

FENCHEL-NIELSEN COORDINATES FOR $\mathrm{SL}(3, \mathbb{C})$ REPRESENTATIONS

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ABSTRACT. We define Fenchel-Nielsen coordinates for representations of surface groups to $\mathrm{SL}(3, \mathbb{C})$. We also show how these coordinates relate to the classical Fenchel-Nielsen coordinates and to their generalisations by Kourouniotis, Tan, Goldman, Zhang and Parker-Platis.

1. INTRODUCTION

The Teichmüller space of a closed, orientable Riemann surface S_g of genus $g \geq 2$ is the space of marked hyperbolic structures on S_g up to isotopy. Fenchel and Nielsen constructed global coordinates on this space. The coordinates depend on a choice \mathcal{L} of $3g - 3$ distinct, non-trivial homotopy classes of simple closed curves on the surface with disjoint representatives. The coordinates are the set of hyperbolic lengths of geodesics in each homotopy class in \mathcal{L} , together with twists around these geodesics. An alternative description of the Teichmüller space of S_g is the space of irreducible, discrete, totally loxodromic representations of $\pi_1(S_g)$ to $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}_+(S_g)$ up to conjugation. In this context, Fenchel-Nielsen length coordinates are equivalent to the $3g - 3$ traces of the elements of $\mathrm{PSL}(2, \mathbb{R})$ representing the curves in \mathcal{L} and the twist coordinates are traces (or eigenvalues) of elements in their centralisers.

The second definition of Teichmüller space can be generalised to representations of $\pi_1(S_g)$ to other groups G and so Fenchel-Nielsen coordinates can be generalised as well. Particular cases where Fenchel-Nielsen coordinates have been constructed are when G is one of $\mathrm{SL}(2, \mathbb{C})$, Kourouniotis [?], Tan [?]; $\mathrm{PSL}(3, \mathbb{R})$, Goldman [?] or $\mathrm{SU}(2, 1)$, Parker and Platis [?]. In the case of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SU}(2, 1)$ there is an important change from the case of $\mathrm{SL}(2, \mathbb{R})$, namely the space of totally loxodromic representations is a proper subset of the corresponding component of the representation variety. Indeed, in the $\mathrm{SL}(2, \mathbb{C})$ case, the space of discrete, faithful, totally loxodromic representations has fractal boundary, and this boundary can intersect itself in complicated ways [?]. Therefore the allowable values of these coordinates describe a larger space containing the space of discrete, faithful, totally loxodromic representations as a proper subspace.

In this paper we construct Fenchel-Nielsen coordinates to the case where $G = \mathrm{SL}(3, \mathbb{C})$. Both $\mathrm{SL}(3, \mathbb{R})$ and $\mathrm{SU}(2, 1)$ are subgroups of $\mathrm{SL}(3, \mathbb{C})$ and, after taking the irreducible representation, we can embed $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ as subgroups of $\mathrm{SL}(3, \mathbb{C})$ as well. We show how our coordinates are related to those constructed in [?, ?, ?, ?]. Our motivation for considering $\mathrm{SL}(3, \mathbb{C})$ representations of surface groups is the study of complex Kleinian groups, which are discrete subgroups of $\mathrm{SL}(3, \mathbb{C})$; see the book [?] by

Cano, Navarrete, Seade. Much of the focus of [?] and related papers has been the case of elementary or Fuchsian groups. We hope that defining Fenchel-Nielsen coordinates for $\mathrm{SL}(3, \mathbb{C})$ representations of surface groups will facilitate the study of their action on \mathbb{CP}^2 , as studied in [?].

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2. STATEMENT OF MAIN RESULTS

The Teichmüller space of a surface S_g is defined as follows.

Definition 2.1. *Let S_g be a closed, compact surface of genus $g \geq 2$. The Teichmüller space of S_g is the quotient $\mathcal{T}(S_g) = \{(X, f)\} / \sim$, where*

- X is S_g together with a hyperbolic structure.
- $f : S_g \rightarrow X$ is a homeomorphism called a marking.
- $(X, f) \sim (Y, g)$ if and only if there exists an isometry $\phi : X \rightarrow Y$ such that $\phi \circ f$ is isotopic to g .

Our results depend on a choice of a curve system on S_g .

Definition 2.2. *Let S_g be a closed, oriented surface of genus $g \geq 2$. A curve system on S_g denoted $\mathcal{L} = \{[\gamma_1], \dots, [\gamma_{3g-3}]\}$ is a maximal set of distinct, non-trivial homotopy classes of simple closed curves in S_g with simple, disjoint representatives γ_j . The set $S_g \setminus \bigcup_{j=1}^{3g-3} \gamma_j$ is a decomposition of the surface into $2g - 2$ pairs of pants (three holed spheres) Y_1, \dots, Y_{2g-2} . Each curve γ_j is the boundary of precisely two pairs of pants (including the case when a curve corresponds two different boundary curves of the same pair of pants).*

When we consider $X = f(S_g)$, that is S_g together with a hyperbolic structure, we always assume that f sends γ_j to the geodesic on X in the homotopy class $[\gamma_j]$. By a mild abuse of notation, we call this image γ_j as well.

We consider representations to $\mathrm{SL}(3, \mathbb{C})$ where $A_j = \rho([\gamma_j])$ is strongly loxodromic, that is the eigenvalues of A_j have pairwise different absolute values, see Theorem ?? below. We are now in a position to give our main definition, which should be thought of as an extension to $\mathrm{SL}(3, \mathbb{C})$ of the classical definition of quasi-Fuchsian representations of surface groups.

Definition 2.3. *Let S_g be a closed, oriented surface of genus $g \geq 2$ and let \mathcal{L} be a curve system on S_g . Let $[\gamma_j]$ for $j = 1, \dots, 3g - 3$ and Y_k for $k = 1, \dots, 2g - 2$ be as above. Let $\mathcal{Y}(S_g, \mathcal{L})$ be the space of conjugacy classes of representations $\rho : \pi_1(S_g) \rightarrow \mathrm{SL}(3, \mathbb{C})$ so that*

- (1) *for each $j = 1, \dots, 3g - 3$ the curve γ_j is represented by a strongly loxodromic map $A_j = \rho([\gamma_j])$; and*
- (2) *for each $k = 1, \dots, 2g - 2$ the restriction of ρ to $\pi_1(Y_k)$ is irreducible.*

Our first result gives coordinates on the representations in $\mathcal{Y}(S_g, \mathcal{L})$. The relevant definitions may be found later in the paper, as follows: shape invariants in Definition ??; commutator equations (??) from Theorem ??; twist-bend and bulge-turn parameters in Definition ??.

Theorem 2.4. *Let S_g and \mathcal{L} be as above. Let $\rho \in \mathcal{Y}(S_g, \mathcal{L})$ and write $\Gamma = \rho(\pi_1(S_g))$. For $j = 1, \dots, 3g - 3$ write $A_j = \rho([\gamma_j])$, write $(t + i\theta)_\Gamma(\gamma_j)$ for the complex twist-bend along γ_j and write $(s + i\phi)_\Gamma(\gamma_j)$ for the complex bulge-turn along γ_j . For $k = 1, \dots, 2g - 2$ write $\sigma_+(Y_k)$ and $\sigma_-(Y_k)$ for the shape invariants of Y_k . Then Γ is determined up to conjugation by*

- (1) *the traces $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$ and $\mathrm{tr}(A_1^{-1}), \dots, \mathrm{tr}(A_{3g-3}^{-1})$;*
- (2) *the shape invariants $\sigma_+(Y_1), \dots, \sigma_+(Y_{2g-2})$ and $\sigma_-(Y_1), \dots, \sigma_-(Y_{2g-2})$;*
- (3) *the choice of a root of the commutator equations $Q(Y_1), \dots, Q(Y_{2g-2})$;*
- (4) *the twist-bend parameters $(t + i\theta)_\Gamma(\gamma_1), \dots, (t + i\theta)_\Gamma(\gamma_{3g-3})$ and the bulge-turn parameters $(s + i\phi)_\Gamma(\gamma_1), \dots, (s + i\phi)_\Gamma(\gamma_{3g-3})$.*

Our main interest is in the space of conjugacy classes of discrete, faithful, totally strongly loxodromic representations of $\pi_1(S_g)$ to $\mathrm{SL}(3, \mathbb{C})$, or to Lie subgroups G of $\mathrm{SL}(3, \mathbb{C})$. Clearly, such representations form a proper subset of $\mathcal{Y}(S_g, \mathcal{L})$. There are several reasons for this. The first reason is that, because in that definition, we have only assumed $3g - 3$ conjugacy classes (and hence their powers) are strongly loxodromic. It may be that other elements in the group are represented by non-loxodromic maps. Secondly, even if the group is totally strongly loxodromic, it may be non-discrete. To see this, we observe that in $\mathrm{SL}(2, \mathbb{C})$, and hence in its image in $\mathrm{SL}(3, \mathbb{C})$ under the irreducible representation, there are non-discrete totally (strongly) loxodromic representations of surface groups.

Furthermore, we chose to consider representations to $\mathrm{SL}(3, \mathbb{C})$, or to a Lie subgroup, that may be continuously deformed through discrete, faithful, totally strongly loxodromic representations from the space of Fuchsian representations to $\mathrm{SO}_0(2, 1)$. We call such representations quasi-Fuchsian. It would also be possible to consider other components of the space of discrete, faithful representations of $\pi_1(S_g)$, see for example Goldman, Kapovich, Leeb [?], but in this case, care would be needed when considering \mathbb{C} -Fuchsian pairs of pants.

Definition 2.5. *The Fuchsian space, denoted $\mathcal{F}(S_g)$, for the surface S_g is the subspace of $\mathrm{Hom}(\pi_1(S_g), \mathrm{SO}_0(2, 1)) / \mathrm{SO}_0(2, 1)$ of irreducible, discrete, faithful, totally loxodromic representations $\rho : \pi_1(S_g) \rightarrow \mathrm{SO}_0(2, 1)$ up to conjugation.*

We remark that $\mathrm{SO}_0(2, 1)$ and $\mathrm{PSL}(2, \mathbb{R})$ are isomorphic (for a concrete isomorphism, see Section ?? below), and both are the orientation preserving isometry groups of the hyperbolic plane. Note that $\mathcal{F}(S_g)$ is a copy of Teichmüller space $\mathcal{T}(S_g)$ from Definition ?. We now give an $\mathrm{SO}_0(2, 1)$ version of the theorem of Fenchel and Nielsen, see Section ?? below.

Theorem 2.6 (Fenchel-Nielsen). *Let S_g and \mathcal{L} be as above. Let $\rho \in \mathcal{F}(S_g)$. For $j = 1, \dots, 3g - 3$ write $A_j = \rho([\gamma_j])$ and write $t_\Gamma(\gamma_j)$ for the twist along γ_j . Then Γ is determined up to conjugation by*

- (1) *the traces $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$, where each trace lies in $[3, \infty)$,*
- (2) *the twists $t_\Gamma(\gamma_1), \dots, t_\Gamma(\gamma_{3g-3})$, where each twist lies in \mathbb{R} .*

In order to relate this result to Theorem ?? we observe that for such representations: the traces are real and greater than 3 and satisfy $\mathrm{tr}(A_j) = \mathrm{tr}(A_j^{-1})$; the bend, bulge

and turn parameters $\theta_\Gamma(\gamma_j)$, $s_\Gamma(\gamma_j)$, $\phi_\Gamma(\gamma_j)$ are all zero, the commutator equations have repeated roots and the shape invariants satisfy

$$(2.1) \quad \sigma_+(Y_k) = \sigma_-(Y_k) = \operatorname{tr}(A_{k_1}) + \operatorname{tr}(A_{k_2}) + \operatorname{tr}(A_{k_3}) + 1 \\ + 2\sqrt{(\operatorname{tr}(A_{k_1}) + 1)(\operatorname{tr}(A_{k_2}) + 1)(\operatorname{tr}(A_{k_3}) + 1)},$$

We can now define quasi-Fuchsian space for $\mathrm{SL}(3, \mathbb{C})$ and its Lie subgroups.

Definition 2.7. *Let G be a group satisfying $\mathrm{SO}_0(2, 1) < G < \mathrm{SL}(3, \mathbb{C})$. Let S_g be a closed, oriented surface of genus $g \geq 2$ and let \mathcal{L} be a curve system on S_g . Define the G -quasi-Fuchsian space of S_g with respect to \mathcal{L} , written $\mathcal{Q}(S_g, \mathcal{L}, G)$, to be the collection of $\mathrm{SL}(3, \mathbb{C})$ -conjugacy classes of representations $\rho : \pi_1(S_g) \rightarrow G$ satisfying*

- (1) *each representation in $\mathcal{Q}(S_g, \mathcal{L}, G)$ is discrete, faithful and totally strongly loxodromic.*
- (2) *$\mathcal{Q}(S_g, \mathcal{L}, G)$ contains the Fuchsian space $\mathcal{F}(S_g)$. That is, the $\mathrm{SL}(3, \mathbb{C})$ -conjugacy class of each Fuchsian representation $\rho_0 : \pi_1(S_g) \rightarrow \mathrm{SO}_0(2, 1) < G$ lies in $\mathcal{Q}(S_g, \mathcal{L}, G)$.*
- (3) *$\mathcal{Q}(S_g, \mathcal{L}, G)$ is path connected. That is, each representation $\rho \in \mathcal{Q}(S_g, \mathcal{L}, G)$ may be connected to a Fuchsian representation by a path of representations lying in $\mathcal{Q}(S_g, \mathcal{L}, G)$.*

We remark that in the case of $G = \mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SU}(2, 1)$ the space $\mathcal{Q}(S_g, \mathcal{L}, G)$ is not a component of the representation variety. In these cases, we may move towards the boundary of the space of discrete and faithful representations by pinching a simple closed curve on S_g so that in the limit is represented by a parabolic map and then beyond to representations where it becomes elliptic of infinite order, and so the representation ceases to be discrete.

Moreover, the requirement (3) means that a representation in $\mathcal{Q}(S_g, \mathcal{L}, G)$ should be connected by a path of representations in $\mathcal{Q}(S_g, \mathcal{L}, G)$ to a Fuchsian representation in $\mathcal{F}(S_g)$. This is more restrictive than simply requiring a representation whose image lies in G . In particular, when $G = \mathrm{SL}(3, \mathbb{R})$ then each loxodromic map must have positive eigenvalues. This is because all eigenvalues of loxodromic maps in $\mathrm{SO}_0(2, 1)$ are positive, and when continuously deforming through loxodromic maps we cannot have eigenvalue 0. Similarly, when $G = \mathrm{SU}(2, 1)$ we need to be in the component of the deformation space containing $\mathrm{SO}_0(2, 1)$ representations, and hence the Toledo invariant must be zero.

Since $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{C}))$ is a subset of $\mathcal{Y}(S_g, \mathcal{L})$ we can repeat Theorem ?? word for word. We are interested in $\mathcal{Q}(S_g, \mathcal{L}, G)$ in the cases where G is one of $\mathrm{SO}(3; \mathbb{C})$ (that is the irreducible representation of $\mathrm{PSL}(2, \mathbb{C})$), $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{SU}(2, 1)$. We now give Fenchel-Nielsen coordinates for $\mathcal{Q}(S_g, \mathcal{L}, G)$ in each of these cases. In our observations following Theorem ?? we observed that a representation being Fuchsian placed certain restrictions on the parameters from Theorem ??. Similar restrictions apply for each of the other groups G . These restrictions cut out subspaces of $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{C}))$ which are all linear, apart from the condition (??) for the shape invariant when $G = \mathrm{SO}_0(2, 1)$ or $\mathrm{SO}(3, \mathbb{C})$.

Our next result concerns the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ to $\mathrm{SL}(3, \mathbb{C})$. The image of this representation is $\mathrm{SO}(3; \mathbb{C})$, see below. The next three results are re-interpretations of the theorems of Kourouniotis [?], Tan [?], of Goldman [?] and of Parker-Platis [?] respectively.

Theorem 2.8. *Let S_g and \mathcal{L} be as above. Let $\rho \in \mathcal{Q}(S_g, \mathcal{L}, \mathrm{SO}(3; \mathbb{C}))$ and write $\Gamma = \rho(\pi_1(S_g))$. For $j = 1, \dots, 3g - 3$ write $A_j = \rho([\gamma_j])$, write $(t + i\theta)_\Gamma(\gamma_j)$ for the complex twist bend along γ_j . For $k = 1, \dots, 2g - 2$ write $\sigma_+(Y_k)$ and $\sigma_-(Y_k)$ for the shape invariants of Y_k . Then Γ is determined up to conjugation by*

- (1) *the traces $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$;*
- (2) *the twist-bends $(t + i\theta)_\Gamma(\gamma_1), \dots, (t + i\theta)_\Gamma(\gamma_{3g-3})$.*

We remark that $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SO}(3; \mathbb{C}))$ is contained in the subset of $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{C}))$ where

- (1) $\mathrm{tr}(A_j) = \mathrm{tr}(A_j^{-1}) \in \mathbb{C}$ for $j = 1, \dots, 3g - 3$;
- (2) if $\gamma_{k_1}, \gamma_{k_2}, \gamma_{k_3}$ are the boundary curves of Y_k then the shape invariants $\sigma_+(Y_k) = \sigma_-(Y_k)$ and satisfy equation (??).
- (3) the commutator equations Q_k all have repeated roots;
- (4) the bulge-turn parameters $s_\Gamma(\gamma_j), \phi_\Gamma(\gamma_j)$ are all zero.

Theorem 2.9. *Let S_g and \mathcal{L} be as above. Let $\rho \in \mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{R}))$ and write $\Gamma = \rho(\pi_1(S_g))$. For $j = 1, \dots, 3g - 3$ write $A_j = \rho([\gamma_j])$, write $t_\Gamma(\gamma_j)$ for the twist bend γ_j and write $s_\Gamma(\gamma_j)$ for the bulge along γ_j . For $k = 1, \dots, 2g - 2$ write $\sigma_+(Y_k)$ and $\sigma_-(Y_k)$ for the shape invariants of Y_k . Then Γ is determined up to conjugation by*

- (1) *the traces $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$ and $\mathrm{tr}(A_1^{-1}), \dots, \mathrm{tr}(A_{3g-3}^{-1})$;*
- (2) *the shape invariants $\sigma_+(Y_1), \dots, \sigma_+(Y_{2g-2})$ and $\sigma_-(Y_1), \dots, \sigma_-(Y_{2g-2})$;*
- (3) *the choice of a root of the commutator equations $Q(Y_1), \dots, Q(Y_{2g-2})$;*
- (4) *the twists $t_\Gamma(\gamma_1), \dots, t_\Gamma(\gamma_{3g-3})$ and the bulges $s_\Gamma(\gamma_1), \dots, s_\Gamma(\gamma_{3g-3})$.*

We remark that $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{R}))$ is (contained in) the subset of $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{C}))$ where

- (1) $\mathrm{tr}(A_j)$ and $\mathrm{tr}(A_j^{-1}) \in \mathbb{R}_+$ for $j = 1, \dots, 3g - 3$;
- (2) the shape invariants $\sigma_+(Y_k)$ and $\sigma_-(Y_k)$ are real;
- (3) the bend and turn parameters $\theta_\Gamma(\gamma_j), \phi_\Gamma(\gamma_j)$ are all zero.

Theorem 2.10. *Let S_g and \mathcal{L} be as above. Let $\rho \in \mathcal{Q}(S_g, \mathcal{L}, \mathrm{SU}(2, 1))$ and write $\Gamma = \rho(\pi_1(S_g))$. For $j = 1, \dots, 3g - 3$ write $A_j = \rho([\gamma_j])$, write $t_\Gamma(\gamma_j)$ for the twist along γ_j and $\phi_\Gamma(\gamma_j)$ for the turn along γ_j . For $k = 1, \dots, 2g - 2$ write $\sigma_+(Y_k)$ and $\sigma_-(Y_k)$ for the shape invariants of Y_k . Then Γ is determined up to conjugation by*

- (1) *the traces $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$;*
- (2) *the shape invariants $\sigma_+(Y_1), \dots, \sigma_+(Y_{2g-2})$;*
- (3) *the choice of a root of the commutator equations $Q(Y_1), \dots, Q(Y_{2g-2})$;*
- (4) *the twists $t_\Gamma(\gamma_1), \dots, t_\Gamma(\gamma_{3g-3})$ and turns $\phi_\Gamma(\gamma_1), \dots, \phi_\Gamma(\gamma_{3g-3})$.*

We remark that $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SU}(2, 1))$ is contained in the subset of $\mathcal{Q}(S_g, \mathcal{L}, \mathrm{SL}(3, \mathbb{C}))$ where

- (1) $\mathrm{tr}(A_j^{-1}) = \overline{\mathrm{tr}(A_j)}$ for $j = 1, \dots, 3g - 3$;
- (2) $\sigma_-(Y_k) = \overline{\sigma_+(Y_k)}$ for $k = 1, \dots, 2g - 2$;
- (3) the bend and bulge parameters $\theta_\Gamma(\gamma_j), s_\Gamma(\gamma_j)$ are all zero.

We summarise the above results in the following table.

G	Parameters	Equations
$\mathrm{SO}_0(2, 1)$	$\mathrm{tr}(A_j)$ $t_\Gamma(\gamma_j)$	$\mathrm{tr}(A_j) = \overline{\mathrm{tr}(A_j)} = \mathrm{tr}(A_j^{-1}) = \overline{\mathrm{tr}(A_j^{-1})}$ $\sigma_+(Y_k) = \sigma_-(Y_k)$ given by (??) $Q(Y_k)$ repeated roots $\theta_\Gamma(\gamma_j) = s_\Gamma(\gamma_j) = \phi_\Gamma(\gamma_j) = 0$
$\mathrm{SO}(3; \mathbb{C})$	$\mathrm{tr}(A_j)$ $t_\Gamma(\gamma_j), \theta_\Gamma(\gamma_j)$	$\mathrm{tr}(A_j) = \mathrm{tr}(A_j^{-1}), \overline{\mathrm{tr}(A_j)} = \mathrm{tr}(A_j^{-1})$ $\sigma_+(Y_k) = \sigma_-(Y_k)$ given by (??) $Q(Y_k)$ repeated roots $s_\Gamma(\gamma_j) = \phi_\Gamma(\gamma_j) = 0$
$\mathrm{SL}(3, \mathbb{R})$	$\mathrm{tr}(A_j), \mathrm{tr}(A_j^{-1})$ $\sigma_+(Y_k), \sigma_-(Y_k)$ root of $Q(Y_k)$ $t_\Gamma(\gamma_j), s_\Gamma(\gamma_j)$	$\mathrm{tr}(A_j) = \overline{\mathrm{tr}(A_j)}, \mathrm{tr}(A_j^{-1}) = \overline{\mathrm{tr}(A_j^{-1})}$ $\sigma_+(Y_k) = \overline{\sigma_+(Y_k)}, \sigma_-(Y_k) = \overline{\sigma_-(Y_k)}$ $\theta_\Gamma(\gamma_j) = \phi_\Gamma(\gamma_j) = 0$
$\mathrm{SU}(2, 1)$	$\mathrm{tr}(A_j)$ $\sigma_+(Y_k)$ root of $Q(Y_k)$ $t_\Gamma(\gamma_j), \phi_\Gamma(\gamma_j)$	$\mathrm{tr}(A_j) = \overline{\mathrm{tr}(A_j^{-1})}, \mathrm{tr}(A_j^{-1}) = \overline{\mathrm{tr}(A_j)}$ $\sigma_+(Y_k) = \overline{\sigma_-(Y_k)}, \sigma_-(Y_k) = \overline{\sigma_+(Y_k)}$ $\theta_\Gamma(\gamma_j) = s_\Gamma(\gamma_j) = 0$
$\mathrm{SL}(3, \mathbb{C})$	$\mathrm{tr}(A_j), \mathrm{tr}(A_j^{-1})$ $\sigma_+(Y_k), \sigma_-(Y_k)$ root of $Q(Y_k)$ $t_\Gamma(\gamma_j), \theta_\Gamma(\gamma_j), s_\Gamma(\gamma_j), \phi_\Gamma(\gamma_j)$	

We note that the conditions above essentially characterise the representations of each pants group $\rho(\pi_1(Y_k))$. To see this, we use the following theorem of Acosta.

Proposition 2.11 (Theorem 1.1 of Acosta [?]). *Let Γ be a finitely generated group and let $\rho : \Gamma \longrightarrow \mathrm{SL}(3, \mathbb{C})$ be an irreducible representation of Γ . Then*

- (1) *If $\mathrm{tr}(A) \in \mathbb{R}$ for all $A \in \rho(\Gamma)$ then $\rho(\Gamma)$ is conjugate to a representation of γ to $\mathrm{SL}(3, \mathbb{R})$.*
- (2) *If $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)}$ for all $A \in \rho(\Gamma)$ then $\rho(\Gamma)$ is conjugate to a representation in $\mathrm{SU}(3)$ or $\mathrm{SU}(2, 1)$. In particular, if $\rho(\Gamma)$ contains loxodromic maps then it is conjugate to a representation in $\mathrm{SU}(2, 1)$.*

Our result is

Theorem 2.12. *Let $\Gamma = \langle A, B, C : CBA = I \rangle$ be an irreducible subgroup of $\mathrm{SL}(3, \mathbb{C})$. Let σ_+ and σ_- be the shape invariants given by (??) and (??).*

- (1) *If $\mathrm{tr}(A) = \mathrm{tr}(A^{-1}), \mathrm{tr}(B) = \mathrm{tr}(B^{-1}), \mathrm{tr}(C) = \mathrm{tr}(C^{-1}), \sigma_+ = \sigma_-$ and $Q(\Gamma)$ has repeated roots, then up to conjugacy $\Gamma < \mathrm{SO}(3; \mathbb{C})$;*
- (2) *If $\mathrm{tr}(A), \mathrm{tr}(A^{-1}), \mathrm{tr}(B), \mathrm{tr}(B^{-1}), \mathrm{tr}(C), \mathrm{tr}(C^{-1}), \sigma_+$ and σ_- are all real then up to conjugacy $\Gamma < \mathrm{SL}(3, \mathbb{R})$;*
- (3) *If $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)}, \mathrm{tr}(B^{-1}) = \overline{\mathrm{tr}(B)}, \mathrm{tr}(C^{-1}) = \overline{\mathrm{tr}(C)}$ and $\sigma_- = \overline{\sigma_+}$ then up to conjugacy $\Gamma < \mathrm{SU}(2, 1)$.*

3. BACKGROUND ON FENCHEL-NIELSEN COORDINATES

3.1. Geometrical Fenchel-Nielsen co-ordinates. In [?] Fenchel and Nielsen construct global coordinates for $\mathcal{T}(S_g)$, giving it the structure of a differentiable manifold. These coordinates depend on a choice of a curve system \mathcal{L} on S_g as given in Definition ???. Fenchel-Nielsen coordinates consist of $3g - 3$ length coordinates and $3g - 3$ twist coordinates. The length coordinates $(\ell_X(\gamma_1), \dots, \ell_X(\gamma_{3g-3}))$ for a surface X in Teichmüller space are the hyperbolic lengths of the geodesics γ_j measured using the hyperbolic structure on X . In order to define the twists, we need to do a little more work. Consider $[\gamma_j] \in \mathcal{L}$. Then γ_j either lies on the boundary of two distinct pairs of pants Y and Y' or it corresponds to two boundary curves of a single pair of pants Y . Let α_j be a homotopically non-trivial simple closed curve in $Y \cup Y'$ (respectively Y) intersecting γ_j minimally, that is in two points (respectively one point). We construct a piecewise geodesic curve in the homotopy class of α_j as follows. It consists of (a) two arcs δ_j and δ'_j (respectively a single arc δ_j) contained in γ_j and (b) two simple geodesic arcs $\beta_j \subset Y$, $\beta'_j \subset Y'$ (respectively a single simple geodesic arc $\beta_j \subset Y$) meeting γ_j orthogonally at their endpoints. Elementary hyperbolic geometry shows that δ_j and δ'_j have the same length. The twist $t_X(\gamma_j)$ is the signed difference between the hyperbolic length of δ_j (measured using the hyperbolic structure on X) and the same length on some fixed reference surface X_0 , and where the sign of $t_X(\gamma_j)$ is determined by a choice of orientation on γ_j . For example we could take X_0 to be the surface where each δ_j has length zero, that is where α_j and γ_j are orthogonal simple closed geodesics, but this choice is not necessary. As a relative invariant, the twist is independent of the choice of α_j .

We define the Fenchel-Nielsen coordinates of $\mathcal{T}(S_g)$ with respect to a given curve system $\mathcal{L} = \{[\gamma_1], \dots, [\gamma_{3g-3}]\}$ to be the map $FN : \mathcal{T}(S_g) \rightarrow \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ given by

$$FN_X = (\ell_X(\gamma_1), \dots, \ell_X(\gamma_{3g-3}), t_X(\gamma_1), \dots, t_X(\gamma_{3g-3})).$$

The theorem of Fenchel and Nielsen says that these are global coordinates in the sense that two marked surfaces with distinct hyperbolic structures give different values of these parameters, and each value of the parameters gives a hyperbolic structure on the surface.

3.2. Algebraic Fenchel-Nielsen coordinates. We now reinterpret Fenchel-Nielsen coordinates in terms of $\mathrm{Hom}(\pi_1(S_g), \mathrm{SL}(2, \mathbb{R})) / \mathrm{SL}(2, \mathbb{R})$, the deformation space of representations of $\pi_1(S_g)$ to $\mathrm{SL}(2, \mathbb{R})$ up to conjugacy. This is equivalent to the Fuchsian space from Definition ??? under the identification of $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{SO}_0(2, 1)$, see Section ???. Let Y be one of the pairs of pants, that is Y is a component of $S_g \setminus \bigcup_{j=1}^{3g-3} \gamma_j$. Then $\pi_1(Y)$ is a free group on two generators. We can take the generators to be the homotopy classes of curves corresponding to two of the boundary curves. Then the third boundary curve corresponds to the product of these two generators. In fact, it is more convenient to regard $\pi_1(Y)$ as having three generators, corresponding to the three boundary components, with a single relation that their product is the identity. That is, if the homotopy classes of ∂Y are $[\alpha]$, $[\beta]$, $[\gamma]$ then

$$\pi_1(Y) = \langle [\alpha], [\beta], [\gamma] : [\gamma][\beta][\alpha] = id \rangle.$$

Consider a representation $\rho : \pi_1(Y) \longrightarrow \mathrm{SL}(2, \mathbb{R})$ and write $A = \rho([\alpha])$, $B = \rho([\beta])$ and $C = \rho([\gamma])$. We then have $CBA = I$. In other words,

$$\rho(\pi_1(Y)) = \Gamma = \langle A, B, C : CBA = I \rangle.$$

A classical theorem of Fricke and Vogt (see Theorem ?? below for a precise statement) says that if ρ is irreducible then $\rho(\pi_1(Y))$ is completely determined up to conjugacy by $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(C)$. Furthermore, in order for A , B and C to represent the boundaries of a pair of pants, they must all be hyperbolic elements, their axes should be disjoint and not separate each other. In this case, a well known result, see Gilman and Maskit [?], says that $\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) < 0$. We therefore normalise the representation be supposing that each of $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(C)$ lies in the interval $(-\infty, -2)$. We then note that $\mathrm{tr}(A) = -2 \cosh(\ell_X(\alpha)/2)$ where, as above, $\ell_X(\alpha)$ is the length with respect to the hyperbolic metric on X of the geodesic α in the homotopy class $[\alpha]$, and similarly for $\mathrm{tr}(B) = -2 \cosh(\ell_X(\beta)/2)$ and $\mathrm{tr}(C) = -2 \cosh(\ell_X(\gamma)/2)$.

We now discuss the algebraic interpretation of how to attach two pairs of pants and how to close a handle, see Parker and Platis [?]. First consider attaching two pairs of pants. Suppose that Y and Y' are two pairs of pants with a hyperbolic structure and geodesic boundary. Write $\Gamma = \rho(\pi_1(Y))$ and $\Gamma' = \rho(\pi_1(Y'))$, with

$$\Gamma = \langle A, B, C : CBA = I \rangle, \quad \Gamma' = \langle A', B', C' : C'B'A' = I \rangle,$$

for the images of their fundamental groups under ρ . We want to glue them along the boundary curves α and α' . In order to do so, α and α' must have the same length and opposite orientation. Algebraically, this says that if $A = \rho([\alpha])$ and $A' = \rho([\alpha'])$ then A' is conjugate to A^{-1} . Without loss of generality, we assume $A' = A^{-1}$. This gives a representation of $\pi_1(Y \cup_\alpha Y')$ as the free product with amalgamation along $\langle A \rangle = \langle A' \rangle$:

$$\begin{aligned} \rho(\pi_1(Y \cup_\alpha Y')) &= \Gamma *_{\langle A \rangle} \Gamma' \\ &= \langle A, B, C : CBA = I \rangle *_{\langle A \rangle} \langle A', B', C' : C'B'A' = I \rangle \\ &= \langle B, C, B', C' : CBC'B' = I \rangle. \end{aligned}$$

To obtain the relation, we combine $AA' = CBA = C'B'A' = I$:

$$(CB)(C'B') = A^{-1}A'^{-1} = (A'A)^{-1} = I.$$

Now consider closing a handle. Suppose Y is a pair of pants with a hyperbolic structure and geodesic boundary. Write $\Gamma = \langle A, B, C : CBA = I \rangle$ for the image of its fundamental group under ρ . We want to glue two boundary components α and β . We write them as $A = \rho([\alpha])$ and $B = \rho([\beta])$. As above, this means α and β have the same length and opposite orientation. Algebraically, this means B is conjugate to A^{-1} . Suppose that the conjugating map is denoted by D , so $B = DA^{-1}D^{-1}$. We can therefore form the HNN extension

$$\begin{aligned} \Gamma *_{\langle D \rangle} &= \langle A, (DA^{-1}D^{-1}), C : C(DA^{-1}D^{-1})A = I \rangle *_{\langle D \rangle} \\ &= \langle A, C, D : C[D, A^{-1}] = I \rangle. \end{aligned}$$

Suppose \mathcal{L} is chosen in such a way S_g is obtained using the following process. First, attach $2g - 2$ pairs of pants to form a $2g$ -holed sphere, so that the $2g$ boundary curves form g pairs where each pair is in the same pair of pants. Secondly, close the g handles

by identifying curves in the same pair of pants. Using induction on the attaching step above, we see that the $2g$ -holed sphere is represented by a group

$$\langle B_1, C_1, \dots, B_g, C_g : C_1 B_1 \cdots B_g C_g = I \rangle.$$

Closing the g handles replaces each pair B_k, C_k with a commutator. Thus we obtain the standard presentation for a surface group. In what follows, it is not necessary to make this choice. The main difference would be that we would close handles by identifying boundary curves in different pairs of pants.

We now discuss how to interpret the Fenchel-Nielsen twist $t_\Gamma(\alpha)$ parameter around α . In the above construction we made a choice when we performed the gluing. The ambiguity in that choice is exactly given by an element K of the centraliser of A . That is, K commutes with A and so must have the same eigenvectors. In the first case, we can conjugate $\langle A', B', C' : C'B'A' = I \rangle$ by K to obtain

$$\Gamma *_{\langle A \rangle} K \Gamma' K^{-1} = \langle B, C, K B' K^{-1}, K C' K^{-1} : C B (K C' K^{-1}) (K B' K^{-1}) = I \rangle.$$

In the second case, we replace the conjugating map D with DK to obtain

$$\Gamma *_{\langle DK \rangle} = \langle A, C, DK : C[DK, A^{-1}]C = I \rangle.$$

Since K commutes with A we still have $A' = K A' K^{-1} = K A^{-1} K^{-1} = A^{-1}$ and $B = D K A^{-1} (D K)^{-1} = D A^{-1} D^{-1}$.

In order to relate K to the twist $t_\Gamma(\alpha)$ we could use the trace of K in the same way that we related $\ell_X(\alpha)$ to $\text{tr}(A)$. However, that does not capture the sign of the twist. Instead, we use an eigenvalue. Since A is hyperbolic (loxodromic) and K is in its centraliser $Z(A)$, they must have the same eigenvectors. The eigenvalues of A are $-e^{\ell_X(\alpha)/2}$ and $-e^{-\ell_X(\alpha)/2}$ and we denote the associated eigenvectors by $\mathbf{v}_+(A)$ and $\mathbf{v}_-(A)$, respectively. We suppose $\text{tr}(K) > 0$ and define the twist by saying the eigenvalue λ_K of K associated to the eigenvector $\mathbf{v}_+(A)$ is $e^{t_\Gamma(\alpha)/2}$. Note that the choice of this eigenvalue is equivalent to a choice of orientation of α . The twist parameter could also be parametrised using traces. For example, in [?] Maskit computes the Fenchel-Nielsen coordinates explicitly using matrices.

In the above definition, we made a choice of Y rather than Y' . We now show $t_\Gamma(\alpha)$ is independent of this choice. Swapping the roles of Y and Y' , means we conjugate $\langle A, B, C : CBA = I \rangle$ by K^{-1} to obtain

$$\langle K^{-1}BK, K^{-1}CK, B', C' : (K^{-1}CK)(K^{-1}BK)C'B' = I \rangle.$$

Thus the twist is given by the eigenvalue $\lambda_{K^{-1}}$ of K^{-1} associated to $\mathbf{v}_+(A') = \mathbf{v}_-(A)$, and this eigenvalue is again $e^{t_\Gamma(\alpha)}$. Thus this definition of $t_\Gamma(\alpha)$ does not depend on a choice of Y or Y' . Similarly, when closing a handle the definition does not depend on the choices we made.

Combining all of the pairs of pants associated to the curve system $\mathcal{L} = \{[\gamma_1], \dots, [\gamma_{3g-3}]\}$ on S_g , we obtain algebraic Fenchel Nielsen coordinates associated to the representation $\Gamma = \rho(\pi_1(S_g))$ as

$$FN_\rho = (\text{tr}(A_1), \dots, \text{tr}(A_{3g-3}), t_\Gamma(\gamma_1), \dots, t_\Gamma(\gamma_{3g-3}))$$

where $A_j = \rho([\gamma_j])$ and $K_j \in Z(A_j)$ has eigenvalue $e^{t_\Gamma(\gamma_j)/2}$ associated to the eigenvector of A_j with eigenvalue of largest absolute value.

Next, we mention some examples of our interest for the develop of this project:

- For $G = \mathrm{SL}(2, \mathbb{C})$ Kourouniotis [?] and Tan [?] (independently) generalise the Fenchel-Nielsen coordinates for quasi-Fuchsian representations.
- For $G = \mathrm{SL}(3, \mathbb{R})$ Goldman in [?] generalises the Fenchel-Nielsen coordinates for the space of convex projective structures.
- For $G = \mathrm{SU}(2, 1)$ Parker and Platis in [?] generalise the Fenchel-Nielsen coordinates for the space of complex hyperbolic quasi-Fuchsian representations.

We remark that the space of convex projective structures studied by Goldman is the Hitchin component of the $\mathrm{SL}(3, \mathbb{R})$ character variety of $\pi_1(S_g)$ [?]. In his PhD thesis [?], Tengren Zhang defined Fenchel-Nielsen coordinates for the Hitchin component of the $\mathrm{SL}(n, \mathbb{R})$ character variety for all $n \geq 2$.

In this paper we generalise Fenchel-Nielsen coordinates for the case when $G = \mathrm{SL}(3, \mathbb{C})$. All of the four cases mentioned above give representations of $\pi_1(\Sigma_g)$ to subgroups of $\mathrm{SL}(3, \mathbb{C})$ our coordinates should be a direct generalisation in each case. Since we lose many of the geometric features, we are going to use the algebraic version using as a main tools traces and eigenvalues of the representations.

Discrete subgroups of $\mathrm{SL}(3, \mathbb{C})$ and their action on \mathbb{CP}^2 have been studied by Cano, Navarrete and Seade [?]. In particular, Cano, Parker and Seade [?] showed that the action on \mathbb{CP}^2 of a discrete subgroup of $\mathrm{SO}_0(2, 1)$ (the image under the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$) has three components in its domain of discontinuity and a connected limit set. It would be interesting to investigate how this varies for a surface group as we vary the representation away from $\mathrm{SO}_0(2, 1)$. Our Fenchel-Nielsen coordinates give a framework for doing so.

4. COMPLEX PROJECTIVE FENCHEL-NIELSEN COORDINATES

In this section we are going to mimic the construction from Section ?? but for the space $\mathrm{Hom}(\pi_1(S_g), \mathrm{SL}(3, \mathbb{C})) / \mathrm{SL}(3, \mathbb{C})$ of representations of $\pi_1(S_g)$ to $\mathrm{SL}(3, \mathbb{C})$ up to conjugation. The classical trichotomy of elements of $\mathrm{SL}(2, \mathbb{C})$, can be generalised to $\mathrm{SL}(3, \mathbb{C})$ as follows, see Theorem 4.3.1 on page 112 of Cano, Navarrete and Seade [?]:

Theorem 4.1. *Every element in $\mathrm{SL}(3, \mathbb{C}) \setminus \{I\}$ is one and only one of the following classes: elliptic (diagonalizable with unitary eigenvalues), parabolic (non-diagonalizable) or loxodromic (diagonalizable with non-unitary eigenvalues).*

- (i) *An elliptic transformation belongs to one and only one of the following classes: regular (it has pairwise different eigenvalues) or conjugate to a complex reflection (two eigenvalues are repeated).*
- (ii) *A parabolic transformation belongs to one and only one of the following classes: unipotent (it has eigenvalues equal to one), or ellipto-parabolic (it is not unipotent).*
- (iii) *A loxodromic element belongs to one and only one of the following four classes: loxo-parabolic (only have two eigenvalues with different modulus), complex homothety, screw (different eigenvalues but two of them have the same modulus) or strongly loxodromic (different eigenvalues with different modulus).*

We are going to be interested in irreducible, faithful and discrete representations of the fundamental group of a surface in $\mathrm{SL}(3, \mathbb{C})$ where all the elements of the representation will be strongly loxodromic maps.

Using a result of Navarrete, Theorem 7.3 of [?], see also Theorem 4.3.3 of Cano-Navarrete-Seade [?] we can use $\mathrm{tr}(A)$ and $\mathrm{tr}(A^{-1})$ to determine whether or not $A \in \mathrm{SL}(3, \mathbb{C})$ is strongly loxodromic.

Proposition 4.2. *Define*

$$F(x, y) = x^2y^2 - 4(x^3 + y^3) + 18xy - 27.$$

The map $A \in \mathrm{SL}(3, \mathbb{C})$ is strongly loxodromic if and only if

- (1) *either $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)}$ and $F(\mathrm{tr}(A), \mathrm{tr}(A^{-1})) > 0$,*
- (2) *or $\mathrm{tr}(A^{-1}) \neq \overline{\mathrm{tr}(A)}$ and $F(\mathrm{tr}(A), \mathrm{tr}(A^{-1})) \neq 0$.*

4.1. Polynomial matrix relations. The goal of this section is to extend the work of Fricke and Vogt, see Theorem A of Goldman [?], to two generator subgroups of $\mathrm{SL}(3, \mathbb{C})$ following the work of Lawton [?], see also Will [?] and Parker [?].

We define two polynomials in eight variables:

$$(4.1) \quad S_0(\mathbf{x}) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8 + x_1x_2x_5x_6$$

$$(4.2) \quad P_0(\mathbf{x}) = x_1^2x_2x_5^2x_6 + x_1x_2^2x_5x_6^2 + x_1^2x_2^2x_3 + x_5^2x_6^2x_7 + x_1^2x_6^2x_8 + x_2^2x_4x_5^2 \\ - x_1^2x_2x_5x_7 - x_1x_3x_5^2x_6 - x_1^2x_4x_5x_6 - x_1x_2x_5^2x_8 \\ - x_2^2x_5x_6x_8 - x_1x_2x_4x_6^2 - x_1x_2^2x_6x_7 - x_2x_3x_5x_6^2 \\ - x_1^3x_2x_6 - x_2x_5^3x_6 - x_1x_2^3x_5 - x_1x_6^3x_5 \\ - x_1x_2x_3x_4x_5 - x_1x_5x_6x_7x_8 - x_1x_2x_3x_6x_8 - x_2x_4x_5x_6x_7 \\ + x_1^2x_2x_8 + x_4x_5^2x_6 + x_1^2x_3x_6 + x_2x_5^2x_7 + x_1^2x_4x_7 + x_3x_5^2x_8 \\ + x_1x_2^2x_4 + x_5x_6^2x_8 + x_2^2x_3x_5 + x_1x_6^2x_7 + x_2^2x_7x_8 + x_3x_4x_6^2 \\ + x_3^2x_4x_5 + x_1x_7^2x_8 + x_3^2x_6x_8 + x_2x_4x_7^2 \\ + x_1x_3x_4^2 + x_5x_7x_8^2 + x_4^2x_6x_7 + x_2x_3x_8^2 \\ - 2x_1x_2x_3^2 - 2x_5x_6x_7^2 - 2x_2x_4^2x_5 - 2x_1x_6x_8^2 \\ + x_1x_2x_5x_6 + x_1x_3x_5x_7 + x_1x_4x_5x_8 \\ + x_2x_3x_6x_7 + x_2x_4x_6x_8 + x_3x_4x_7x_8 \\ + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + x_8^3 \\ - 3x_1x_3x_8 - 3x_4x_5x_7 - 3x_2x_3x_4 - 3x_6x_7x_8 \\ + 3x_1x_4x_6 + 3x_2x_5x_8 + 3x_1x_2x_7 + 3x_3x_5x_6 \\ - 6x_1x_5 - 6x_2x_6 - 6x_3x_7 - 6x_4x_8 + 9.$$

Theorem 4.3 (Lawton [?]). *Let $\mathbf{x} = (x_1, \dots, x_8)$ be any vector in \mathbb{C}^8 . Then:*

- (1) *There exist $A, B \in \mathrm{SL}(3, \mathbb{C})$ so that*

$$(4.3) \quad \begin{aligned} x_1 &= \mathrm{tr}(A), & x_2 &= \mathrm{tr}(B), & x_3 &= \mathrm{tr}(AB), & x_4 &= \mathrm{tr}(A^{-1}B), \\ x_5 &= \mathrm{tr}(A^{-1}), & x_6 &= \mathrm{tr}(B^{-1}), & x_7 &= \mathrm{tr}(B^{-1}A^{-1}), & x_8 &= \mathrm{tr}(B^{-1}A). \end{aligned}$$

(2) If A and B are as in part (1) then

$$\mathrm{tr}[A, B] + \mathrm{tr}[B, A] = S_0(\mathbf{x}), \quad \mathrm{tr}[A, B]\mathrm{tr}[B, A] = P_0(\mathbf{x})$$

where S_0 and P_0 are the polynomials defined by (??) and (??) evaluated at the point \mathbf{x} given by (??).

In particular, $\mathrm{tr}[A, B]$ and $\mathrm{tr}[B, A]$ are the roots of the polynomial

$$(4.4) \quad Q_0(X) = X^2 - S_0(\mathbf{x})X + P_0(\mathbf{x})$$

whose coefficients only depend on the eight traces in (??)

(3) Let $A, B \in \mathrm{SL}(3, \mathbb{C})$ be as in part (1). If the group $\langle A, B \rangle$ is irreducible, then it is determined up to conjugation within $\mathrm{SL}(3, \mathbb{C})$ by

$$\begin{aligned} &\mathrm{tr}(A), \quad \mathrm{tr}(B), \quad \mathrm{tr}(AB), \quad \mathrm{tr}(A^{-1}B), \quad \mathrm{tr}[A, B], \\ &\mathrm{tr}(A^{-1}), \quad \mathrm{tr}(B^{-1}), \quad \mathrm{tr}(B^{-1}A^{-1}), \quad \mathrm{tr}(B^{-1}A). \end{aligned}$$

In other words, if $\langle A, B \rangle$ is irreducible then it is determined by the point $\mathbf{x} \in \mathbb{C}^8$ from (??) together with a choice of root of the quadratic polynomial (??).

Note that part (3) means that if $\langle A', B' \rangle$ is any representation (possibly reducible) so that the eight traces in (??) agree with those of $\langle A, B \rangle$ and that we choose the same root of the quadratic (??) for both groups, then $\langle A', B' \rangle$ is irreducible and conjugate to $\langle A, B \rangle$.

If $\langle A, B \rangle$ is reducible, then A and B share an eigenvector. It is clear that this vector is an eigenvector of the commutator $[A, B]$ with eigenvalue 1; see Lemma ?? below. From this it follows that $\mathrm{tr}[A, B] = \mathrm{tr}[B, A]$ and so $\langle A, B \rangle$ is in the branching locus of the quadratic Q_0 . That is $S_0^2 - 4P_0 = 0$. We will see later that the converse is not true, namely there are irreducible groups in the branching locus, for example when $\langle A, B \rangle$ is in the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ to $\mathrm{SL}(3, \mathbb{C})$.

Given $\langle A, B, C : CBA = I \rangle$, the coordinates of Fricke and Vogt for $\mathrm{SL}(2, \mathbb{R})$ representations of this group are symmetric in cyclic permutation of A , B and C . This is not the case with Lawton's parameters for $\mathrm{SL}(3, \mathbb{C})$ representations of the group. We now show how to symmetrise Lawton's parameters.

First we observe that

$$x_3 = \mathrm{tr}(AB) = \mathrm{tr}(C^{-1}), \quad x_7 = \mathrm{tr}(B^{-1}A^{-1}) = \mathrm{tr}(C).$$

Symmetrising $x_4 = \mathrm{tr}(A^{-1}B)$ and $x_8 = \mathrm{tr}(B^{-1}A)$ is slightly more difficult.

Lemma 4.4. *Let $A \in \mathrm{SL}(3, \mathbb{C})$. Then the characteristic polynomial of A is*

$$(4.5) \quad \chi_A(x) = x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A^{-1})x - 1$$

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A . Then $\lambda_1\lambda_2\lambda_3 = \det(A) = 1$ which is the constant term of the characteristic polynomial. We know that the quadratic term is $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A)$. Using $\lambda_1\lambda_2\lambda_3 = 1$ the linear term is

$$\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2 = \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}.$$

Since $\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}$ are the eigenvalues of A^{-1} , we see that the linear term is $\mathrm{tr}(A^{-1})$, as claimed. \square

The lemma below was proved in [?] for $\mathrm{SU}(2, 1)$, but in fact is valid for $\mathrm{SL}(3, \mathbb{C})$.

Lemma 4.5. *Let $A, B, C \in SL(3, \mathbb{C})$ with $CBA = I$, then*

$$(4.6) \quad \begin{aligned} \operatorname{tr}(A^{-1}B) - \operatorname{tr}(A^{-1})\operatorname{tr}(B) &= \operatorname{tr}(C^{-1}A) - \operatorname{tr}(C^{-1})\operatorname{tr}(A) \\ &= \operatorname{tr}(B^{-1}C) - \operatorname{tr}(B^{-1})\operatorname{tr}(C), \end{aligned}$$

$$(4.7) \quad \begin{aligned} \operatorname{tr}(AB^{-1}) - \operatorname{tr}(B^{-1})\operatorname{tr}(A) &= \operatorname{tr}(CA^{-1}) - \operatorname{tr}(A^{-1})\operatorname{tr}(C) \\ &= \operatorname{tr}(BC^{-1}) - \operatorname{tr}(C^{-1})\operatorname{tr}(B). \end{aligned}$$

Proof. From Lemma ?? and the Cayley-Hamilton theorem we have

$$(4.8) \quad A^3 - \operatorname{tr}(A)A^2 + \operatorname{tr}(A^{-1})A - I = O$$

We multiply equation (??) on the left by BA^{-1} to get

$$BA^2 - \operatorname{tr}(A)BA + \operatorname{tr}(A^{-1})B - BA^{-1} = O.$$

Since $CBA = I$ then $C^{-1} = BA$ and we substitute

$$C^{-1}A - \operatorname{tr}(A)C^{-1} + \operatorname{tr}(A^{-1})B - BA^{-1} = O$$

and taking traces we get

$$\operatorname{tr}(C^{-1}A) - \operatorname{tr}(A)\operatorname{tr}(C^{-1}) + \operatorname{tr}(A^{-1})\operatorname{tr}(B) - \operatorname{tr}(BA^{-1}) = 0.$$

Rearranging gives

$$\operatorname{tr}(C^{-1}A) - \operatorname{tr}(C^{-1})\operatorname{tr}(A) = \operatorname{tr}(A^{-1}B) - \operatorname{tr}(A^{-1})\operatorname{tr}(B).$$

Cyclically permuting A, B and C gives

$$\operatorname{tr}(A^{-1}B) - \operatorname{tr}(A^{-1})\operatorname{tr}(B) = \operatorname{tr}(B^{-1}C) - \operatorname{tr}(B^{-1})\operatorname{tr}(C).$$

This gives (??)

Starting from (??) and multiplying on the right by $A^{-1}C$, a similar argument gives (??). \square

Definition 4.6. *Define the shape invariants of the triple A, B, C , or of the group $\Gamma = \langle A, B, C : CBA = I \rangle$ they generate, as*

$$(4.9) \quad \sigma_+ = \sigma_+(A, B, C) = \sigma_+(\Gamma) := \operatorname{tr}(A^{-1}B) - \operatorname{tr}(A^{-1})\operatorname{tr}(B),$$

$$(4.10) \quad \sigma_- = \sigma_-(A, B, C) = \sigma_-(\Gamma) := \operatorname{tr}(B^{-1}A) - \operatorname{tr}(B^{-1})\operatorname{tr}(A).$$

From Lemma ??, we see that the shape invariants are invariant under cyclic permutation of A, B and C .

We also remark that

$$[A, B] = ABA^{-1}B^{-1} = ABC$$

and

$$[B, A] = BAB^{-1}A^{-1} = C^{-1}B^{-1}A^{-1} = (ABC)^{-1}.$$

It is easy to see that this implies

$$\operatorname{tr}[A, B] = \operatorname{tr}[B, C] = \operatorname{tr}[C, A], \quad \operatorname{tr}[B, A] = \operatorname{tr}[C, B] = \operatorname{tr}[A, C].$$

This implies that equation (??) is invariant under cyclic permutation of A, B, C and so this must be true of the polynomials $S_0(\mathbf{x})$ and $P_0(\mathbf{x})$. Following Proposition 4.10 of [?], the easiest way to see this is to use the variables $\mathbf{y} = (y_1, \dots, y_8)$ where

$$(4.11) \quad \begin{aligned} y_1 &= \text{tr}(A), & y_2 &= \text{tr}(B), & y_3 &= \text{tr}(C), & y_4 &= \sigma_+(A, B, C), \\ y_5 &= \text{tr}(A^{-1}), & y_6 &= \text{tr}(B^{-1}), & y_7 &= \text{tr}(C^{-1}), & y_8 &= \sigma_-(A, B, C). \end{aligned}$$

In particular, $x_3 = y_7$ and

$$x_4 = \text{tr}(A^{-1}B) = \sigma_+(A, B, C) + \text{tr}(A^{-1})\text{tr}(B) = y_4 + y_2y_5.$$

Using this substitution, we define

$$\begin{aligned} S(y_1, \dots, y_8) &= S_0(y_1, y_2, y_7, (y_4 + y_2y_5), y_6, y_7, y_3, (y_8 + y_1y_6)), \\ P(y_1, \dots, y_8) &= P_0(y_1, y_2, y_7, (y_4 + y_2y_5), y_6, y_7, y_3, (y_8 + y_1y_6)). \end{aligned}$$

Specifically, performing the substitution we obtain:

$$(4.12) \quad S(\mathbf{y}) = y_1y_5 + y_2y_6 + y_3y_7 + y_4y_8 - y_1y_2y_3 - y_5y_6y_7 - 3,$$

$$(4.13) \quad \begin{aligned} P(\mathbf{y}) &= y_1y_2y_3y_5y_6y_7 \\ &\quad + y_1^2y_2^2y_7 + y_3y_5^2y_6^2 + y_1^2y_3^2y_6 + y_2y_5^2y_7^2 + y_2^2y_3^2y_5 + y_1y_6^2y_7^2 \\ &\quad + y_1y_2y_5y_6 + y_2y_3y_6y_7 + y_1y_3y_5y_7 \\ &\quad - 2y_1y_2y_7^2 - 2y_3^2y_5y_6 - 2y_1y_3y_6^2 - 2y_2^2y_5y_7 - 2y_2y_3y_5^2 - 2y_1^2y_6y_7 \\ &\quad + y_1^3 + y_2^3 + y_3^3 + y_5^3 + y_6^3 + y_7^3 \\ &\quad + 3y_1y_2y_3 + 3y_5y_6y_7 - 6y_1y_5 - 6y_2y_6 - 6y_3y_7 \\ &\quad + y_1y_2y_4y_5y_7 + y_1y_3y_4y_6y_7 + y_2y_3y_4y_5y_6 \\ &\quad + y_1y_2^2y_4 + y_4y_5^2y_6 + y_1^2y_3y_4 + y_4y_5y_7^2 + y_2y_3^2y_4 + y_4y_6^2y_7 \\ &\quad + y_1y_3y_5y_6y_8 + y_2y_3y_5y_7y_8 + y_1y_2y_6y_7y_8 \\ &\quad + y_5y_6^2y_8 + y_1^2y_2y_8 + y_3y_5^2y_6 + y_1y_3^2y_8 + y_6y_7^2y_8 + y_2^2y_3y_8 \\ &\quad + (y_4^2 - 3y_8)(y_1y_7 + y_2y_5 + y_3y_6) \\ &\quad + (y_8^2 - 3y_4)(y_1y_6 + y_2y_7 + y_3y_5) \\ &\quad + y_4y_8(y_1y_5 + y_2y_6 + y_3y_7 - 6) + y_4^3 + y_8^3 + 9. \end{aligned}$$

It is easy to see that cyclic permutation of A, B and C gives a permutation of (y_1, \dots, y_8) that preserves $S(\mathbf{y})$ and $P(\mathbf{y})$. Therefore, we can rewrite Lawton's theorem as follows, which generalises the theorem of Fricke and Vogt to our case:

Theorem 4.7. *Let $\mathbf{y} = (y_1, \dots, y_8)$ be any vector in \mathbb{C}^8 . Then:*

(1) *There exist $A, B, C \in \text{SL}(3, \mathbb{C})$ with $CBA = I$ so that*

$$(4.14) \quad \begin{aligned} y_1 &= \text{tr}(A), & y_2 &= \text{tr}(B), & y_3 &= \text{tr}(C), & y_4 &= \sigma_+(A, B, C), \\ y_5 &= \text{tr}(A^{-1}), & y_6 &= \text{tr}(B^{-1}), & y_7 &= \text{tr}(C^{-1}), & y_8 &= \sigma_-(A, B, C), \end{aligned}$$

where σ_+ and σ_- are given by (??) and (??).

(2) *If A, B, C are as in part (1) then*

$$\text{tr}[A, B] + \text{tr}[B, A] = S(\mathbf{y}), \quad \text{tr}[A, B]\text{tr}[B, A] = P(\mathbf{y})$$

where S and P are the polynomials defined by (??) and (??) evaluated at the point \mathbf{y} given by (??).

In particular, $\mathrm{tr}[A, B]$ and $\mathrm{tr}[B, A]$ are the roots of the polynomial

$$(4.15) \quad Q(X) = X^2 - S(\mathbf{y})X + P(\mathbf{y})$$

whose coefficients only depend on the traces and shape invariants in (??).

- (3) Let $A, B, C \in \mathrm{SL}(3, \mathbb{C})$ with $CBA = I$ be as in part (1). If the group generated by A, B and C is irreducible, then it is determined up to conjugation within $\mathrm{SL}(3, \mathbb{C})$ by

$$\begin{aligned} &\mathrm{tr}(A), \quad \mathrm{tr}(B), \quad \mathrm{tr}(C), \quad \sigma_+(\Gamma), \quad \mathrm{tr}[A, B], \\ &\mathrm{tr}(A^{-1}), \quad \mathrm{tr}(B^{-1}), \quad \mathrm{tr}(C^{-1}), \quad \sigma_-(\Gamma). \end{aligned}$$

In other words, if $\langle A, B \rangle$ is irreducible then it is determined by the eight traces from (??) together with a choice of root of the quadratic polynomial Q , from (??).

4.2. Twist-bend-bulge-turn parameter. Once again, we use the free product with amalgamation of $\Gamma = \rho(\pi_1(Y))$ and $\Gamma' = \rho(\pi_1(Y'))$ along $A' = A^{-1}$ and we use the HNN extension to glue $\Gamma = \rho(\pi_1(Y))$ along two conjugate peripheral curves A and $B = DA^{-1}D^{-1}$ to obtain

$$\begin{aligned} \Gamma *_{\langle A \rangle} \Gamma' &= \langle B, C, B', C' : CBC'B' = I \rangle, \\ \Gamma *_{\langle D \rangle} &= \langle A, C, D : C[D, A^{-1}] = I \rangle. \end{aligned}$$

As in Section ??, there are further parameters that capture the freedom we have when taking the free product with amalgamation and the HNN extension. Namely, in each case we take $K \in Z(A)$, the centraliser of A . Given such a K we obtain

$$\begin{aligned} \Gamma *_{\langle A \rangle} (K\Gamma'K^{-1}) &= \langle B, C, KB'K^{-1}, KC'K^{-1} : CB(KC'K^{-1})(KB'K^{-1}) = I \rangle, \\ \Gamma *_{\langle DK \rangle} &= \langle A, C, DK : C[DK, A^{-1}] = I \rangle. \end{aligned}$$

We now explain how to parameterise $Z(A)$. Since we assumed that A is strongly loxodromic, it has three distinct eigenvalues and hence has three (complex) one dimensional eigenspaces. Thus, its centraliser $Z(A)$ consists of all elements of $\mathrm{SL}(3, \mathbb{C})$ preserving each of these eigenspaces. This space has two complex dimensions and we parametrise using complex twist-bend and bulge-turn parameters.

It is easiest to define the twist-bend and bulge turn parameters when A is diagonal; see Goldman [?]. Suppose the strongly loxodromic map A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that $|\lambda_1| > |\lambda_2| > |\lambda_3|$. Let $\mathbf{v}_+(A)$, $\mathbf{v}_0(A)$ and $\mathbf{v}_-(A)$ be eigenvectors associated to $\lambda_1, \lambda_2, \lambda_3$ respectively.

Conjugating if necessary, assume that $\mathbf{v}_+(A)$, $\mathbf{v}_0(A)$, $\mathbf{v}_-(A)$ are the standard basis vectors, and so A is a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Clearly the centraliser $Z(A)$ of A is the set of all diagonal matrices in $SL(3, \mathbb{C})$. This is the direct product of the two one-parameters subgroups

$$(4.16) \quad T^u = \begin{pmatrix} e^u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-u} \end{pmatrix}, \quad U^v = \begin{pmatrix} e^{-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-v} \end{pmatrix},$$

where $u, v \in \mathbb{C}$. Write $K \in Z(A)$ as

$$K = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}.$$

If $K = T^u U^v$ then

$$\kappa_1 = e^{u-v}, \quad \kappa_2 = e^{2v}, \quad \kappa_3 = e^{-u-v}.$$

That is, we can define u and v in a conjugation invariant way as follows. We will include the dependence on Γ and α which we use later.

Definition 4.8. *First, the bulge-turn parameter $v = (s + i\phi)_\Gamma(\alpha)$ is defined by*

$$\kappa_2 = e^{2v} = e^{2(s+i\phi)_\Gamma(\alpha)},$$

which is the eigenvalue of K corresponding to the eigenvector $\mathbf{v}_0(A)$. In order to make v well defined, we suppose $\phi_\Gamma(\alpha) \in \mathbb{R}/\pi\mathbb{Z}$. Next, we define the twist-bend $u = (t + i\theta)_\Gamma(\alpha)$ by

$$\kappa_1 = e^{u-v} = e^{(t+i\theta)_\Gamma(\alpha) - (s+i\phi)_\Gamma(\alpha)}$$

which is the eigenvalue of K corresponding to the eigenvector $\mathbf{v}_+(A)$. In order to make u well defined, we suppose $\theta_\Gamma(\alpha) \in \mathbb{R}/2\pi\mathbb{Z}$.

For clarity, we have

- (i) the *twist* along α is $t_\Gamma(\alpha) = \text{Re}(u) \in \mathbb{R}$,
- (ii) the *bend* along α is $\theta_\Gamma(\alpha) = \text{Im}(u) \in \mathbb{R}/2\pi\mathbb{R}$,
- (iii) the *bulge* along α is $s_\Gamma(\alpha) = \text{Re}(v) \in \mathbb{R}$,
- (iv) the *turn* along α is $\phi_\Gamma(\alpha) = \text{Im}(v) \in \mathbb{R}/\pi\mathbb{R}$.

Using the decomposition of S_g along $\mathcal{L} = \{[\gamma_1], \dots, [\gamma_{3g-3}]\}$ into three holed spheres $\{Y_1, \dots, Y_{2g-2}\}$ these two operations allow us to construct a representation of $\pi_1(S_g)$ to $SL(3, \mathbb{C})$. This yields the following parameters:

- (1) $6g - 6$ complex trace parameters arising from the curves $\gamma_1, \dots, \gamma_{3g-3}$, namely $\text{tr}(A_1), \dots, \text{tr}(A_{3g-3})$ and $\text{tr}(A_1^{-1}), \dots, \text{tr}(A_{3g-3}^{-1})$, where $A_j = \rho([\gamma_j])$,
- (2) $4g - 4$ complex shape parameters arising from the pairs of pants Y_1, \dots, Y_{2g-2} , namely $\sigma_+(Y_1), \dots, \sigma_+(Y_{2g-2})$ and $\sigma_-(Y_1), \dots, \sigma_-(Y_{2g-2})$,
- (3) choices of a root of the for each of the $2g - 2$ polynomials $Q(Y_1), \dots, Q(Y_{2g-2})$,
- (4) $3g - 3$ complex twist-bend parameters $(t + i\theta)_\Gamma(\gamma_1), \dots, (t + i\theta)_\Gamma(\gamma_{3g-3})$ and $3g - 3$ complex bulge-turn parameters $(s + i\phi)_\Gamma(\gamma_1), \dots, (s + i\phi)_\Gamma(\gamma_{3g-3})$.

This proves Theorem ??.

5. $\mathrm{SL}(2, \mathbb{K})$ COORDINATES

In this section, we consider representations of $\pi_1(S_g)$ to $\mathrm{SL}(3, \mathbb{C})$ that factor through $\mathrm{SL}(2, \mathbb{K})$ where \mathbb{K} is either \mathbb{R} or \mathbb{C} . Most of the construction works in both cases and so it is convenient to cover them together. We will highlight the places where there is a difference. We are most interested in the case where the inclusion of $\mathrm{SL}(2, \mathbb{K})$ in $\mathrm{SL}(3, \mathbb{C})$ is via the irreducible representation. It will be useful to also briefly consider a particular type of reducible representation.

5.1. Representations where $\mathrm{tr}(A) = \mathrm{tr}(A^{-1})$. It is well known that if $A \in \mathrm{SL}(2, \mathbb{K})$ then $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$. This will also be true of the images of $\mathrm{SL}(2, \mathbb{K})$ in $\mathrm{SL}(3, \mathbb{C})$ we consider. The following lemma is a simple consequence of this fact.

Lemma 5.1. *Suppose that A is an element of $\mathrm{SL}(3, \mathbb{C})$ for which $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$. Then A has 1 as an eigenvalue.*

Proof. Using Lemma ?? we see that the characteristic polynomial of A is

$$\begin{aligned}\chi_A(x) &= x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A)x - 1 \\ &= (x - 1)(x^2 - (\mathrm{tr}(A) - 1)x + 1).\end{aligned}$$

The result follows. \square

Consider a subgroup $\Gamma = \langle A, B, C : CBA = I \rangle$ of $\mathrm{SL}(3, \mathbb{C})$ where all elements have trace equal to the trace of their inverse. This means we must be in the branching locus of the quadratic Q given by (??) whose roots are $\mathrm{tr}[A, B]$ and $\mathrm{tr}[B, A] = \mathrm{tr}([A, B]^{-1})$. The following proposition shows that, in such a group, the equation of the branching locus factorises. In subsequent sections we will characterise the different factors.

Theorem 5.2. *Suppose that $A, B, C \in \mathrm{SL}(3, \mathbb{C})$ with $CBA = I$ satisfy*

$$\begin{aligned}\mathrm{tr}(A) &= \mathrm{tr}(A^{-1}), & \mathrm{tr}(B) &= \mathrm{tr}(B^{-1}), & \mathrm{tr}(C) &= \mathrm{tr}(C^{-1}), \\ \sigma_+(A, B, C) &= \sigma_-(A, B, C), & \mathrm{tr}[A, B] &= \mathrm{tr}([A, B]^{-1}).\end{aligned}$$

where σ_+, σ_- are given by (??) and (??). Write $a = \mathrm{tr}(A)$, $b = \mathrm{tr}(B)$, $c = \mathrm{tr}(C)$. Then

(1) either $\sigma_+ = \sigma_- = 3 - a - b - c$ and

$$\mathrm{tr}[A, B] = -abc + a^2 + b^2 + c^2 + ab + bc + ac - 3a - 3b - 3c + 3.$$

(2) or $\sigma_+ = \sigma_-$ is a root of the polynomial

$$\begin{aligned}T_2(t) &= t^2 - 2(a + b + c + 1)t \\ &\quad - 4abc + a^2 + b^2 + c^2 - 2ab - 2bc - 2ac - 2a - 2b - 2c - 3.\end{aligned}$$

and

$$\mathrm{tr}[A, B] = (a + b + c + 1)\sigma_+ + (a + 1)(b + 1)(c + 1) - 1.$$

Proof. Setting $y_1 = y_5 = a$, $y_2 = y_6 = b$, $y_3 = y_7 = c$ and $y_4 = y_8 = t$ in the polynomials S and P from (??) and (??) gives

$$\begin{aligned} S &= t^2 - 2abc + a^2 + b^2 + c^2 - 3, \\ P &= a^2b^2c^2 + 2a^2b^2c + 2a^2bc^2 + 2ab^2c^2 + a^2b^2 + b^2c^2 + a^2c^2 \\ &\quad - 4abc^2 - 4ab^2c - 4a^2bc + 2a^3 + 2b^3 + 2c^3 + 6abc \\ &\quad + 2a^2bct + 2abc^2t + 2ab^2ct + 2ab^2t + 2a^2bt + 2a^2ct + 2ac^2t + 2bc^2t + 2b^2ct \\ &\quad + 2(ab + bc + ac)(t^2 - 3t) + (a^2 + b^2 + c^2 - 6)t^2 + 2t^3 + 9. \end{aligned}$$

Since $\text{tr}[A, B] = \text{tr}[B, A]$ the polynomial Q from (??) has repeated roots. This means that $0 = S^2 - 4P$. Substituting from the above expressions we find

$$\begin{aligned} S^2 - 4P &= (t^2 - 2abc + a^2 + b^2 + c^2 - 3)^2 \\ &\quad - 4a^2b^2c^2 - 8a^2b^2c - 8a^2bc^2 - 8ab^2c^2 - 4a^2b^2 - 4b^2c^2 - 4a^2c^2 \\ &\quad + 16abc^2 + 16ab^2c + 16a^2bc - 8a^3 - 8b^3 - 8c^3 - 24abc \\ &\quad - 8a^2bct - 8abc^2t - 8ab^2ct \\ &\quad - 8ab^2t - 8a^2bt - 8a^2ct - 8ac^2t - 8bc^2t - 8b^2ct \\ &\quad - 8(ab + bc + ac)(t^2 - 3t) - 4(a^2 + b^2 + c^2 - 6)t^2 + 2t^3 - 36 \\ &= t^4 - 8t^3 - 2(2abc + a^2 + b^2 + c^2 + 4ab + 4bc + 4ac - 9)t^2 \\ &\quad - 8(a^2bc + ab^2c + abc^2 + a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2 - 3ab - 3bc - 3ac)t \\ &\quad - 4a^3bc - 4ab^3c - 4abc^3 - 8a^2b^2c - 8a^2bc^2 - 8ab^2c^2 \\ &\quad + a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2a^2c^2 + 16a^2bc + 16ab^2c + 16abc^2 \\ &\quad - 8a^3 - 8b^3 - 8c^3 + 18a^2 + 18b^2 + 18c^2 - 27 \\ &= (t + a + b + c - 3)^2 \\ &\quad \cdot (t^2 - 2(a + b + c + 1)t - 4abc + a^2 + b^2 + c^2 - 2(ab + bc + ac + a + b + c) - 3). \end{aligned}$$

Therefore the possible values of $t = \sigma_+ = \sigma_-$ correspond to the two cases in the statement of the theorem. Substituting these into $\text{tr}[A, B] = S/2$ gives the values of $\text{tr}[A, B]$. \square

5.2. Two-generator subgroups of $\text{SL}(2, \mathbb{C})$. We will use the following classical theorem of Fricke and Vogt; see Theorem A of Goldman [?]:

Theorem 5.3 (Fricke, Vogt). *Let $f : \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow \mathbb{C}$ be a regular function that is invariant under the action of $\text{SL}(2, \mathbb{C})$ by conjugation. Then there exists a polynomial function $F(x, y, z) \in \mathbb{C}[x, y, z]$ so that*

$$f(A, B) = F(\text{tr}(A), \text{tr}(B), \text{tr}(AB)).$$

Furthermore, for all $(x, y, z) \in \mathbb{C}^3$ there exist $A, B \in \text{SL}(2, \mathbb{C})$ so that

$$\text{tr}(A) = x, \quad \text{tr}(B) = y, \quad \text{tr}(AB) = z.$$

In particular, Fricke and Vogt show we can express $\mathrm{tr}(A^{-1}B)$ or $\mathrm{tr}[A, B]$ as the following polynomials in $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(AB)$:

$$(5.1) \quad \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB),$$

$$(5.2) \quad \mathrm{tr}[A, B] = \mathrm{tr}^2(A) + \mathrm{tr}^2(B) + \mathrm{tr}^2(AB) - 2 - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB).$$

This theorem almost says that the traces $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(AB)$ determine the pair (A, B) up to conjugation. In fact to get this statement we need to exclude the case where A and B commute.

Proposition 5.4 (Section 2.2 of Goldman [?]). *Let $A, B, A', B' \in \mathrm{SL}(2, \mathbb{C})$. Suppose*

$$\mathrm{tr}(A) = \mathrm{tr}(A'), \quad \mathrm{tr}(B) = \mathrm{tr}(B'), \quad \mathrm{tr}(AB) = \mathrm{tr}(A'B')$$

and $\mathrm{tr}[A, B] \neq 2$ (so also $\mathrm{tr}[A', B'] \neq 2$). Then there exists $D \in \mathrm{SL}(2, \mathbb{C})$ so that $A' = DAD^{-1}$ and $B' = DBD^{-1}$.

In the case where A, B, C are loxodromic (hyperbolic) elements of $\mathrm{SL}(2, \mathbb{R})$ satisfying $CBA = I$ there are various possibilities for the configuration of their axes in the hyperbolic plane. We are interested in the case where the axes are pairwise disjoint and bound a common region. We can characterise this configuration using traces.

Proposition 5.5 (Gilman and Maskit [?]). *Let A, B, C be hyperbolic elements of $\mathrm{SL}(2, \mathbb{R})$ with $CBA = I$. Suppose that the axes of A, B and C are pairwise disjoint and that they bound a region in the hyperbolic plane. Then*

$$\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) < 0$$

5.3. Reducible representations. Suppose that $\hat{A}, \hat{B}, \hat{C}$ are elements of $\mathrm{SL}(2, \mathbb{C})$ with $\hat{C}\hat{B}\hat{A} = I$. Then we define the following block diagonal elements of $\mathrm{SL}(3, \mathbb{C})$:

$$(5.3) \quad A = \begin{pmatrix} \hat{A} & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \hat{B} & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \hat{C} & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly we have $CBA = I$. It is also clear that $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$, $\mathrm{tr}(B^{-1}) = \mathrm{tr}(B)$ and $\mathrm{tr}(C^{-1}) = \mathrm{tr}(C)$. Similarly $\mathrm{tr}([A, B]^{-1}) = \mathrm{tr}[A, B]$. Note that in this case the traces do not determine the group up to conjugation. In order to see this, observe that if \mathbf{a} and \mathbf{b} are any column vectors in \mathbb{C}^2 then

$$A' = \begin{pmatrix} \hat{A} & \mathbf{a} \\ 0 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} \hat{B} & \mathbf{b} \\ 0 & 1 \end{pmatrix}, \quad C' = \begin{pmatrix} \hat{C} & -\hat{C}\hat{B}\mathbf{a} - \hat{C}\mathbf{b} \\ 0 & 1 \end{pmatrix}$$

satisfy $C'B'A' = I$ and $\mathrm{tr}(A') = \mathrm{tr}(A)$ etc.

Note that there are other reducible representations, for example in (??) we can multiply \hat{A} by $\lambda \in \mathbb{C} - \{0\}$ and in A make the bottom right hand entry λ^{-2} instead of 1. Similarly for B and C . Such representations do not satisfy $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$, and we will not consider them here.

It is straightforward to use equations (??) to write $\mathrm{tr}(A^{-1}B)$, and hence the shape invariant $\sigma_+ = \sigma_-$ in terms of $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(C)$.

Lemma 5.6. *Let $A, B, C \in \mathrm{SL}(3, \mathbb{C})$ with $CBA = I$ be as given in (??). Let σ_+ and σ_- be given by (??) and (??). Then*

$$(5.4) \quad \sigma_+ = \sigma_- = 3 - \mathrm{tr}(A) - \mathrm{tr}(B) - \mathrm{tr}(C).$$

Proof. First observe that $\text{tr}(A) = \text{tr}(\hat{A}) + 1$ and so on. Therefore, using (??) we have

$$\begin{aligned}\sigma_+ &= \text{tr}(A^{-1}B) - \text{tr}(A^{-1})\text{tr}(B) \\ &= (\text{tr}(\hat{A}^{-1}\hat{B}) + 1) - (\text{tr}(\hat{A}) + 1)(\text{tr}(\hat{B}) + 1) \\ &= \text{tr}(\hat{A})\text{tr}(\hat{B}) - \text{tr}(\hat{C}) + 1 - \text{tr}(\hat{A})\text{tr}(\hat{B}) - \text{tr}(\hat{A}) - \text{tr}(\hat{B}) - 1 \\ &= 3 - \text{tr}(A) - \text{tr}(B) - \text{tr}(C).\end{aligned}$$

Since each of the traces of A , B and $A^{-1}B$ equals the trace of its inverse we have $\sigma_- = \sigma_+$. \square

Theorem 5.7. *Let A, B, C be any elements of $\text{SL}(3, \mathbb{C})$ satisfying:*

- (a) $CBA = I$,
- (b) $\text{tr}(A^{-1}) = \text{tr}(A)$, $\text{tr}(B^{-1}) = \text{tr}(B)$, $\text{tr}(C^{-1}) = \text{tr}(C)$ and $\text{tr}([A, B]^{-1}) = \text{tr}[A, B]$,
- (c) $\sigma_+(A, B, C) = \sigma_-(A, B, C)$.

If $\sigma_+(A, B, C) = 3 - \text{tr}(A) - \text{tr}(B) - \text{tr}(C)$ then the group $\langle A, B, C : CBA = I \rangle$ is reducible.

Proof. Define $a := \text{tr}(A)$, $b := \text{tr}(B)$, $c := \text{tr}(C)$.

Using the theorem of Fricke and Vogt, Theorem ?? we can find $\hat{A}, \hat{B}, \hat{C} \in \text{SL}(2, \mathbb{C})$ with $\hat{C}\hat{B}\hat{A} = I$ and so that $\text{tr}(\hat{A}) = a - 1$, $\text{tr}(\hat{B}) = b - 1$, $\text{tr}(\hat{C}) = c - 1$. Thus, we can construct A_0, B_0, C_0 in $\text{SL}(3, \mathbb{C})$ of the form (??) with $\text{tr}(A_0) = a$, $\text{tr}(B_0) = b$ and $\text{tr}(C_0) = c$. Using Lemma ?? we have

$$(\sigma_0)_+ = (\sigma_0)_- = 3 - \text{tr}(A_0) - \text{tr}(B_0) - \text{tr}(C_0)$$

Therefore, there is a reducible representation $\langle A_0, B_0, C_0 : C_0 B_0 A_0 = I \rangle$ for which the eight traces agree with those of $\langle A, B, C : CBA = I \rangle$. Hence, the latter group must also be reducible (see Lawton's theorem, Theorem ?? (3), and the remark following this theorem). \square

5.4. Irreducible representations. In this section, we consider the irreducible representation of $\text{SL}(2, \mathbb{K})$ to $\text{SL}(3, \mathbb{C})$ for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

Consider the following map from \mathbb{K}^2 to \mathbb{K}^3 :

$$\Phi : \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} -w_1^2 \\ \sqrt{2}w_1w_2 \\ w_2^2 \end{pmatrix}.$$

Writing $z_1 = -w_1^2$, $z_2 = \sqrt{2}w_1w_2$ and $z_3 = w_2^2$ we see that the image of Φ satisfies $2z_1z_3 + z_2^2 = 0$. We can write the latter equation in terms of a quadratic form. Let

$$(5.5) \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then we can write

$$2z_1z_3 + z_2^2 = (z_1 \quad z_2 \quad z_3) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \mathbf{z}^t J \mathbf{z}.$$

Therefore $\Phi(\mathbf{w})$ lies in the zero set of the quadratic form defined by J . It is easy to check J has signature $(2, 1)$, that is it has two positive eigenvalues (both $+1$) and one negative eigenvalue (which is -1).

Now consider $\mathrm{SL}(2, \mathbb{K})$. It acts naturally by left multiplication on \mathbb{K}^2 . Applying Φ enables to construct a map $\Phi_* : \mathrm{SL}(2, \mathbb{K}) \mapsto \mathrm{SL}(3, \mathbb{K})$ as follows

$$\Phi_*(\hat{A})\Phi(\mathbf{w}) = \Phi(\hat{A}\mathbf{w}).$$

A short calculation gives

$$(5.6) \quad \Phi_* : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & -\sqrt{2}ab & -b^2 \\ -\sqrt{2}ac & ad+bc & \sqrt{2}bd \\ -c^2 & \sqrt{2}cd & d^2 \end{pmatrix}.$$

It is not hard to check that Φ_* is a homomorphism whose kernel is $\{\pm I\}$ and whose image is contained in $\mathrm{SO}(J; \mathbb{K})$. In fact, it is not hard to check that when $\mathbb{K} = \mathbb{C}$ then Φ_* maps $\mathrm{SL}(2, \mathbb{C})$ onto $\mathrm{SO}(J; \mathbb{C})$ and when $\mathbb{K} = \mathbb{R}$ then Φ_* maps $\mathrm{SL}(2, \mathbb{R})$ onto the identity component $\mathrm{SO}_0(J; \mathbb{R})$ of $\mathrm{SO}(J; \mathbb{R})$. Note that since J has signature $(2, 1)$, this means Φ_* is a bijection between $\mathrm{SO}_0(2, 1; \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$. When $\mathbb{K} = \mathbb{C}$ the signature of J is not well defined.

Lemma 5.8. *Let $\hat{A} \in \mathrm{SL}(2, \mathbb{K})$ with eigenvalues λ and λ^{-1} . Then the eigenvalues of $A = \Phi_*(\hat{A})$ are $\lambda^2, 1$ and λ^{-2} . In particular, $\mathrm{tr}(A) = \mathrm{tr}^2(\hat{A}) - 1$. Moreover, if $\mathbf{u} \in \mathbb{K}^2$ is an eigenvector of \hat{A} with eigenvalue λ , then $\Phi(\mathbf{u})$ is an eigenvector of A with eigenvalue λ^2 .*

Proof. It is not hard to see from the (??) that $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$. From Lemma ?? we see that A has 1 as an eigenvalue. Now suppose $\mathbf{u} \in \mathbb{K}^2$ is an eigenvector of \hat{A} with eigenvalue λ . Then

$$A\Phi(\mathbf{u}) = \Phi_*(\hat{A})\Phi(\mathbf{u}) = \Phi(\hat{A}\mathbf{u}) = \Phi(\lambda\mathbf{u}) = \lambda^2\Phi(\mathbf{u}).$$

This gives the second part. Thus, the eigenvalues are $\lambda^2, 1$ and λ^{-2} . Therefore

$$\mathrm{tr}(A) = \lambda^2 + 1 + \lambda^{-2} = (\lambda + \lambda^{-1})^2 - 1 = \mathrm{tr}^2(\hat{A}) - 1.$$

□

We now consider \hat{A}, \hat{B} and \hat{C} in $\mathrm{SL}(2, \mathbb{K})$ with $\hat{C}\hat{B}\hat{A} = I$ and the corresponding $A = \Phi_*(\hat{A}), B = \Phi_*(\hat{B}), C = \Phi_*(\hat{C})$ in $\mathrm{SL}(3, \mathbb{C})$. Since Φ_* is a homomorphism we have $CBA = I$. We know that $\langle \hat{A}, \hat{B}, \hat{C} : \hat{C}\hat{B}\hat{A} = I \rangle < \mathrm{SL}(2, \mathbb{K})$ is determined up to conjugation by $\mathrm{tr}(\hat{A}), \mathrm{tr}(\hat{B})$ and $\mathrm{tr}(\hat{C})$ using the theorem of Fricke and Vogt, Theorem ??. Therefore, it is tempting to say that its image under Φ_* , namely $\langle A, B, C : CBA = I \rangle$, is determined up to conjugation by $\mathrm{tr}(A), \mathrm{tr}(B)$ and $\mathrm{tr}(C)$. However, this is not quite true. Consider $\hat{A}_1, \hat{B}_1, \hat{C}_1$ in $\mathrm{SL}(2, \mathbb{K})$ with $\hat{C}_1\hat{B}_1\hat{A}_1 = I$ and $\mathrm{tr}(\hat{A}_1) = -\mathrm{tr}(\hat{A}), \mathrm{tr}(\hat{B}_1) = -\mathrm{tr}(\hat{B}), \mathrm{tr}(\hat{C}_1) = -\mathrm{tr}(\hat{C})$ (such matrices exist by the theorem of Fricke and Vogt). As we have already seen, $\mathrm{tr}(\hat{A})\mathrm{tr}(\hat{B})\mathrm{tr}(\hat{C})$ is independent of the choice of lift of \hat{A} and \hat{B} from $\mathrm{PSL}(2, \mathbb{K})$ to $\mathrm{SL}(2, \mathbb{K})$. Since

$$\mathrm{tr}(\hat{A}_1)\mathrm{tr}(\hat{B}_1)\mathrm{tr}(\hat{C}_1) = -\mathrm{tr}(\hat{A})\mathrm{tr}(\hat{B})\mathrm{tr}(\hat{C}),$$

the two groups $\langle \widehat{A}, \widehat{B}, \widehat{C} : \widehat{C}\widehat{B}\widehat{A} = I \rangle$ $\langle \widehat{A}_1, \widehat{B}_1, \widehat{C}_1 : \widehat{C}_1\widehat{B}_1\widehat{A}_1 = I \rangle$ correspond to different subgroups of $\mathrm{PSL}(2, \mathbb{K})$.

Write $A_1 = \Phi_*(\widehat{A}_1)$, $B_1 = \Phi_*(\widehat{B}_1)$, $C_1 = \Phi_*(\widehat{C}_1)$. Then

$$\mathrm{tr}(A_1) = (\mathrm{tr}(\widehat{A}_1))^2 - 1 = (-\mathrm{tr}(\widehat{A}))^2 - 1 = \mathrm{tr}(A),$$

and similarly $\mathrm{tr}(B_1) = \mathrm{tr}(B)$ and $\mathrm{tr}(C_1) = \mathrm{tr}(C)$. However $\langle A, B, C : CBA = I \rangle$ and $\langle A_1, B_1, C_1 : C_1B_1A_1 = I \rangle$ are not conjugate. The ambiguity is captured by looking at $\mathrm{tr}(A^{-1}B)$ and $\mathrm{tr}(A_1^{-1}B_1)$, or in a more invariant way by the shape invariants $\sigma_+ = \sigma_- = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$ and $(\sigma_1)_+ = (\sigma_1)_- = \mathrm{tr}(A_1^{-1}B_1) - \mathrm{tr}(A_1^{-1})\mathrm{tr}(B_1)$ which are the two roots of the quadratic polynomial in the following lemma.

Lemma 5.9. *Suppose that A, B and C are all in the image of Φ_* and satisfy $CBA = I$. Write $a = \mathrm{tr}(A)$, $b = \mathrm{tr}(B)$ and $c = \mathrm{tr}(C)$. Let σ_+ and σ_- be given by (??) and (??). Then $\sigma_+ = \sigma_-$ is a root of the polynomial*

$$X^2 - 2(a + b + c + 1)X - 4abc + a^2 + b^2 + c^2 - 2ab - 2ac - 2bc - 2a - 2b - 2c - 3.$$

That is,

$$X = a + b + c + 1 \pm 2\sqrt{(a+1)(b+1)(c+1)}.$$

Proof. Writing \widehat{A}, \widehat{B} for matrices that are sent to A and B by Φ_* , we have

$$\begin{aligned} \mathrm{tr}(A^{-1}B) &= \mathrm{tr}^2(\widehat{A}^{-1}\widehat{B}) - 1 \\ &= \left(\mathrm{tr}(\widehat{A}\widehat{B}) - \mathrm{tr}(\widehat{A})\mathrm{tr}(\widehat{B}) \right)^2 - 1 \\ &= (\mathrm{tr}(AB) + 1) - 2\mathrm{tr}(\widehat{A})\mathrm{tr}(\widehat{B})\mathrm{tr}(\widehat{A}\widehat{B}) + (\mathrm{tr}(A) + 1)(\mathrm{tr}(B) + 1) - 1 \\ &= \mathrm{tr}(A)\mathrm{tr}(B) + \mathrm{tr}(A) + \mathrm{tr}(B) + \mathrm{tr}(C) + 1 - 2\mathrm{tr}(\widehat{A})\mathrm{tr}(\widehat{B})\mathrm{tr}(\widehat{A}\widehat{B}). \end{aligned}$$

Thus

$$(5.7) \quad \sigma_+ = \sigma_- = \mathrm{tr}(A) + \mathrm{tr}(B) + \mathrm{tr}(C) + 1 - 2\mathrm{tr}(\widehat{A})\mathrm{tr}(\widehat{B})\mathrm{tr}(\widehat{A}\widehat{B}).$$

Therefore

$$\begin{aligned} &(\sigma_{\pm} - \mathrm{tr}(A) - \mathrm{tr}(B) - \mathrm{tr}(C) - 1)^2 \\ &= 4\mathrm{tr}^2(\widehat{A})\mathrm{tr}^2(\widehat{B})\mathrm{tr}^2(\widehat{A}\widehat{B}) \\ &= 4(\mathrm{tr}(A) + 1)(\mathrm{tr}(B) + 1)(\mathrm{tr}(C) + 1). \end{aligned}$$

The result follows by rearranging this expression. \square

Corollary 5.10. *Suppose that $A, B, C \in \mathrm{SL}(3, \mathbb{C})$ with $CBA = I$. Suppose that*

$$\begin{aligned} \mathrm{tr}(A) &= \mathrm{tr}(A^{-1}), & \mathrm{tr}(B) &= \mathrm{tr}(B^{-1}), & \mathrm{tr}(C) &= \mathrm{tr}(C^{-1}), \\ \sigma_+(A, B, C) &= \sigma_-(A, B, C), & \mathrm{tr}[A, B] &= \mathrm{tr}([A, B]^{-1}). \end{aligned}$$

where σ_+, σ_- are given by (??) and (??). Then either $\langle A, B, C : CBA = I \rangle$ is reducible or else, up to conjugacy, it is in the image of the map Φ_* from (??).

Proof. From Theorem ?? either the traces of A, B, C and the shape invariants satisfy $\sigma_+ = \sigma_- = 3 - \text{tr}(A) - \text{tr}(B) - \text{tr}(C)$, in which case $\langle A, B, C : CBA = I \rangle$ is reducible by Theorem ??, or else they satisfy

$$\begin{aligned} 0 &= t^2 - 2(a + b + c + 1)t \\ &\quad - 4abc + a^2 + b^2 + c^2 - 2ab - 2bc - 2ac - 2a - 2b - 2c - 3 \end{aligned}$$

where $a = \text{tr}(A)$, $b = \text{tr}(B)$, $c = \text{tr}(C)$ and $t = \sigma_+(A, B, C)$. In this case there are matrices \hat{A} , \hat{B} and \hat{C} in $SL(2, \mathbb{C})$ whose images under Φ_* have the desired traces. Providing this representation is irreducible, it is determined by these traces up to conjugation, see Theorem ?? (3). This gives the result. \square

In our application to three holed spheres, there is a consistent choice of root of the equation from Lemma ?. Let Y be a three holed sphere with boundary curves α , β and γ . We are interested in Fuchsian representations of $\pi_1(Y)$ in the case where $\mathbb{K} = \mathbb{R}$ and quasi-Fuchsian representations in the case where $\mathbb{K} = \mathbb{C}$. In the first case, these are representations $\rho : \pi_1(Y) \mapsto \Gamma$, where Γ is a subgroup of $\text{Isom}_+(\mathbf{H}_{\mathbb{R}}^2)$, the orientation preserving isometries of the hyperbolic plane, with the property that $\mathbf{H}_{\mathbb{R}}^2/\Gamma$ is homeomorphic to Y . Necessarily this means that α , β and γ are represented by hyperbolic (loxodromic) maps.

Proposition 5.11. *Let Y be a three holed sphere with boundary curves α , β , γ and let $\rho : \pi_1(Y) \mapsto \Gamma < \text{SO}_0(2, 1)$ be a Fuchsian representation of its fundamental group. Let $A = \rho([\alpha])$, $B = \rho([\beta])$ and $C = \rho([\gamma])$. Then the shape invariants σ_{\pm} of Γ and the trace of $[A, B]$ are given by*

$$\begin{aligned} \sigma_+ &= \sigma_- = \text{tr}(A) + \text{tr}(B) + \text{tr}(C) + 1 + 2\sqrt{(\text{tr}(A) + 1)(\text{tr}(B) + 1)(\text{tr}(C) + 1)}, \\ \text{tr}[A, B] &= \left(\text{tr}(A) + \text{tr}(B) + \text{tr}(C) + 1 + \sqrt{(\text{tr}(A) + 1)(\text{tr}(B) + 1)(\text{tr}(C) + 1)} \right)^2 - 1 \end{aligned}$$

where we take the positive square root.

Proof. By construction, there exist $\hat{A}, \hat{B} \in SL(2, \mathbb{R})$ so that $A = \Psi_*(\hat{A})$, $B = \Psi_*(\hat{B})$, $C = (BA)^{-1} = \Psi_*(\hat{A}^{-1}\hat{B}^{-1})$. Hence

$$\text{tr}(A) + 1 = \text{tr}^2(\hat{A}), \quad \text{tr}(B) + 1 = \text{tr}^2(\hat{B}), \quad \text{tr}(C) + 1 = \text{tr}^2(\hat{A}\hat{B}).$$

Using Proposition ?? we have

$$\text{tr}(\hat{A})\text{tr}(\hat{B})\text{tr}(\hat{A}\hat{B}) < 0.$$

Therefore, taking the positive square root, we have

$$\text{tr}(\hat{A})\text{tr}(\hat{B})\text{tr}(\hat{A}\hat{B}) = -\sqrt{(\text{tr}(A) + 1)(\text{tr}(B) + 1)(\text{tr}(C) + 1)}.$$

We obtain the result by substituting this into equation (??). \square

The space of quasi-Fuchsian representations of $\pi_1(Y)$ is a connected set that contains the Fuchsian representations and on which A , B and C are always loxodromic. A consequence of the latter condition is that for all quasi-Fuchsian representations $\text{tr}(A) \neq -1$, $\text{tr}(B) \neq -1$ and $\text{tr}(C) \neq -1$. Thus, on the space of quasi-Fuchsian representations

there is a well defined branch of $\sqrt{(\operatorname{tr}(A) + 1)(\operatorname{tr}(B) + 1)(\operatorname{tr}(C) + 1)}$ that agrees with the positive square root when the three traces are real and positive. This branch is obtained by analytic continuation along paths of quasi-Fuchsian representations.

Corollary 5.12. *Let Y be a three holed sphere with boundary curves α, β, γ and let $\rho : \pi_1(Y) \rightarrow \Gamma < \operatorname{SO}(3; \mathbb{C})$ be a quasi-Fuchsian representation of its fundamental group. Let $A = \rho([\alpha])$, $B = \rho([\beta])$ and $C = \rho([\gamma])$. Then the shape invariants σ_{\pm} of Γ and the trace of $[A, B]$ are given by*

$$\begin{aligned} \sigma_+ &= \sigma_- = \operatorname{tr}(A) + \operatorname{tr}(B) + \operatorname{tr}(C) + 1 + 2\sqrt{(\operatorname{tr}(A) + 1)(\operatorname{tr}(B) + 1)(\operatorname{tr}(C) + 1)}, \\ \operatorname{tr}[A, B] &= \left(\operatorname{tr}(A) + \operatorname{tr}(B) + \operatorname{tr}(C) + 1 + \sqrt{(\operatorname{tr}(A) + 1)(\operatorname{tr}(B) + 1)(\operatorname{tr}(C) + 1)} \right)^2 - 1 \end{aligned}$$

where the square root is a well defined branch that agrees with the positive square root when all three traces are real and positive.

We now consider the twist-bend-bulge-turn parameters associated to the loxodromic map A . As before, we assume that $\mathbf{v}_+(A)$, $\mathbf{v}_0(A)$, $\mathbf{v}_-(A)$ are the standard basis vectors, and so A is a diagonal matrix

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

where $\lambda \in \mathbb{K}$ has $|\lambda| > 1$. We then consider $K \in \operatorname{SO}(J; \mathbb{K})$ in the centraliser $Z(A)$ of A . This has the form

$$K = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}.$$

Since K is in the image of Φ_* , we see that $\kappa_2 = 1$ and $\kappa_3 = \kappa_1^{-1}$. Thus $K = T^u$ for some $u \in \mathbb{K}$. Hence the bulge and the turn are both zero.

Summarising, when $\mathbb{K} = \mathbb{R}$ a representation of $\pi_1(S_g)$ in $\mathcal{T}(S_g)$ is determined by the following parameters

- (1) $3g - 3$ real trace parameters $\operatorname{tr}(A_1), \dots, \operatorname{tr}(A_{3g-3})$,
- (2) $3g - 3$ real twist parameters $t_{\Gamma}(\gamma_1), \dots, t_{\Gamma}(\gamma_{3g-3})$.

Moreover, the shape invariants are determined by the trace parameters using equation (??) and the commutator polynomials $Q(Y_1), \dots, Q(Y_{2g-2})$ all have repeated roots. Also the real bend parameters $\theta_{\Gamma}(\gamma_1), \dots, \theta_{\Gamma}(\gamma_{3g-3})$ and the complex bulge-turn parameters $(s + i\phi)_{\Gamma}(\gamma_1), \dots, (s + i\phi)_{\Gamma}(\gamma_{3g-3})$ are all zero. This proves Theorem ??.

Likewise, when $\mathbb{K} = \mathbb{C}$, a representation $\pi_1(S_g)$ in $\mathcal{Q}(S, \mathcal{L}, \operatorname{SO}(3; \mathbb{C}))$ is determined by the following parameters

- (1) $3g - 3$ complex trace parameters $\operatorname{tr}(A_1), \dots, \operatorname{tr}(A_{3g-3})$,
- (2) $3g - 3$ complex twist-bend parameters $(t + i\theta)_{\Gamma}(\gamma_1), \dots, (t + i\theta)_{\Gamma}(\gamma_{3g-3})$.

Moreover, the shape invariants are determined by the trace parameters using equation (??) and the commutator polynomials $Q(Y_1), \dots, Q(Y_{2g-2})$ all have repeated roots. Finally, the complex bulge-turn parameters $(s + i\phi)_{\Gamma}(\gamma_1), \dots, (s + i\phi)_{\Gamma}(\gamma_{3g-3})$ are all zero. This proves Theorem ??.

Finally, consider an irreducible subgroup of $\Gamma = \langle A, B, C : CBA = I \rangle$ where the traces A , B and C all equal the traces of their respective inverses and also $\sigma_+ = \sigma_-$, so $\mathrm{tr}(A^{-1}B) = \mathrm{tr}(B^{-1}A)$ and where Q has repeated roots, so $\mathrm{tr}[A, B] = \mathrm{tr}([A, B]^{-1})$. Using Theorem ?? we see that either $\sigma_+ = 3 - \mathrm{tr}(A) - \mathrm{tr}(B) - \mathrm{tr}(C)$ or else it is a root of a particular quadratic polynomial T_2 . Since the group is assumed to be irreducible, the former cannot happen, using Theorem ?. Hence the traces must satisfy T_2 . Using Lawton's theorem, Γ is uniquely determined up to conjugation by these traces and by Lemma ?? there is a representation in the image of Φ_* with the same values for the traces. Hence Γ is a subgroup of $\mathrm{SO}(3; \mathbb{C})$. This proves Theorem ?? (1).

6. $\mathrm{SL}(3, \mathbb{R})$ -COORDINATES

In this section we consider totally loxodromic representations of $\pi_1(S_g)$ to $\mathrm{SL}(3, \mathbb{R})$. We are interested in those representations that can be connected to the Fuchsian representations, that is those whose image lies in $\mathrm{SO}_0(2, 1)$, through a continuous path of representations. Choi and Goldman [?] showed that the component of $\mathrm{SL}(3, \mathbb{R})$ representations containing the Fuchsian representations corresponds to the space of convex real projective structures on S_g . This component is called the Hitchin component. Since any loxodromic element of $\mathrm{SO}_0(2, 1)$ has positive eigenvalues, each non-trivial element of $\pi_1(S_g)$ will be represented by a loxodromic map with positive eigenvalues. In [?], Goldman defined Fenchel-Nielsen coordinates for such representations. His parameters are boundary parameters, internal parameters and twist-bulge parameters. These correspond to our trace parameters, shape invariants and twist-bulge parameters respectively. Goldman's internal parameters were not symmetric under cyclic permutation of the boundary curves of each three-holed sphere, but later Zhang [?] showed how to symmetrise them. We will use Zhang's parameters.

6.1. Fenchel-Nielsen coordinates for $\mathrm{SL}(3, \mathbb{R})$. It is clear that for representations to $\mathrm{SL}(3, \mathbb{R})$ the trace parameters $\mathrm{tr}(A_j^{\pm 1})$ and the shape invariants $\sigma_{\pm}(Y_k)$ are all real. Using Lawton's theorem, we see that $\rho(\pi_1(Y_k))$ is determined by these parameters together with a choice of root of the commutator quadratic $Q(Y_k)$.

Now consider the twist-bend-bulge-turn parameters. Recall from Section ?? that if A is loxodromic then an element K of the centraliser $Z(A)$ of $A = \rho([\alpha])$ can be written as $T^u U^v$ and has eigenvalues e^{u-v} , e^{2v} and e^{-u-v} . For representations to $\mathrm{SL}(3, \mathbb{R})$ these all need to be real. Hence the bend $\theta_{\Gamma}(\alpha) = \mathrm{Im}(u)$ and the turn $\phi_{\Gamma}(\alpha) = \mathrm{Im}(v)$ are zero. We are left with the twist and bulge parameters $t_{\Gamma}(\alpha) = \mathrm{Re}(u)$ and $s_{\Gamma}(\alpha) = \mathrm{Re}(v)$, see Section 5.5 of Goldman [?]. Note that in the next section we will use s and t to denote Goldman's internal parameters rather than the bulge and twist. The context will make this clear.

Thus we have proved that $\rho : (\pi_1(S_g)) \longrightarrow \mathrm{SL}(3, \mathbb{R})$ is determined by

- (1) $6g - 6$ real trace parameters $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$ and $\mathrm{tr}(A_1^{-1}), \dots, \mathrm{tr}(A_{3g-3}^{-1})$;
- (2) $4g - 4$ real shape invariants $\sigma_+(Y_1), \dots, \sigma_+(Y_{2g-2})$ and $\sigma_-(Y_1), \dots, \sigma_-(Y_{2g-2})$;
- (3) a choice of root for each of the $2g - 2$ polynomials $Q(Y_1), \dots, Q(Y_{2g-2})$;
- (4) $3g - 3$ real twist parameters $t_{\Gamma}(\gamma_1), \dots, t_{\Gamma}(\gamma_{3g-3})$ and $3g - 3$ real bulge parameters $s_{\Gamma}(\gamma_1), \dots, s_{\Gamma}(\gamma_{3g-3})$.

This proves Theorem ??.

Moreover, consider a subgroup $\langle A, B, C : CBA = I \rangle$ of $\mathrm{SL}(3, \mathbb{C})$ where $\mathrm{tr}(A^{\pm 1})$, $\mathrm{tr}(B^{\pm 1})$, $\mathrm{tr}(C^{\pm 1})$, and σ_{\pm} are all real. Using Lawton's theorem all traces in this group are determined by real polynomial functions of these traces, and so must themselves be real. Hence, using Acosta's theorem we see that the group is conjugate to a subgroup of $\mathrm{SL}(3, \mathbb{R})$. This proves Theorem ?? (2).

6.2. Goldman-Zhang coordinates. Now we relate our method of parameterising loxodromic maps with Goldman's. This relates our trace parameters and shape invariants to Goldman's boundary and internal parameters. (Our twist and bulge parameters agree with his.)

Suppose that $A \in \mathrm{SL}(3, \mathbb{R})$ is loxodromic with (real) eigenvalues λ_A, μ_A, ν_A satisfying $0 < \lambda_A < \mu_A < \nu_A$. Note this implies $0 < \lambda_A < 1$. Goldman defines $\tau_A = \mu_A + \nu_A$ and he shows that $2/\sqrt{\lambda_A} < \tau_A < \lambda_A + \lambda_A^{-2}$. Since the eigenvalues of A^{-1} are λ_A^{-1} , $\mu_A^{-1} = \lambda_A \nu_A$ and $\nu_A^{-1} = \lambda_A \mu_A$ we see

$$\mathrm{tr}(A) = \lambda_A + \tau_A, \quad \mathrm{tr}(A^{-1}) = \lambda_A^{-1} + \lambda_A \tau_A.$$

It is easy to see that the Jacobian of the map $(\lambda_A, \tau_A) \mapsto (\mathrm{tr}(A), \mathrm{tr}(A^{-1}))$ is zero if and only if $\tau_A = \lambda_A + \lambda_A^{-2}$. Hence when $\tau_A < \lambda_A + \lambda_A^{-2}$ there is a bijection between the parameters $(\mathrm{tr}(A), \mathrm{tr}(A^{-1}))$ and (λ_A, τ_A) . The inverse map can be constructed by solving the characteristic polynomial of A , whose coefficients are determined by $\mathrm{tr}(A)$ and $\mathrm{tr}(A^{-1})$.

Now consider a triple of loxodromic maps A, B, C in $\mathrm{SL}(3, \mathbb{R})$ with positive eigenvalues and satisfying $CBA = I$. Goldman parameterises this triple by $(\lambda_A, \tau_A, \lambda_B, \tau_B, \lambda_C, \tau_C)$, which he calls boundary invariants, and two further parameters s and t , called internal parameters. Goldman's internal parameter s is invariant under cyclic permutation of A, B, C , but t is not. Zhang [?] remedied this by defining a parameter r . We will relate our parameters σ_{\pm} with Zhang's parameters s and r .

Let $\mathbf{r}_A, \mathbf{r}_B$ and \mathbf{r}_C be vectors in \mathbb{R}^3 corresponding to the repelling fixed points of A, B and C . The parameter s may be expressed in terms of certain $\mathrm{SL}(3, \mathbb{R})$ invariant cross-ratios denoted $(a, b, c, d)_e$ as follows, see Section 4 of [?] or equation 2.2 of [?].

$$\begin{aligned} \rho_A(s) &:= (A^{-1}\mathbf{r}_B, \mathbf{r}_C, \mathbf{r}_B, A\mathbf{r}_C)_{\mathbf{r}_A} &= 1 + \sqrt{\frac{\lambda_C \lambda_A}{\lambda_B}} \tau_A s + \frac{\lambda_C}{\lambda_B} s^2, \\ \rho_B(s) &:= (B^{-1}\mathbf{r}_C, \mathbf{r}_A, \mathbf{r}_C, B\mathbf{r}_A)_{\mathbf{r}_B} &= 1 + \sqrt{\frac{\lambda_A \lambda_B}{\lambda_C}} \tau_B s + \frac{\lambda_A}{\lambda_C} s^2, \\ \rho_C(s) &:= (C^{-1}\mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_A, C\mathbf{r}_B)_{\mathbf{r}_C} &= 1 + \sqrt{\frac{\lambda_B \lambda_C}{\lambda_A}} \tau_C s + \frac{\lambda_C}{\lambda_B} s^2, \end{aligned}$$

This defines the internal parameter s . Following Zhang, Proposition 2.19 of [?], we define the internal parameter r by

$$\begin{aligned} r &= \left((B^{-1}\mathbf{r}_C, \mathbf{r}_B, B\mathbf{r}_A, \mathbf{r}_C)_{\mathbf{r}_A} - 1 \right) \left(C^{-1}\mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_A, C\mathbf{r}_B \right)_{\mathbf{r}_C} \\ &= \left((A^{-1}\mathbf{r}_B, \mathbf{r}_A, A\mathbf{r}_C, \mathbf{r}_B)_{\mathbf{r}_C} - 1 \right) \left(B^{-1}\mathbf{r}_C, \mathbf{r}_A, \mathbf{r}_C, B\mathbf{r}_A \right)_{\mathbf{r}_B} \\ &= \left((C^{-1}\mathbf{r}_A, \mathbf{r}_C, C\mathbf{r}_B, \mathbf{r}_A)_{\mathbf{r}_B} - 1 \right) \left(A^{-1}\mathbf{r}_B, \mathbf{r}_C, \mathbf{r}_B, A\mathbf{r}_C \right)_{\mathbf{r}_A}. \end{aligned}$$

Note that Goldman's internal parameter t is given by $t = r/\rho_B(s)$.

In [?] Goldman gives implicit matrices for the representation of the three boundary elements of a pair of pants Y , this matrices are

$$\begin{aligned} A &= \begin{pmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 a_3 c_2 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{pmatrix}, \\ B &= \begin{pmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ \alpha_2 b_1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{pmatrix}, \\ C &= \begin{pmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{pmatrix} \end{aligned}$$

Where

$$\begin{aligned} \alpha_1 \beta_1 \gamma_1 &= \det(A) = 1, & \lambda_A &= \alpha_1, & \tau_A &= -\beta_1 + \gamma_1(b_3 c_2 - 1), \\ \alpha_2 \beta_2 \gamma_2 &= \det(B) = 1, & \lambda_B &= \beta_2, & \tau_B &= -\gamma_2 + \alpha_2(a_3 c_1 - 1), \\ \alpha_3 \beta_3 \gamma_3 &= \det(C) = 1, & \lambda_C &= \gamma_3, & \tau_C &= -\alpha_3 + \beta_3(a_2 b_1 - 1). \end{aligned}$$

The inverses of A , B and C are given by

$$\begin{aligned} A^{-1} &= \begin{pmatrix} \alpha_1^{-1} & \beta_1^{-1} a_2 & \alpha_1^{-1} a_3 + \beta_1^{-1} a_2 b_3 \\ 0 & -\beta_1^{-1} & -\beta_1^{-1} b_3 \\ 0 & \beta_1^{-1} c_2 & -\gamma_1^{-1} + \beta_1^{-1} b_3 c_2 \end{pmatrix}, \\ B^{-1} &= \begin{pmatrix} -\alpha_2^{-1} + \gamma_2^{-1} a_3 c_1 & 0 & \gamma_2^{-1} a_3 \\ \beta_2^{-1} b_1 + \gamma_2^{-1} b_3 c_1 & \beta_2^{-1} & \gamma_2^{-1} b_3 \\ -\gamma_2^{-1} c_1 & 0 & -\gamma_2^{-1} \end{pmatrix}, \\ C^{-1} &= \begin{pmatrix} -\alpha_3^{-1} & -\alpha_3^{-1} a_2 & 0 \\ \alpha_3^{-1} b_1 & -\beta_3^{-1} + \alpha_3^{-1} a_2 b_1 & 0 \\ \alpha_3^{-1} c_1 & \gamma_3^{-1} c_2 + \alpha_3^{-1} a_2 c_1 & \gamma_3^{-1} \end{pmatrix}. \end{aligned}$$

Since we have the presentation $CBA = I$ then

$$\alpha_1 \alpha_2 \alpha_3 = \beta_1 \beta_2 \beta_3 = \gamma_1 \gamma_2 \gamma_3 = 1.$$

Changing variables as Goldman does and using Zhang's symmetrised coordinates, we have

$$\begin{aligned}\alpha_1 &= \lambda_A, & \alpha_2 &= \sqrt{\frac{\lambda_C}{\lambda_A \lambda_B}} \frac{1}{s}, & \alpha_3 &= \sqrt{\frac{\lambda_B}{\lambda_A \lambda_C}} s, \\ \beta_1 &= \sqrt{\frac{\lambda_C}{\lambda_A \lambda_B}} s, & \beta_2 &= \lambda_B, & \beta_3 &= \sqrt{\frac{\lambda_A}{\lambda_B \lambda_C}} \frac{1}{s}, \\ \gamma_1 &= \sqrt{\frac{\lambda_B}{\lambda_A \lambda_C}} \frac{1}{s}, & \gamma_2 &= \sqrt{\frac{\lambda_A}{\lambda_B \lambda_C}} s, & \gamma_3 &= \lambda_C\end{aligned}$$

and

$$\begin{aligned}a_2 &= \frac{r}{\rho_B(s)}, & a_3 &= 2, & b_1 &= \frac{\rho_B(s)\rho_C(s)}{r}, \\ b_3 &= 2, & c_1 &= \frac{\rho_B(s)}{2}, & c_2 &= \frac{\rho_A(s)}{2}.\end{aligned}$$

Then a lengthy, but straightforward calculation yields

$$\begin{aligned}\sigma_+ &= \left(\sqrt{\lambda_A \lambda_B \lambda_C} + \frac{1}{\sqrt{\lambda_A \lambda_B \lambda_C}} \right) s + \left(r + \frac{\rho_A(s)\rho_B(s)\rho_C(s)}{r} \right) \frac{1}{s^2} \\ &\quad + \left(\sqrt{\frac{\lambda_C}{\lambda_B}} \sqrt{\lambda_A} \tau_A + \sqrt{\frac{\lambda_A}{\lambda_C}} \sqrt{\lambda_B} \tau_B + \sqrt{\frac{\lambda_B}{\lambda_A}} \sqrt{\lambda_C} \tau_C \right) \frac{1}{s} + \frac{2}{s^2}, \\ \sigma_- &= \left(\sqrt{\lambda_A \lambda_B \lambda_C} + \frac{1}{\sqrt{\lambda_A \lambda_B \lambda_C}} \right) \frac{1}{s} + \left(\sqrt{\lambda_A \lambda_B \lambda_C} r + \frac{\rho_A(s)\rho_B(s)\rho_C(s)}{\sqrt{\lambda_A \lambda_B \lambda_C} r} \right) \frac{1}{s} \\ &\quad + \left(\sqrt{\frac{\lambda_B}{\lambda_C}} \sqrt{\lambda_A} \tau_A + \sqrt{\frac{\lambda_C}{\lambda_A}} \sqrt{\lambda_B} \tau_B + \sqrt{\frac{\lambda_A}{\lambda_B}} \sqrt{\lambda_C} \tau_C \right) s + 2s^2.\end{aligned}$$

7. $SU(2, 1)$ -COORDINATES

In [?] Parker and Platis constructed Fenchel Nielsen coordinates for surface groups. Much of the construction we have given in previous sections is modelled on their coordinates. However, there is one big difference. Parker and Platis did not give coordinates for $\pi_1(Y)$ that are invariant under cyclic permutation of the three boundary curves. Instead they focussed on two of them, α and β , represented by A and B respectively. They then used $\text{tr}(A)$, $\text{tr}(B)$ and the cross-ratios of the fixed points of A and B . In this section we will show that there is a bijection between our coordinates and the Parker-Platis coordinates.

7.1. Hermitian forms and $SU(2, 1)$. Consider a Hermitian form $\langle \cdot, \cdot \rangle$ on \mathbb{C}^3 . We can write this form in terms of a matrix J and we suppose this form has signature $(2, 1)$. In what follows, we suppose J is given by (??). The group $SU(J)$ is the group of matrices with determinant 1 that preserve the form $\langle \cdot, \cdot \rangle$. From this it follows that any A in $SU(J)$ satisfies $A^* J A = J$ where A^* is the conjugate transpose matrix of A . That is, $A^{-1} = J^{-1} A^* J$. Since $\text{tr}(A^*) = \overline{\text{tr}(A)}$, we make the important observation that $\text{tr}(A^{-1}) = \overline{\text{tr}(A)}$. Applying Lawton's theorem, we immediately see that $\rho : \pi_1(Y) \rightarrow SU(J)$ is determined up to conjugation by $\text{tr}(A)$, $\text{tr}(B)$, $\text{tr}(C)$ and σ_+ together with a

root of the quadratic $Q(Y)$. Since the roots of the latter are the traces of $[A, B]$ and its inverse, these roots are complex conjugates of each other, see Parker [?].

Suppose that $A \in \mathrm{SU}(J)$ is loxodromic, that is its eigenvalues λ, μ, ν satisfy $|\lambda| > |\mu| > |\nu|$ and $\lambda\mu\nu = 1$. Now the eigenvalues of A^{-1} are the same as those of A^* . The former are $\lambda^{-1}, \mu^{-1}, \nu^{-1}$ and those of the latter are $\bar{\lambda}, \bar{\mu}$ and $\bar{\nu}$. By looking at their absolute values, we immediately see that $\lambda^{-1} = \bar{\nu}$, $\mu^{-1} = \bar{\mu}$ and $\nu^{-1} = \bar{\lambda}$. Thus, $\nu = \bar{\lambda}^{-1}$ and $\mu = \lambda^{-1}\nu^{-1} = \lambda^{-1}\bar{\lambda}$. Hence the trace of A is completely determined by λ . Indeed, for loxodromic maps there is a bijection between $\mathrm{tr}(A)$ and λ , see Lemma 4.1 of Parker and Platis [?].

7.2. Fenchel-Nielsen coordinates for $\mathrm{SU}(2, 1)$. It is clear that for representations to $\mathrm{SU}(2, 1)$ the trace parameters satisfy $\mathrm{tr}(A_j^{-1}) = \overline{\mathrm{tr}(A_j)}$ and the shape invariants satisfy $\sigma_-(Y_k) = \overline{\sigma_+(Y_k)}$. Using Lawton's theorem, we see that $\rho(\pi_1(Y_k))$ is determined by these parameters together with a choice of root of the commutator quadratic $Q(Y_k)$.

Now consider the twist-bend-bulge-turn parameters. Suppose that $A \in \mathrm{SU}(2, 1)$ is loxodromic and K is in the centraliser $Z(A)$ of $A = \rho([\alpha])$. Without loss of generality, we can write

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\bar{\lambda} & 0 \\ 0 & 0 & \bar{\lambda}^{-1} \end{pmatrix}, \quad K = T^u U^v = \begin{pmatrix} e^{u-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-u-v} \end{pmatrix}.$$

The e^{2v} eigenspace of K is the same as the $\lambda^{-1}\bar{\lambda}$ eigenspace of A , which is in V_+ . Therefore $|e^{2v}| = 1$ and so v is purely imaginary. In particular, the bulge $s_\Gamma(\alpha) = \mathrm{Re}(v)$ is zero. Furthermore, we have $e^{-u-v} = \overline{e^{u-v}}^{-1} = e^{-\bar{u}+\bar{v}} = e^{-\bar{u}-v}$. In turn, this implies that $u = \bar{u}$ and so u is real. In particular the bend $\theta_\Gamma(\alpha) = \mathrm{Im}(u)$ is zero. Therefore, we only have twist and turn parameters $t_\Gamma(\alpha) = \mathrm{Re}(u)$ and $\phi_\Gamma(\alpha) = \mathrm{Im}(v)$. Note that Parker and Platis used the word bend for what we are calling turn.

Thus we have proved that $\rho : (\pi_1(S_g)) \rightarrow \mathrm{SU}(2, 1)$ is determined by

- (1) $3g - 3$ complex trace parameters $\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_{3g-3})$;
- (2) $2g - 2$ complex shape invariants $\sigma_+(Y_1), \dots, \sigma_+(Y_{2g-2})$;
- (3) a choice of root for each of the $2g - 2$ polynomials $Q(Y_1), \dots, Q(Y_{2g-2})$;
- (4) $3g - 3$ real twist parameters $t_\Gamma(\gamma_1), \dots, t_\Gamma(\gamma_{3g-3})$ and $3g - 3$ real turn parameters $\phi_\Gamma(\gamma_1), \dots, \phi_\Gamma(\gamma_{3g-3})$.

This proves Theorem ??.

Moreover, consider a subgroup $\langle A, B, C : CBA = I \rangle$ of $\mathrm{SL}(3, \mathbb{C})$ where $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)}$, $\mathrm{tr}(B^{-1}) = \overline{\mathrm{tr}(B)}$, $\mathrm{tr}(C^{-1}) = \overline{\mathrm{tr}(C)}$ and $\sigma_- = \overline{\sigma_+}$. Using Lawton's theorem all traces in this group are determined by functions of these traces satisfy $\mathrm{tr}(W^{-1}) = \overline{\mathrm{tr}(A)}$. Hence, using Acosta's theorem we see that the group is conjugate to a subgroup of $\mathrm{SU}(2, 1)$. This proves Theorem ?? (3).

7.3. Parker-Platis coordinates. It remains to discuss the relationship between our method of parametrisation of $\rho(\pi_1(Y)) = \langle A, B, C : CBA = I \rangle$ and that of Parker and Platis.

The main difference between our parameterisation and that of Parker and Platis is that they use cross-ratios. Suppose that $\mathbf{r}_A, \mathbf{a}_A$ be repulsive and attractive eigenvectors

of A and \mathbf{r}_B , \mathbf{a}_B be repulsive and attractive eigenvectors of B respectively. Following Section 6.1 of Parker and Platis [?] we define three cross-ratios associated to A and B as follows

$$(7.1) \quad \mathbb{X}_1 = \frac{\langle \mathbf{r}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{a}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{r}_A, \mathbf{a}_A \rangle},$$

$$(7.2) \quad \mathbb{X}_2 = \frac{\langle \mathbf{a}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{a}_A, \mathbf{r}_A \rangle},$$

$$(7.3) \quad \mathbb{X}_3 = \frac{\langle \mathbf{a}_B, \mathbf{a}_A \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_A \rangle \langle \mathbf{a}_B, \mathbf{r}_A \rangle}.$$

Falbel [?] showed they satisfy the following equations, see also Proposition 5.2 of [?]

$$\begin{aligned} |\mathbb{X}_2| &= |\mathbb{X}_1| |\mathbb{X}_3|, \\ 2|\mathbb{X}_1|^2 \operatorname{Re}(\mathbb{X}_3) &= |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 + 1 - 2\operatorname{Re}(\mathbb{X}_1 + \mathbb{X}_2). \end{aligned}$$

Note that these two equations determine $|\mathbb{X}_3|$ and $\operatorname{Re}(\mathbb{X}_3)$ in terms of \mathbb{X}_1 and \mathbb{X}_2 . But there remains an ambiguity in the choice of sign of $\operatorname{Im}(\mathbb{X}_3)$.

In [?] Parker and Platis use $(\lambda_A, \lambda_B, \mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ to parametrise $\rho(\pi_1(Y)) = \langle A, B \rangle$.

Proposition 7.1. *Suppose that A and B are loxodromic elements of $\operatorname{SU}(J)$. Let \mathbb{X}_j for $j = 1, 2, 3$ be the cross-ratios of their eigenvectors as defined by (??), (??), (??). Write $C = (AB)^{-1}$ and $\sigma_+ = \operatorname{tr}(A^{-1}B) - \operatorname{tr}(A^{-1})\operatorname{tr}(B)$. There is a bijection depending only on the eigenvalues of A and B between $(\lambda_A, \lambda_B, \mathbb{X}_1, \mathbb{X}_2)$ and $(\operatorname{tr}(A), \operatorname{tr}(B), \operatorname{tr}(C), \sigma_+)$. Moreover, fixing the other parameters, the sign of the imaginary part of \mathbb{X}_3 is determined by the choice of a root of the commutator quadratic Q .*

Proof. Write the eigenvalues of A and B as λ_A, μ_A and ν_A and λ_B, μ_B and ν_B with $|\lambda_A| > |\mu_A| > |\nu_A|$ and $|\lambda_B| > |\mu_B| > |\nu_B|$.

First, the eigenvalues of A are the roots of the characteristic polynomial

$$\chi_A(x) = x^3 - \operatorname{tr}(A)x^2 + \overline{\operatorname{tr}(A)}x - 1.$$

Thus there is a bijection between $\operatorname{tr}(A)$ and the ordered set eigenvalues of A . Now suppose that $A \in \operatorname{SU}(J)$ is loxodromic. Since we have $\mu_A = \lambda_A^{-1}\bar{\lambda}_A$ and $\nu_A = \bar{\lambda}_A^{-1}$, we see that there is a bijection between the set of possible values of $\operatorname{tr}(A)$ and the set of possible values of λ_A , see Lemma 4.1 of Parker and Platis.

We know that $\operatorname{tr}(C) = \operatorname{tr}(A^{-1}B^{-1})$ and $\operatorname{tr}(C^{-1}) = \operatorname{tr}(BA)$. Also

$$\sigma_- = \operatorname{tr}(B^{-1}A) - \operatorname{tr}(B^{-1})\operatorname{tr}(A) = \bar{\sigma}_+.$$

Therefore it suffices to show there is a bijection between the two sets $(\mathbb{X}_1, \bar{\mathbb{X}}_1, \mathbb{X}_2, \bar{\mathbb{X}}_2)$ and $(\operatorname{tr}(BA), \operatorname{tr}(A^{-1}B^{-1}), \operatorname{tr}(A^{-1}B), \operatorname{tr}(AB^{-1}))$.

As above, write the eigenvalues of A and B as λ_A, μ_A and ν_A and λ_B, μ_B and ν_B with $|\mu_A| = |\mu_B| = 1$. Then from Proposition 7.6 of Parker-Platis [?] we have

$$\begin{aligned}
& \operatorname{tr}(BA) - (\lambda_A + \nu_A)\mu_B - \mu_A(\lambda_B + \nu_B) + \mu_A\mu_B \\
&= (\nu_A - \mu_A)(\nu_B - \mu_B)\mathbb{X}_1 + (\lambda_A - \mu_A)(\lambda_B - \mu_B)\overline{\mathbb{X}}_1 \\
&\quad + (\lambda_A - \mu_A)(\nu_B - \mu_B)\mathbb{X}_2 + (\nu_A - \mu_A)(\lambda_B - \mu_B)\overline{\mathbb{X}}_2, \\
& \operatorname{tr}(A^{-1}B^{-1}) - (\lambda_A^{-1} + \nu_A^{-1})\mu_B^{-1} - \mu_A^{-1}(\lambda_B^{-1} + \nu_B^{-1}) + \mu_A^{-1}\mu_B \\
&= (\nu_A^{-1} - \mu_A^{-1})(\nu_B^{-1} - \mu_B^{-1})\mathbb{X}_1 + (\lambda_A^{-1} - \mu_A^{-1})(\lambda_B^{-1} - \mu_B^{-1})\overline{\mathbb{X}}_1 \\
&\quad + (\lambda_A^{-1} - \mu_A^{-1})(\nu_B^{-1} - \mu_B^{-1})\mathbb{X}_2 + (\nu_A^{-1} - \mu_A^{-1})(\lambda_B^{-1} - \mu_B^{-1})\overline{\mathbb{X}}_2, \\
& \operatorname{tr}(A^{-1}B) - (\lambda_A^{-1} + \nu_A^{-1})\mu_B - \mu_A^{-1}(\lambda_B + \nu_B) + \mu_A^{-1}\mu_B \\
&= (\nu_A^{-1} - \mu_A^{-1})(\nu_B - \mu_B)\mathbb{X}_1 + (\lambda_A^{-1} - \mu_A^{-1})(\lambda_B - \mu_B)\overline{\mathbb{X}}_1 \\
&\quad + (\lambda_A^{-1} - \mu_A^{-1})(\nu_B - \mu_B)\mathbb{X}_2 + (\nu_A^{-1} - \mu_A^{-1})(\lambda_B - \mu_B)\overline{\mathbb{X}}_2, \\
& \operatorname{tr}(B^{-1}A) - (\lambda_A + \nu_A)\mu_B^{-1} - \mu_A(\lambda_B^{-1} + \nu_B^{-1}) + \mu_A\mu_B^{-1} \\
&= (\nu_A - \mu_A)(\nu_B^{-1} - \mu_B^{-1})\mathbb{X}_1 + (\lambda_A - \mu_A)(\lambda_B^{-1} - \mu_B^{-1})\overline{\mathbb{X}}_1 \\
&\quad + (\lambda_A - \mu_A)(\nu_B^{-1} - \mu_B^{-1})\mathbb{X}_2 + (\nu_A - \mu_A)(\lambda_B^{-1} - \mu_B^{-1})\overline{\mathbb{X}}_2.
\end{aligned}$$

This forms a set of linear equations in $\mathbb{X}_1, \overline{\mathbb{X}}_1, \mathbb{X}_2$ and $\overline{\mathbb{X}}_2$. We can solve for the cross-ratios provided the determinant of the corresponding matrix is non-zero. A brief calculation shows that this determinant is

$$\begin{aligned}
\Delta &= \left((\lambda_A - \mu_A)(\nu_A^{-1} - \mu_A^{-1}) - (\nu_A - \mu_A)(\lambda_A^{-1} - \mu_A^{-1}) \right)^2 \\
&\quad \cdot \left((\lambda_B - \mu_B)(\nu_B^{-1} - \mu_B^{-1}) - (\nu_B - \mu_B)(\lambda_B^{-1} - \mu_B^{-1}) \right)^2 \\
&= (\lambda_A - \nu_A)^2 (\lambda_A - \mu_A)^2 (\nu_A - \mu_A)^2 (\lambda_B - \nu_B)^2 (\lambda_B - \mu_B)^2 (\nu_B - \mu_B)^2.
\end{aligned}$$

On the last line we used $\lambda_A\mu_A\nu_A = \lambda_B\mu_B\nu_B = 1$. Since A and B were assumed to be loxodromic we see they do not have repeated eigenvalues, and hence $\Delta \neq 0$.

Furthermore, given $(\lambda_A, \lambda_B, \mathbb{X}_1, \mathbb{X}_2)$, or equivalently $(\operatorname{tr}(A), \operatorname{tr}(B), \operatorname{tr}(C), \sigma_+)$, using Corollary 6.5 of [?], we have

$$\mathbb{X}_3 = \frac{F(\lambda_A, \lambda_B, \mathbb{X}_1, \mathbb{X}_2) - \operatorname{tr}[A, B]}{|\mathbb{X}_1|^2 |\lambda_A|^2 |\nu_A - \mu_A|^2 |\lambda_A|^2 |\nu_B - \mu_B|^2}$$

where $F(\lambda_A, \lambda_B, \mathbb{X}_1, \mathbb{X}_2)$ is a real valued, real analytic function. Thus, the ambiguity in the roots of the commutator equation is the same as the ambiguity in the sign of the imaginary part of \mathbb{X}_3 . This completes the proof. \square

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