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On the continuity of Følner averages ☆

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ABSTRACT

It is known that if each point x of a dynamical system is generic for some invariant measure μ_x, then there is a strong connection between certain ergodic and topological properties of that system. In particular, if the acting group is abelian and the map $x \mapsto \mu_x$ is continuous, then every orbit closure is uniquely ergodic.

In this note, we show that if the acting group is not abelian, orbit closures may well support more than one ergodic measure even if $x \mapsto \mu_x$ is continuous. We provide examples of such a situation via actions of the group of all orientation-preserving homeomorphisms on the unit interval as well as the Lamplighter group. To discuss these examples, we need to extend the existing theory of weakly mean equicontinuous group actions to allow for multiple ergodic measures on orbit closures and to allow for actions of general amenable groups.

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These extensions are achieved by adopting an operator-theoretic approach.

1. Introduction

Temporal averages and their relation to invariant measures of a dynamical system are fundamental in ergodic theory. A classical notion intimately linked to this relationship is that of generic points [16]. Given a dynamical system (X, f) with X a compact metric space and $f : X \to X$ a homeomorphism, a point $x \in X$ is called *generic* for a measure μ (which is invariant under f) if for every continuous function $\varphi : X \to \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_X \varphi \, d\mu.$$

It is well known that for each ergodic measure μ there is a set of full μ-measure all of whose elements are generic for μ. One may ask what happens if all points of a dynamical system are generic for some invariant measure. To the authors' knowledge, the first to investigate this question were Dowker and Lederer [4]. As they point out, already the classical theory of Kryloff and Bogoliouboff gives that genericity of every point together with minimality of the system implies unique ergodicity, see also [20]. Indeed, weakening the assumption of minimality, they prove the following intriguing statement.

Theorem 1.1 *([4]). Assume that (X, f) has a unique minimal set and that all points are generic for some invariant measure. Then (X, f) is either uniquely ergodic or there exist infinitely many ergodic measures.*

In 1981, Katznelson and Weiss [14] improved on this result by showing that in the second case, there must exist uncountably many ergodic measures. Against the backdrop of this interesting interplay between ergodic and topological properties, it is natural to look for other assumptions which have similar structural consequences when all points are generic.

A most natural such assumption is continuity of the map $x \mapsto \mu_x$ which sends each point $x \in X$ to the invariant measure μ_x it is generic for. A priori, we can expect one of the following (non-exclusive) cases: (a) no further rigidity (some but not all μ_x are ergodic), (b) all μ_x are ergodic or (c) each orbit closure is uniquely ergodic. Recent work of Downarowicz and Weiss [5], Cai, Kwietniak, Li and Pourmand [2] as well as Xu and Zheng [27], shows the following.

Theorem 1.2 *([2,5,27]). Suppose all points of (X, f) are generic and that the map $x \mapsto \mu_x$ is continuous. Then each orbit closure of (X, f) is uniquely ergodic.*

In fact, this strong rigidity observed for $\mathbb{Z}$-actions extends to actions by (countable) abelian groups, see [2] and [27] (note that strictly speaking, [2] only considers integer actions but—as the authors point out—their techniques straightforwardly extend to the general abelian setting). The main contribution of the present work is to show that this is no longer the case when we consider actions by non-abelian groups. Indeed, we provide concrete counter-examples involving the group of all orientation preserving homeomorphisms on the unit interval as well as the Lamplighter group showing that from the above mentioned cases also (a) and (b) (without pointwise unique ergodicity) can occur.

Before formulating our main result, we introduce some terminology. To be able to average along orbits (and to be able to make sense of the notion of genericity), we need the notion of a (left) Følner sequence $(F_n)_{n\in\mathbb{N}}$ in the acting group G. The precise nature of (F_n) depends on the properties of G, see Section 2.2. When G is a countable discrete amenable group (such as the Lamplighter group), the sets F_n are finite sets and a point $x \in X$ is *μ-generic* along $(F_n)_{n\in\mathbb{N}}$ (with μ an invariant measure) if

$$\lim_{n\to\infty} 1/|F_n| \cdot \sum_{g\in F_n} \varphi(gx) = \int_X \varphi \, d\mu$$

for every continuous function $\varphi : X \to \mathbb{R}$. Clearly, this is equivalent to

$$\lim_{n\to\infty} 1/|F_n| \cdot \sum_{g\in F_n} \delta_{gx} = \mu,$$

where the limit is taken in the weak*-topology.

With this, we can state our main result, see Section 4 for the details.

Theorem 1.3. *There exists an effective transitive action of the Lamplighter group G on a compact metric space X with two trivial ergodic measures $\delta_{\hat\infty}$ and $\delta_{\check\infty}$ where for each of the following (mutually exclusive) alternatives, there is a left Følner sequence $(F_n)_{n\in\mathbb{N}}$ in G such that $x \mapsto \lim_{n\to\infty} 1/|F_n| \cdot \sum_{g\in F_n} \delta_{gx} = \mu_x$ is well-defined and*

- *$x \mapsto \mu_x$ is not continuous or*
- *$x \mapsto \mu_x$ is continuous and μ_x is not ergodic for any $x \in X \setminus \{\hat\infty, \check\infty\}$ or*
- *$x \mapsto \mu_x$ is continuous and μ_x is ergodic for some but not all $x \in X \setminus \{\hat\infty, \check\infty\}$ or*
- *$x \mapsto \mu_x$ is continuous and μ_x is ergodic for every $x \in X$.*

Moreover, if $(F_n)_{n\in\mathbb{N}}$ is a right Følner sequence, then Følner averages along $(F_n)_{n\in\mathbb{N}}$ are not continuous. Specifically,

$$\lim_{n\to\infty} 1/|F_n| \cdot \sum_{g\in F_n} \delta_{gx} = 1/2 \cdot \delta_{\hat\infty} + 1/2 \cdot \delta_{\check\infty} \qquad (x \in X \setminus \{\hat\infty, \check\infty\}).$$

En passant, the above answers a question by Li, Ye and Yu [17, Question 7] in the negative: for general actions by locally compact σ-compact amenable groups, the notions of Besicovitch-mean equicontinuity and Weyl-mean equicontinuity do not coincide (for a background, see [10,17] and references therein). Further, our example also shows that the assumptions in [27, Theorem 1.7] are necessary.

The last part of Theorem 1.3 has to be seen against the background of a general structural result, see Theorem 3.12: if each $x \in X$ is μ_x-generic along a fixed *right* Følner sequence and the map $x \mapsto \mu_x$ is continuous, then the conclusions of Theorem 1.2 still hold. In more general terms, this shows that already for the very basic ergodic properties considered in this article, there are fundamental differences between left and right Følner sequences.

Let us point out that we provide a second (and simpler) example which shows that Theorem 1.2 cannot hold in general (that is, beyond actions of abelian groups), see Example 3.8. That example, however, does not allow for the case where all μ_x are ergodic (case (b)) and furthermore, involves the less standard concept of thin Følner sequences (introduced in [24]), see Section 2.2.2 and the appendix for a background.

Finally, letting $\mathcal{F}$ be a fixed Følner sequence, the property that every point $x \in X$ is μ_x-generic along $\mathcal{F}$ with $x \mapsto \mu_x$ continuous is intimately linked to what is referred to as $\mathcal{F}$-weak mean equicontinuity, see Section 3. This notion was introduced by Zheng and Zheng for $\mathbb{Z}$-actions, see [28] and further [2,27]. We generalize (parts of) [2, Theorem 4.4] as well as [27, Theorems 3.5 and 4.3] beyond the setting of countable discrete abelian/amenable groups, see Theorems 3.4 and 3.12. This allows us to discuss the novel examples from above in a unified framework. We achieve this unification by associating to a general $\mathcal{F}$-weakly mean equicontinuous action a natural bounded linear operator S on $C(X)$ (the space of all continuous functions on X). This operator is a positive contractive projection (meaning that $S\varphi \geq 0$ if $\varphi \geq 0$, $S\mathbf{1} \leq 1$ and $S^2 = S$) and we obtain

Theorem 1.4. *The cases (a)–(c) from above translate to*

(a) $S(\varphi S\psi) = S(S\varphi S\psi)$ *for all* $\varphi, \psi \in C(X)$ *(*Seever's identity*);*
(b) $S(\varphi S\psi) = S\varphi S\psi$ *for all* $\varphi, \psi \in C(X)$ *(S is an* averaging operator*) iff* μ_x *is ergodic for all* $x \in X$*;*
(c) $T_g S = S$ *for all* $g \in G$ *iff each orbit closure is uniquely ergodic.*

Here, $T_g\varphi = \varphi \circ g$ for each $g \in G$ and $\varphi \in C(X)$, see Theorem 3.14 for the details. As an immediate consequence of this operator-theoretic perspective, we get that in general, the space of all invariant measures of a $\mathcal{F}$-weak mean equicontinuous action is a Bauer simplex, see Corollary 3.16.

2. Preliminaries

We start by briefly reviewing some basics from topological dynamics and ergodic theory. For a thorough discussion of these topics from a functional analytical perspective (which we frequently assume in this note), see [7].

A *topological dynamical system* (or simply a *system*) is a triple (X, G, α) where G is a topological group, X is a compact metrizable space and $\alpha\colon G \times X \to X$ is a jointly continuous left action of G on X. We often just write gx for $\alpha(g, x)$ and keep α implicit by simply referring to (X, G) as a system. For $x \in X$, we denote by $Gx = \{gx\colon g \in G\}$ the *orbit* of x.

Given a system (X, G), the action of G on X canonically defines the so-called *Koopman representation* of G on $C(X)$—the space of all continuous functions on X equipped with the sup-norm $\|\cdot\|_\infty$. Specifically, given $f \in C(X)$ and $g \in G$, we write $g.f$ for the mapping $x \mapsto f(gx)$ in $C(X)$. Through the dual of this representation, G further acts on $\mathcal{M}(X)$—the collection of all Borel probability measures on X. Specifically, given $\mu \in \mathcal{M}(X)$ and $g \in G$, we write $g^*\mu$ for the measure $\nu \in \mathcal{M}(X)$ with $\nu(f) = \mu(g.f)$ for $f \in C(X)$. If $\mu \in \mathcal{M}(X)$ satisfies $g^*\mu = \mu$, we call μ *invariant*. The collection of all invariant measures of (X, G) is denoted by $\mathcal{M}(X, G)$. Recall that a measure $\mu \in \mathcal{M}(X, G)$ is *ergodic* if each essentially invariant Borel set $A \subseteq X$ satisfies $\mu(A) \in \{0, 1\}$. Here, A is *essentially invariant* if $\mu(A \triangle gA) = 0$ for each $g \in G$ with $\triangle$ the symmetric difference.

2.1. Wasserstein distance

We throughout consider $\mathcal{M}(X)$ and $\mathcal{M}(X, G)$ equipped with the weak*-topology. Among the metrics which are compatible with this topology, we will make use of the *Wasserstein distance*.

Given $\mu, \nu \in \mathcal{M}(X)$, recall that $\iota \in \mathcal{M}(X^2)$ is a *coupling* of μ with ν, if the push-forwards of the projections π_1 and π_2 to the first and second coordinate, respectively, satisfy $\pi_1^*\iota = \mu$ and $\pi_2^*\iota = \nu$. The Wasserstein distance $W\colon \mathcal{M}(X)^2 \to [0, \infty)$ is defined through

$$W(\mu, \nu) = \inf_{\iota} \iota(d),$$

where the infimum is taken over all couplings ι of μ with ν, and d is some compatible metric on X. It is well known that W is a metric on $\mathcal{M}(X)$ which induces the weak*-topology, see [26, Chapter 7].

In fact, we will frequently utilize an alternative way of computing W via

$$W(\mu, \nu) = \sup_f \left| \int_X f d\mu - \int_X f d\nu \right|,$$

where the supremum is taken over all Lipschitz functions f on X with $\|f\|_{Lip} \leq 1$ [26, Remark 7.5]. Here, $\|f\|_{Lip}$ denotes the infimum of all possible Lipschitz constants for f.

2.2. Følner sequences

In order to study statistical properties along the orbits of a system (X, G), one often averages along Følner sequences.

Besides the standard Følner sequences—which we may refer to as *thick* Følner sequences in the following—we also deal with so-called *thin* Følner sequences, which allow to deal with amenable groups which are not locally compact [24]. Note that whenever we speak of Følner sequences (thin or thick), we actually refer to *left* Følner sequences. At times, we will also deal with *right* Følner sequences but whenever we do, we explicitly mention the attribute *right*. The definition of right Følner sequences is the same as for the left ones only that below—in (2.1) and (2.2)—one has to multiply from the right by K and g, respectively. Further, in (2.1), the Haar measure θ has to be replaced by a right Haar measure θ_r while in (2.2), the definition of $\mathfrak{m}_V$ has to be adjusted in the obvious way.

2.2.1. (Thick) Følner sequences

Let G be a locally compact group with a (left) Haar measure θ. A sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty compact subsets in G with positive Haar measure is called a *(thick) Følner sequence* if

$$\lim_{n\to\infty} \frac{\theta(F_n \Delta K F_n)}{\theta(F_n)} = 0, \tag{2.1}$$

for all compact $K \subseteq G$.

It is well-known that a locally compact group is σ-compact and amenable if and only if it contains a (thick) Følner sequence [8, Theorem 3.2.1].

Proposition 2.1. *If $(\mathcal{F}^{(m)})_{m\in\mathbb{N}}$ is a countable family of thick Følner sequences in a locally compact topological group G, then there exists a thick Følner sequence $\mathcal{F}$ such that $\mathcal{F}$ and $\mathcal{F}^{(m)}$ have a common subsequence for any $m \in \mathbb{N}$.*

Proof. As G allows for a (thick) Følner sequence, it is σ-compact. That is, there is a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets with $\bigcup_{n\in\mathbb{N}} K_n = G$ and $K_n \subseteq K_{n+1}$. Let $(m_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that for each $m \in \mathbb{N}$, we have $m_n = m$ infinitely often. We pick F_n from $\mathcal{F}^{(m_n)}$ with $\theta(K_n F_n \Delta F_n)/\theta(F_n) \leq 1/n$. $\square$

2.2.2. Thin Følner sequences

Let G be a topological group. Consider finite subsets $F, E \subseteq G$ and a neighbourhood V of the neutral element $e_G \in G$. The *V-matching number* $\mathfrak{m}_V(F, E)$ of F and E is the maximal cardinality of a subset M of F with an injection $\phi\colon M \to E$ such that for every $f \in M$, we have $\phi(f) \in Vf$.

The following concept was introduced in [24]. A sequence $(F_n)_{n\in\mathbb{N}}$ of finite non-empty subsets of G is said to be a *thin Følner sequence* in G if for every $g \in G$ and every open neighbourhood V of e_G, we have

$$\lim_{n\to\infty} \frac{\mathfrak{m}_V(F_n, gF_n)}{|F_n|} = 1. \tag{2.2}$$

Remark 2.2. If G is a countable discrete group, then the concepts of thin and thick Følner sequences coincide.

Remark 2.3. Note that a topological group that allows for a thin Følner sequence is separable. Indeed, if $(F_n)_{n\in\mathbb{N}}$ is such a sequence, the countable set $\bigcup_{n\in\mathbb{N}} F_n F_n^{-1}$ is dense.

Remark 2.4. As much as it is straightforward, it is important to observe that given a thin Følner sequence (F_n) and some sequence (g_n) in G, we have $\mathfrak{m}_V(F_n, gF_n) = \mathfrak{m}_V(F_n g_n, gF_n g_n)$ for each $g \in G$ and each neighbourhood V of the neutral element. In particular, with (F_n) also $(F_n g_n)$ is Følner.

A second countable topological group G is amenable if and only if it allows for a thin Følner sequence. Indeed, in [24, Remark 4.6], it is shown that whenever there is a thin Følner sequence, then G is amenable—see also Corollary A.2 below. To observe the converse, let $\{g_n \in G : n \in \mathbb{N}\}$ be a countable dense subset of G and let V_n be the open ball of radius $1/n$ centred at the neutral element e_G. With [24, Theorem 4.5], we get that for each $n \in \mathbb{N}$, there exists a finite set $F_n \subseteq G$ such that $\mathfrak{m}_{V_n}(F_n, g_k F_n)/|F_n| \geq 1 - 1/n$ for $k = 1, \ldots, n$. It is straightforward to see that $(F_n)_{n\in\mathbb{N}}$ is a thin Følner sequence in G.

Example 2.5. Examples of amenable not locally compact groups are given by *extremely amenable groups* [12]. These include the orientation-preserving homeomorphisms $\mathrm{Hom}_+(\mathbb{I})$ (equipped with the topology of uniform convergence) on the closed unit interval $\mathbb{I}$ [21]. By the above, there is hence a thin Følner sequence (F_n) in $\mathrm{Hom}_+(\mathbb{I})$.

Similarly to Proposition 2.1, we have

Proposition 2.6. *If $(\mathcal{F}^{(m)})_{m\in\mathbb{N}}$ is a countable family of thin Følner sequences in a second countable topological group G, then there exists a thin Følner sequence $\mathcal{F}$ such that $\mathcal{F}$ and $\mathcal{F}^{(m)}$ have a common subsequence for any $m \in \mathbb{N}$.*

Remark 2.7. Note that a topological group is second countable if and only if it is metrizable and separable. First, due to the Birkhoff-Kakutani Theorem, a topological group is metrizable if and only if it is first countable [19, Section 1.22]. Second, a metric space is separable if and only if it is second countable.

Proof of Proposition 2.6. Let $\{g_n \in G : n \in \mathbb{N}\}$ be dense in G and let $(m_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that for each $m \in \mathbb{N}$, we have $m_n = m$ infinitely often. Denote by V_n the open ball of radius $1/n$ centred at e_G. We pick F_n from $\mathcal{F}^{(m_n)}$ with $\mathfrak{m}_{V_n}(F_n, g_k F_n)/|F_n| \geq 1 - 1/n$ for all $k = 1, \dots, n$. $\square$

Standing assumptions Throughout this work and without further mentioning, G is assumed to be a σ-compact locally compact amenable group or a second countable amenable group. When speaking of a Følner sequence in G (without mentioning thin or thick), we refer to whatever concept of Følner sequence (thin or thick) is available in G. In fact, for the sake of a concise presentation, we formulate most statements without specifying the kind of the involved Følner sequences and readers may interpret each such statement as two statements—one in which all the occurring Følner sequences are thin and another one in which all are thick. In a similar vein, given a (thin or thick) Følner sequence $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ in G and some map f from G into $\mathbb{R}$, we may write the average of f over F_n as $1/\theta(F_n) \cdot \int_{F_n} f(g)\, d\theta(g)$ where θ denotes the counting measure in case (F_n) is a thin Følner sequence while it denotes a Haar measure if (F_n) is thick.

A background on ergodic theory with (thin) Følner sequences is given in Appendix A.

3. Weak mean equicontinuity

Given a dynamical system (X, G) and a Følner sequence $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ in G, for each $n \in \mathbb{N}$, we define a positive bounded operator $S_n = S_n^{\mathcal{F}} : C(X) \to C(X)$ through

$$S_n : f \mapsto \Big(x \mapsto 1/\theta(F_n) \int_{F_n} f(gx)\, d\theta(g)\Big).$$

Following [27], based on the dual S_n^* and the Wasserstein distance W, we introduce the pseudometrics

$$\mathcal{W}_{\mathcal{F}}(x, y) = \varlimsup_{n\to\infty} W(S_n^*\delta_x, S_n^*\delta_y)$$

and $\mathcal{W}(x, y) = \sup_{\mathcal{F}} \mathcal{W}_{\mathcal{F}}(x, y)$, where the supremum is taken over all Følner sequences. While most of the time, we are interested in left Følner sequences in this work, we may also consider the above objects (in particular, S_n and $\mathcal{W}_{\mathcal{F}}$) to be defined for right Følner sequences. Formally, the respective definitions remain the same after replacing the (left) Haar measure θ by a right Haar measure θ_r.

Remark 3.1. As in [27, Appendix], one can see that if $\mathcal{F}$ is a thin Følner sequence in G, then

$$\mathcal{W}_{\mathcal{F}}(x, y) = \varlimsup_{n\to\infty} \inf_{\sigma} \frac{1}{|F_n|} \sum_{g\in F_n} d(gx, \sigma(g)y), \tag{3.1}$$

where the infimum is taken over all permutations σ of F_n. Indeed, for $G = \mathbb{Z}$ and $\mathcal{F}$ the standard Følner sequence in $\mathbb{Z}$, (3.1) was the original definition for the pseudometric $\mathcal{W}_{\mathcal{F}}$ as introduced in [28] (motivated by the idea that statistical properties of the long time behaviour should not depend on the dynamical order on orbits). Shortly after, the above equation was established in [2] for countable abelian groups and in [27] for countable discrete amenable groups.

In the present work, we study the relationship between the continuity of the above pseudometrics and the continuity of Følner averages.

Definition 3.2. We say (X, G) is *$\mathcal{F}$-weakly mean equicontinuous* if $\mathcal{W}_{\mathcal{F}} \in C(X^2)$. We call (X, G) *weakly mean equicontinuous* if $\mathcal{W} \in C(X^2)$.

The next statement shows that this terminology is in line with [27].

Proposition 3.3. *A system (X, G) is weakly mean equicontinuous if and only if $\mathcal{W}_{\mathcal{F}} \in C(X^2)$ for every Følner sequence $\mathcal{F}$.*

Proof. Since $\mathcal{W}(x, y) \geq \mathcal{W}_{\mathcal{F}}(x, y)$, one implication is obvious. For the converse, suppose $\mathcal{W}_{\mathcal{F}}$ is continuous for all Følner sequences $\mathcal{F}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $x_n \to x \in X$. Since $\mathcal{W}$ is a pseudometric, it suffices to show that $\mathcal{W}(x_n, x) \to 0$. For $n \in \mathbb{N}$, there exists a Følner sequence $\mathcal{F}^{(n)}$ such that $\mathcal{W}_{\mathcal{F}^{(n)}}(x_n, x) + 1/n \geq \mathcal{W}(x_n, x)$ and such that $\mathcal{W}_{\mathcal{F}^{(n)}}(x_n, x)$ is a limit. By Propositions 2.1 and 2.6 (in the thick and thin case, respectively), there exists a Følner sequence $\mathcal{F}$ such that $\mathcal{F}$ and $\mathcal{F}^{(n)}$ have a common subsequence $\tilde{\mathcal{F}}^{(n)}$ for all $n \in \mathbb{N}$. Thus $\mathcal{W}(x_n, x) \leq \mathcal{W}_{\mathcal{F}^{(n)}}(x_n, x) + 1/n = \mathcal{W}_{\tilde{\mathcal{F}}^{(n)}}(x_n, x) + 1/n \leq \mathcal{W}_{\mathcal{F}}(x_n, x) + 1/n \to 0$. $\square$

Theorem 3.4. *Let (X, G) be a topological dynamical system. Suppose there is a left or right Følner sequence $\mathcal{F}$ such that*

(i) (X, G) is $\mathcal{F}$-weakly mean equicontinuous.

Then there is a subsequence $\mathcal{F}'$ of $\mathcal{F}$ such that

(ii) Every $x \in X$ is μ_x-generic along $\mathcal{F}'$ for some measure $\mu_x \in \mathcal{M}(X)$ and the map $X \ni x \mapsto \mu_x \in \mathcal{M}(X)$ is continuous.
(iii) There is a bounded linear operator $S\colon C(X) \to C(X)$ such that $S_n^{\mathcal{F}'} f(x)$ converges to $Sf(x)$ for each $f \in C(X)$ and $x \in X$.

Moreover, (ii) holds (for some $\mathcal{F}'$) if and only if (iii) holds (for the same sequence $\mathcal{F}'$) and further, (ii) and (iii) imply (i) with $\mathcal{F} = \mathcal{F}'$.

Remark 3.5. Recall that $x \in X$ is *μ-generic along* $\mathcal{F}$ if for each $f \in C(X)$, the sequence $S_n(f)(x)$ converges to $\mu(f)$. Clearly, if x is μ-generic along some Følner sequence, then $\mathrm{supp}(\mu) \subseteq \overline{Gx}$.

Proof. For (i) $\Rightarrow$ (ii), given $n \in \mathbb{N}$, let $X_n \subseteq X$ be some finite set which is $1/n$-dense in X. Due to compactness of $\mathcal{M}(X)$, we can recursively define for each $n \in \mathbb{N}$ a subsequence $\mathcal{F}^{(n)}$ of $\mathcal{F}^{(n-1)}$ (starting with $\mathcal{F}^{(0)} = \mathcal{F}$) such that each $x \in X_n$ is μ_x-generic along $\mathcal{F}^{(n)}$ for some $\mu_x \in \mathcal{M}(X)$. A diagonal argument then gives a subsequence $\mathcal{F}'$ of $\mathcal{F}$ such that each x in the dense set $\tilde{X} = \bigcup_{n\in\mathbb{N}} X_n$ is μ_x-generic along $\mathcal{F}'$ for some $\mu_x \in \mathcal{M}(X)$.

Now, as $\mathcal{W}_{\mathcal{F}'}$ is continuous (since we assume $\mathcal{W}_{\mathcal{F}}$ to be continuous), we have

$$\overline{\lim_{n\to\infty}} |S_n^{\mathcal{F}'} f(x) - S_n^{\mathcal{F}'} f(y)| \leq \overline{\lim_{n\to\infty}} W(S_n^{\mathcal{F}'^*}\delta_x, S_n^{\mathcal{F}'^*}\delta_y) = \mathcal{W}_{\mathcal{F}'}(x,y) \tag{3.2}$$

for each Lipschitz function f with $\|f\|_{Lip} \leq 1$ and all $x, y \in X$.

As a consequence, given such Lipschitz function f, some $y \in X$ and $\varepsilon > 0$, we can pick $x \in \tilde{X}$ with $\mathcal{W}_{\mathcal{F}'}(x,y) < \varepsilon$ to obtain that for all $n, m > N$ (with N such that for $n, m > N$ we have $|S_n^{\mathcal{F}'} f(x) - S_m^{\mathcal{F}'} f(x)| < \varepsilon$ and $|S_n^{\mathcal{F}'} f(x) - S_n^{\mathcal{F}'} f(y)| \leq \mathcal{W}_{\mathcal{F}'}(x,y) + \varepsilon$)

$$\begin{aligned}
&|S_n^{\mathcal{F}'} f(y) - S_m^{\mathcal{F}'} f(y)| \\
&\leq |S_n^{\mathcal{F}'} f(y) - S_n^{\mathcal{F}'} f(x)| + |S_n^{\mathcal{F}'} f(x) - S_m^{\mathcal{F}'} f(x)| + |S_m^{\mathcal{F}'} f(x) - S_m^{\mathcal{F}'} f(y)| \leq 5\varepsilon.
\end{aligned}$$

In other words, $S_n^{\mathcal{F}'} f(y)$ converges for all $y \in X$. It follows that for any function g in the span of the Lipschitz functions f with $\|f\|_{Lip} \leq 1$, $S_n^{\mathcal{F}'} g(y)$ converges for each $y \in X$. The collection of all such g is dense in $C(X)$ (see [3, Chapter 12]) and it is not hard to see that hence, $S_n^{\mathcal{F}'} f(x)$ converges for each $f \in C(X)$ and $x \in X$. We write $\mu_x(f) = \lim_{n\to\infty} S_n^{\mathcal{F}'} f(x)$. Clearly, $\mu_x \colon C(X) \ni f \mapsto \mu_x(f)$ lies in $\mathcal{M}(X)$. Further, the map $x \mapsto \mu_x$ is continuous since (X, G) is $\mathcal{F}'$-weakly mean equicontinuous.

For (ii) $\Rightarrow$ (i), let $\mathcal{F}'$ be a Følner sequence such that (ii) holds. Then, for $x, y \in X$, we have $S_n^*\delta_x \to \mu_x$ and $S_n^*\delta_y \to \mu_y$. Thus, $\mathcal{W}_{\mathcal{F}'}(x,y) = \overline{\lim}_{n\to\infty} W(S_n^*\delta_x, S_n^*\delta_y) = W(\mu_x, \mu_y)$, which is continuous in $(x, y) \in X^2$.

We next show that (ii) (for a given Følner sequence $\mathcal{F}'$) implies (iii) (with the same Følner sequence). To that end, set $S\colon C(X) \to C(X)$, $f \mapsto \mu_{(\cdot)}(f)$ with $\mu_{(\cdot)}$ given by (ii). Due to (ii), S is well defined. It is straightforward to see that S is bounded and linear. Moreover, due to (ii), for every $f \in C(X)$, $S_n^{\mathcal{F}'} f$ converges pointwise to Sf.

Finally, for (iii) $\Rightarrow$ (ii), we define $\mu_x = S^*\delta_x$ for $x \in X$. Clearly, the mapping $x \mapsto \mu_x$ is continuous. Further,

$$S_n^{\mathcal{F}'} f(x) \to Sf(x) = (S^*\delta_x)(f) = \mu_x(f).$$

We hence conclude that x is μ_x-generic along $\mathcal{F}'$. $\square$

Remark 3.6. In the above statement, if $\mathcal{F}'$ is a left Følner sequence, then $\mu_x \in \mathcal{M}(X,G)$ by a standard Krylov-Bogolyubov argument. We will see later (last part of Theorem 3.12) that the same holds true if $\mathcal{F}'$ is a right Følner sequence.

Corollary 3.7. *Suppose (X,G) is $\mathcal{F}$-weakly mean equicontinuous for some Følner sequence $\mathcal{F}$. Then the operator $S : C(X) \to C(X)$ from Theorem 3.4 is a positive contractive projection, that is, $Sf \geq 0$ if $f \geq 0$, $S\mathbf{1} \leq \mathbf{1}$ and $S^2 = S$.*

Proof. Recall that S is the limit of a subsequence of $(S_n^{\mathcal{F}})_{n\in\mathbb{N}}$. For notational convenience, we denote this subsequence simply by (S_n). The positivity and $S\mathbf{1} \leq \mathbf{1}$ are straightforwardly inherited from the elements of (S_n).

To see that S is a projection, first observe

$$(S_n^*\mu)f = \mu(S_nf) \xrightarrow{n\to\infty} \mu(Sf) = (S^*\mu)f \quad \text{for all} \quad \mu \in \mathcal{M}(X), f \in C(X),$$

using $\lim_{n\to\infty} S_nf(x) = Sf(x)$ for all $x \in X$ and dominated convergence. Accordingly, by a standard Krylov-Bogolyubov argument, $S^*\mu \in \mathcal{M}(X,G)$. Now, due to the invariance of $S^*\mu$, we have $S_n^*(S^*\mu) = S^*\mu$ for each $n \in \mathbb{N}$. This in turn gives for $\mu \in \mathcal{M}(X)$ and $f \in C(X)$

$$(S^*(S^*\mu))f = \lim_{n\to\infty} (S_n^*(S^*\mu))f = (S^*\mu)f,$$

that is, S^* is a projection. Finally, due to the Hahn-Banach Theorem, $\mathcal{M}(X)$ separates points in $C(X)$ so that the above implies $S^2 = S$. $\square$

Example 3.8. We next describe a Følner sequence $\hat{\mathcal{F}}$ with respect to which $(\mathbb{I}, G)$—where $G = \mathrm{Hom}_+(\mathbb{I})$ acts on $\mathbb{I}$ in the obvious way—is $\hat{\mathcal{F}}$-weakly mean equicontinuous.

To that end, recall from Example 2.5 that there actually is some thin Følner sequence (F_n) in G. Let (h_n) be a dense sequence in G. In the following, we may assume without loss of generality (by possibly going over to a subsequence) that $\mathfrak{m}_{B_{1/n}(\mathrm{Id})}(F_N, h_nF_N)/|F_N| \geq 1 - 1/N$ whenever $N \geq n$, where $B_{1/n}(\mathrm{Id})$ is the $1/n$-ball centred at the identity Id.

Now, given $x \in \mathbb{I}$ and $\varepsilon > 0$, let us refer to $g \in G$ as (x,ε)-*repelling* if $gy < \varepsilon$ whenever $y < x - \varepsilon$ and $gy > 1 - \varepsilon$ whenever $y > x + \varepsilon$. Note that for each $x \in \mathbb{I}$ and each $n \in \mathbb{N}$, there is $g_n^x \in G$ such that each element in $F_ng_n^x$ is $(x, 1/n^2)$-repelling. Indeed, with $\delta > 0$ such that $g\delta < 1/n^2$ and $g(1-\delta) > 1 - 1/n^2$ for all $g \in F_n$, we may choose g_n^x to be any $(x, \min\{\delta, 1/n^2\})$-repelling element in G.

Define $\hat{\mathcal{F}} = (\hat{F}_n)$ by $\hat{F}_n = \bigcup_{x=0,\frac{1}{n},\ldots,1} F_ng_n^x$. Then $\mathfrak{m}_{B_{1/n}(\mathrm{Id})}(\hat{F}_N, h_n\hat{F}_N)/|\hat{F}_N| \geq 1 - 1/N$ if $N \geq n$ and it is easy to see that $\hat{\mathcal{F}}$ is hence Følner. Further, for each $y \in \mathbb{I}$, we have

$$\frac{\{g \in \hat{F}_n : gy \leq 1/n^2\}}{|\hat{F}_n|} \to 1 - y \qquad \text{and} \qquad \frac{\{g \in \hat{F}_n : gy \geq 1 - 1/n^2\}}{|\hat{F}_n|} \to y.$$

In other words, for each $f \in C(\mathbb{I})$ and each $y \in \mathbb{I}$, we have $S^{\hat{\mathcal{F}}} f(y) = (1-y)f(0) + yf(1)$, that is, y is μ_y-generic along $\hat{\mathcal{F}}$, with $\mu_y = (1-y)\delta_0 + y\delta_1$.

Lemma 3.9. *Let (X, G) be a system and let $\mathcal{F}$ be a Følner sequence in G. If (X, G) is $\mathcal{F}$-weakly mean equicontinuous, then the support of each ergodic measure is uniquely ergodic.*

Proof. Consider two ergodic measures $\mu, \nu \in \mathcal{M}(X, G)$ and assume without loss of generality that $\mathrm{supp}(\nu) \subseteq \mathrm{supp}(\mu)$. By Corollary A.7, there exists a subsequence $\mathcal{F}'$ of $\mathcal{F}$ such that μ-almost every point is μ-generic along $\mathcal{F}'$ and ν-almost every point is ν-generic along $\mathcal{F}'$. In particular, there is $y \in X$ which is ν-generic along $\mathcal{F}'$ and a sequence $(x_n)_{n\in\mathbb{N}}$ where each x_n is μ-generic along $\mathcal{F}'$ and $x_n \to y$. Hence, $\mathcal{W}_{\mathcal{F}}(x_n, y) \geq \mathcal{W}_{\mathcal{F}'}(x_n, y) = W(\mu, \nu)$ and the continuity of $\mathcal{W}_{\mathcal{F}}$ implies $\mu = \nu$. $\square$

For the convenience of the reader, we include a proof of the next statement; for $\mathbb{Z}$-actions, see also [11, Proposition 3.9].

Lemma 3.10. *Let (X, G) be a system. For every ergodic measure μ and every transitive point $x \in X$, there is a Følner sequence $\mathcal{F}$ such that x is μ-generic along $\mathcal{F}$.*

Remark 3.11. Recall that $x \in X$ is *transitive* if $\overline{Gx} = X$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be dense in $C(X)$. By Corollary A.7, there is a Følner sequence $(F_n)_{n\in\mathbb{N}}$ and a point $y \in X$ such that y is μ-generic along $(F_n)_{n\in\mathbb{N}}$. By possibly restricting to a subsequence, we may assume without loss of generality that $\left|1/\theta(F_n) \int_{F_n} f_j(gy)\, d\theta(g) - \mu(f_j)\right| \leq 1/(2n)$ for all $j = 1, \dots, n$. For $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that for all $z \in B_{\varepsilon_n}(y)$ and $j \in \{1, \dots, n\}$, we have $\left|1/\theta(F_n) \int_{F_n} f_j(gz)\, d\theta(g) - \mu(f_j)\right| \leq 1/n$. By transitivity of x, there exists $g_n \in G$ such that $g_n x \in B_{\varepsilon_n}(y)$. Therefore,

$$\left| \frac{1}{\theta(F_n g_n)} \int\limits_{F_n g_n} f_j(gx)\, d\theta(g) - \mu(f_j) \right| = \left| \frac{1}{\theta(F_n)} \int\limits_{F_n} f_j(g g_n x)\, d\theta(g) - \mu(f_j) \right| \leq \frac{1}{n}.$$

This shows that x is μ-generic along the Følner sequence $(F_n g_n)_{n\in\mathbb{N}}$. $\square$

The next assertion generalizes Theorems 3.5 and 4.3 in [27] beyond the setting of countable discrete amenable groups.

Theorem 3.12. *Let (X, G) be a system. The following statements are equivalent.*

(i) *(X, G) is weakly mean equicontinuous.*

(ii) For all $x \in X$, the orbit closure $\overline{Gx}$ is uniquely ergodic with an ergodic measure μ_x and the map $x \mapsto \mu_x$ is continuous.

(iii) For every Følner sequence $\mathcal{F}$ in G, $S_n^{\mathcal{F}}$ converges in the strong operator topology.

Moreover, if (X, G) is $\mathcal{F}$-weakly mean equicontinuous for some Følner sequence $\mathcal{F}$ and there is an invariant measure μ with full support, that is, $\operatorname{supp}(\mu) = X$, *then (i)–(iii) are satisfied.*

Finally, if $\mathcal{W}_{\mathcal{F}}$ is continuous for some right Følner sequence $\mathcal{F}$ in G, then (i)–(iii) are satisfied.

Proof. Note that (iii) $\Rightarrow$ (i) follows from Theorem 3.4 (in combination with Proposition 3.3).

We first discuss (i) $\Rightarrow$ (ii). Due to Theorem 3.4 and Remark 3.6, it suffices to show that for every $x \in X$, the orbit closure $\overline{Gx}$ is uniquely ergodic. To that end, consider $\mu, \nu \in \mathcal{M}(\overline{Gx}, G)$. Let $\varepsilon > 0$, $y \in \operatorname{supp}(\mu)$ and pick $z \in Gx$ such that $\mathcal{W}(y, z) < \varepsilon$. By Lemma 3.10, there exists a Følner sequence $\mathcal{F}$ such that z is ν-generic along $\mathcal{F}$. By Corollary A.7, there exists a subsequence $\mathcal{F}'$ of $\mathcal{F}$ such that μ-almost every point is μ-generic along $\mathcal{F}'$. Accordingly, there is $y' \in \operatorname{supp}(\mu)$ such that y' is μ-generic along $\mathcal{F}'$ and $\mathcal{W}(y, y') < \varepsilon$. Now,

$$2\varepsilon \geq \mathcal{W}(y', y) + \mathcal{W}(y, z) \geq \mathcal{W}(y', z) \geq \mathcal{W}_{\mathcal{F}'}(y', z) = W(\mu, \nu),$$

where we used that W is compatible with the weak*-topology in the last step. As $\varepsilon > 0$ was arbitrary, we conclude $\mu = \nu$.

For (ii) $\Rightarrow$ (iii), we proceed in a similar spirit as in the proof of [2, Theorem 4.4]. First, note that for each $f \in C(X)$, and each Følner sequence $\mathcal{F}$, Theorem A.3 and point (ii) give that $x \mapsto S_n^{\mathcal{F}} f(x) = S_n f(x)$ converges pointwise to the continuous function $x \mapsto \mu_x(f)$. Note that strong convergence of S_n is equivalent to uniformity of this convergence (for each f). Hence, we assume for a contradiction that there is $f \in C(X)$ and $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there is $n \geq N$ and $x_n \in X$ with $|S_n f(x_n) - \mu_{x_n}(f)| > \varepsilon$. By possibly going over to a subsequence, we may assume without loss of generality that $S_n^* \delta_{x_n}$ converges to some $\nu \in \mathcal{M}(X, G)$ (using Krylov-Bogolyubov) and that x_n converges to some $x \in X$. Note that $|\nu(f) - \mu_x(f)| \geq \varepsilon$.

The ergodic decomposition of ν reads $\nu = \int_{\mathcal{M}(X,G)} \mu \, d\lambda(\mu)$ (see e.g. [22, page 77]) where λ-almost every measure μ is ergodic and necessarily satisfies $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\nu)$. Among those measures, there must be some μ_0 with $|\mu_0(f) - \mu_x(f)| \geq \varepsilon$. Pick some $y \in \operatorname{supp}(\mu_0)$ and observe that by (ii), $\mu_0 = \mu_y$.

Finally, by the Portmanteau Theorem (and since $y \in \operatorname{supp}(\nu) \supseteq \operatorname{supp}(\mu_y)$), for every open neighbourhood U of y, we have

$$\varliminf_{n\to\infty} S_n^* \delta_{x_n}(U) = \varliminf_{n\to\infty} 1/\theta(F_n) \cdot \theta(\{g x_n \in U : g \in F_n\}) \geq \nu(U) > 0.$$

As a consequence, there is a sequence g_n in G with $g_n x_n \to y$ so that, due to the continuity of $x \mapsto \mu_x$, we have $\mu_{g_n x_n} \to \mu_y$. At the same time, due to the unique ergodicity of orbit closures, $\mu_{g_n x_n} = \mu_{x_n} \to \mu_x$. It follows that $\mu_x = \mu_y$ in contradiction to the assumptions on $\mu_y = \mu_0$.

To see the "moreover"-part, given $x \in X$, observe that due to [9, Lemma 6] (or, alternatively, the ergodic decomposition of μ), there is a sequence (μ_n) of ergodic measures in $\mathcal{M}(X,G)$ and a sequence (x_n) in X with $x_n \in \operatorname{supp}(\mu_n)$ such that $x_n \to x$. By Lemma 3.9, $\mathcal{W}_{\mathcal{F}}(x_n, gx_n) = 0$ for each $g \in G$ so that continuity of $\mathcal{W}_{\mathcal{F}}$ gives

$$\mathcal{W}_{\mathcal{F}}(x, gx) = \lim_{n\to\infty} \mathcal{W}_{\mathcal{F}}(x_n, gx_n) = 0. \tag{3.3}$$

Due to Theorem 3.4, we may assume without loss of generality that for each $f \in C(X)$, $y \mapsto S^{\mathcal{F}} f(y) = \lim_{n\to\infty} S_n^{\mathcal{F}} f(y) = \mu_y(f)$ is well-defined and continuous. With (3.3), this gives that $y \mapsto S^{\mathcal{F}} f(y)$ is actually constant on $\overline{Gx}$. Now, Theorem A.3 and Tietze's Extension Theorem give that $\overline{Gx}$ is uniquely ergodic. As x was arbitrary, this shows (ii).

To see the last part, recall from Theorem 3.4 that there is a bounded linear operator $S^{\mathcal{F}'} : C(X) \to C(X)$ and a subsequence $\mathcal{F}'$ of $\mathcal{F}$ such that $S_n^{\mathcal{F}'} f(x)$ converges to $S^{\mathcal{F}'} f(x)$ for each $f \in C(X)$ and $x \in X$. Observe that since $\mathcal{F}'$ is right Følner, we have for each $f \in C(X)$ and all $g \in G$ that $S^{\mathcal{F}'} f(x) = S^{\mathcal{F}'} f(gx)$. As $S^{\mathcal{F}'} f$ is continuous, we therefore have that $S^{\mathcal{F}'} f$ is constant on orbit closures. With Proposition A.4 and Tietze's Extension Theorem, this implies (ii) (and $S^{\mathcal{F}'} f(x) = \mu_x(f)$ for each $x \in X$). □

Example 3.13. As a consequence of the above, a transitive system is weakly mean equicontinuous if and only if it is uniquely ergodic. In particular, the system $(\mathbb{I}, G)$ from Example 3.8 is not weakly mean equicontinuous. However, the canonical action of $G = \mathrm{Hom}_+(\mathbb{I})$ on $\mathbb{R}/\mathbb{Z}$—obtained by identifying 0 and 1—clearly is weakly mean equicontinuous.

We end this section by combining the above results with some aspects of the general theory of positive contractive projections and averaging operators, see [23, Section 11.3] as well as [13,15,25] (and references therein).

Theorem 3.14. *Let $\mathcal{F}$ be a left or right Følner sequence and suppose that (X,G) is $\mathcal{F}$-weakly mean equicontinuous. Then there exists a subsequence (F_n) of $\mathcal{F}$ and a bounded linear operator $S\colon C(X) \to C(X)$ such that for all $f \in C(X)$ and $x \in X$*

$$\lim_{n\to\infty} 1/\theta(F_n) \cdot \int_{F_n} f(gx)\, d\theta(g) = Sf(x). \tag{3.4}$$

Vice versa, if there is a Følner sequence (F_n) and a bounded linear operator $S\colon C(X) \to C(X)$ satisfying (3.4), then (X,G) is (F_n)-weakly mean equicontinuous. Moreover, S is a positive contractive projection and the following statements hold.

(a) $S(fSh) = S(SfSh)$ *for all* $f, h \in C(X)$ *(*Seever's identity*).*
(b) $S(fSh) = SfSh$ *for all* $f, h \in C(X)$ *(*S *is an* averaging operator*) if and only if* μ_x *is ergodic for all* $x \in X$*.*
(c) $T_gS = S$ *for all* $g \in G$ *(where* $T_gf = g.f$ *for* $g \in G$ *and* $f \in C(X)$*) if and only if each orbit closure is uniquely ergodic.*

Proof. The first part is Theorem 3.4. Further, if $\mathcal{F}$ is left Følner, then Corollary 3.7 gives that S is a positive contractive projection. If, alternatively, $\mathcal{F}$ is right Følner, then it is again obvious that S is positive and contractive (as the expression on the left in (3.4) is positive and contractive for each n). Furthermore, as shown in the last part of the proof of Theorem 3.12, $Sf(x) = \mu_x(f)$ for all $x \in X$ and $f \in C(X)$ with $\mu_x \in \mathcal{M}(X, G)$ and $x \mapsto \mu_x$ constant on orbit closures. This immediately implies $S^2 = S$, that is, S is also a positive contractive projection for right Følner sequences.

Now, positive contractive projections on $C(X)$ always fulfil Seever's identity, see [25, Theorem 1]. This gives item (a).

For item (c), we make use of the fact that $Sf(x) = \mu_x(f)$ for some $\mu_x \in \mathcal{M}(X, G)$ (see Theorem 3.4 and Remark 3.6 for left Følner sequences or again, the last part of the proof of Theorem 3.12 for right Følner sequences). Let us first assume that each orbit closure is uniquely ergodic. Then, for all $f \in C(X)$ and $x \in X$ we get that $T_gSf(x) = \mu_{gx}(f) = \mu_x(f) = Sf(x)$. Note that in the other direction, we only have to consider left Følner sequences due to the last part of Theorem 3.12. Now, if S is invariant under all T_g's, we have that $\mu_{gx}(f) = T_gSf(x) = Sf(x) = \mu_x(f)$ for all $f \in C(X)$, $x \in X$ and $g \in G$. That is, $x \mapsto \mu_x$ is constant along orbits. As furthermore, $x \mapsto \mu_x$ is continuous (Theorem 3.4), we have that $x \mapsto \mu_x$ is constant on orbit closures and we obtain unique ergodicity as in the proof of Theorem 3.12 (using Corollary A.7).

Finally, to prove (b), we make use of the following characterization, see [15, Theorem 2.2]: the operator S is averaging if and only if for each $x \in X$ the support of μ_x is contained in the set

$$D_x = \{y \in X : Sf(y) = Sf(x) \text{ for all } f \in C(X)\} = \{y \in X : \mu_y = \mu_x\}.$$

First, observe that if each orbit closure is uniquely ergodic, then (b) is trivial since in this case, μ_x is obviously ergodic and $\operatorname{supp}(\mu_x) \subseteq \overline{Gx} \subseteq D_x$. Due to the last part of Theorem 3.12, we are hence left to show (b) only for left Følner sequences.

Now, according to Lemma 3.9, we have $\operatorname{supp}(\mu_x) \subseteq D_x$ in case that μ_x is ergodic. Accordingly, S is averaging if all μ_x are ergodic. For the other direction, assume that $\operatorname{supp}(\mu_x)$ is contained in D_x. Since $\operatorname{supp}(\mu_x)$ is closed and invariant, there is an ergodic measure μ with $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu_x)$. By Corollary A.7, there exists a subsequence (F'_n) of (F_n) such that μ-almost every point is μ-generic along (F'_n). In particular, there exists $y \in \operatorname{supp}(\mu)$ with

$$\mu(f) = \lim_{n\to\infty} S_n^{(F'_n)} f(y) = \lim_{n\to\infty} S_n^{(F_n)} f(y) = \mu_y(f) = \mu_x(f),$$

for all $f \in C(X)$. Accordingly, all μ_x are ergodic. $\square$

Remark 3.15. Note that the relation $T_g S = S$ (for $g \in G$) in Theorem 3.14 (c) is equivalent to $T_g S = S T_g$ (for $g \in G$) since $\mu_x \in \mathcal{M}(X, G)$ (for $x \in X$) is equivalent to $S T_g = S$ (for $g \in G$).

Recall that $\mathcal{M}(X, G)$ is a simplex and that a simplex whose extreme points form a closed set is called a *Bauer simplex*, see [22]. We can conclude from Theorem 3.14 together with [18, Theorem 3] the following rigidity result which is also indirectly contained in [2,27] for countable discrete abelian/amenable groups and $\mathcal{F}$ a two-sided Følner sequence.

Corollary 3.16. *Let (X, G) be a topological dynamical system which is $\mathcal{F}$-weakly mean equicontinuous for some left or right Følner sequence $\mathcal{F}$. Then $\mathcal{M}(X, G)$ is a Bauer simplex.*

4. $\mathcal{F}$-weak mean equicontinuity versus weak mean equicontinuity

In Example 3.8, we already saw that in general, $\mathcal{F}$-weak mean equicontinuity for some Følner sequence $\mathcal{F}$ does not imply weak mean equicontinuity. As much as this phenomenon somehow appears to be related to the *size* of the acting group, it is not merely a consequence of the lack of local compactness. In fact, in this section, we give a complete description of the Følner averages of an action of the Lamplighter group—which, in contrast to $\mathrm{Hom}_+(\mathbb{I})$, is locally compact. Among others, we will see that even if a system is $\mathcal{F}$-weakly mean equicontinuous and every point is generic for an *ergodic* measure along $\mathcal{F}$, the system does not need to be weakly mean equicontinuous.

Consider $\hat{X} = (\mathbb{Z} \cup \{\infty\}) \times \{1\}$ and $\check{X} = (\mathbb{Z} \cup \{\infty\}) \times \{0\}$ and write $\hat{s} = (s, 1)$ and $\check{s} = (s, 0)$ for $s \in \mathbb{Z} \cup \{\infty\}$. We equip $\hat{X}$ with some metric $\hat{d}$ which induces the topology of the one-point compactification (of $\mathbb{Z}$) on $\hat{X}$ and metrize $\check{X}$ similarly (by $\check{d}$). Finally, we equip $X = \check{X} \cup \hat{X}$ with the metric d that restricts to $\check{d}$ and $\hat{d}$ on $\check{X}$ and $\hat{X}$, respectively, and which satisfies $d(x, y) = 1$ for $x \in \hat{X}$ and $y \in \check{X}$.

Let $f\colon X \to X$ be the transposition $f = (\hat{0}, \check{0})$, that is, f is the identity on $X \setminus \{\check{0}, \hat{0}\}$ and satisfies $f(\hat{0}) = \check{0}$, $f(\check{0}) = \hat{0}$. Let $\sigma\colon X \to X$ be the shift map with $\sigma(\check{s}) = \widecheck{(s-1)}$ and $\sigma(\hat{s}) = \widehat{(s-1)}$, where $\infty - 1 = \infty$. Then the countable discrete group $G = \langle f, \sigma \rangle$ is (isomorphic to) the Lamplighter group and acts on X by homeomorphisms. Defining $f_n = (\hat{n}, \check{n})$, observe that $f_n = \sigma^{-n} \circ f \circ \sigma^n$ and $f_b \circ \sigma^a = \sigma^a \circ \sigma^{-b-a} \circ f \circ \sigma^{a+b} = \sigma^a \circ f_{a+b}$. Further, $f_n^2 = \mathrm{Id}$ and

$$f_n \circ f_m = f_m \circ f_n \qquad \text{for all } n, m \in \mathbb{Z}. \tag{4.1}$$

We denote by $[\mathbb{Z}]^{<\omega}$ the collection of all finite subsets of $\mathbb{Z}$; elements of $[\mathbb{Z}]^{<\omega}$ are denoted by bold face lower case letters. Given $\mathbf{b} = \{b_1, \ldots, b_k\} \in [\mathbb{Z}]^{<\omega}$, we write $f_{\mathbf{b}} = f_{b_k} \circ \cdots \circ f_{b_1}$. Note that due to (4.1), $f_{\mathbf{b}}$ is well-defined. For the convenience of the reader, we recall the following basic fact about G. For more information regarding the Lamplighter group, see for instance [1, Section 4].

Lemma 4.1. *The mapping $\mathbb{Z} \times [\mathbb{Z}]^{<\omega} \to G$ defined by $(a, \mathbf{b}) \mapsto \sigma^a \circ f_{\mathbf{b}}$ is bijective.*

Proof. From $f^2 = \mathrm{Id}$ we observe that every element $g \in G$ is of the form

$$g = \sigma^{a_k} \circ f \circ \sigma^{a_{k-1}} \circ f \circ \cdots \circ f \circ \sigma^{a_0},$$

where $a_0, \ldots, a_k \in \mathbb{Z}$. We write $b_i = \sum_{j=0}^{i-1} a_j$ for $i = 1, \ldots, k+1$. With $a = b_{k+1}$ and $\mathbf{b} = \{b_i \colon i = 1, \ldots, k\}$, a straightforward computation shows $g = \sigma^a \circ f_{\mathbf{b}}$. This yields surjectivity of the mapping under consideration.

To show injectivity, consider distinct $(a, \mathbf{b}), (a', \mathbf{b}') \in \mathbb{Z} \times [\mathbb{Z}]^{<\omega}$. Write $g = \sigma^a \circ f_{\mathbf{b}}$ and $g' = \sigma^{a'} \circ f_{\mathbf{b}'}$. If $a \neq a'$, clearly $g \neq g'$. If $a = a'$, then $\mathbf{b} \neq \mathbf{b}'$. From $\mathbf{b} = \{b \in \mathbb{Z} \colon g(\hat{b}) \in \check{X}\}$ and a similar statement about $\mathbf{b}'$ and g', we obtain $g \neq g'$ from $\mathbf{b} \neq \mathbf{b}'$. □

4.1. Left Følner sequences

We next discuss certain left Følner sequences (F_n) in G, where for each $n \in \mathbb{N}$, we will obtain F_n as a product of elements of a Følner sequence in $\mathbb{Z}$ and subsets of

$$I_n = \mathcal{P}([-2^n, 2^n]),$$

that is, subsets of the power set of $[-2^n, 2^n]$. Here and in the following, unless stated otherwise, given $m, n \in \mathbb{Z}_{\geq 0}$, we denote by $[-m, n]$ the respective interval in $\mathbb{Z}$, that is, $[-m, n] = \{-m, -m+1, \ldots, n\}$. We will make use of the following auxiliary statement whose proof we include for the convenience of the reader.

Lemma 4.2. *Let $(r_\ell)_{\ell \in \mathbb{Z}}$ be a real-valued sequence with $0 \leq r_\ell \leq 1$. Then for each $n \in \mathbb{N}$, there is a collection $V_n \subseteq \mathcal{P}([-n, n])$ such that*

$$\left| \frac{|\{\mathbf{v} \in V_n \colon \ell \in \mathbf{v}\}|}{|V_n|} - r_\ell \right| \leq 1/n \qquad (\ell = -n, \ldots, n). \tag{4.2}$$

Proof. We construct $(V_n)_{n \in \mathbb{N}}$ recursively in a way which ensures $|V_{n+1}| > |V_n| \geq n$ and $|r_\ell^{(n)} - r_\ell| \leq 1/|V_n|$ for $\ell = -n, \ldots, n$, where $r_\ell^{(n)} = |\{\mathbf{v} \in V_n \colon \ell \in \mathbf{v}\}|/|V_n|$.

The case $n = 1$ is trivial. Assuming we have already constructed V_n, we define $\mathbf{w} = \{\ell \in [-n, n] \colon r_\ell^{(n)} < r_\ell\}$.

We now enlarge some elements of $V_n \cup \{\mathbf{w}\}$ by adding $-(n+1)$. Specifically, we pick $U \subseteq V_n$ such that $\mathbf{w} \notin U$ and $r_{-(n+1)} - 1/|V_n| \leq |U|/|V_n| \leq r_{-(n+1)}$. We set

$$V_n' = \{\mathbf{u} \cup \{-n-1\} \colon \mathbf{u} \in U \cup \{\mathbf{w}\}\} \cup V_n \setminus U.$$

Similarly, we enlarge some sets of V_n' by adding $n+1$ to obtain $V_{n+1} \in \mathcal{P}([-n-1, n+1])$ such that $|V_{n+1}| > |V_n|$ and $|r_\ell^{(n+1)} - r_\ell| \leq 1/|V_{n+1}|$ holds for $\ell = -(n+1), \dots, n+1$. The statement follows. $\square$

Given a real-valued sequence $r = (r_\ell)_{\ell \in \mathbb{Z}}$ with $0 \leq r_\ell \leq 1$, we choose V_n as in Lemma 4.2 and set $B_n = B_n^r = \{\mathbf{b} \in I_n \colon \mathbf{b} \cap [-n, n] \in V_n\}$ as well as

$$F_n = F_n^r = \{\sigma^a \circ f_{\mathbf{b}} \in G \colon a \in [-2^n, 2^n],\, \mathbf{b} \in B_n\}. \tag{4.3}$$

Note that due to (4.2), for $\ell = -n, \dots, n$, we have

$$\left| \frac{|\{\mathbf{b} \in B_n \colon \ell \in \mathbf{b}\}|}{|B_n|} - r_\ell \right| \leq 1/n. \tag{4.4}$$

Lemma 4.3. *We have that $(F_n)_{n\in\mathbb{N}}$ with F_n as in (4.3) is left Følner in G.*

Proof. Consider $g = \sigma^c \circ f_{\mathbf{d}} \in G$ so that $g \circ \sigma^a \circ f_{\mathbf{b}} = \sigma^{c+a} \circ f_{\mathbf{d}+a} \circ f_{\mathbf{b}}$, where we write $\mathbf{d} + a = \{d + a \colon d \in \mathbf{d}\}$. Further, observe that

$$\begin{aligned}
(g \circ F_n) \setminus F_n &\subseteq g \circ \bigcup_{t \in \mathbf{d}} \{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon t + a \notin [-2^n, 2^n] \setminus [-n, n]\} \\
&\quad \cup g \circ \{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon c + a \notin [-2^n, 2^n]\} \\
&\subseteq g \circ \bigcup_{k \in \mathbf{d} \cup \{c\}} \{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon a \notin ([-2^n, 2^n] \setminus [-n, n]) - k\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{|(g \circ F_n) \setminus F_n|}{|F_n|} &\leq \sum_{k \in \mathbf{d} \cup \{c\}} \frac{|B_n| \cdot |[-2^n, 2^n] \setminus ([-2^n, 2^n] \setminus [-n, n]) - k|}{|B_n| \cdot |[-2^n, 2^n]|} \\
&= \sum_{k \in \mathbf{d} \cup \{c\}} \frac{|[-2^n, 2^n] \setminus [-2^n, 2^n] - k|}{|[-2^n, 2^n]|} + \frac{|[-2^n, 2^n] \cap [-n, n] - k|}{|[-2^n, 2^n]|}
\end{aligned}$$

which tends to 0 as $n \to \infty$ since $[-2^n, 2^n]$ defines a Følner sequence in $\mathbb{Z}$. $\square$

Remark 4.4. For future reference, observe that a straightforward adaption of the above proof shows that for any sequence $\left(r^{(n)}\right)_{n\in\mathbb{N}}$ of sequences $r^{(n)}$ with values in the real interval $[0, 1]$, we have that $F_n = F_n^{r^{(n)}}$ defines a (left) Følner sequence $(F_n)_{n\in\mathbb{N}}$.

Remark 4.5. Some authors prefer to understand Følner sequences to be monotone and exhausting. Clearly, one can always obtain a monotone and exhausting Følner sequence (F_n') from (F_n^r) with the same asymptotics. Indeed, given a strictly increasing and sufficiently sparse sequence (n_k) in $\mathbb{N}$, we may just set $F_k' = F_{n_{k+1}}^r \cup \{\sigma^a \circ f_{\mathbf{b}} \colon a \in [-2^{n_k}, 2^{n_k}],\, \mathbf{b} \in I_{n_k}\}$.

Lemma 4.6. *Let $r = (r_\ell)_{\ell\in\mathbb{Z}}$ be a real-valued sequence with $0 \leq r_\ell \leq 1$. Consider a (left) Følner sequence (F_n) given by (4.3). Then for $x = (b, i) \in X$, we have the following convergence (with respect to the weak*-topology) as $n \to \infty$*

$$\frac{1}{|F_n|} \cdot \sum_{g\in F_n} \delta_{gx} \to \begin{cases} (1 - r_b) \cdot \delta_{\hat\infty} + r_b \cdot \delta_{\check\infty} & \text{if } x \in \hat X, \\ r_b \cdot \delta_{\hat\infty} + (1 - r_b) \cdot \delta_{\check\infty} & \text{if } x \in \check X. \end{cases} \tag{4.5}$$

Proof. W.l.o.g. we assume $x \in \hat X$, i.e. that $x = \hat b$. With $A_n = [-2^n, 2^n]$, we have

$$\begin{aligned} \frac{1}{|F_n|} \sum_{g\in F_n} \delta_{gx} &= \frac{1}{|A_n| \cdot |B_n|} \sum_{\mathbf{b}\in B_n} \sum_{a\in A_n} \delta_{\sigma^a\circ f_{\mathbf{b}} x} \\ &= \frac{1}{|A_n| \cdot |B_n|} \sum_{\substack{a\in A_n \\ \mathbf{b}\in B_n,\, b\notin\mathbf{b}}} \delta_{\sigma^a \hat b} + \frac{1}{|A_n| \cdot |B_n|} \sum_{\substack{a\in A_n \\ \mathbf{b}\in B_n,\, b\in\mathbf{b}}} \delta_{\sigma^a \check b}, \end{aligned}$$

for each $n \in \mathbb{N}$. Observe that $(\hat X, \langle\sigma\rangle)$ and $(\check X, \langle\sigma\rangle)$, respectively, are uniquely ergodic so that $\hat b$ ($\check b$) is $\delta_{\hat\infty}$-generic ($\delta_{\check\infty}$-generic) along $(A_n)_{n\in\mathbb{N}}$. Hence, $\frac{1}{|A_n|} \sum_{a\in A_n} \delta_{\sigma^a \hat b} \to \delta_{\hat\infty}$ and $\frac{1}{|A_n|} \sum_{a\in A_n} \delta_{\sigma^a \check b} \to \delta_{\check\infty}$. Now, (4.5) follows from $|\{\mathbf{b} \in B_n : b \in \mathbf{b}\}|/|B_n| \to r_b$, see (4.4). □

Remark 4.7. Note that by considering Følner sequences $(F_n)_{n\in\mathbb{N}}$ as in Remark 4.4, we can enforce essentially any kind of convergence or divergence of the measures $\frac{1}{|F_n|} \sum_{g\in F_n} \delta_{gx}$. This shows the first half of Theorem 1.3. However, there is one inherent and unavoidable symmetry. As the computation in the above proof shows, along *each* Følner sequence in G, $\frac{1}{|F_n|} \sum_{g\in F_n} \delta_{g(b,1)}$ accumulates at $\lambda\delta_{\hat\infty} + (1 - \lambda)\delta_{\check\infty}$ if and only if $\frac{1}{|F_n|} \sum_{g\in F_n} \delta_{g(b,0)}$ accumulates at $(1 - \lambda)\delta_{\hat\infty} + \lambda\delta_{\check\infty}$.

4.2. Right Følner sequences

Observe that unless $r_\ell = 1/2$ for all $\ell \in \mathbb{Z}$, (F^r_n) as defined in (4.3) is not right Følner. Indeed, given $b \in \mathbb{Z}$, we have

$$\varliminf_{n\to\infty} |(F^r_n \circ f_b) \triangle F^r_n|/|F^r_n| \geq |2r_b - 1|.$$

Note that, a priori, for a right Følner sequence (F_n), the average $1/|F_n| \cdot \sum_{g\in F_n} \delta_{gx}$ does not need to converge to an invariant measure. Yet, in our example, we obtain the following.

Lemma 4.8. *Assume (F_n) is a right Følner sequence. Then* *(i) For each* $b \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{|\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \in \mathbf{b}\}|}{|F_n|} = \lim_{n\to\infty} \frac{|\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \notin \mathbf{b}\}|}{|F_n|} = \frac{1}{2}.$$

(ii) For each $x \in X \setminus \{\hat{\infty}, \check{\infty}\}$, *we have* $1/|F_n| \cdot \sum_{g\in F_n} \delta_{gx} \to 1/2 \cdot \delta_{\hat{\infty}} + 1/2 \cdot \delta_{\check{\infty}}$.

Proof. Given any sequence (F_n) of subsets in G, for each $b \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} |F_n \circ f_b \setminus F_n| &\geq \big||\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \in \mathbf{b}\}| - |\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \notin \mathbf{b}\}|\big| \\ &= \big||F_n| - 2|\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \in \mathbf{b}\}|\big|. \end{aligned}$$

Hence, if (F_n) is right Følner, i.e. $|F_n \circ f_b \setminus F_n|/|F_n| \to 0$, we obtain (i).

Towards (ii), assume w.l.o.g. that $x \in \hat{X}$, i.e. that $x = \hat{b}$ for some $b \in \mathbb{Z}$. We have

$$\frac{1}{|F_n|} \cdot \sum_{g\in F_n} \delta_{gx} = \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \notin \mathbf{b}}} \delta_{\sigma^a \hat{b}} + \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \in \mathbf{b}}} \delta_{\sigma^a \check{b}}, \tag{4.6}$$

for each $n \in \mathbb{N}$. Observe that since (F_n) is (right) Følner, we have that for each $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, there is n_0 such that for all $n \geq n_0$,

$$|\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon |a| < k_0\}|/|F_n| < \varepsilon. \tag{4.7}$$

Now, given some continuous function h on X and $\varepsilon > 0$ (we may assume $\varepsilon < 1/4$), choose $k_0 \in \mathbb{N}$ such that $|h(\hat{\infty}) - h(\widehat{b-a})| < \varepsilon$ and $|h(\check{\infty}) - h(\widetilde{b-a})| < \varepsilon$ for all $a \in \mathbb{Z}$ with $|a| \geq k_0$ and let n_0 be such that (4.7) holds for all $n \geq n_0$. Due to (i), we may also assume without loss of generality that $\big|1/2 - |\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \in \mathbf{b}\}|/|F_n|\big| < \varepsilon$ and $\big|1/2 - |\{\sigma^a \circ f_{\mathbf{b}} \in F_n \colon b \notin \mathbf{b}\}|/|F_n|\big| < \varepsilon$ for all $n \geq n_0$. With (4.6), we obtain for all $n \geq n_0$,

$$\begin{aligned} &|1/2 \cdot \delta_{\hat{\infty}}(h) + 1/2 \cdot \delta_{\check{\infty}}(h) - 1/|F_n| \cdot \sum_{g\in F_n} \delta_{gx}(h)| \\ &= \Bigg|1/2 \cdot h(\hat{\infty}) + 1/2 \cdot h(\check{\infty}) - \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \notin \mathbf{b}}} h(\widehat{b-a}) - \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \in \mathbf{b}}} h(\widetilde{b-a})\Bigg| \\ &\leq \Bigg|1/2 \cdot h(\hat{\infty}) - \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \notin \mathbf{b}}} h(\widehat{b-a})\Bigg| + \Bigg|1/2 \cdot h(\check{\infty}) - \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \in \mathbf{b}}} h(\widetilde{b-a})\Bigg| \\ &\leq \Bigg|1/2 \cdot h(\hat{\infty}) - \frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \notin \mathbf{b}}} h(\hat{\infty})\Bigg| + \Bigg|\frac{1}{|F_n|} \sum_{\substack{\sigma^a \circ f_{\mathbf{b}} \in F_n \\ b \notin \mathbf{b}}} h(\hat{\infty}) - h(\widehat{b-a})\Bigg| + \Big|\ldots\Big| \\ &\leq \varepsilon \cdot (h(\hat{\infty}) + h(\check{\infty}) + 2). \end{aligned}$$

As ε and h were arbitrary, this finishes the proof. □

Remark 4.9. For the example under consideration, Lemma 4.8 rules out continuity of the map $x \mapsto \mu_x$ if μ_x is obtained through averaging along a right Følner sequence. This shows the second half of Theorem 1.3.

Appendix A. Ergodic theory with thin Følner sequences

The goal of this appendix is to convince the reader that some relevant parts of the basic machinery from ergodic theory are also available when working with thin Følner sequences. The discussed statements are well-known for σ-compact locally compact topological groups and (thick) Følner sequences, see for example [6]. While most parts of their proofs immediately carry over to the case of thin Følner sequences, we provide some discussion of the slight deviations from the standard situation (and whenever there is no such discussion, the proofs are literally the same in both cases).

In all of the following, we adopt the notation and the standing assumptions from the main body of this article.

Lemma A.1. *Given a system (X, G), for $f \in C(X)$ and $g \in G$, we have $\|S_n(g.f - f)\|_\infty \to 0$.*

Proof. Let $\varepsilon > 0$. By continuity, there exists an open neighbourhood V of e_G such that for all $g' \in V$, we have $\|g'.f - f\|_\infty < \varepsilon/2$. By definition of the V-matching numbers, for $n \in \mathbb{N}$, there exists $E_n \subseteq F_n$ with $|E_n| = \mathfrak{m}_V(F_n, gF_n)$ and a bijection $\phi_n \colon F_n \to gF_n$ such that for all $g' \in E_n$, we have $\phi_n(g'){g'}^{-1} \in V$ and hence,

$$\|\phi_n(g').f - g'.f\|_\infty = \|{g'}^{-1}.(\phi_n(g').f) - f\|_\infty = \|\phi_n(g'){g'}^{-1}.f - f\|_\infty \leq \varepsilon/2.$$

Further, for large enough $n \in \mathbb{N}$, we have $\mathfrak{m}_V(gF_n, F_n)/|F_n| \geq 1 - \varepsilon/(4\|f\|_\infty)$ and hence,

$$\frac{|F_n \setminus E_n|}{|F_n|} = 1 - \frac{|E_n|}{|F_n|} < \frac{\varepsilon}{4\|f\|_\infty}.$$

For such n, we compute

$$\begin{aligned}
\|S_n(g.f - f)\|_\infty &= \frac{1}{|F_n|}\Big\| \sum_{g' \in F_n} g'.(g.f) - g'.f \Big\|_\infty = \frac{1}{|F_n|}\Big\| \sum_{g' \in F_n} gg'.f - g'.f \Big\|_\infty \\
&\leq \frac{1}{|F_n|} \sum_{g' \in E_n} \|\phi_n(g').f - g'.f\|_\infty + \frac{1}{|F_n|} \sum_{g' \in F_n \setminus E_n} 2\|f\|_\infty \\
&\leq \frac{|E_n|}{|F_n|}\frac{\varepsilon}{2} + \frac{|F_n \setminus E_n|}{|F_n|} 2\|f\|_\infty \leq \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the statement follows. □

Corollary A.2 *(Krylov-Bogolyubov). Given a dynamical system (X, G) and $x \in X$, then every limit point of $(S_n^* \delta_x)$ is contained in $\mathcal{M}(X, G)$.*

Proof. Let μ be a limit point and $f \in C(X)$. We have

$$\begin{aligned} |g^*\mu(f) - \mu(f)| &\leq |g^*\mu(f) - g^* S_n^* \delta_x(f)| + |g^* S_n^* \delta_x(f) - S_n^* \delta_x(f)| + |S_n^* \delta_x(f) - \mu(f)| \\ &\leq |\mu(g.f) - S_n^* \delta_x(g.f)| + \|S_n(g.f - f)\|_\infty + |S_n^* \delta_x(f) - \mu(f)| \to 0. \end{aligned}$$

Thus, $g^*\mu = \mu$ as claimed. □

Based on the above, one obtains

Theorem A.3. *Let (X, G) be a dynamical system. The following are equivalent.*

(i) The system (X, G) is uniquely ergodic.
(ii) For every $f \in C(X)$, there is a constant c such that for some Følner sequence $(F_n)_{n\in\mathbb{N}}$ and every $x \in X$, we have $(S_n f)(x) \to c$.

Further, if one of the above conditions holds, then the convergence in (ii) is uniform in $x \in X$, independent of the specific Følner sequence $(F_n)_{n\in\mathbb{N}}$, and we have $c = \mu(f)$.

Proposition A.4 *(cf. [10, Proposition 2.3]). Let (X, G) be a dynamical system. Suppose for each $f \in C(X)$ there is a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ and a constant c with*

$$\lim_{n\to\infty} \frac{1}{\theta_r(F_n)} \int_{F_n} f(gx)\, d\theta_r(g) = c,$$

for all $x \in X$. Then (X, G) has a unique G-invariant measure μ and $\mu(f) = c$.

In the following, we denote by $C(\mu)$ the image of $C(X)$ under the canonical mapping into $L^2(\mu)$.[1] For $p = 1, 2$, denote by $I^p(\mu)$ the set of all $f \in L^p(\mu)$ with $g.f = f$ for all $g \in G$. Denote by P_μ the projection onto the subspace $I^2(\mu)$ in $L^2(\mu)$.

For the convenience of the reader, we briefly discuss the part of the proof of the next statement where the case of thin Følner sequences slightly deviates from the standard case.

Theorem A.5 *(Mean Ergodic Theorem). Let (X, G) be a dynamical system and $\mu \in \mathcal{M}(X, G)$. For $f \in L^2(\mu)$, we have $S_n f \to P_\mu f$ in $L^2(\mu)$.*

[1] Note that this mapping is not necessarily injective and hence, not necessarily an embedding.

Proof. Define $A = \{g.f - f : f \in C(\mu), g \in G\}$. Since $C(\mu)$ is dense in $L^2(\mu)$, one can show that $L^2(\mu) = I^2(\mu) \oplus \overline{A}$, where $\overline{A}$ denotes the closure of A. Now, recall that $\|f\|_2 \leq \|f\|_\infty$ for all $f \in C(\mu)$. Thus, with Lemma A.1, one can show that for $f \in \overline{A}$, we have $\|S_n f\|_2 \to 0$. Since $S_n f = f$ for all $f \in I^2(\mu)$, we have $S_n f \to P_\mu f$ for each $f \in L^2(\mu)$. □

Corollary A.6 *(L^1-Mean Ergodic Theorem). Let (X, G) be a dynamical system and $\mu \in \mathcal{M}(X, G)$. For $f \in L^1(\mu)$, we have that $S_n f$ converges in $L^1(\mu)$ to an element in $I^1(\mu)$. In particular, if μ is ergodic, then $S_n f \to \mu(f)$.*

Corollary A.7 *([10, Theorem 2.4]). Let (X, G) be a dynamical system with an ergodic measure μ. Then every Følner sequence $\mathcal{F}$ allows for a subsequence $\mathcal{F}'$ such that μ-almost every point is μ-generic along $\mathcal{F}'$.*

Data availability

No data was used for the research described in the article.

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