(\eta)$$

$$
\begin{aligned}
&= \int_{\Upsilon(B_r^c)} \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} W_n^\eta) f_{\eta,r} d\pi_{B_r} \pi_{B_r^c}(\eta) \\
&= \int_{\Upsilon(B_r^c)} \left(\sum_{j=1}^n \chi_{C_j}(\eta) \int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} V_j) f_{\eta,r} d\pi_{B_r} \right) d\pi_{B_r^c}(\eta) \\
&\geq (1-\varepsilon) \int_{\Upsilon(B_r^c)} \left(\sum_{j=1}^n \chi_{C_j}(\eta) \mathcal{V}_{\Upsilon(B_r)}(f_{\eta,r}) \wedge \varepsilon^{-1} \right) d\pi_{B_r^c} \\
&= (1-\varepsilon) \int_{\cup_{j=1}^n C_j} \mathcal{V}_{\Upsilon(B_r)}(f_{\eta,r}) \wedge \varepsilon^{-1} d\pi_{B_r^c}(\eta).
\end{aligned} \tag{5.12}
$$

By Lemma 5.6,

$$
\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* W_n^i) f d\pi = \left(\int_{\Upsilon(B_r)} (\nabla^*_{\Upsilon(B_r)} W_n^i(\cdot + \eta)) f_{\eta,r} d\pi_{B_r} \right) d\pi_{B_r^c}(\eta), \tag{5.13}
$$

which along with (5.12) gives the claimed inequality by letting $i \to \infty$ and $n \to \infty$. □

5.2 Relaxation approach

In this subsection we introduce a second notion of functions with bounded variations. We rely on a relaxation approach.

Definition 5.7 (*BV functions II: relaxation*) Let $F \in L^1(\Upsilon(\mathbb{R}^n), \pi)$, we define *the total variation of F* by

$$
|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) := \inf\{\liminf_{n\to\infty} \|\nabla F_n\|_{L^1(T\Upsilon)} : F_n \to F \text{ in } L^1(\Upsilon(\mathbb{R}^n), \pi),\ F_n \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\}. \tag{5.14}
$$

If $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) < \infty$, we say that F has finite *relaxed total variation*.

Definition 5.8 (*Total variation pre-measure*) If $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) < \infty$, we define a map

$$
|\mathsf{D}_* F| : \{G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)) : G \text{ is non-negative}\} \to \mathbb{R},
$$

$$
|\mathsf{D}_* F|[G] := \inf \left\{ \liminf_{n\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G |\nabla F_n|_{T\Upsilon} d\pi : F_n \to F \text{ in } L^1(\Upsilon(\mathbb{R}^n), \pi),\ F_n \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \right\}. \tag{5.15}
$$

Notice that $|\mathsf{D}_* F|[G] \leq \|G\|_{L^\infty} |\mathsf{D}_* F|$ and $|\mathsf{D}_* F|[G_1 + G_2] \geq |\mathsf{D}_* F|[G_1] + |\mathsf{D}_* F|[G_2]$. By construction, $|\mathsf{D}_* F|[G]$ is the lower semi-continuous envelope of the functional $\mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \ni F \mapsto \int_{\Upsilon(\mathbb{R}^n)} G |\nabla F|_{T\Upsilon} d\pi$. Therefore, the map $F \mapsto |\mathsf{D}_* F|(G)$ is lower semi-continuous with respect to the L^1-convergence for any non-negative $G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$.

It will be shown in Corollary 7.4 that $|\mathsf{D}_* F|$ is represented by a finite measure $|\mathsf{D} F|$, i.e.

$$
|\mathsf{D}_* F|[G] = \int_{\Upsilon(\mathbb{R}^n)} G d|\mathsf{D} F| \quad \text{for any non-negative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)).
$$

5.3 Heat semigroup approach

In this subsection we present the third approach to BV functions. We employ the heat semigroup to define the total variation of a function $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$, $p > 1$.

Proposition 5.9 *Let* $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$. *Then* $\|\nabla T_t F\|_{L^1} < \infty$ *for* $t > 0$ *and the following limit exists*

$$\mathcal{T}(F) := \lim_{t\to 0} \|\nabla T_t F\|_{L^1} . \tag{5.16}$$

Definition 5.10 (*BV functions III: heat semigroup*) A function $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$ is *BV in the sense of the heat semigroup* if $\mathcal{T}(F) < \infty$. We define *the total variation of* F by $\mathcal{T}(F)$.

To prove Proposition 5.9, we need the *Bakry–Émery inequality* with exponent $q = 1$, i.e. for any $t, s > 0$, $F \in \cup_{p>1} L^p(\Upsilon(\mathbb{R}^n), \pi)$, it holds

$$\int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F| d\pi < \infty , \qquad |\nabla T_{t+s} F| \le T_t |\nabla T_s F| \quad \pi\text{-a.e.} . \tag{5.17}$$

The inequality (5.17) will be proven in Corollary 5.16 in Sect. 5.4. Let us now use it to show Proposition 5.9.

Proof of Proposition 5.9 Let $F \in L^p(\Upsilon, \pi)$ for $p > 1$. By (5.17), we see that

$$\|\nabla T_t F\|_{L^1} \le \liminf_{s\to 0} \|\nabla T_{t+s} F\|_{L^1} \le \liminf_{s\to 0} \|\nabla T_s F\|_{L^1} .$$

By taking $\limsup_{t\to 0}$, we obtain $\limsup_{t\to 0} \|\nabla T_t F\|_{L^1} \le \liminf_{s\to 0} \|\nabla T_s F\|_{L^1}$, which concludes the proof. □

5.4 *p*-Bakry–Émery inequality

In order to complete the proof of Proposition 5.9, we show the p-Bakry–Émery inequality for the *Hodge heat flow*, which implies in turn the scalar version (5.17) of the p-Bakry–Émery inequality. It will play a significant role also in the proof of Theorem 5.18. Recall that, for $F = \Phi(f_1^*, \ldots, f_k^*) \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$,

$$\nabla F(\gamma, x) = \sum_{i=1}^k \partial_i \Phi(f_1^*\gamma, \ldots, f_k^*\gamma) \nabla_{\mathbb{R}^n} f_i(x) ,$$

$$\Delta F(\gamma) = \sum_{i,j=1}^k \partial_{ij}^2 \Phi(f_1^*\gamma, \ldots, f_k^*\gamma) \langle \nabla_{\mathbb{R}^n} f_i, \nabla_{\mathbb{R}^n} f_j \rangle_{T_\gamma \Upsilon}$$

$$+ \sum_{i=1}^k \partial_i \Phi(f_1^*\gamma, \ldots, f_k^*\gamma)(\Delta_{\mathbb{R}^n} f_i)^* \gamma , \tag{5.18}$$

where $\langle \nabla_{\mathbb{R}^n} f_i, \nabla_{\mathbb{R}^n} f_j \rangle_{T_\gamma \Upsilon} := (\langle \nabla_{\mathbb{R}^n} f_i, \nabla_{\mathbb{R}^n} f_j \rangle_{T_x \Upsilon})^* \gamma := \int_{\mathbb{R}^n} \langle \nabla_{\mathbb{R}^n} f_i, \nabla_{\mathbb{R}^n} f_j \rangle_{\mathbb{R}^n}(x) d\gamma(x)$. See e.g., [2, (4.7)] for the proofs.

Definition 5.11 (*Hodge Laplacian*) For $V = \sum_{k=1}^m F_k v_k$ with $F_k = \Phi_k((f_1^k)^*, \dots, (f_\ell^k)^*)$, define Hodge Laplacian of V as

$$\Delta_H V(\gamma, x) := \sum_{k=1}^m \sum_{i,j=1}^\ell \partial_{ij}^2 \Phi_k\big((f_1^k)^*\gamma, \dots, (f_\ell^k)^*\gamma\big)\Big(\big\langle \nabla_{\mathbb{R}^n} f_i^k, \nabla_{\mathbb{R}^n} f_j^k\big\rangle_{T_x\mathbb{R}^n}\Big)^* \gamma \; v_k(x) \tag{5.19}$$

$$+ \sum_{k=1}^m \sum_{i=1}^\ell \partial_i \Phi_k\big((f_1^k)^*\gamma, \dots, (f_\ell^k)^*\gamma\big)(\Delta_{\mathbb{R}^n} f_k(x))^* \gamma \; v_k(x)$$

$$+ \sum_{k=1}^m \Phi_k\big((f_1^k)^*\gamma, \dots, (f_\ell^k)^*\gamma\big)\Delta_{H,\mathbb{R}^n} v_k(x),$$

$$+ 2\sum_{k=1}^m \sum_{i=1}^\ell \partial_i \Phi_k\big((f_1^k)^*\gamma, \dots, (f_\ell^k)^*\gamma\big)(\nabla_{\mathbb{R}^n} f_i^k \cdot \nabla_{\mathbb{R}^n}) v_k(x) \tag{5.20}$$

where $\Delta_{H,\mathbb{R}^n} v_k$ is the Hodge Laplacian of $v_k \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, and $(\nabla_{\mathbb{R}^n} f_i^k \cdot \nabla_{\mathbb{R}^n}) v_k(x)$ is the vector field whose ith coordinate coincides with $\big\langle \nabla_{\mathbb{R}^n} f_i^k, \nabla_{\mathbb{R}^n}(v_k)_i\big\rangle_{T_x\mathbb{R}^n}$. It turns out that $\Delta_H V$ does not depend on the choice of both the representative of V and the inner and outer functions of F_k (see [1, Theorem 3.5]).

For the proof of Theorem 5.13 below, we introduce the following space of exponential cylinder functions *with Schwartz inner functions*:

$$\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n)) := \mathrm{Span}_{\mathbb{R}}\Big\{\exp\big\{\log(1+f)^*\big\} \;:\; f \in \mathcal{S},\; -\delta \le f \le 0 \text{ for some } \delta \in (0,1)\Big\},$$

where $\mathcal{S}$ is the space of Schwartz functions in $\mathbb{R}^n$ (i.e., functions in $\mathbb{R}^n$ whose derivatives are all rapidly decreasing). We note that $T_t\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n)) \subset \mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ for every $t > 0$, and that $(\Delta, \mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n)))$ is essentially self-adjoint in $L^2(\Upsilon(\mathbb{R}^n), \pi)$ exactly by the same proof as in [2, Theorem 4.2].

Remark 5.12 Exponential cylinder functions have been originally discussed in [2], where they choose a larger class of inner functions. We introduced $\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ with inner functions in the space $\mathcal{S}$ of Schwartz functions for the proof of Theorem 5.13, where we need to choose a smaller class of inner functions to approximate $\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ by cylinder functions in a sufficiently good way. See the last paragraph of the proof of Theorem 5.13.

We define the corresponding energy functional:

$$\mathcal{E}_H(V, W) := \langle -\Delta_H V, W\rangle_{L^2(T\Upsilon,\pi)} = \int_{\Upsilon(\mathbb{R}^n)} \mathbf{\Gamma}^\Upsilon(V, W)\, d\pi\,, \quad V, W \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,, \tag{5.21}$$

where $\mathbf{\Gamma}^\Upsilon$ denotes the square field operator associated with Δ_H. By [1, Theorem 3.5], the form $\mathcal{E}_H$ is closable on $\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ and the corresponding closure is denoted by $\mathcal{D}(\mathcal{E}_H)$ and the corresponding (Friedrichs) extension of $\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ is denoted by $\mathcal{D}(\Delta_H)$. Let $\{\mathbf{T}_t\}$ denote the corresponding L^2-semigroup. It holds that

$$\mathbf{T}_t V \in \mathcal{D}(\mathcal{E}_H)\,, \quad \text{for any } t \ge 0 \text{ and } V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)). \tag{5.22}$$

The following intertwining property holds.

Theorem 5.13 $\nabla T_t F = \mathbf{T}_t \nabla F$ *for any* $t \geq 0$ *and for any* $F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$.

Proof We apply [42, Theorem 2.1] with $\mathcal{D} = \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$, $D = \nabla$, $A = \Delta$, $\hat{A} = \Delta_H$, $\hat{T}_t = \mathbf{T}_t$, $R = 0$, which concludes the sought statement. To do so, we verify Conditions (i)–(iv) of [42, Theorem 2.1]. Condition (i) and (ii) are straightforward by construction. Using the commutation $\nabla_{\mathbb{R}^n} \Delta_{\mathbb{R}^n} = \Delta_{H,\mathbb{R}^n} \nabla_{\mathbb{R}^n}$ and the representation (5.18) and (5.19), we can readily verify Condition (iv), i.e., $\nabla \Delta F = \Delta_H \nabla F$ for any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$.

We now verify Condition (iii), viz., $(\lambda - \Delta)\mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \subset H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ is dense for sufficiently large $\lambda > 0$. We prove it with $\lambda = 0$, viz., $\Delta\mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \subset H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ is dense. We first prove that $\Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n)) \subset H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ is dense. Define $L := \{F \in \Delta\mathcal{D}(\Delta) : F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)\}$. By Lemma 5.14 below, $\Delta\mathcal{D}(\Delta) \subset L^2(\Upsilon(\mathbb{R}^n), \pi)$ is dense. Furthermore,

$$T_t \Delta\mathcal{D}(\Delta) = \Delta T_t \mathcal{D}(\Delta) \subset \Delta\mathcal{D}(\Delta) \cap H^{1,2}(\Upsilon(\mathbb{R}^n), \pi).$$

In particular, $T_t \Delta\mathcal{D}(\Delta) \subset L$. Combining [13, (4.26)] with the fact that $\mathcal{E}$ coincides with the Cheeger energy associated with the L^2-transportation distance d_Υ and the Poisson measure π (see [26, Proposition 2.3]), we have the following regularisation inequality

$$\mathcal{E}(T_t F) \leq \frac{\|F\|_{L^2}^2}{2t} \qquad t > 0\,. \tag{5.23}$$

Therefore, combined with the density $\Delta\mathcal{D}(\Delta) \subset L^2(\Upsilon(\mathbb{R}^n), \pi)$, the space $\mathcal{T} := \cup_{t>0} T_t \Delta\mathcal{D}(\Delta)$ is weakly dense in $H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$. As $\mathcal{T}$ is a convex subset in $H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$, by Mazur's lemma,

$$\mathcal{T} \text{ is strongly dense in } H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)\,. \tag{5.24}$$

For every $G \in \mathcal{T} = \cup_{t>0} T_t \Delta\mathcal{D}(\Delta) = \cup_{t>0} \Delta T_t \mathcal{D}(\Delta)$ with an expression $G = \Delta T_t F$ with $F \in \mathcal{D}(\Delta)$ for some $t > 0$, we can take $F_n \in \mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ so that

$$\|\Delta F_n - \Delta F\|_{L^2} + \|F_n - F\|_{L^2} \to 0 \tag{5.25}$$

by the essential self-adjointness of $(\Delta, \mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n)))$. Furthermore, it can be readily verified that

$$\|\Delta T_t F_n - \Delta T_t F\|_{L^2} + \|T_t F_n - T_t F\|_{L^2} \to 0 \tag{5.26}$$

by the L^2-contraction property of T_t and the commutation $\Delta T_t = T_t \Delta$ for $t > 0$. Noting $T_t F_n \in \mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ by the stability of $\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ under the action of T_t, the formula (5.26) particularly shows that the sequence $(\Delta T_t F_n)_{n\in\mathbb{N}} \subset \Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ approximates $G = \Delta T_t F \in \mathcal{T}$ in the strong L^2-topology. Furthermore, by using (5.23) again, we have the uniform energy bound:

$$\sup_{n\in\mathbb{N}} \mathcal{E}(\Delta T_t F_n) = \sup_{n\in\mathbb{N}} \mathcal{E}(T_t \Delta F_n) \leq \sup_{n\in\mathbb{N}} \frac{1}{2t}\|\Delta F_n\| < \infty. \tag{5.27}$$

For every $H \in \mathcal{D}(\Delta)$,

$$\begin{aligned}
&\int_{\Upsilon(\mathbb{R}^n)} \langle \nabla(\Delta T_t F_n - G), \nabla H \rangle_{T_\gamma \Upsilon} d\pi(\gamma) + \int_{\Upsilon(\mathbb{R}^n)} (\Delta T_t F_n - G) H d\pi \\
&\quad = -\int_{\Upsilon(\mathbb{R}^n)} (\Delta T_t F_n - \Delta T_t F)\Delta H d\pi + \int_{\Upsilon(\mathbb{R}^n)} (\Delta T_t F_n - \Delta T_t F) H d\pi \\
&\quad \xrightarrow{n\to\infty} 0\,.
\end{aligned} \tag{5.28}$$

By the uniform bound (5.27) and the fact that $\mathcal{D}(\Delta)$ is dense in $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$, (5.28) shows that $(\Delta T_t F_n)_{n\in\mathbb{N}} \subset \Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ converges to $G = \Delta T_t F \in \mathcal{T}$ weakly in $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$. Thus, $\Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ approximates $\mathcal{T}$ in the weak $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$ topology. By (5.24) and the fact that $\Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ is a convex subspace in $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$, by applying Mazur's lemma again, we conclude that $\Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ is strongly dense in $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$.

Therefore, to complete the verification of Condition (iii), it suffices to prove that $\Delta\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ approximates $\Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ in $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$. The idea of the proof is, however, the same as in that of [2, Proposition 4.1]: for $F = \exp\{\log(1+f)^*\} \in \mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$, we can take an approximation $f_n \in C_c^\infty(\mathbb{R}^n)$ of the inner function $f \in \mathcal{S}$ so that $F_n = \exp\{\log(1+f_n)^*\} \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ converges to F in a sufficiently good way to conclude that $\Delta\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ approximates $\Delta\mathrm{ECylF}_{\mathcal{S}}(\Upsilon(\mathbb{R}^n))$ in $H^{1,2}(\Upsilon(\mathbb{R}^n),\pi)$. As this proof is mostly a repetition of [2, Proposition 4.1], we omit the details here. □

Lemma 5.14 *For $F \in L^2(\Upsilon(\mathbb{R}^n),\pi)$, there exists $F_n \in \mathcal{D}(\Delta)$ so that $\|\Delta F_n - F\|_{L^2} \to 0$.*

Proof We first show that $\Delta G_\alpha F \to \Delta G_\beta F$ in $L^2(\Upsilon(\mathbb{R}^n),\pi)$ for every $F \in L^2(\Upsilon(\mathbb{R}^n),\pi)$ as $\alpha \to \beta$ for $\alpha,\beta > 0$. By the resolvent equality $G_\alpha - G_\beta = (\beta-\alpha)G_\alpha G_\beta$, we have that

$$\|\Delta(G_\alpha - G_\beta)F\|_{L^2} = (\beta-\alpha)\|\Delta G_\alpha G_\beta F\|_{L^2} = (\beta-\alpha)\|G_\alpha \Delta G_\beta F\|_{L^2}.$$

By the L^2-contraction of αG_α, we obtain

$$(\beta-\alpha)\|G_\alpha \Delta G_\beta F\|_{L^2} \le \frac{\beta-\alpha}{\alpha^2}\|\Delta G_\beta F\|_{L^2} \to 0, \quad \alpha \to \beta.$$

Thus, $\Delta G_\alpha F \to \Delta G_\beta F$ as $\alpha \to \beta$ in $L^2(\Upsilon(\mathbb{R}^n),\pi)$.

We now prove the sought statement. Let $F_n := (1/(\alpha-1))G_{\alpha+1/n}F \in \mathcal{D}(\Delta)$. Then, by the general identity $(\alpha-\Delta)G_\alpha = \mathrm{Id}$, and by the convergence $\Delta G_\alpha F \to \Delta G_\beta F$ in $L^2(\Upsilon(\mathbb{R}^n),\pi)$ proven above, we have

$$\Delta F_n = \frac{1}{\alpha-1}\Delta G_{\alpha+1/n}F \xrightarrow{n\to\infty} \frac{1}{\alpha-1}\Delta G_\alpha F = \frac{(\alpha-1)}{(\alpha-1)}F = F, \quad F \in L^2(\Upsilon(\mathbb{R}^n),\pi). \qquad \square$$

Theorem 5.15 *Let $F \in \mathcal{D}(\mathcal{E}_H)$. Then $|\mathbf{T}_t F|_{T\Upsilon} \le T_t|F|_{T\Upsilon}$ π-a.e. for every $t \ge 0$. In particular $\mathbf{T}_t$ can be extended to the L^p-velocity fields $L^p(T\Upsilon(\Omega),\pi_\Omega)$ for every $1 \le p < \infty$.*

Proof By the Weitzenböck formula [1, Theorem 3.7] on $\Upsilon(\mathbb{R}^n)$, we can express $\Delta_H = \nabla^*\nabla + R^\Upsilon$, where R^Υ is the lifted curvature tensor from the base space $\mathbb{R}^n$. Since $\mathbb{R}^n$ is flat, we can easily deduce $R^\Upsilon = 0$.

Now, setting $\mathbf{\Gamma}(V,W) := \mathbf{\Gamma}^\Upsilon(V,W) + 2R^\Upsilon(V,W) = \mathbf{\Gamma}^\Upsilon(V,W)$ we can apply [43, Theorem 3.1] (see the proof of [43, Theorem 3.1] for $p=1$) and [43, Proposition 3.5], to get the sought conclusion of the first assertion.

We now prove the second assertion. Let $V \in L^p(T\Upsilon(\Omega),\pi_\Omega)$. Then, the density of cylinder vector fields gives the existence of a sequence $V_n \in \mathrm{CylF}(\mathbb{R}^n) \subset \mathcal{D}(\mathcal{E}_H)$ such that $|V_n - V|_{T\Upsilon} \to 0$ in $L^p(\Upsilon(\mathbb{R}^n),\pi)$ as $n \to \infty$. We can define

$$\mathbf{T}_t V := \lim_{n\to\infty} \mathbf{T}_t V_n\,. \tag{5.29}$$

The existence of the limit follows from

$$|\mathbf{T}_t V_n - \mathbf{T}_t V_m|_{T\Upsilon} \le T_t|V_n - V_m|_{T\Upsilon}\,, \tag{5.30}$$

as well as the independence of the limit from the approximating sequence $(V_n)_{n\in\mathbb{N}}$. □

Theorem 5.16 (p-Bakry–Émery estimate) *Let $p > 1$. The following assertions hold:*

(i) $T_t : H^{1,p}(\Upsilon(\mathbb{R}^n), \pi) \to H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$ *is a continuous operator for every* $t > 0$.

(ii) *For every* $F \in H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$,

$$|\nabla T_t F|^p_{T\Upsilon} \le T_t |\nabla F|^p_{T\Upsilon} \quad \pi\text{-a.e.}\,. \tag{5.31}$$

(iii) *Let* $1 < p \le 2$. *For every* $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ *it holds that*

$$\|\nabla T_t F\|_{L^p(T\Upsilon)} \le C(p) t^{-1/2} \|F\|_{L^p}, \quad t > 0. \tag{5.32}$$

In particular, $T_t : L^p(\Upsilon(\mathbb{R}^n), \pi) \to H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$ *is a continuous operator for every* $t > 0$.

(iv) *For every* $t, s > 0$, $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$, *it holds that* $\|\nabla T_t F\|_{L^1(T\Upsilon(\mathbb{R}^n))} < \infty$ *and*

$$|\nabla T_{t+s} F|_{T\Upsilon} \le T_t |\nabla T_s F|_{T\Upsilon} \quad \pi\text{-a.e.}\,. \tag{5.33}$$

Proof **(i).** By Theorems 5.13 and 5.15, for any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds that

$$|\nabla T_t F|_{T\Upsilon} = |\mathbf{T}_t \nabla F|_{T\Upsilon} \le T_t |\nabla F|_{T\Upsilon} \quad \pi\text{-a.e.}\,. \tag{5.34}$$

A simple application of Jensen's inequality to (5.34) gives

$$|\nabla T_t F|^p_{T\Upsilon} \le T_t |\nabla F|^p_{T\Upsilon}, \quad \text{for } F \in \mathrm{CylF}(\Upsilon) \text{ and } p \ge 1\,. \tag{5.35}$$

Let $F_n \in \mathrm{CylF}(\Upsilon)$ be a $H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$-Cauchy sequence. Then, by (5.35) and the invariance $\pi(T_t f) = \pi(f)$,

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t (F_n - F_m)|^p_{T\Upsilon} d\pi &\le \int_{\Upsilon(\mathbb{R}^n)} T_t |\nabla (F_n - F_m)|^p_{T\Upsilon} d\pi \\ &= \int_{\Upsilon(\mathbb{R}^n)} |\nabla (F_n - F_m)|^p_{T\Upsilon} d\pi \to 0. \end{aligned}\tag{5.36}$$

Since $H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$ is the closure of $\mathrm{CylF}(\Upsilon)$ w.r.t. the norm $\|\nabla \cdot\|_{L^p(T\Upsilon)} + \|\cdot\|_{L^p(\Upsilon,\pi)}$, by (5.36), the operator T_t is extended to $H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$ continuously. The proof of the first assertion is complete.

(ii). Let $F \in H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$ and take $F_n \in \mathrm{CylF}(\Upsilon)$ converging to F in $H^{1,p}(\Upsilon(\mathbb{R}^n), \pi)$. Then, by the lower semi-continuity of $|\nabla \cdot|^p_{T\Upsilon}$ w.r.t. the L^p-strong convergence, the continuity of the L^p-semigroup T_t and the inequality (5.35), we obtain

$$\begin{aligned}|\nabla T_{t+s} F|^p_{T\Upsilon} = |\nabla T_t T_s F|^p_{T\Upsilon} &\le \liminf_{n\to\infty} |\nabla T_t T_s F_n|^p_{T\Upsilon} \\ &\le \liminf_{n\to\infty} T_t |\nabla T_s F_n|^p_{T\Upsilon} \le T_t |\nabla T_s F|^p_{T\Upsilon}.\end{aligned}$$

Here the last equality follows from the assertion (i).

(iii). Let $p > 1$ be fixed. For any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ satisfying $F \ge 0$, it holds

$$\begin{aligned}p(p-1)\int_*^t \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_s F|^2_{T\Upsilon} |T_s F|^{p-2} d\pi ds &= \int_{\Upsilon(\mathbb{R}^n)} |F|^p d\pi - \int_{\Upsilon(\mathbb{R}^n)} |T_t F|^p d\pi \\ &\le \int_{\Upsilon(\mathbb{R}^n)} |F|^p d\pi\,,\end{aligned}$$

where the first equality follows by the following argument:

$$\frac{d}{dt}\int_{\Upsilon(\mathbb{R}^n)} |T_t F|^p d\pi = p \int_{\Upsilon(\mathbb{R}^n)} |T_t F|^{p-1} \Delta T_t F d\pi$$

$$
\begin{aligned}
&= -p\int_{\Upsilon(\mathbb{R}^n)}\Big\langle\nabla|T_t(F)|^{p-1},\nabla T_tF\Big\rangle_{T_\gamma\Upsilon}d\pi(\gamma)\\
&= -p(p-1)\int_{\Upsilon(\mathbb{R}^n)}\Big\langle|T_tF|^{p-2}\nabla T_tF,\nabla T_tF\Big\rangle_{T_\gamma\Upsilon}d\pi(\gamma)\\
&= -p(p-1)\int_{\Upsilon(\mathbb{R}^n)}|T_tF|^{p-2}\big|\nabla T_tF\big|^2_{T_\gamma\Upsilon}d\pi(\gamma)\,.
\end{aligned}
$$

By the contraction property of T_t, we obtain

$$
\begin{aligned}
\int_*^t\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_sF|^p_{T\Upsilon}d\pi ds &\le \Big(\int_*^t\int_{\Upsilon(\mathbb{R}^n)}|T_sF|^pd\pi ds\Big)^{\frac{2-p}{2}}\\
&\quad\Big(\int_*^t\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_sF|^2_{T\Upsilon}|T_sF|^{p-2}d\pi ds\Big)^{\frac{p}{2}}\\
&\le Ct^{\frac{2-p}{2}}\|F\|^p_{L^p}\,.
\end{aligned}
$$

We now employ the Bakry–Émery inequality (5.35) combined with the contraction property of T_t to show that $s\to\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_sF|^p_{T\Upsilon}d\pi$ is non-increasing, which yields

$$
t\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_tF|^p_{T\Upsilon}d\pi\le\int_*^t\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_sF|^p_{T\Upsilon}d\pi ds\le Ct^{\frac{2-p}{2}}\|F\|^p_{L^p}\,. \tag{5.37}
$$

This implies our conclusion for cylinder functions. We extended it to any $F\in L^p(\Upsilon(\mathbb{R}^n),\pi)$ by means of a density argument. Indeed, given $F\in L^p(\Upsilon(\mathbb{R}^n),\pi)$, we can find $F_n\in\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ such that $F_n\to F$ in L^p. The continuity of the semigroup T_t gives $T_tF_n\to T_tF$ in L^p, while the lower semi-continuity of the functional $G\to\int_{\Upsilon(R^n)}|\nabla G|^p_{T\Upsilon(\mathbb{R}^n)}d\pi$ with respect to the L^p convergence for $p>1$ yields

$$
\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_tF|^p_{T\Upsilon}d\pi\le\liminf_{n\to\infty}\int_{\Upsilon(\mathbb{R}^n)}|\nabla T_tF_n|^p_{T\Upsilon}d\pi\le Ct^{-1/2}\|F\|_{L^p}\,.
$$

(iv). Note that the assertion in the case of $1<p\le2$ implies the one in the case of $p>2$ by $L^p(\Upsilon(\mathbb{R}^n),\pi)\subset L^q(\Upsilon(\mathbb{R}^n),\pi)$ whenever $1\le q\le p$. Thus, we only need to prove it in the case of $1<p\le2$. Let $F\in L^p(\Upsilon(\mathbb{R}^n),\pi)$. Then, by the assertion (iii), $T_sF\in H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$. Take G_n converging to T_sF in $H^{1,p}(\Upsilon(\mathbb{R}^n),\pi)$. Then, up to taking a subsequence from $\{G_n\}$, and by making use of (5.34), we conclude that

$$
|\nabla T_{t+s}F|_{T\Upsilon}=|\nabla T_tT_sF|_{T\Upsilon}=\lim_{n\to\infty}|\nabla T_tG_n|_{T\Upsilon}\le\lim_{n\to\infty}T_t|\nabla G_n|_{T\Upsilon}=T_t|\nabla T_sF|_{T\Upsilon}\,.\square
$$

Remark 5.17 In [26] (see also [24]), the 2-Bakry–Émery estimate was proved in the case of the configuration space over a complete Riemannian manifold with Ricci curvature bound. For the purpose of the current paper, however, we need a stronger estimate, i.e., the p-Bakry–Émery estimate (5.31) for arbitrary $1<p<\infty$ and also the regularity estimate (5.32) of the heat semigroup, both of which do not follow only from the 2-Bakry–Émery inequality.

5.5 Equivalence of BV functions

In Sect. 5, we introduced the three different definitions (the variational/the relaxation/the semigroup approaches) of BV functions. In this section we show that the three different definitions of BV functions are equivalent.

Theorem 5.18 (Equivalence of BV functions) *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. Then,*

$$\mathcal{V}(F) = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) = \mathcal{T}(F)\,.$$

The proof of Theorem 5.18 will be given later in this section. Thanks to Theorem 5.18, we can introduce a universal definition of BV functions for $L^2(\Upsilon(\mathbb{R}^n), \pi)$-functions.

Definition 5.19 (*BV functions*) A function $F \in L^2(\Upsilon(\mathbb{R}^n))$ belongs to $\mathrm{BV}(\Upsilon(\mathbb{R}^n))$ if

$$\mathcal{V}(F) = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) = \mathcal{T}(F) < \infty\,.$$

We prepare several lemmas for the proof of Theorem 5.18.

Lemma 5.20 *For any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ and $t \geq 0$ it holds*

$$(\nabla^* \mathbf{T}_t V) = T_t(\nabla^* V)\,. \tag{5.38}$$

In particular $(\nabla^* \mathbf{T}_t V) \in L^p(\Upsilon(\mathbb{R}^n))$ for every $1 < p < \infty$.

Proof Let $F \in \mathrm{CylF}(\Upsilon)$. By the π-symmetry of T_t and Theorem 5.13, we have that

$$\begin{aligned}
\int_{\Upsilon(\mathbb{R}^n)} F\, T_t(\nabla^* V) d\pi &= \int_{\Upsilon(\mathbb{R}^n)} T_t F\, (\nabla^* V) d\pi = -\int_{\Upsilon(\mathbb{R}^n)} \langle V(\gamma, \cdot), \nabla T_t F(\gamma)\rangle_{T\Upsilon} d\pi \\
&= -\int_{\Upsilon(\mathbb{R}^n)} \langle V(\gamma, \cdot), \mathbf{T}_t \nabla F(\gamma)\rangle_{T\Upsilon} d\pi \\
&= -\int_{\Upsilon(\mathbb{R}^n)} \langle \mathbf{T}_t V(\gamma, \cdot), \nabla F(\gamma)\rangle_{T\Upsilon} d\pi \\
&= \int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* \mathbf{T}_t V) d\pi,
\end{aligned}$$

which immediately implies (5.38). □

Let us now introduce $\mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi)$, the space of vector fields with divergence in $L^p(\Upsilon(\mathbb{R}^n), \pi)$, as the closure of $\mathrm{CylV}(\Upsilon(\mathbb{R}^n)) \subset L^p(T\Upsilon(\mathbb{R}^n), \pi)$ with respect to the norm $\|V\|_{L^p} + \|\nabla^* V\|_{L^p}$.

In the case $p = 2$, we have the following inclusion

$$\mathcal{D}(\mathcal{E}_H) \subset \mathbf{D}^2(T\Upsilon(\mathbb{R}^n), \pi)\,, \tag{5.39}$$

as a consequence of the inequality $\|\nabla^* V\|_{L^2} \leq \mathcal{E}_H(V, V)$ for every $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$.

Lemma 5.21 *Let $1 < p < \infty$ and $1 < p' < \infty$ such that $1/p + 1/p' = 1$. If $F \in L^{p'}(\Upsilon(\mathbb{R}^n), \pi)$ then*

$$\mathcal{V}(F) = \sup\left\{ \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F d\pi : V \in \mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi),\ |V|_{T\Upsilon} \leq 1 \right\}. \tag{5.40}$$

Proof Let $V \in \mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi)$ with $|V|_{T\Upsilon} \leq 1$, to conclude the proof we just need to build a sequence $(W_n)_{n\in\mathbb{N}} \subset \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|W_n| \leq 1$ and $\|\nabla^* V - \nabla^* W_n\|_{L^p} \to 0$ as $n \to \infty$. To that aim we first consider a sequence $V_n \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $\|V - V_n\|_{L^p} + \|\nabla^* V - \nabla^* V_n\|_{L^p} \to 0$ as $n \to \infty$, which exists by definition. We now define W_n by cutting V_n of as we did in (5.10) in the proof of Proposition 5.5. □

Proof of Theorem 5.18 We first show the inequality $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) \leq \mathcal{V}(F)$ for $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. We assume without loss of generality that $\mathcal{V}(F) < \infty$. Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. Set $F_n = T_{1/n} F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$. By the symmetry of $\mathbf{T}_t$ in $L^2(T\Upsilon, \pi)$ and Lemma 5.20, we have that, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \leq 1$, it holds

$$\int_{\Upsilon(\mathbb{R}^n)} F_n \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} T_{1/n}(\nabla^* V) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \nabla^*(\mathbf{T}_{1/n} V) F d\pi. \tag{5.41}$$

The inclusion (5.22) and (5.39) imply that $\mathbf{T}_{1/n} V \in \mathbf{D}^2(T\Upsilon(\mathbb{R}^n), \pi)$, while Theorem 5.15 ensures that $|\mathbf{T}_{1/n} V|_{T\Upsilon} \leq T_{1/n} |V|_{T\Upsilon} \leq 1$.

Therefore, we can apply Lemma 5.21 to (5.41) to obtain $\|\nabla F_n\|_{L^1} \leq \mathcal{V}(F)$. Since $F_n \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ and $\mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ is dense in $H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$, we have $|\mathsf{D}_* F_n|(\Upsilon(\mathbb{R}^n)) \leq \|\nabla F_n\|_{L^1}$, by definition. By the lower semi-continuity of $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$ with respect to the L^2-convergence, it holds

$$|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) \leq \liminf_{n\to\infty} |\mathsf{D}_* F_n|(\Upsilon(\mathbb{R}^n)) \leq \liminf_{n\to\infty} \|\nabla F_n\|_{L^1(T\Upsilon,\pi)} \leq \mathcal{V}(F).$$

We now prove $\mathcal{T}(F) \leq |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. Let $F_n \in \mathrm{CylF}(\Upsilon)$ such that $F_n \to F$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ and $\|\nabla F_n\|_{L^1(T\Upsilon)} \to |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. Then, by the 1-Bakry–Émery inequality (5.34) on cylinder functions,

$$\|\nabla T_t F\|_{L^1} \leq \liminf_{n\to\infty} \|\nabla T_t F_n\|_{L^1} \leq \liminf_{n\to\infty} \|\nabla F_n\|_{L^1} = |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)).$$

Thus, $\mathcal{T}(F) \leq |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$.

Finally we prove $\mathcal{V}(F) \leq \mathcal{T}(F)$ for every $F \in L^p(\Upsilon(\mathbb{R}^n))$. For $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ and $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \leq 1$, we have that

$$\int_{\Upsilon(\mathbb{R}^n)} T_t F \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle \nabla T_t F, V \rangle d\pi \leq \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F|_{T\Upsilon} d\pi.$$

Since $T_t F \to F$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$, we obtain that

$$\int_{\Upsilon(\mathbb{R}^n)} F \nabla^* V d\pi \leq \lim_{t\to 0} \int_{\Upsilon(\mathbb{R}^n)} |\nabla T_t F|_{T\Upsilon} d\pi.$$

Thus, we conclude $\mathcal{V}(F) \leq \mathcal{T}(F)$.

The proof of Theorem 5.18 was given above. However, for the sake of completeness, we include a proof of the inequality $\mathcal{V}(F) \leq |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$, which holds in the more general case of $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ with $1 < p \leq \infty$.

Let $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$ for some $p > 1$ and $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) < \infty$. Let $F_n \in \mathrm{CylF}(\Upsilon)$ such that $F_n \to F$ in $L^1(\Upsilon(\mathbb{R}^n))$ and $\|\nabla F_n\|_{L^1(T\Upsilon)} \to |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. Let $F_{n,M} := (F_n \vee -M) \wedge M$ and $F_M := (F \vee -M) \wedge M$. Then, $F_{n,M} \to F_M$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ and $\|\nabla F_{n,M}\|_{L^1(T\Upsilon)} \leq \|\nabla F_n\|_{L^1(T\Upsilon)}$. Thus, $\limsup_{n\to\infty} \|\nabla F_{n,M}\|_{L^1(T\Upsilon)} \leq |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))$. By the integration by parts formula (2.13), it holds

$$\int_{\Upsilon(\mathbb{R}^n)} F_{n,M} \nabla^* V d\pi = -\int_{\Upsilon(\mathbb{R}^n)} \langle V, \nabla F_{n,M} \rangle_{T\Upsilon} d\pi \leq \|\nabla F_{n,M}\|_{L^1(T\Upsilon)} \leq \|\nabla F_n\|_{L^1(T\Upsilon)},$$

for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \leq 1$. By taking a (non-relabelled) subsequence from $\{F_{n,M}\}$ so that $F_{n,M} \to F_M$ π-a.e.,

and using the dominated convergence theorem (note that $|F_{n,M} \nabla^* V| \leq M |\nabla^* V| \in L^1(\Upsilon(\mathbb{R}^n), \pi)$ uniformly in n), we obtain that

$$\int_{\Upsilon(\mathbb{R}^n)} F_M \nabla^* V d\pi = \lim_{n\to\infty} \int_{\Upsilon(\mathbb{R}^n)} F_{n,M} \nabla^* V d\pi \leq \liminf_{n\to\infty} \|\nabla F_n\|_{L^1(T\Upsilon)} \leq |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)),$$

for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$. Since $F_M \to F$ in $L^p(\Upsilon(\mathbb{R}^n), \pi)$ as $M \to \infty$ by the hypothesis $F \in L^p(\Upsilon(\mathbb{R}^n), \pi)$, we conclude $\mathcal{V}(F) \le |\mathsf{D}F|(\Upsilon(\mathbb{R}^n))$. □

Remark 5.22 The proof of all the inequalities except $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) \le \mathcal{V}(F)$ remains true for every $1 < p < \infty$. In order to prove the inequality $|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) \le \mathcal{V}(F)$ in full generality following the same strategy we need show that $\mathbf{T}_t V \in \mathbf{D}^p(T\Upsilon(\mathbb{R}^n), \pi)$ for $1 < p < \infty$ and $V \in \mathrm{CylV}(T\Upsilon)$. This should follow, for instance, from the L^p-boundedness of vector-valued Riesz transforms, and will be addressed in a future work.

6 Sets of finite perimeter

In this section we introduce and study the notion of *set with finite perimeter*. Let us begin with a definition

Definition 6.1 (*Sets of finite perimeter*) Let $\Omega \subset \mathbb{R}^n$ be either a closed domain or the Euclidean space $\mathbb{R}^n$. A Borel set $E \subset \Upsilon(\Omega)$ is said to have finite perimeter if $\mathcal{V}_{\Upsilon(\Omega)}(\chi_E) < \infty$.

We refer the reader to Definition 5.1 for the introduction of the total variation $\mathcal{V}_{\Upsilon(\Omega)}(\cdot)$.

6.1 Sets of finite perimeter in $\Upsilon(B_r)$

We first develop the necessary theory in the configuration space $\Upsilon(B_r)$, in which every argument essentially comes down to finite-dimensional geometric analysis since only finitely many particles are allowed to belong to B_r.

Let us recall the decomposition $\Upsilon(B_r) = \bigsqcup_{k \ge 0} \Upsilon^k(B_r)$, where $(\Upsilon^k(B_r), \mathsf{d}_{\Upsilon^k}, \pi^k_{B_r})$ is the k-particle configuration space $\Upsilon^k(B_r)$ over B_r equipped with the L^2-transportation distance d_{Υ^k} and $\pi^k_{B_r} := \pi_{B_r}|_{\Upsilon^k(B_r)}$. We introduce the reduced boundary in $\Upsilon(B_r)$.

Definition 6.2 (*Reduced boundary in* $\Upsilon(B_r)$) Fix $r > 0$. Given $E \subset \Upsilon(B_r)$, set $E^k := E \cap \Upsilon^k(B_r)$ and define

$$\partial^*_{\Upsilon(B_r)} E := \bigsqcup_{k \ge 0} \partial^*_{\Upsilon^k(B_r)} E^k ,$$

$$\partial^*_{\Upsilon^k(B_r)} E^k := \left\{ \gamma \in \Upsilon^k(B_r) \,:\, \limsup_{s \to 0} \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma) \cap E^k)}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma))} > 0, \quad \limsup_{s \to 0} \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma) \setminus E^k)}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma))} > 0 \right\} ,$$

where $\mathsf{B}^k_s(\gamma)$ denotes the metric ball of radius $s > 0$ centred at $\gamma \in \Upsilon^k(B_r)$ w.r.t. d_{Υ^k}.

We can readily show that the m-codimensional Hausdorff measure $\rho^m_{\Upsilon^k(B_r)}$ w.r.t. d_{Υ^k} coincides with the push-forward measure of the m-codimensional spherical Hausdorff measure $\rho^m_{B_r^{\times k}}$ on $B_r^{\times k}$ w.r.t. the quotient map $\mathbf{s}_k$:

$$\rho^m_{\Upsilon^k(B_r)} = (\mathbf{s}_k)_\# \rho^m_{B_r^{\times k}} = (\mathbf{s}_k)_\# \mathbf{S}^{nk-m}_{B_r^{\times k}} , \tag{6.1}$$

where $\mathbf{S}^{nk-m}_{B_r^{\times k}}$ is the m-codimensional spherical Hausdorff measure on $B_r^{\times k}$ and $\mathbf{s}_k$ is the quotient map $B_r^{\times k} \to \Upsilon^k(B_r)$ as defined in Sect. 2. Having this in mind, we prove the following Gauß–Green formula in $\Upsilon(B_r)$.

Proposition 6.3 (Gauß–Green formula in $\Upsilon(B_r)$) *Fix $r>0$. If $E \subset \Upsilon(B_r)$ is a set of finite perimeter then there exists a vector field $\sigma_E : \Upsilon(B_r) \to T\Upsilon(B_r)$ such that $|\sigma_E|_{T\Upsilon(B_r)} = 1$ $\rho^1_{\Upsilon(B_r)}$-a.e. on $\partial^*_{\Upsilon(B_r)} E$, and*

$$\int_E (\nabla^* V) d\pi_{B_r} = \int_{\partial^*_{\Upsilon(B_r)} E} \langle V, \sigma_E \rangle d\rho^1_{\Upsilon(B_r)} \quad \textit{for } V \in \mathrm{CylV}(\Upsilon(B_r)). \tag{6.2}$$

Moreover $\mathcal{V}_{\Upsilon(B_r)}(\chi_E) = \rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)} E)$.

Proof Exploiting the decomposition $\Upsilon(B_r) = \bigsqcup_{k\geq 0} \Upsilon^k(B_r)$, where each $\Upsilon^k(B_r)$ is a connected component, we reduce our analysis to the study of $E^k := E \cap \Upsilon^k(B_r)$.

Set $\mathbf{E}^k := \mathbf{s}_k^{-1}(E^k)$. Given

$$V = \sum_{k=1}^m \Phi(f^*_{1,k}, \ldots, f^*_{n_k,k}) v_k \in \mathrm{CylV}(\Upsilon(B_r))\,,$$

we can define $\mathbf{V} \in C^\infty_*(B_r^{\times k}; \mathbb{R}^{nk})$ as

$$\begin{aligned}\mathbf{V}(x_1, \ldots, x_k) = \sum_{k=1}^m & \Phi(f_{1,k}(x_1) + \cdots + f_{n_k,k}(x_k), \ldots, f_{n_k,k}(x_1) + \cdots \\ & + f_{n_k,k}(x_k)) v_k(x_1, \ldots, x_k)\,.\end{aligned}$$

Notice that $|\mathbf{V}|_{\mathbb{R}^{nk}} \leq 1$ whenever $|V|_{T\Upsilon} \leq 1$. It is now immediate that $\mathbf{E}^k$ is of finite perimeter on $B_r^{\times k}$. Thus, standard results og geometric measure theory on the Euclidean space $\mathbb{R}^{nk}$ (see e.g., [47, Thm. 5.8.2]), we obtain

$$\int_{\mathbf{E}^k} (\nabla^* \mathbf{V}) d\mathbf{S}^{nk}_{B_r^{\times k}} = \int_{\partial^*_{B_r^{\times k}} \mathbf{E}^k} \langle \mathbf{V}, \sigma_{\mathbf{E}^k} \rangle d\rho^1_{B_r^{\times k}} \quad \text{for } \mathbf{V} \in C^\infty_*(B_r^{\times k}; \mathbb{R}^{nk})\,. \tag{6.3}$$

Here $\sigma_{\mathbf{E}^k}$ is a vector field $\sigma_{\mathbf{E}^k} : B_r^{\times k} \to \mathbb{R}^{nk}$ such that $|\sigma_{\mathbf{E}^k}|_{\mathbb{R}^{nk}} = 1$ $\rho^1_{B_r^{\times k}}$-a.e. on $\partial^*_{B_r^{\times k}} \mathbf{E}^k$. By passing to the quotient by means of the map $\mathbf{s}_k$ in both sides of (6.3) and using (6.1), we get the sought conclusion. □

Remark 6.4 An alternative proof of Proposition 6.3 can be given by employing the theory of RCD spaces (see [8] and references therein). Indeed $(\Upsilon^k(B_r), \mathsf{d}^k, \pi^k_{B_r})$ is an $\mathrm{RCD}(0, kn)$ space and E^k is of finite perimeter. Hence we can apply [19, Theorem 2.2] to get the integration by parts formula, written in terms of the total variation measure $|D\chi_{E^k}|$. From [9, Corollary 4.7] we deduce the identity $|D\chi_{E^k}| = \rho^1_{\Upsilon(B_r)}|_{\partial^*_{\Upsilon^k(B_r)} E^k}$.

Let us now prove a measurability statement. The proof follows arguing exactly in the same way as in the proof of Proposition 3.6, thus, we omit it.

Lemma 6.5 *Fix $r>0$. If $F : \Upsilon(\mathbb{R}^n) \to \mathbb{R}$ is a Borel function, then*

$$\Upsilon(B_r^c) \ni \eta \to \int_{\Upsilon(B_r)} F_{\eta,r} d\rho^1_{\Upsilon(B_r)} \quad \textit{is } \pi_{B_r^c}\textit{-measurable}\,.$$

6.2 Sets of finite perimeter on $\Upsilon(\mathbb{R}^n)$

We now study sets of finite perimeter on the configuration space $\Upsilon(\mathbb{R}^n)$ by employing the already developed theory for the space $\Upsilon(B_r)$. The main idea is to reduce a set $E \subset \Upsilon(\mathbb{R}^n)$ to its sections $E_{\eta,r} \subset \Upsilon(B_r)$ and apply the results for sets of finite perimeter in $\Upsilon(B_r)$, combined with the disintegration argument. We finally let $r \to \infty$ to recover the information on the perimeter of the original set E.

Let us begin by introducing the definition of the reduced boundary in $\Upsilon(\mathbb{R}^n)$.

Definition 6.6 (*Reduced boundary in* $\Upsilon(\mathbb{R}^n)$) Let $E \subset \Upsilon(\mathbb{R}^n)$ be a Borel set. For every $r > 0$ we set

$$\partial_r^* E := \{\gamma \in \Upsilon(\mathbb{R}^n) \, : \, \gamma|_{B_r} \in \partial^*_{\Upsilon(B_r)} E_{\gamma|_{B_r^c},r}\} \,. \tag{6.4}$$

The reduced boundary of E is defined as

$$\partial^* E := \liminf_{i\to\infty,\, i\in\mathbb{N}} \partial_i^* E = \bigcup_{i>0} \bigcap_{j>i,\, j\in\mathbb{N}} \partial_j^* E \,. \tag{6.5}$$

Remark 6.7 We defined $\partial^* E$ by taking the liminf along the sequence $\{\partial_i^* E\}_{i\in\mathbb{N}}$. This choice is completely arbitrary and, as we will see in the sequel (cf. Theorem 6.15), if we change the defining sequence, then the reduced boundary can change, but only up to an $\|E\|$-negligible set, where $\|E\|$ is the perimeter measure that will be defined later. Thus, the reduced boundary is well-defined up to $\|E\|$-negligible sets.

Notice that, for every $\eta \in \Upsilon(B_r^c)$ it holds

$$(\partial_r^* E)_{\eta,r} = \partial^*_{\Upsilon(B_r)} E_{\eta,r} \,. \tag{6.6}$$

Lemma 6.8 *If E is a Borel subset of $\Upsilon(\mathbb{R}^n)$, then $\partial_r^* E$ and $\partial^* E$ are Borel.*

Proof Since $\partial^* E = \liminf_{r\to\infty} \partial_r^* E$, it suffices to show the Borel measurability of $\partial_r^* E$ for every $r > 0$.

Step 1: We prove the following statement: for every $k \in \mathbb{N}$ and $s > 0$ the function

$$\{\gamma \in \Upsilon(\mathbb{R}^n) \, : \, \gamma(B_r) = k\} \ni \gamma \mapsto \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} \tag{6.7}$$

is Borel.

Since the Borel measurability of the map $\gamma \mapsto \pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))$ is easy, we only give a proof of the Borel measurability of the map $\gamma \mapsto \pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})$.

Let us identify $\{\gamma \in \Upsilon(\mathbb{R}^n) \, : \, \gamma(B_r) = k\} \simeq \Upsilon^k(B_r) \times \Upsilon(B_r^c)$. It allows us to introduce the product topology τ_p on $\{\gamma \in \Upsilon(\mathbb{R}^n) \, : \, \gamma(B_r) = k\}$, that is coarser than the vague topology τ_v as a consequence of the following observation: since B_r^c is open, the vague topology τ_v on $\Upsilon(B_r^c)$ coincides with the relative topology induced by $\Upsilon(\mathbb{R}^n)$. Thus, it suffices to see that the vague topology on $\Upsilon(B_r)$ is coarser than the relative topology induced by $\Upsilon(\mathbb{R}^n)$. For this purpose, we only need to show that, for any $\phi \in C_c(B_r)$ (note that ϕ does not necessarily vanish at the boundary of B_r), there exists an extension $\widetilde{\phi} \in C_c(\mathbb{R}^n)$ so that $\widetilde{\phi} = \phi$ on B_r. Given $\phi \in C_c(B_r)$, we take $\Phi \in C(\mathbb{R}^n)$ which is the extension of ϕ to $\mathbb{R}^n$ given by the Tietze extension theorem. Let us now pick $\kappa \in C_c(\mathbb{R}^n)$ such that $\kappa = 1$ on B_r and $\kappa = 0$ on B_{2r}^c. Then, it holds $\widetilde{\phi} := \kappa\Phi \in C_c(\mathbb{R}^n)$ and $\widetilde{\phi} = \phi$ in B_r, which concludes the sought statement.

By the inclusion $\tau_p \subset \tau_v$ of the topologies, we have the inclusion of the corresponding Borel σ-algebras $\mathscr{B}(\tau_p) \subset \mathscr{B}(\tau_v)$. Since the map

$$\Upsilon^k(B_r) \times \Upsilon^k(B_r) \times \Upsilon(B_r^c) \ni (\gamma_1, \gamma_2, \eta) \to \chi_E(\gamma_1 + \eta)\chi_{\mathsf{B}_s^k(\gamma_2)}(\gamma_1)\,, \tag{6.8}$$

is $\mathscr{B}(\tau_p)$-measurable, it is also $\mathscr{B}(\tau_v)$-measurable. Hence, Fubini's theorem gives that

$$\begin{aligned}\Upsilon^k(B_r) \times \Upsilon(B_r^c) \ni (\gamma_2, \eta) \to & \int_{\Upsilon^k(B_r)} \chi_E(\gamma_1 + \eta)\chi_{\mathsf{B}_s^k(\gamma_2)}(\gamma_1) d\pi^k_{\mathsf{B}_r}(\gamma_1) \\ &= \pi^k_{B_r}(\mathsf{B}_s^k(\gamma_2|_{B_r}) \cap E^k_{\eta,r})\,,\end{aligned} \tag{6.9}$$

is $\mathscr{B}(\tau_v)$-measurable as well.

Step 2: Fix $k \in \mathbb{N}$ and set

$$A_1^{k,r} := \left\{ \gamma \in \Upsilon(\mathbb{R}^n) \,:\, \limsup_{s\to 0} \frac{\pi^k_{B_r}(\mathsf{B}_s^k(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}_s(\gamma|_{B_r}))} > 0 \right\}\,,$$

$$A_2^{k,r} := \left\{ \gamma \in \Upsilon(\mathbb{R}^n) \,:\, \limsup_{j\to \infty} \frac{\pi^k_{B_r}(\mathsf{B}_{2^{-j}}^k(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))} > 0 \right\}\,.$$

Then $A_1^{k,r} = A_2^{k,r}$.

Observe that $A_2^{k,r} \subset A_1^{k,r}$. The converse inequality follows from the following observation. If $2^{-j} \le s \le 2^{-j+1}$ then

$$\begin{aligned}\frac{\pi^k_{B_r}(\mathsf{B}_s^k(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} &\ge \frac{\pi^k_{B_r}(\mathsf{B}_{2^{-j}}^k(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))} \frac{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} \\ &\ge C(k,n) \frac{\pi^k_{B_r}(\mathsf{B}_{2^{-j}}^k(\gamma|_{B_r}) \cap E^k_{\gamma|_{B_r^c},r})}{\pi^k_{B_r}(\mathsf{B}^k_{2^{-j}}(\gamma|_{B_r}))}\,,\end{aligned}$$

where we used the estimate $C(n,k)^{-1} e^{-\mathbf{L}^n(B_r)} s^{nk} \le \pi^k_{B_r}(\mathsf{B}^k_s(\gamma)) \le C(n,k) e^{-\mathbf{L}^n(B_r)} s^{nk}$ for any $s < r/5$, $\gamma \in \Upsilon(B_r)$ and some constant $C(n,k) \ge 1$ depending only on n and k. Indeed, the latter estimate can be obtained by the following observation: letting $\gamma = \{x_1, \ldots, x_k\}$, we have

$$B_r^{\times k} \cap \mathbf{s}_k^{-1}(\mathsf{B}^k_s(\gamma)) = B_r^{\times k} \cap \bigcup_{\sigma_k \in \mathfrak{S}_k} B_s(\mathbf{x}_{\sigma_k})\,,$$

hence

$$\pi^k_{B_r}(\mathsf{B}^k_s(\gamma)) = \frac{e^{-\mathbf{L}^n(B_r)}}{k!} \mathbf{L}^{kn}(B_r^{\times k} \cap \mathbf{s}_k^{-1}(\mathsf{B}^k_s(\gamma))) \le e^{-\mathbf{L}^n(B_r)} C(n,k) s^{nk}\,,$$

recall that $\mathbf{L}^n$ denotes the n-dimensional Lebesgue measure. The opposite inequality follows from

$$\begin{aligned}\pi^k_{B_r}(\mathsf{B}^k_s(\gamma)) &= \frac{e^{-\mathbf{L}^n(B_r)}}{k!} \mathbf{L}^{kn}(B_r^{\times k} \cap \mathbf{s}_k^{-1}(\mathsf{B}^k_s(\gamma))) \\ &\ge \frac{e^{-\mathbf{L}^n(B_r)}}{k!} \mathbf{L}^{kn}(B_r^{\times k} \cap B_s(\mathbf{x}_{\sigma_k})) \ge e^{-\mathbf{L}^n(B_r)} C(n,k) s^{nk}\,.\end{aligned}$$

Step 3: We conclude the proof. Thanks to Step 1 and Step 2 we know that $A_1^{k,r}$ is Borel for every $k \in \mathbb{N}$ and $r > 0$. The same arguments as in Step 1 and Step 2 apply to the Borel measurability for the following set:

$$\left\{ \gamma \in \Upsilon(\mathbb{R}^n) \, : \, \limsup_{s \to 0} \frac{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}) \setminus E^k_{r,\gamma|_{B_r^c}})}{\pi^k_{B_r}(\mathsf{B}^k_s(\gamma|_{B_r}))} > 0 \right\} , \tag{6.10}$$

hence, $\partial_r^* E$ is a Borel set. □

6.3 Perimeter measures

In this subsection, based on the variational approach, we introduce the perimeter measure $\|E\|$ for a set $E \subset \Upsilon(\mathbb{R}^n)$ satisfying $\mathcal{V}(\chi_E) < \infty$. In order to construct $\|E\|$, we first introduce a localised perimeter measure $\|E\|_r$ on $\Upsilon(\mathbb{R}^n)$, and show the monotonicity of $\|E\|_r$ as $r \to \infty$.

Definition 6.9 For every Borel set $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}_r(\chi_E) < \infty$, we define

$$\|E\|_r := \rho^1_{\Upsilon(B_r)}|_{(\partial_r^* E)_{\eta,r}} \otimes \pi_{B_r^c}(\eta) \quad \text{on } \Upsilon(\mathbb{R}^n) \, , \tag{6.11}$$

which is equivalently defines as follows: for every bounded Borel measurable function F on $\Upsilon(\mathbb{R}^n)$,

$$\int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r := \int_{\Upsilon(B_r^c)} \left(\int_{\Upsilon(B_r)} F_{\eta,r} d\rho^1_{\Upsilon(B_r)}|_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \right) d\pi_{B_r^c}(\eta). \tag{6.12}$$

Lemma 6.10 *Let $r > 0$. For every Borel set $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E) < \infty$, $\|E\|_r$ is a well-defined finite Borel measure.*

Proof Let us first show that $\|E\|_r$ is well-defined. The map $\gamma \mapsto F_{\eta,r}(\gamma)$ is $\rho^1_{\Upsilon(B_r)}|_{\partial^* E_{\eta,r}}$-measurable by Lemma 3.1. On account of the definition (6.12), we only need to show that the map

$$\Upsilon(B_r^c) \ni \eta \to \int_{\Upsilon(B_r)} F_{\eta,r} d\rho^1_{\Upsilon(B_r)}|_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \, , \tag{6.13}$$

is $\pi_{B_r^c}$-measurable for any Borel function $F : \Upsilon(\mathbb{R}^n) \to \mathbb{R}$. To show it, we use (6.6) and rewrite

$$\int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} F_{\eta,r} \, d\rho^1_{\Upsilon(B_r)} = \int_{(\partial_r^* E)_{\eta,r}} F_{\eta,r} \, d\rho^1_{\Upsilon(B_r)} = \int_{\Upsilon(B_r)} (\chi_{\partial_r^* E} F)_{\eta,r} \, d\rho^1_{\Upsilon(B_r)} \, .$$

Now, the claimed conclusion follows from Lemma 6.5 by observing that $\chi_{\partial_r^* E} F$ is a Borel function.

The finiteness of the measure $\|E\|_r$ is immediate by Propositions 6.3 and 5.5, indeed

$$\|E\|_r(\Upsilon(\mathbb{R}^n)) = \int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}((\chi_E)_{\eta,r}) d\pi_{B_r^c}(\eta) = \mathcal{V}_r(\chi_E) \le \mathcal{V}(\chi_E) < \infty. \qquad □$$

Lemma 6.11 *Let $r > 0$. For every Borel set $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}_r(\chi_E) < \infty$, there exists a vector field $\sigma_{E,r} : \Upsilon(\mathbb{R}^n) \to T\Upsilon(\mathbb{R}^n)$ such that*

(i) $\sigma_{E,r}(\gamma) \in T_\gamma \Upsilon(\mathbb{R}^n)$ *satisfies* $\sigma_{E,r}(\gamma, x) = 0$ *for* $x \in B_r^c$*;*
(ii) $|\sigma_{E,r}|_{T\Upsilon} = 1$, $\|E\|_r$*-a.e.;*

(iii) *for every* $V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$,

$$\int_E (\nabla^* V) d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r \,. \tag{6.14}$$

(iv) $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$, *and for every non-negative function* $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ *it holds*

$$\int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r = \sup\left\{\int_E (\nabla^* F V) d\pi \ : \ V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon} \le 1\right\} . \tag{6.15}$$

Proof By Proposition 5.5, there exists a measurable set $\Omega_r \subset \Upsilon(B_r^c)$ so that $\pi_{B_r^c}(\Omega_r) = 1$ and $\mathcal{V}_{\Upsilon(B_r)}(\chi_{E_{\eta,r}}) < \infty$ for every $\eta \in \Omega_r$. By Proposition 6.3, for every $\eta \in \Omega_r$, there exists a unique $T\Upsilon(B_r)$-valued Borel measurable map $\sigma_{\eta,r}$ on $\Upsilon(B_r)$ so that $|\sigma_{\eta,r}|_{T\Upsilon(B_r)} = 1$ $\rho^1_{\Upsilon(B_r)}|_{\partial^* E_{\eta,r}}$-a.e., and

$$\int_{E_{\eta,r}} (\nabla^* V_{\eta,r}) d\pi_{B_r} = \int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)} d\rho^1_{\Upsilon(B_r)}, \quad V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))\,, \tag{6.16}$$

where we used $V_{\eta,r} \in \mathrm{CylV}_*(\Upsilon(B_r))$ whenever $V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$. By taking the integral with respect to $\pi_{B_r^c}$, and arguing as in (4.9) we obtain

$$\begin{aligned}\int_E (\nabla^* V) d\pi &= \int_{\Upsilon(B_r^c)} \int_{E_{\eta,r}} (\nabla_r^* V_{\eta,r}) d\pi_{B_r} d\pi_{B_r^c}(\eta) \\ &= \int_{\Upsilon(B_r^c)} \int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)} d\rho^1_{\Upsilon(B_r)} d\pi_{B_r^c}(\eta)\,. \end{aligned} \tag{6.17}$$

Note that the map $\eta \mapsto \int_{\partial^*_{\Upsilon(B_r)} E_{\eta,r}} \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)} d\rho^1_{\Upsilon(B_r)}$ is $\pi_{B_r^c}$-measurable since, in view of (6.16), it is equal to a $\pi_{B_r^c}$-measurable function, and therefore, the argument (6.17) is justified. For $\gamma \in \Upsilon(\mathbb{R}^n)$ we define

$$\sigma_{E,r}(\gamma) := \begin{cases} \sigma_{\gamma|_{B_r^c},r}(\gamma|_{B_r}) & \text{if } \gamma|_{B_r^c} \in \Omega_r\,, \\ \sigma_r(\gamma) = 0 & \text{otherwise}\,. \end{cases} \tag{6.18}$$

Let us now observe that, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, we have

$$(\langle V, \sigma_{E,r}\rangle_{T\Upsilon(\mathbb{R}^n)})_{\eta,r} = \langle V_{\eta,r}, \sigma_{\eta,r}\rangle_{T\Upsilon(B_r)}\,. \tag{6.19}$$

By combining the definition (6.11) of $\|E\|_r$ with (6.17), (6.18) and (6.19), we deduce the assertion (iii).

The assertion (i) follows from the definition (6.18), and the assertion (ii) follows from

$$(|\sigma_{E,r}|_{T\Upsilon(\mathbb{R}^n)})_{\eta,r} = |\sigma_{\eta,r}| = 1, \quad \rho^1_{\Upsilon(B_r)}|_{\partial^* E_{\eta,r}}\text{-a.e..}$$

We now prove (iv). We first prove the equality $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$. From (iii) and (ii) we deduce

$$\begin{aligned}\mathcal{V}_r(\chi_E) &= \sup\left\{\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) f d\pi : V \in \mathrm{CylV}^r(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon(\mathbb{R}^n)} \le 1\right\} \\ &= \sup\left\{\int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r : V \in \mathrm{CylV}^r(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon(\mathbb{R}^n)} \le 1\right\}\end{aligned}$$

$$\leq \|E\|_r(\Upsilon(\mathbb{R}^n))\,.$$

Furthermore, Proposition 5.5 and Lemma 6.3 imply

$$\begin{aligned}\mathcal{V}_r(\chi_E) &\geq \int_{\Upsilon(B_r^c)} \mathcal{V}_{\Upsilon(B_r)}((\chi_E)_{\eta,r})d\pi_{B_r^c}(\eta)\\ &= \int_{\Upsilon(B_r^c)} \rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)}E_{\eta,r})d\pi_{B_r^c}(\eta) = \|E\|_r(\Upsilon(\mathbb{R}^n))\,. \end{aligned} \tag{6.20}$$

Thus, the proof of the equality $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$ is complete.

Let us finally address (6.15). From the equality $\mathcal{V}_r(\chi_E) = \|E\|_r(\Upsilon(\mathbb{R}^n))$, we deduce the existence of a sequence $V_k \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$ such that $|V_k|_{T\Upsilon} \leq 1$, and

$$\lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} \langle V_k, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r = \int_{\Upsilon(\mathbb{R}^n)} d\|E\|_r\,,$$

hence,

$$\begin{aligned}\lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} |V_k - \sigma_{E,r}|^2_{T\Upsilon} d\|E\|_r &= \lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} (|V_k|^2_{T\Upsilon} + |\sigma_{E,r}|^2_{T\Upsilon} - 2\langle V_k, \sigma_{E,r}\rangle_{T\Upsilon}) d\|E\|_r\\ &\leq \lim_{k\to\infty} 2\int_{\Upsilon(\mathbb{R}^n)} (1 - \langle V_k, \sigma_{E,r}\rangle_{T\Upsilon}) d\|E\|_r = 0\,.\end{aligned}$$

Therefore, for every $F \in \mathrm{CylF}(\Upsilon)$

$$\lim_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} F\langle V_k, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r = \int_{\Upsilon(\mathbb{R}^n)} F\, d\|E\|_r\,,$$

in particular, by making use of (6.14) with $V = FV_k$, it holds that

$$\int_{\Upsilon(\mathbb{R}^n)} F\, d\|E\|_r \leq \sup\left\{\int_E (\nabla^* FV)d\pi \;:\; V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n)),\; |V|_{T\Upsilon} \leq 1\right\}. \tag{6.21}$$

The converse inequality follows form $|\sigma_{E,r}|_{T\Upsilon} = 1$ $\|E\|_r$-a.e. and the fact that F is non-negative:

$$\begin{aligned}\int_E (\nabla^* FV)d\pi &= \int_{\Upsilon(\mathbb{R}^n)} F\langle V_k, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r\\ &\leq \int_{\Upsilon(\mathbb{R}^n)} |F| d\|E\|_r = \int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r\,.\end{aligned} \tag{6.22}$$

□

Corollary 6.12 *If $\mathcal{V}(\chi_E) < \infty$, then $r \mapsto \|E\|_r(A)$ is monotone non-decreasing for every Borel measurable set A.*

Proof In view of the density of cylinder functions on $L^2(\Upsilon(\mathbb{R}^n), \pi)$ it is enough to check that

$$r \to \int_{\Upsilon(\mathbb{R}^n)} F d\|E\|_r \quad \text{is non-decreasing}\,,$$

for every non-negative $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$, which easily follows from (6.15) and the inclusion $\mathrm{CylV}^s_*(\Upsilon(\mathbb{R}^n)) \subset \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$ for $s \leq r$. □

By the monotonicity of $r \mapsto \|E\|_r$ in Corollary 6.12, we may define the limit measure as follows:

Definition 6.13 (*Perimeter measure*) Given $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E) < \infty$, we define the perimeter measure as

$$\|E\|(A) := \lim_{r\to\infty} \|E\|_r(A) \quad \text{for every Borel set } A\,. \tag{6.23}$$

We finally obtain the Gauß–Green formula for the perimeter measure $\|E\|$. For a Borel set $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E) < \infty$, let $L^2(T\Upsilon, \|E\|)$ be the completion of $\mathrm{CylV}(\Upsilon)$ with respect to $\|\cdot\|_{L^2(T\Upsilon,\|E\|)}$ analogously in (2.10).

Theorem 6.14 (Gauß–Green formula for $\|E\|$) *For a Borel set $E \subset \Upsilon(\mathbb{R}^n)$ with $\mathcal{V}(\chi_E) < \infty$, there exists a unique element $\sigma_E \in L^2(T\Upsilon, \|E\|)$ such that $|\sigma_E|_{T\Upsilon} = 1$ $\|E\|$-a.e. and*

$$\int_E \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_E\rangle_{T\Upsilon} d\|E\| \quad V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)). \tag{6.24}$$

Proof Note that, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, there exists $r > 0$ so that $V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$. Thus, by (iii) in Lemma 6.11, for any $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, there exists $r > 0$ and $\sigma_{E,r} : \Upsilon(\mathbb{R}^n) \to T\Upsilon$ so that $|\sigma_{E,r}| = 1$ $\|E\|$-a.e., and

$$\begin{aligned}\int_E \nabla^* V d\pi &= \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{E,r}\rangle_{T\Upsilon} d\|E\|_r \\ &\le \|E\|(\Upsilon(\mathbb{R}^n))^{1/2}\|V\|_{L^2(T\Upsilon,\|E\|_r)} \\ &\le \|E\|(\Upsilon(\mathbb{R}^n))^{1/2}\|V\|_{L^2(T\Upsilon,\|E\|)}.\end{aligned}$$

The last inequality followed from the monotonicity in Corollary 6.12.

In particular, the linear operator L defined as

$$L : L^2(T\Upsilon(\mathbb{R}^n), \|E\|) \to \mathbb{R}\,, \quad L^2(T\Upsilon(\mathbb{R}^n), \|E\|) \ni V \mapsto L(V) := \int_E \nabla^* V d\pi\,, \tag{6.25}$$

is a well-defined continuous operator on the Hilbert space $L^2(T\Upsilon(\mathbb{R}^n), \|E\|)$ and satisfies $\|L\| \le \|E\|(\Upsilon(\mathbb{R}^n))^{1/2}$. Therefore, the Riesz representation theorem in the Hilbert space $L^2(T\Upsilon(\mathbb{R}^n), \|E\|)$ gives the existence of $\sigma_E \in L^2(T\Upsilon(\mathbb{R}^n), \|E\|)$ so that

$$\begin{aligned}&\|\sigma_E\|_{L^2(T\Upsilon,\|E\|)} \le \|E\|(\Upsilon(\mathbb{R}^n))^{1/2}\,, \\ &\int_E \nabla^* V d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma\rangle d\|E\| \quad V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,.\end{aligned}$$

It suffices to show that $|\sigma|_{T\Upsilon} = 1$ $\|E\|$-a.e. By (iv) in Lemma 6.11 and Corollary 6.12, we deduce that

$$\begin{aligned}\|E\|(\Upsilon(\mathbb{R}^n)) &= \lim_{r\to\infty}\|E\|_r(\Upsilon(\mathbb{R}^n)) = \lim_{r\to\infty}\mathcal{V}_r(\chi_E) \\ &= \lim_{r\to\infty} \sup_{V\in\mathrm{CylV}^r_r(\Upsilon(\mathbb{R}^n)),\,|V|_{T\Upsilon}\le 1} \int_E \nabla^* V d\pi \\ &\le \int_{\Upsilon(\mathbb{R}^n)} |\sigma|_{T\Upsilon} d\|E\| \le \|E\|(\Upsilon(\mathbb{R}^n))^{1/2}\|\sigma\|_{L^2(T\Upsilon,\|E\|)} \\ &\le \|E\|(\Upsilon(\mathbb{R}^n))\,,\end{aligned}$$

which yields $|\sigma|_{T\Upsilon} = 1$ $\|E\|$-a.e. as a consequence of the characterisation of the equality for the Hölder inequality. $\square$

6.4 Perimeters and one-codimensional Poisson measures

In this subsection, we prove one of the main results in this paper. Namely, the perimeter measure $\|E\|$ based on the variational approach (Definition 6.13) coincides with the 1-codimensinal Poisson measure ρ^1 (Definition 3.8) restricted to the reduced boundary $\partial^* E$ of E (Definition 6.6).

Theorem 6.15 *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a set with $\mathcal{V}(\chi_E) < \infty$. Then,*

$$\|E\| = \rho^1|_{\partial^* E}.$$

Before giving the proof, we prove a lemma.

Lemma 6.16 *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a set with $\mathcal{V}(\chi_E) < \infty$. Then, for any $r > 0$, $\varepsilon > 0$, it holds*

$$(\partial_r^* E)_{\eta,r} \subset (\partial_{r+\varepsilon}^* E)_{\eta,r} \quad \text{up to } \rho^1_{\Upsilon(B_r)}\text{-negligible sets for } \pi_{B_r^c}\text{-a.e. } \eta. \tag{6.26}$$

Namely, there exists a measurable set $\Omega_{r,\varepsilon} \subset \Upsilon(\mathbb{R}^n)$ so that $\pi_{B_r^c}(\Omega_{r,\varepsilon}) = 1$ and for any $\eta \in \Omega_{r,\varepsilon}$, it holds that

$$\rho^1_{\Upsilon(B_r)}\Big((\partial_r^* E)_{\eta,r} \setminus \partial_{r+\varepsilon}^* E)_{\eta,r}\Big) = 0\,. \tag{6.27}$$

Proof By (6.6) and the definition (6.11) of the perimeter measure $\|E\|_r$, we see that

$$\begin{aligned}
\infty > \|E\|(A) \geq \|E\|_{r+\varepsilon}(A) &= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}(\partial^*_{\Upsilon(B_{r+\varepsilon})} E_{\eta,r+\varepsilon} \cap A_{\eta,r+\varepsilon}) d\pi_{B^c_{r+\varepsilon}}(\eta) \\
&= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon} E)_{\eta,r+\varepsilon} \cap A_{\eta,r+\varepsilon}\big) d\pi_{B^c_{r+\varepsilon}}(\eta) \\
&= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon} E \cap A)_{\eta,r+\varepsilon}\big) d\pi_{B^c_{r+\varepsilon}}(\eta)\,.
\end{aligned} \tag{6.28}$$

By the monotonicity $\|E\|_{r+\varepsilon}(A) \geq \|E\|_r(A)$ in Corollary 6.12, we obtain that

$$\int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon} E \cap A)_{\eta,r+\varepsilon}\big) d\pi_{B^c_{r+\varepsilon}}(\eta) \geq \int_{\Upsilon(B^c_r)} \rho^1_{\Upsilon(B_r)}\big((\partial^*_r E \cap A)_{\eta,r}\big) d\pi_{B^c_r}(\eta).$$

Taking $A = \Upsilon(\mathbb{R}^n) \setminus \partial^*_{r+\varepsilon} E$, we have that

$$\begin{aligned}
0 &= \int_{\Upsilon(B^c_{r+\varepsilon})} \rho^1_{\Upsilon(B_{r+\varepsilon})}\big((\partial^*_{r+\varepsilon} E \cap A)_{\eta,r+\varepsilon}\big) d\pi_{B^c_{r+\varepsilon}}(\eta) \\
&\geq \int_{\Upsilon(B^c_r)} \rho^1_{\Upsilon(B_r)}\big((\partial^*_r E \cap A)_{\eta,r}\big) d\pi_{B^c_r}(\eta)\,.
\end{aligned}$$

Thus, $\rho^1_{\Upsilon(B_r)}\big((\partial^*_r E \cap A)_{\eta,r}\big) = 0$ for $\pi_{B^c_r}$-a.e. η, which implies that

$$\big(\partial^*_r E \cap (\Upsilon(\mathbb{R}^n) \setminus \partial^*_{r+\varepsilon} E)\big)_{\eta,r} = (\partial^*_r E)_{\eta,r} \setminus \big((\partial^*_{r+\varepsilon} E)_{\eta,r} \cap (\partial^*_r E)_{\eta,r}\big)$$

is $\rho^1_{\Upsilon(B_r)}$-negligible for $\pi_{B^c_r}$-a.e. η. □

Proof of Theorem 6.15 Fix $r > 0$ and $\eta \in \Upsilon(B^c_r)$. It holds

$$(\partial^* E)_{\eta,r} := \left(\bigcup_{i>0} \bigcap_{j>i} \partial^*_j E\right)_{\eta,r} = \bigcup_{i>0} \bigcap_{j>i} (\partial^*_j E)_{\eta,r}\,. \tag{6.29}$$

The monotonicity formula (6.27) in Lemma 6.16 gives the existence of $\Omega_{r,j} \subset \Upsilon(\mathbb{R}^n)$ so that $\pi_{B_r^c}(\Omega_{r,j}) = 1$, and for any $\eta \in \Omega_{r,j}$

$$(\partial_r^* E)_{\eta,r} \subset (\partial_j^* E)_{\eta,r} \quad j \geq r \quad \text{up to a } \rho^1_{\Upsilon(B_r)}\text{-negligible set}\,.$$

Take $\Omega_r = \cap_{j \geq r, j \in \mathbb{N}} \Omega_{r,j}$. Then $\pi_{B_r^c}(\Omega_r) = 1$, and by using (6.6), we obtain that for any $\eta \in \Omega_r$,

$$\partial^*_{\Upsilon(B_r)} E_{\eta,r} = (\partial_r^* E)_{\eta,r} \subset (\partial^* E)_{\eta,r} \quad \text{up to a } \rho^1_{\Upsilon(B_r)}\text{-negligible set}\,.$$

This implies that for any Borel set $A \subset \Upsilon(\mathbb{R}^n)$,

$$\rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)} E_{\eta,r} \cap A_{\eta,r}) \leq \rho^1_{\Upsilon(B_r)}((\partial^* E \cap A)_{\eta,r}), \quad \eta \in \Omega_r.$$

Thus, by noting that $\pi_{B_r^c}(\Omega_r) = 1$ and recalling Definitions 6.13 and 3.8, we obtain

$$\begin{aligned}
\|E\|(A) &:= \lim_{r\to\infty} \|E\|_r(A) \\
&= \lim_{r\to\infty} \int_{\Upsilon(B_r^c)} \rho^1_{\Upsilon(B_r)}(\partial^*_{\Upsilon(B_r)} E_{\eta,r} \cap A_{\eta,r}) d\pi_{B_r^c}(\eta) \\
&\leq \lim_{r\to\infty} \int_{\Upsilon(B_r^c)} \rho^1_{\Upsilon(B_r)}((\partial^* E \cap A)_{\eta,r}) d\pi_{B_r^c}(\eta) \\
&= \rho^1(A \cap \partial^* E)\,.
\end{aligned}$$

In order to conclude the proof, it is enough to check that

$$\|E\|(\Upsilon(\mathbb{R}^n)) \geq \rho^1(\partial^* E)\,. \tag{6.30}$$

Indeed, given any Borel set A, by making use of the already proven inequality $\|E\| \leq \rho^1|_{\partial^* E}$, we obtain

$$\begin{aligned}
\|E\|(\Upsilon(\mathbb{R}^n)) &= \|E\|(A) + \|E\|(A^c) \leq \rho^1(A \cap \partial^* E) + \rho^1(A^c \cap \partial^* E) \\
&= \rho^1(\partial^* E) \leq \|E\|(\Upsilon(\mathbb{R}^n))\,.
\end{aligned}$$

Thus, $\|E\|(A) + \|E\|(A^c) = \rho^1(A \cap \partial^* E) + \rho^1(A^c \cap \partial^* E)$ for any Borel set A. Assume that there exists a Borel set A so that $\|E\|(A) < \rho^1(A \cap \partial^* E)$. Since $\|E\| \leq \rho^1(\cdot \cap \partial^* E)$, it implies

$$\|E\|(A) + \|E\|(A^c) < \rho^1(A \cap \partial^* E) + \rho^1(A^c \cap \partial^* E),$$

which is a contradiction.

We now prove (6.30). Let $s < r$. By recalling Definitions 6.9, 3.5 of $\|E\|_r$ and ρ^1_r respectively and using the monotonicity of ρ^1_r in Theorem 3.7, we have

$$\|E\|_r(\Upsilon(\mathbb{R}^n)) = \int_{\Upsilon(B_r^c)} \rho^1_{\Upsilon(B_r)}((\partial_r^* E)_{\eta,r}) d\pi_{B_r^c}(\eta) = \rho^1_r(\partial_r^* E) \geq \rho^1_s(\partial_r^* E)\,,$$

hence

$$\|E\|(\Upsilon(\mathbb{R}^n)) = \lim_{i\to\infty} \|E\|_i(\Upsilon(R^n)) \geq \liminf_{i\to\infty} \rho^1_s(\partial_i^* E) \geq \rho^1_s(\liminf_{i\to\infty} \partial_i^* E) = \rho_s(\partial^* E)\,.$$

Passing to the limit $s \to \infty$, we conclude (6.30). $\square$

7 Total variation and Gauß–Green formula

In this section, we prove a relation between the coarea with respect to the perimeter measure $\|E\|$ and the variation $|\mathsf{D}_* F|$ obtained via relaxation of Cylinder functions. As an application, we introduce the total variation *measure* $|\mathsf{D}F|$ for BV functions F, and prove the Gauß–Green formula.

7.1 Total variation measures via coarea formula

Recall that, for $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, the map $\mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \ni G \mapsto |\mathsf{D}_* F|[G]$ is defined by the relaxation approach in Definition 5.7. The main result of this subsection is the following formula:

Theorem 7.1 *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$. Then,*

$$\mathcal{V}(\chi_{\{F>t\}}) < \infty \quad \textit{a.e. } t \in \mathbb{R}, \tag{7.1}$$

and the following formula holds:

$$\int_{-\infty}^{\infty} \left(\int_{\Upsilon(\mathbb{R}^n)} G d \big\| \{F > t\} \big\| \right) dt = |\mathsf{D}_* F|[G], \quad \textit{for any non-negative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)) \,. \tag{7.2}$$

The proof of Theorem 7.1 will be given later in this section. Before discussing the proof, we study several consequences of Theorem 7.1. By (7.1), the left-hand side of (7.2) makes sense with $G \equiv 1$ since the right-hand side $|\mathsf{D}_* F|[1] < \infty$ is finite due to $F \in \mathrm{BV}(\Upsilon(\mathbb{R}^n))$ and Theorem 5.18. This leads us to provide the following definition of the total variation measure.

Definition 7.2 (*Total variation measure*) For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, define *the total variation measure* $|\mathsf{D}F|$ as follows:

$$|\mathsf{D}F| := \int_{-\infty}^{\infty} \big\| \{F > t\} \big\| dt. \tag{7.3}$$

We now investigate relations between the total variation measure $|\mathsf{D}\chi_E|$ and the perimeter measure $\|E\|$ defined in Definition 6.13 and the $(1, 2)$-capacity $\mathrm{Cap}_{1,2}$ defined in Definition 4.2.

Corollary 7.3 (Total variation and perimeters) *Let $E \subset \Upsilon(\mathbb{R}^n)$ satisfy $|\mathsf{D}\chi_E|(\Upsilon(\mathbb{R}^n)) < \infty$. Then,*

$$|\mathsf{D}\chi_E| = \|E\| \quad \textit{as measures} \,.$$

Proof By Theorem 5.18, $\mathcal{V}(\chi_E) < \infty$ and $\|E\|$ is well-defined. Noting that

$$\{\chi_E > t\} = \begin{cases} \Upsilon(\mathbb{R}^n) & t \le 0; \\ E & 0 < t \le 1; \\ \emptyset & t > 1, \end{cases}$$

and $\|\Upsilon(\mathbb{R}^n)\| = 0$ and $\|\emptyset\| = 0$, we obtain that

$$|\mathsf{D}\chi_E|(A) = \int_{-\infty}^{\infty} \big\| \{\chi_E > t\} \big\| (A) dt = 0 + \|E\|(A) + 0 = \|E\|(A) \quad \text{for every Borel set A .} \square$$

Corollary 7.4 (Total variation and capacity) *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$. For any Borel set $A \subset \Upsilon(\mathbb{R}^n)$,*

$$\mathrm{Cap}_{1,2}(A) = 0 \implies |\mathrm{D}F|(A) = 0.$$

Proof Let $\mathrm{Cap}_{1,2}(A) = 0$. By Theorems 7.1 and 6.15, we can write

$$|\mathrm{D}F|(A) = \int_{-\infty}^{\infty} \|\{F > t\}\|(A)dt = \int_{-\infty}^{\infty} \rho^1(\partial^*\{F > t\} \cap A)dt\,, \tag{7.4}$$

hence it suffices to show that $\rho^1(\partial^*\{F > t\} \cap A) = 0$. This follows from the absolute continuity of ρ^1 with respect to $\mathrm{Cap}_{1,2}$ obtained in Theorem 4.3. $\square$

7.2 Proof of Theorem 7.1

This subsection is devoted to the proof of Theorem 7.1. Let us begin with two propositions.

Proposition 7.5 *Let $E \subset \Upsilon(\mathbb{R}^n)$ be a set with $\mathcal{V}(\chi_E) < \infty$. Then, for every non-negative function $G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds*

$$\int_{\Upsilon(\mathbb{R}^n)} G d\|E\| = \sup\left\{\int_E (\nabla^* G V)d\pi \ :\ V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon} \le 1\right\}. \tag{7.5}$$

In particular, the following hold:

(i) *if $F_k \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$, and $F_k \to \chi_E$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ as $k \to \infty$, then*

$$\liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_k|_{T\Upsilon} d\pi \ge \int_{\Upsilon(\mathbb{R}^n)} G d\|E\|\,, \quad \textit{for non-negative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,;$$

(ii) *if $\chi_{E_k} \to \chi_E$ in $L^1(\Upsilon(\mathbb{R}^n), \pi)$ as $k \to \infty$, where $(E_k)_k$ are sets of finite perimeter, then*

$$\liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|E_k\| \ge \int_{\Upsilon(\mathbb{R}^n)} F d\|E\|\,, \quad \textit{for non-negative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))\,.$$

Proof Fix $\varepsilon > 0$. We pick $r > 0$ such that $\int_{\Upsilon(\mathbb{R}^n)} G d\|E\|_r \ge \int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon$. From (6.15) we deduce the existence of $V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n))$ with $|V|_{T\Upsilon} \le 1$ such that $\int_E (\nabla^* G V)d\pi \ge \int_{\Upsilon(\mathbb{R}^n)} G d\|E\|_r - \varepsilon$, yielding

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} G d\|E\| &\le \int_E (\nabla^* G V)d\pi + 2\varepsilon \\ &\le \sup\left\{\int_E (\nabla^* G V)d\pi \ :\ V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon} \le 1\right\} + 2\varepsilon\,.\end{aligned}$$

By taking $\varepsilon \to 0$, the one inequality is proved.

We now prove the converse inequality. Take a representative $G = \Phi(f_1^*, \ldots, f_k^*)$ and take $r > 0$ so that $\cup_{i=1}^k \mathrm{supp}[f_i] \subset B_r$. By the divergence formula (2.14), we can easily see

$$\begin{aligned}&\sup\left\{\int_E (\nabla^* G V)d\pi \ :\ V \in \mathrm{CylV}^r_*(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon} \le 1\right\} \\ &= \sup\left\{\int_E (\nabla^* G V)d\pi \ :\ V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n)),\ |V|_{T\Upsilon} \le 1\right\}.\end{aligned}$$

By combining it with the formula (6.15) and the monotonicity of $r \mapsto \|E\|_r$ in Corollary 6.12, the converse inequality is proved.

Let us now prove (i) and (ii). Fix $\varepsilon > 0$. By Theorem 6.14, we can take $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|V|_{T\Upsilon} \leq 1$ and

$$\int_E (\nabla^* G V) d\pi \geq \int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon .$$

Let k_j be a subsequence such that $\lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_{k_j}|_{T\Upsilon} d\pi = \liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_k|_{T\Upsilon} d\pi$, it holds

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon \leq \int_E (\nabla^* G V) d\pi &= \lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} F_{k_j} (\nabla^* G V) d\pi \\ &= \lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G \langle \nabla F_{k_j}, V \rangle_{T\Upsilon} d\pi \\ &\leq \liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_k|_{T\Upsilon} d\pi .\end{aligned}$$

Furthermore, by using Theorem 6.14 with V being GV, we deduce that

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} G d\|E\| - \varepsilon \leq \int_E (\nabla^* G V) d\pi &= \lim_{j\to\infty} \int_{E_{k_j}} (\nabla^* G V) d\pi \\ &= \lim_{j\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G \langle V, \sigma_{E_{k_j}} \rangle_{T\Upsilon} d\|E_{k_j}\| \\ &\leq \liminf_{k\to\infty} \int_{\Upsilon(\mathbb{R}^n)} G d\|E_{k_j}\| . \square\end{aligned}$$

Proposition 7.6 *For any $F \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ it holds*

$$\begin{aligned}&\int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} G\, d\big\|\{F > t\}\big\| dt \\ &= \int_{\Upsilon(\mathbb{R}^n)} G\, |\nabla F|_{T\Upsilon} d\pi\ , \quad \textit{for non-negative } G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n)) .\end{aligned} \tag{7.6}$$

Proof The map

$$\mathbb{R} \ni t \to m(t) := \int_{\{F>t\}} G|\nabla F|_{T\Upsilon} d\pi \tag{7.7}$$

is monotone and finite since $|\nabla F|_{T\Upsilon} \in L^1(\Upsilon(\mathbb{R}^n))$. Let $t \in \mathbb{R}$ be a point on which the map $t \mapsto m(t)$ is differentiable and set

$$g_\varepsilon(s) := \begin{cases} 1 & s \leq t \\ \varepsilon^{-1}(t-s)+1 & t \leq s \leq t+\varepsilon \\ 0 & s > t+\varepsilon . \end{cases} \tag{7.8}$$

Notice that $g_\varepsilon \circ F \to \chi_{\{F>t\}}$ in $L^p(\Upsilon(\mathbb{R}^n))$ for any $p \in [1,\infty)$ as $\varepsilon \to 0$. Indeed,

$$\int_{\Upsilon(\mathbb{R}^n)} |g_\varepsilon \circ F - \chi_{\{F>t\}}|^p d\pi \leq 2^p \pi(\{t \leq F \leq t+\varepsilon\}) \to 0, \quad \text{as } \varepsilon \to 0 . \tag{7.9}$$

Standard calculus rules give

$$\int_{\Upsilon(\mathbb{R}^n)} G|\nabla(g_\varepsilon\circ F)|_{T\Upsilon} d\pi \le \varepsilon^{-1}\int_{\{t<F\le t+\varepsilon\}} G|\nabla F|_{T\Upsilon} d\pi \le \frac{m(t+\varepsilon)-m(t)}{\varepsilon} \tag{7.10}$$

while (7.5) in Proposition 7.5 implies

$$\int_{\Upsilon(\mathbb{R}^n)} G\, d\|\{F>t\}\| \le \liminf_{\varepsilon\to 0}\int_{\Upsilon(\mathbb{R}^n)} G|\nabla(g_\varepsilon\circ F)|_{T\Upsilon} d\pi = m'(t)\,. \tag{7.11}$$

Since m is differentiable for a.e. $t\in\mathbb{R}$, the one inequality comes by integrating (7.11).

Let us prove the converse inequality. Let $V\in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|V|_{T\Upsilon}\le 1$. Then, by Theorem 6.14, we deduce

$$\begin{aligned}\int_{\Upsilon(\mathbb{R}^n)} F(G\nabla^* V) d\pi &= \int_{-\infty}^{\infty}\int_{\{F>t\}} (G\nabla^* V) d\pi\, dt\\ &\le \int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} G d\|\{F>t\}\| dt\,,\end{aligned}$$

which easily yields the sought conclusion. □

Proof of Theorem 7.1 Let $F\in L^2(\Upsilon(\mathbb{R}^n),\pi)$ such that $|\mathsf{D}F|(\Upsilon(\mathbb{R}^n))<\infty$ and $G\in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$ be non-negative. By definition there exists a sequence $(F_n)\subset \mathrm{CylF}(\Upsilon)$ such that $F_n\to F$ in $L^1(\Upsilon(\mathbb{R}^n),\pi)$ and $\int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_n|_{T\Upsilon} d\pi \to |\mathsf{D}_* F|[G]$. From Proposition 7.6 we get

$$\int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} G d\|\{F_n>t\}\| dt = \int_{\Upsilon(\mathbb{R}^n)} G|\nabla F_n|_{T\Upsilon} d\pi\,, \tag{7.12}$$

and passing to the limit for $n\to\infty$ we deduce

$$\int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} G d\|\{F>t\}\| dt \le |\mathsf{D}_* F|[G]\,, \tag{7.13}$$

as a consequence of (ii) in Proposition 7.5 and Fatou's Lemma. In particular $\{F>t\}$ is of finite perimeter for a.e.-$t\in\mathbb{R}$.

Let us now fix $\varepsilon>0$ and consider $V\in\mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ such that $|V|_{T\Upsilon}\le 1$ and $\mathcal{V}(F)-\varepsilon\le \int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* V)d\pi$. By Theorem 6.14, we have

$$\begin{aligned}|\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n))-\varepsilon = \mathcal{V}(F)-\varepsilon \le \int_{\Upsilon(\mathbb{R}^n)} F(\nabla^* V) d\pi &= \int_{-\infty}^{\infty}\int_{\{F>t\}}(\nabla^* V) d\pi\, dt\\ &\le \int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} d\|\{F>t\}\| dt\,,\end{aligned}$$

which easily yields

$$\int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} d\|\{F>t\}\| dt \ge |\mathsf{D}_* F|(\Upsilon(\mathbb{R}^n)) = |\mathsf{D}_* F|[1]\,.$$

The sought conclusion follows now by recalling that $|\mathsf{D}_* F|[G_1+G_2]\ge |\mathsf{D}_* F|[G_1]+|\mathsf{D}_* F|[G_2]$ and by the same argument in the paragraph after (6.30). Indeed,

$$\begin{aligned}|\mathsf{D}_* F|[G]+|\mathsf{D}_* F|[1-G]\le |\mathsf{D}_* F|[1] &\le \int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} d\|\{F>t\}\| dt\\ &= \int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} G d\|\{F>t\}\| dt + \int_{-\infty}^{\infty}\int_{\Upsilon(\mathbb{R}^n)} (1-G) d\|\{F>t\}\| dt\end{aligned}$$

$$\leq |\mathsf{D}_* F|[G] + |\mathsf{D}_* F|[1-G]\,,$$

for any $0 \leq G \leq 1$, $G \in \mathrm{CylF}(\Upsilon(\mathbb{R}^n))$. □

7.3 Gauß–Green formula

We prove the Gauß–Green formula. For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, let $L^2(T\Upsilon, |\mathsf{D}F|)$ denote the completion of $\mathrm{CylV}(\Upsilon)$ with respect to $\|\cdot\|_{L^2(T\Upsilon,|\mathsf{D}F|)}$ analogously in (2.10).

Theorem 7.7 *(Gauß–Green formula) For $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$, there exists a unique element $\sigma_F \in L^2(T\Upsilon, |\mathsf{D}F|)$ such that $|\sigma_F|_{T\Upsilon} = 1$ $|\mathsf{D}F|$-a.e., and*

$$\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F \, d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_F \rangle_{T\Upsilon} d|\mathsf{D}F|, \quad \forall V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,. \tag{7.14}$$

Proof We assume without loss of generality that $|\mathsf{D}F|(\Upsilon(\mathbb{R}^n)) = 1$. By Theorems 6.14 and 7.1, it holds that

$$\begin{aligned}
\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F \, d\pi &= \int_{-\infty}^{\infty} \int_{\{F>t\}} (\nabla^* V) d\pi \, dt \\
&= \int_{-\infty}^{\infty} \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_{\{F>t\}} \rangle_{T\Upsilon} d \big\| \{F > t\} \big\| dt \\
&\leq \int_{\Upsilon(\mathbb{R}^n)} |V|_{T\Upsilon} d|\mathsf{D}F| \\
&\leq \|V\|_{L^2(T\Upsilon, |\mathsf{D}F|)},
\end{aligned}$$

for every $V \in \mathrm{CylV}(\Upsilon)$. In particular, the map L defined by

$$L : L^2(T\Upsilon, |\mathsf{D}F|) \to \mathbb{R}\,, \quad L^2(T\Upsilon, |\mathsf{D}F|) \ni V \mapsto L(V) := \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F \, d\pi\,, \tag{7.15}$$

is a well-defined continuous operator on the Hilbert space $L^2(T\Upsilon, |\mathsf{D}F|)$ and satisfies $\|L\| \leq 1$. Therefore, the Riesz representation theorem on the Hilbert space $L^2(T\Upsilon, |\mathsf{D}F|)$ gives the existence of $\sigma_F \in L^2(T\Upsilon, |\mathsf{D}F|)$ so that

$$\|\sigma_F\|_{L^2(T\Upsilon,|\mathsf{D}F|)} \leq 1\,, \qquad \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F \, d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle V, \sigma_F \rangle d|\mathsf{D}F| \quad V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\,.$$

From Theorems 5.18 and 7.1, we deduce

$$\begin{aligned}
1 = |\mathsf{D}F|(\Upsilon(\mathbb{R}^n)) = |\mathsf{D}_* F|[1] = \mathcal{V}(F) = \sup_{V \in \mathrm{CylV}\,, |V|_{T\Upsilon} \leq 1} \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* V) F \, d\pi \\
\leq \int_{\Upsilon(\mathbb{R}^n)} |\sigma_F|_{T\Upsilon} d|\mathsf{D}F| \leq \|\sigma_F\|_{L^2(T\Upsilon,|\mathsf{D}F|)} \leq 1\,,
\end{aligned}$$

which yields $\|\sigma_F\|_{L^1(T\Upsilon,|\mathsf{D}F|)} = \|\sigma_F\|_{L^2(T\Upsilon,|\mathsf{D}F|)} = 1$, and therefore $|\sigma_F|_{T\Upsilon} = 1$ $|\mathsf{D}F|$-a.e. as a consequence of the characterisation of the equality in Jensen's inequality. □

7.4 BV and Sobolev functions

In this subsection, we discuss the consistency of the just developed theory of BV functions with the $(1, 2)$-Sobolev space $H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$.

Proposition 7.8 *Let $F \in L^2(\Upsilon(\mathbb{R}^n), \pi) \cap \mathrm{BV}(\Upsilon(\mathbb{R}^n))$. Suppose $|\mathrm{D}F| \ll \pi$ with $|\mathrm{D}F| = H \cdot \pi$ and $H \in L^2(\Upsilon(\mathbb{R}^n), \pi)$. Then $F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$ and*

$$H = |\nabla F|, \qquad \sigma_F = \frac{\nabla F}{|\nabla F|} \cdot \chi_{\{|\nabla F| \neq 0\}},$$

where σ_F is the unique element in $L^2(T\Upsilon, |\mathrm{D}F|)$ in the Gauß–Green formula (7.14).

Proof By Theorem 7.7 and recalling $\mathbf{T}_t V \in \mathcal{D}(\mathcal{E}_H) \subset \mathbf{D}^2(T\Upsilon(\mathbb{R}^n), \pi)$ for $V \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$ by (5.39), the approximation of $\mathbf{T}_t V$ by $\mathrm{CylV}(\Upsilon(\mathbb{R}^n), \pi)$ implies that

$$\int_{\Upsilon(\mathbb{R}^n)} (\nabla^* G) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} \langle G, \sigma_F \rangle_{T\Upsilon} F d\pi \quad \forall G \in \mathbf{T}_t \mathrm{CylV}(\Upsilon(\mathbb{R}^n)) \quad \forall t > 0, \tag{7.16}$$

where $\mathbf{T}_t \mathrm{CylV}(\Upsilon(\mathbb{R}^n)) := \{G = \mathbf{T}_t F : F \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))\}$ for $t > 0$. By Lemma 5.20 and the π-symmetry of T_t, for any $U \in \mathrm{CylV}(\Upsilon(\mathbb{R}^n))$, setting $G = \mathbf{T}_t U$, we obtain

$$\begin{aligned}
\int_{\Upsilon(\mathbb{R}^n)} \langle U, \nabla T_t F \rangle d\pi &= \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* U) T_t F d\pi \\
&= \int_{\Upsilon(\mathbb{R}^n)} T_t(\nabla^* U) F d\pi = \int_{\Upsilon(\mathbb{R}^n)} (\nabla^* G) F d\pi \\
&= \int_{\Upsilon(\mathbb{R}^n)} \langle G, \sigma_F \rangle_{T\Upsilon} d|\mathrm{D}F| = \int_{\Upsilon(\mathbb{R}^n)} \langle G, \sigma_F \rangle_{T\Upsilon} H d\pi \\
&= \int_{\Upsilon(\mathbb{R}^n)} \langle U, \mathbf{T}_t(H\sigma_F) \rangle_{T\Upsilon} d\pi .
\end{aligned}$$

Thus, $\mathbf{T}_t(H\sigma_F) = \nabla T_t F$. Letting $t \to 0$, $\mathbf{T}_t(H\sigma_F)$ converges to $H\sigma_F$ in $L^2(T\Upsilon, \pi)$, which implies that $\nabla T_t F$ converges to $H\sigma_F$ in $L^2(T\Upsilon(\mathbb{R}^n), \pi)$. Since $T_t F \to F$ in $L^2(\Upsilon(\mathbb{R}^n), \pi)$, we conclude that $F \in H^{1,2}(\Upsilon(\mathbb{R}^n), \pi)$, and $\nabla F = H\sigma_F$. Therefore, $H \cdot \pi = |\mathrm{D}F| = |\nabla F| \cdot \pi$, and

$$\sigma_F = \frac{\nabla F}{H} \chi_{\{H \neq 0\}} = \frac{\nabla F}{|\nabla F|} \chi_{\{|\nabla F| \neq 0\}} . \square$$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Albeverio, S., Daletskii, A., Lytvynov, E.: Laplace operators on differential forms over configuration spaces. J. Geom. Phys. **37**, 15–46 (2001)
2. Albeverio, S., Kondratiev, Y.-G., Röckner, M.: Analysis and geometry on configuration spaces. J. Funct. Anal. **154**(2), 444–500 (1998)
3. Alonso-Ruiz, P., Baudoin, F., Chen, L., Roger, L.G., Shanmugalingam, N., Teplyaev, A.: Besov class via heat semigroup on Dirichlet spaces I: Sobolev type inequalities. J. Funct. Anal. **278**(11), 108459,48 (2020)
4. Alonso-Ruiz, P., Baudoin, F., Chen, L., Roger, L.G., Shanmugalingam, N., Teplyaev, A.: Besov class via heat semigroup on Dirichlet spaces II: BV functions and Gaussian heat kernel estimates. Calc. Var. Partial Differ. Equ. **59**(3), 10332 (2020)
5. Alonso-Ruiz, P., Baudoin, F., Chen, L., Roger, L.G., Shanmugalingam, N., Teplyaev, A.: Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates. Calc. Var. Partial Differ. Equ. **60**(5), 170,38 (2020)
6. Ambrosio, L.: Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. Adv. Math. **159**, 51–67 (2001)
7. Ambrosio, L.: Fine properties of sets of finite perimeter in doubling metric measure spaces. Set-valued Anal. **10**, 111–128 (2002)
8. Ambrosio, L.: Calculus, heat flow and curvature-dimension bounds in metric measure spaces. In: Proceedings of the ICM 2018 (2018)
9. Ambrosio, L., Brué, E., Semola, D.: Rigidity of the 1-Bakry–Émery inequality and sets of finite perimeter in RCD spaces. Geom. Funct. Anal. **29**, 949–1001 (2019)
10. Ambrosio, L., Di Marino, S.: Equivalent definitions of BV space and of total variation on metric measure spaces. J. Funct. Anal. **266**, 4150–4188 (2014)
11. Ambrosio, L., Figalli, A.: Surface measures and convergence of the Ornstein–Uhlenbeck semigroup in Wiener spaces. Ann. Fac. Sci. Toulouse Math. **20**, 407–438 (2011)
12. Ambrosio, L., Figalli, A., Runa, E.: On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to half spaces. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **24**, 111–122 (2013)
13. Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. **195**, 289–391 (2014)
14. Ambrosio, L., Ghezzi, R., Magnani, V.: BV functions and sets of finite perimeter in sub-Riemannian manifolds. Ann. Inst. H. Poincaré Anal. Nonlinéaire **32**, 489–517 (2015)
15. Ambrosio, L., Maniglia, S., Miranda, M., Jr., Pallara, D.: BV functions in abstract Wiener spaces. J. Funct. Anal. **258**, 785–13 (2010)
16. Ambrosio, L., Miranda, M., Jr., Pallara, D.: Some fine properties of BV functions on Wiener spaces. Anal. Geom. Metr. Spaces **3**, 212–230 (2015)
17. Ambrosio, L., Miranda, M., Jr., Pallara, D.: Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability. Discrete Contin. Dyn. Syst. **28**, 591–606 (2010)
18. Bouleau, N., Hirsch, F.: Dirichlet Forms and Analysis on Wiener Space. De Gruyter, Berlin (1991)
19. Brué, E., Pasqualetto, E., Semola, D.: Rectifiability of the reduced boundary for sets of finite perimeter over RCD(K, N) spaces. Preprint arXiv:1909.00381
20. Burago, D., Burago, Yu., Ivanov, S.: A Course in Metric Geometry, Graduate Studies in Mathematics, vol. 33, AMS (2001)
21. De Giorgi, E.: Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. Ann. Mat. Pura Appl. **36**, 191–213 (1954)
22. De Giorgi, E.: Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. Ricerche Mat. **4**, 95–113 (1955)
23. Dello Schiavo, L., Suzuki, K.: Configuration Spaces Over Singular Spaces I: Dirichlet Form and Metric Measure Geometry. Preprint arXiv:2109.03192
24. Dello Schiavo, L., Suzuki, K.: Configuration Spaces Over Singular Spaces II. Curvature. Preprint arXiv:2205.01379
25. Eberle, A.: Uniqueness and Non-uniqueness of Semigroups Generated by Singular Diffusion Operators, Lecture Notes in Mathematics, vol. 1718, Springer
26. Erbar, M., Huesmann, M.: Curvature bounds for configuration spaces. Calc. Var. **54**, 307–430 (2015)
27. Federer, H.: Geometric measure theory. Grundleherender Mathematics Wissenschaften, vol. 153, Springer (1969)
28. Feyel, D., de la Pradelle, A.: Hausdorff measure on the Wiener space. Potential Anal. **1**, 177–189 (1992)
29. Fremlin, D.H.: In: Fremlin T. (ed.), Measure Theory, vol. IV (2003)

30. Fukushima, M.: On semimartingale characterisations of functionals of symmetric Markov processes. Electron. J. Probab. **4**(18), 1–32 (1999)
31. Fukushima, M.: BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space. J. Funct. Anal. **174**, 227–249 (2000)
32. Fukushima, M., Hino, M.: On the space of BV functions and a related stochastic calculus in infinite dimensions. J. Funct. Anal. **183**, 245–268 (2001)
33. Gelfand, I.M., Vilenkin, N.Y.: Some Applications of Harmonic Analysis. Academic Press, New York (1964)
34. Hino, M.: Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space. J. Funct. Anal. **258**(5), 1656–1681 (2010)
35. Ma, Z.-M., Röckner, M.: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Graduate Studies in Mathematics. Springer (1992)
36. Miranda, M., Jr.: Functions of bounded variation on “good” metric spaces. J. Math. Pures Appl. **82**, 975–1004 (2003)
37. Ouhabaaz, E.-M.: Invariance of closed convex sets and domination criteria for semigroups. Potential Anal. **5**, 611–625 (1996)
38. Reed, M., Simon, B.: Methods of Modern Mathematical Physics: Functional Analysis, vol. 1. Academic Press, San Diego (1977)
39. Röckner, M., Zhu, R.-C., Zhu, X.-C.: BV functions in a Gelfand triple and the stochastic reflection problem on a convex set of a Hilbert space. C. R. Math. **348**(21–22), 1175–1178 (2010)
40. Röckner, M., Zhu, R.-C., Zhu, X.-C.: The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple. Ann. Probab. **40**(4), 1759–1794 (2012)
41. Röckner, M., Zhu, R.-C., Zhu, X.-C.: BV functions in a Gelfand triple for differentiable measure and its applications. Forum Math. **27**, 1657–1687 (2015)
42. Shigekawa, I.: Defective intertwining property and generator domain. J. Funct. Anal. **239**, 357–374 (2006)
43. Shigekawa, I.: L^p contraction semigroups for vector valued functions. J. Funct. Anal. **147**, 69–108 (1997)
44. Suslin, M.: ‘Sur une définition des ensembles mesurables B sans nombres infinis’. C. R. Acad. Sci. (Paris) **164**, 88–91 (1917)
45. Suzuki, K.: On the ergodicity of interacting particle systems under number rigidity. Probab. Theory Relat. Fields **188**, 583–623 (2024)
46. Suzuki, K.: Curvature bound of Dyson Brownian motion. In: Communications in Mathematical Physics. Preprint arXiv:2301.00262 (2025)
47. Ziemer, W.: Weakly Differentiable Functions. Springer, Berlin (1989)