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RESEARCH ARTICLE

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On completely multiplicative ±1 sequences that omit many consecutive +1 values

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Abstract

We say that ± 1-valued completely multiplicative functions are *length-k functions* f if they take the value $+1$ at at most k consecutive integers. We introduce a method to extend the length of f using the idea of the "rotation trick" in [7]. Under the assumption of Elliott's conjecture, this method allows us to construct length-k functions systematically for $k \geqslant 4$ which generalizes the work of Schur for $k = 2$ and Hudson for $k = 3$.

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1 | INTRODUCTION

We say $f : \mathbb{N} \to \mathbb{C}$ is *completely multiplicative* if $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$. Let

$$\mathcal{M}' = \{f : \mathbb{N} \to \{+1, -1\} : f \text{ is completely multiplicative}\}.$$

Recent works in the theory of completely multiplicative functions have proved fruitful in understanding the extremal behavior of bounded sequences, as in the Erdős discrepancy problem. In 1957, Erdős [2] conjectured that for any sequence $\{f(n)\}_{n\in\mathbb{N}}$ consisting of ± 1, there is $x, d \in \mathbb{N}$ such that

$$\sup_{x,d} \left| \sum_{n \leqslant x} f(nd) \right| = \infty.$$

In 2015, Tao [13] proved it by reducing $\{f(n)\}_{n\in\mathbb{N}}$ to $f \in \mathcal{M}'$. Moreover, his result indicated that every $f \in \mathcal{M}'$ satisfies

$$\limsup_{x\to\infty} \left| \sum_{n \leqslant x} f(n) \right| = \infty. \tag{1}$$

One can find more works on classifying functions related to (1) in [1, 5–7].

Our main focus is about the pattern of $\{f(n)\}_{n\in\mathbb{N}}$ for $f \in \mathcal{M}'$. We say that $f \in \mathcal{M}'$ is a *length-k* function if k is the largest positive integer such that there exists $n \in \mathbb{N}$ for which $f(n+1) = \cdots = f(n+k) = +1$. It is natural to try to classify all $f \in \mathcal{M}'$ of length k, for each $k \geqslant 2$.[†] Motivated by the results of Lehmer and Lehmer [9] on the first appearance of consecutive quadratic residues, Mills conjectured [11] that there are only two length-2 functions f_1, f_2, for which

$$f_i(p) = \begin{cases} \left(\frac{p}{3}\right) & \text{if } p \nmid 3, \\ (-1)^i & \text{if } p = 3 \end{cases}$$

where $i = 1, 2$ and $\left(\frac{\cdot}{3}\right)$ is the Legendre symbol mod 3, and Schur [12] confirmed this. For $k = 3$, Hudson [4] conjectured that there are only 13 possibilities. This has recently been proved by Klurman et al. [8].

Theorem 1.1 (Formerly Hudson's conjecture [8, Theorem 2.4]). *Let $q \in \{5, 7, 11, 13, 53\}$ and $i = 1, 2$. Define*

$$f_{(q,i)}(p) = \begin{cases} \left(\frac{p}{q}\right) & \text{if } p \nmid q, \\ (-1)^i & \text{if } p = q, \end{cases} \quad f_{(4,i)}(p) = \begin{cases} \left(\frac{p}{4}\right) & \text{if } p \nmid 4, \\ (-1)^i & \text{if } p = 2, \end{cases} \quad g(p) = \begin{cases} 1 & \text{if } p \nmid 2, \\ -1 & \text{if } p = 2. \end{cases}$$

If $f : \mathbb{N} \to \{+1, -1\}$ is a length-3 function, then f must be one of the above.

In light of Hudson's conjecture, we want to classify functions of higher length, say $k \geqslant 4$. By the above, we learn that there are finitely many length-2 and length-3 functions. For $k \geqslant 4$, conversely, it is possible to construct infinitely many examples. For instance, we can construct a length-4 function f by

$$f(p) = \begin{cases} \left(\frac{p}{5}\right) & \text{if } p \nmid 5 \text{ and } q, \\ -\left(\frac{p}{5}\right) & \text{if } p = q, \\ 1 & \text{if } p = 5, \end{cases}$$

where q can be any odd prime with $q \equiv 2 \bmod 5$. As there are infinitely many choices of q, we can construct infinitely many length-4 functions f. Nevertheless, we believe that such counterexamples can be constructed in a systematic way.

Let χ_q be a real character mod q. We define a *modified character* $\tilde{\chi}_q \in \mathcal{M}'$ at each prime by

$$\tilde{\chi}_q(p) = \begin{cases} \chi(p) & \text{if } p \nmid q, \\ \eta(p) & \text{if } p | q, \text{ where } \eta(p) \in \{+1, -1\}. \end{cases}$$

Our main result shows that the length of $\tilde{\chi}_q$ can be extended to at least k by altering its values at a set of finitely many prime numbers $p > k$ and $p \nmid q$, which we call *modified primes*, whose size is covered by Theorem 1.2. We will use $\delta_q(k)$ to denote the minimal number of modified primes that are needed for the length of $\tilde{\chi}_q$ to be extended to at least k.

[†] There is no length-1 function, as either $f(1) = f(2) = +1$, or $f(4) = f(5) = +1$, or $f(9) = f(10) = +1$.

Theorem 1.2. *Let k be a positive integer and $\tilde{\chi}_q$ be a modified character mod $q > 1$. Then, we have $\delta(k) = \max_{q\in\mathbb{N}} \delta_q(k) = \frac{1}{2}k + O(\log k)$.*

Here, we briefly explain how the modified primes can extend the length of $\tilde{\chi}_q$. Let $I = \{a + 1, \dots, a + k\} \subset \mathbb{N}$ be an interval of length k and $(\tilde{\chi}_q(a + 1), \dots, \tilde{\chi}_q(a + k))$ be the sign pattern of $\tilde{\chi}_q$ on I, denoted by $\tilde{\chi}_q(I)$. We assume that the length of $\tilde{\chi}_q$ is less than k, that is, $\tilde{\chi}_q(I) \neq (+1, \dots, +1)$ for all I. Since $\tilde{\chi}_q$ inherits some amount of periodicity from χ_q, by choosing an appropriate a according to the choice of modified primes, we can find an interval I' of length k such that for all $n \in I'$ with $\tilde{\chi}_q(n) = -1$, there is a unique $p > k, p \nmid q$ such that $p^\nu \parallel n$ for some odd integer $\nu > 0$. By modifying $\tilde{\chi}_q(p)$ to $-\tilde{\chi}_q(p)$ for all modified primes p, all the values of -1 in $\tilde{\chi}_q(I')$ will turn to $+1$, and hence the length of $\tilde{\chi}_q$ is extended to at least k. Lemma 2.1 is based on this idea.

Before giving the application of the extension to $f \in \mathcal{M}'$, we will introduce some concepts from the pretentious approach to analytic number theory of Granville and Soundararajan [3]. Let $\mathcal{M}$ denote the set of multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ with $|f(n)| \leqslant 1$ for all $n \in \mathbb{N}$. Given $x \geqslant 1$, the *pretentious distance* between $f, g \in \mathcal{M}$ is

$$\mathbb{D}(f, g; x) = \left(\sum_{p\leqslant x} \frac{1 - \Re\mathfrak{e}(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

This satisfies the triangle inequality:

$$\mathbb{D}(f_1, h_1; x) + \mathbb{D}(f_2, h_2; x) \geqslant \mathbb{D}(f_1 f_2, h_1 h_2; x) \text{ for } f_1, f_2, h_1, h_2 \in \mathcal{M}. \quad (2)$$

Let $f, g \in \mathcal{M}$. We say that f is *pretentious* to g if

$$\mathbb{D}(f, g; x) = O(1) \text{ as } x \to \infty.$$

We say furthermore that f is a *pretentious function* if f is pretentious to a twisted character $\chi(n)n^{\mathrm{it}}$ where $\chi(n)$ is a Dirichlet character and $t \in \mathbb{R}$, otherwise, it is a *non-pretentious function*. Additionally, if $f \in \mathcal{M}$ is real, f can only be pretentious to a real character (in the proof of Corollary 1.4).

Conjecture 1.3 (Elliott's conjecture [10, Conjecture 1.5]). *For any fixed $a_i, b_i, N \in \mathbb{N}$ such that $a_i b_j \neq a_j b_i$ for all $i, j = 1, 2, \dots, N$ and $i \neq j$, if $f \in \mathcal{M}'$ is non-pretentious*[†]*, then*

$$\sum_{n\leqslant x} \prod_{i=1}^{N} f(a_i n + b_i) = o(x). \quad (3)$$

Under the assumption of Conjecture 1.3, any $f \in \mathcal{M}'$ of finite length is pretentious to a real primitive (not necessarily non-principal) character χ_q (the detailed proof will be given in the next section). As a consequence, the extension of f of finite length can be reduced to the extension of a corresponding modified character $\tilde{\chi}_q$.

[†] The Elliott's conjecture for $f \in \mathcal{M}$ requires a stronger condition which is $\inf_{q\leqslant Q;\chi\ (q);|t|\leqslant x} \mathbb{D}(f, \chi(n)n^{it}; x)^2 \to \infty$ as $x \to \infty$ for each given Q. More discussions can be found in [10].

Corollary 1.4. *Given $k \in \mathbb{N}$, let f be a length-k function.*

(a) *Assuming Conjecture 1.3, there is a real character χ_q mod $q > 1$ such that f is pretentious to χ_q.*
(b) *Suppose f is pretentious to a real character χ_q mod $q > 1$. Let*

$$\mathcal{J}(k) = \{p \in \mathcal{P} : f(p) \neq \chi_q(p) \text{ with } p \nmid q \text{ and } p > k\}. \tag{4}$$

Then, $|\mathcal{J}(k)| \leqslant \frac{1}{2}k$.

By Lemma 2.1, we can see that the length increases with the number of modified primes, and hence Corollary 1.4 follows from Theorem 1.2 immediately.

Remark 1.5. In our extension, we only consider modified primes $p > k$. It remains to investigate a more general case where the modified primes involve primes $p \leqslant k$, that is, modified primes can be any primes, which allows one to construct length-k functions in a more general way. Unlike modified primes $p > k$, the number of flipped values of $f(n)$ in an interval of length k by a modified prime $p \leqslant k$ is more than one. Therefore, it would be fairly tricky to determine $\delta(k)$. If we include $p \leqslant k$ in the count given by $\delta(k)$ then we can currently only bound it crudely by $\frac{k}{\log k}$ using the prime number theorem.

2 | PROOF OF THEOREM 1.2 AND COROLLARY 1.4

We will use auxiliary Lemmas 2.1 and 2.2 to prove Theorem 1.2. Let $S \subset \mathcal{P}$ be a subset of prime numbers. We define $\lambda_S \in \mathcal{M}'$ at each prime by

$$\lambda_S(p) = \begin{cases} 1 & \text{if } p \in \mathcal{P} - S, \\ -1 & \text{if } p \in S. \end{cases}$$

Lemma 2.1. *Let $k \in \mathbb{N}$ and I_k be an interval of length k. Let $\tilde{\chi}_q$ be a modified character mod q and r denote the number of integers $n \in I_k$, where $\tilde{\chi}_q(n) = -1$. Let $P_r = \{p_1, \dots, p_r\}$ be a set of r distinct primes with $p_i > k, p_i \nmid q$ for $i = 1, \dots, r$. We define $f(n) = \tilde{\chi}_q(n)\lambda_{P_r}(n)$. Then, we have $f(I) = (+1, \dots, +1)$ for some interval I of length k.*

Proof. Without loss of generality, we assume that the length of $\tilde{\chi}_q$ is less than k. Let $[k] = \{1, \dots, k\}$ and $I_k(n) = n + [k] = \{n+1, \dots, n+k\}$ be an interval of length k starting from $n+1 \in \mathbb{N}$ with $n + 1 \equiv m \bmod q$ for some $0 \leqslant m < q$. And let $\tilde{\chi}_q(I_k(n)) = (\tilde{\chi}_q(n+1), \dots, \tilde{\chi}_q(n+k))$ denote the sign pattern of $\tilde{\chi}_q$ on $I_k(n)$. Suppose that $\tilde{\chi}_q$ only takes value of -1 at $n + J = \{n + a_1, \dots, n + a_r\} \subset I_k(n)$ where $J = \{a_1, \dots, a_r\}$. Let $p_1, \dots, p_r$ be r distinct primes greater than k and coprime to q. By the Chinese remainder theorem, there exist solutions to the system below:

$$n' + 1 \equiv m \bmod q, \quad n' + a_j \equiv 0 \bmod p_j \text{ for } j = 1, \dots, r.$$

Suppose that the solutions are in the form of $n' = N + \alpha Q$ where $N \in \mathbb{N}$ is fixed, $\alpha \in \mathbb{Z}$, and $Q = q \prod_{j=1}^{r} p_j$. We can choose an α satisfying $p_j^{\nu_j} \parallel n' + a_j$ with ν_j odd for $j = 1, \dots, r$ and $q^{n+1} | n' - n$.

Then, $\tilde{\chi}_q(I_k(n)) = \tilde{\chi}_q(I_k(N + \alpha Q)) = \tilde{\chi}_q(I_k(n'))$ by [8, Lemma 9.4]. Besides, $p_j \nmid n' + a_i$ for all $i \neq j$, and each $1 \leqslant j \leqslant r$ since $p_j > k$ for $1 \leqslant j \leqslant r$. Hence, we have

$$f(n' + a_j) = \tilde{\chi}_q(n' + a_j) = \tilde{\chi}_q(n + a_j) \text{ for } a_j \in [k] - J,$$

$$f(n' + a_j) = -\tilde{\chi}_q(p_j)^{\nu_j}\tilde{\chi}_q\left(\frac{n' + a_j}{p_j^{\nu_j}}\right) = -\tilde{\chi}_q(n' + a_j) = -\tilde{\chi}_q(n + a_j) = 1 \text{ for } a_j \in J.$$

Hence, we have $f(m) = +1$ for all $m \in n' + [k]$. □

Lemma 2.2. *Let $Q > 0$ and χ_q be a non-principal character mod q. Then, we have*

$$\frac{1}{\log Q}\sum_{n \leqslant Q}\frac{\tilde{\chi}_q(n)}{n} \ll \frac{\sqrt{q}\log q}{\log Q}.$$

Proof. Let $x > 0$. We have

$$\sum_{n \leqslant x}\tilde{\chi}_q(n) = \sum_{n \leqslant x}\chi_q(n) + \sum_{\substack{n \leqslant x\\(n,q)>1}}\tilde{\chi}_q(n). \tag{5}$$

Suppose $q = \prod_{i=1}^{r} p_i^{\alpha_i}$ with $r, \alpha_i \in \mathbb{N}$. The second term in (5) is

$$\begin{aligned}\Big|\sum_{\substack{n \leqslant x\\(n,q)>1}}\tilde{\chi}_q(n)\Big| &\leqslant \Big|\sum_{\beta_1,\ldots\beta_r \leqslant \log x}\ \sum_{\substack{n \leqslant x\\ \prod_{i=1}^{r} p_i^{\beta_i}\|n \Rightarrow p_i|q}}\tilde{\chi}_q(n)\Big| = \Big|\sum_{\beta_1,\ldots\beta_r \leqslant \log x}\tilde{\chi}_q\left(\prod_{i=1}^{r} p_i^{\beta_i}\right)\sum_{n \leqslant x/\prod_{i=1}^{r} p_i^{\beta_i}}\chi_q(n)\Big| \\ &\leqslant \sum_{\beta_1,\ldots\beta_r \leqslant \log x}\Big|\sum_{n \leqslant x/\prod_{i=1}^{r} p_i^{\beta_i}}\chi_q(n)\Big|.\end{aligned} \tag{6}$$

Applying Pólya–Vinogradov inequality to (5) and (6), we obtain

$$\sum_{n \leqslant x}\tilde{\chi}_q(n) \ll (\log x)^r\sqrt{q}\log q.$$

By partial summation, we have

$$\frac{1}{\log Q}\sum_{n \leqslant Q}\frac{\tilde{\chi}_q(n)}{n} \ll \frac{\sqrt{q}\log q}{\log Q}.$$ □

Proof of Theorem 1.2. From the proof of Lemma 2.1, we can see that the number of the modified primes is independent of the choice of modified primes $p > k$ and the location of -1s in a sign pattern. It only depends on the number of -1s in an interval of length k. Let $\mathcal{I}_q(k)$ denote an interval of length k such that $\tilde{\chi}_q(\mathcal{I}_q(k))$ contains the least number of -1s. $\delta_q(k)$ is equivalent to the number of -1s in $\tilde{\chi}_q(\mathcal{I}_q(k))$, for which

$$\delta_q(k) = \frac{1}{2}(k - S_q(k)), \text{ where } S_q(k) = \sum_{m\in\mathcal{I}_q(k)} \tilde{\chi}_q(m) = \max_{n\in\mathbb{N}\cup\{0\}} \sum_{m=1}^{k} \tilde{\chi}_q(n+m).$$

In the following, we always assume that the length of $\tilde{\chi}_q$ is less than k, otherwise, $\delta_q(k) = 0$ which is trivial. Next, we will estimate $\delta(k) = \max_{q\in\mathbb{N}} \delta_q(k)$ by investigating its upper and lower bounds. □

2.1 | The upper bound of $\delta(k)$

We will show $\delta(k) \leqslant \frac{1}{2}k$.

2.1.1 | χ_q is the principal character mod q

Let χ_q be a principal character. In this case, it suffices to show

$$A = \lim_{\alpha\to\infty} A(q^\alpha) = \lim_{\alpha\to\infty} \frac{1}{q^\alpha} \sum_{n=1}^{q^\alpha} \sum_{m=0}^{k-1} \tilde{\chi}_q(n+m) > 0,$$

since $S_q(k) \geqslant A(q^\alpha)$ for all $\alpha \in \mathbb{N}$. Let $\alpha > k$ be an integer. After rearranging, we have

$$A(q^\alpha) = \frac{1}{q^\alpha}\left(k \sum_{n=1}^{q^\alpha} \tilde{\chi}_q(n) - \sum_{m=1}^{k-1}(k-m)\tilde{\chi}_q(m) + \sum_{m=1}^{k-1}(k-m)\tilde{\chi}_q(q^\alpha + m) \right) = k\frac{1}{q^\alpha} \sum_{n=1}^{q^\alpha} \tilde{\chi}_q(n),$$

since $\tilde{\chi}_q(m) = \tilde{\chi}_q(q^\alpha + m)$ for all $1 \leqslant m \leqslant (k-1)$ by [8, Lemma 9.4]. By [14, Delange's theorem, III. 4 on p. 326], $\tilde{\chi}_q$ possesses a positive mean value, so we obtain

$$A = \lim_{\alpha\to\infty} A(q^\alpha) = k \lim_{\alpha\to\infty} \frac{1}{q^\alpha} \sum_{n=1}^{q^\alpha} \tilde{\chi}_q(n) = k \prod_p (1 - p^{-1}) \sum_{\nu=0}^{\infty} \tilde{\chi}_q(p)^\nu p^{-\nu} > 0.$$

2.1.2 | χ_q is non-principal mod q

Let $Q \geqslant 1$. By the definition of $\delta_q(k)$, we have

$$\delta_q(k) \leqslant \frac{1}{2}\left(k - \frac{1}{\log Q} \sum_{n=1}^{Q} \frac{1}{n} \sum_{m=0}^{k-1} \tilde{\chi}_q(n+m) \right). \tag{7}$$

Then, it suffices to prove that

$$B = \frac{1}{\log Q} \sum_{n=1}^{Q} \frac{1}{n} \sum_{m=0}^{k-1} \tilde{\chi}_q(n+m) = o(1) \text{ as } Q \to \infty.$$

After rearranging, we obtain

$$\begin{aligned} B &= \frac{1}{\log Q}\left(\sum_{m=0}^{k-1}\sum_{n=1}^{Q}\frac{\tilde{\chi}_q(n+m)}{n+m} + \sum_{m=0}^{k-1} m \sum_{n=1}^{Q}\frac{\tilde{\chi}_q(n+m)}{n(n+m)}\right) \\ &= \frac{1}{\log Q}\left(k\sum_{n=1}^{Q}\frac{\tilde{\chi}_q(n)}{n} - \sum_{n=1}^{k-1}(k-n)\frac{\tilde{\chi}_q(n)}{n} + \sum_{n=1}^{k-1}(k-n)\frac{\tilde{\chi}_q(Q+n)}{(Q+n)} + O(k^2)\right) \\ &= \frac{k}{\log Q}\sum_{n=1}^{Q}\frac{\tilde{\chi}_q(n)}{n} + O\left(\frac{k^2}{\log Q}\right). \end{aligned}$$

If we choose Q to be sufficiently large in terms of k and q, we will obtain $B = o(1)$ by Lemma 2.2. As a result, taking $Q \to \infty$ we obtain $\delta(k) \leqslant \frac{1}{2}k$ by (7).

2.2 | The lower bound of $\delta(k)$

We will bound $\delta(k)$ from below by $\delta_q(k)$ where q is an odd prime with $q < k$, as $\delta(k) \geqslant \delta_q(k)$ for all $q \in \mathbb{N}$.

For simplicity, we assume $\tilde{\chi}_q(q) = 1$. Suppose $k = \sum_{i=0}^{\nu} a_i q^i = k_0$ with $0 \leqslant a_i < q$ for $i = 0, \dots, \nu$, and $a_\nu \neq 0$ where ν is the largest integer such that $q^\nu \leqslant k$. Recall that $\mathcal{I}_q(k)$ denotes an interval of length k such that $\tilde{\chi}_q(\mathcal{I}_q(k))$ contains the least number of -1s. Let k_j denote the number of elements in $\mathcal{I}_q(k_0)$ that are divisible by q^j, that is, $k_j = \sum_{i=j}^{\nu} a_i q^{i-j}$. Suppose $\mathcal{I}_q(k_0) = \{M_0 q - r_0, \dots, N_0 q + (a_0 - r_0)\}$ with $0 \leqslant r_0 \leqslant a_0 < q, 0 < M_0 < N_0$. We can decompose $\mathcal{I}_q(k_0) = \mathcal{C}_q(k_0) \sqcup \mathcal{N}_q(k_0) \sqcup \mathcal{Q}_q(k_0)$ where

$$\begin{aligned} \mathcal{C}_q(k_0) &= \{n \in \{M_0 q, \dots, N_0 q\} : (n,q) = 1\} \\ \mathcal{N}_q(k_0) &= \{M_0 q - r_0, \dots, M_0 q - 1\} \cup \{N_0 q + 1, \dots, N_0 q + (a_0 - r_0)\} \\ \mathcal{Q}_q(k_0) &= \{n \in \{M_0 q, \dots, N_0 q\} : q | n\}. \end{aligned}$$

Given a set Y, let $|Y|^-$ denote the number of -1s in Y. Then, we have

$$\delta_q(k) = |\mathcal{I}_q(k_0)|^- = |\mathcal{C}_q(k_0)|^- + |\mathcal{N}_q(k_0)|^- + |\mathcal{Q}_q(k_0)|^-.$$

As $\tilde{\chi}_q(q) = 1$, $\tilde{\chi}_q(\mathcal{Q}_q(k_0)) = \tilde{\chi}_q(q)\tilde{\chi}_q(\mathcal{K}_q(k_1))$ where $\mathcal{K}_q(k_1) = \{M_0, \dots, N_0\}$ with $|\mathcal{K}_q(k_1)| = k_1$. Following the above decomposition, we obtain

$$|\mathcal{Q}_q(k_0)|^- = |\mathcal{K}_q(k_1)|^- = |\mathcal{C}_q(k_1)|^- + |\mathcal{N}_q(k_1)|^- + |\mathcal{Q}_q(k_1)|^-.$$

By applying this procedure repetitively until k_ν, we have

$$\delta_q(k) = |\mathcal{I}_q(k_0)|^- = \sum_{i=0}^{\nu-1}\left(|\mathcal{C}_q(k_i)|^- + |\mathcal{N}_q(k_i)|^-\right) + |\mathcal{K}_q(k_\nu)|^-.$$

Given a positive integer $l < q$, we define

$$\delta'_q(l) = \frac{1}{2}(l - S'_q(l)) \text{ where } S'_q(l) = \max_{q-l\leqslant n\leqslant q} \sum_{m=0}^{l} \chi_q(n+m).$$

Since

$$|\mathcal{N}_q(k_i)|^- = \sum_{n\in\mathcal{N}_q(k_i)} \chi_q(n) = \sum_{m=0}^{a_i} \chi_q(-r_0+m) \geqslant \delta'_q(a_i) \text{ for } 0 \leqslant i \leqslant \nu, \text{ and } |\mathcal{K}_q(k_\nu)|^- \geqslant \delta_q(a_\nu),$$

we have

$$\begin{aligned}
\delta_q(k) &\geqslant \sum_{i=1}^{\nu} \frac{1}{2}(q-1)k_i + \sum_{j=0}^{\nu-1} \delta'_q(a_i) + \delta_q(a_\nu) \\
&\geqslant \frac{1}{2}(q-1)\sum_{i=1}^{\nu}\sum_{j=i}^{\nu} a_j q^{j-i} + \frac{1}{2}\left(\sum_{i=0}^{\nu} a_i - \sum_{i=0}^{\nu-1} S'_q(a_i) - S_q(a_\nu)\right) \\
&\geqslant \frac{1}{2}k - \frac{1}{2}\left(\sum_{i=0}^{\nu-1} S'_q(a_i) + S_q(a_\nu)\right). \qquad (8)
\end{aligned}$$

If we choose $q = 3$, then we have

$$0 \leqslant S'_3(a_i) \leqslant 1 \text{ for } i = 0, \dots, \nu-1 \text{ and } 1 \leqslant S_3(a_\nu) \leqslant 2. \qquad (9)$$

Applying (9) to (8), we obtain

$$\delta_3(k) \geqslant \frac{1}{2}k - \frac{1}{2}\left(\sum_{i=0}^{\nu-1} 1 + 2\right) = \frac{1}{2}k - \frac{1}{2}\left(\left\lfloor \frac{\log k}{\log 3}\right\rfloor + 2\right) = \frac{1}{2}k + O(\log k).$$

This implies that $\delta(k) \geqslant \frac{1}{2}k + O(\log k)$. Combining with the upper bound from the previous section, we have $\delta(k) = \frac{1}{2}k + O(\log k)$.

Remark 2.3. For the lower bound of $\delta_q(k)$ when $q < k$, one can bound (8) by using the Pólya–Vinogradov inequality instead of choosing $q = 3$. Then, the error term will be $O(\frac{\log k}{\log q} q^{1/2+o(1)}) = o(k)$ instead of $O(\log k)$.

Proof of Corollary 1.4. Since f is of length k, we have

$$S = \sum_{n\in K} \prod_{j=0}^{k} (1 + f(n+j)) = 0 \text{ for any } K \subset \mathbb{N}. \qquad (10)$$

Suppose $K = (0, x] \cap \mathbb{N}$ with $x > 0$ and $I = \{0, 1, \dots, k\}$. Then, expand (10) and take $x \to \infty$, we have

$$0 = \lim_{x\to\infty} S = \lim_{x\to\infty} (x + S_1(x) + S_2(x) + \cdots + S_{k+1}(x)),$$

$$\text{where } S_i(x) = \sum_{\substack{I_i \subseteq I \\ |I_i| = i, 1\leqslant i\leqslant k+1}} \sum_{n\leqslant x} \prod_{j\in I_i} f(n+j).$$

By Conjecture 1.3, if f is a non-pretentious function then $S_i(x) = o(x)$ as $x \to \infty$ for $i = 1, \ldots, k+1$. Then, we have

$$\lim_{x\to\infty} S = \lim_{x\to\infty} x + o(x) \neq 0$$

which contradicts (10). As a consequence, f must be pretentious to a twisted character $\chi(n)n^{it}$. In our case, we may assume χ is real and $t = 0$, in other words, f must be pretentious to a real primitive character or principal character χ. Indeed, since if χ is not real, for $|t| \leqslant x$, we have

$$\mathbb{D}(f, \chi(n)n^{it}; x) \geqslant \frac{1}{4}\sqrt{\log\log x} + O_\chi(1)$$

by [10, Lemma C.1] and $\mathbb{D}(f, \chi(n)n^{it}; x) \to \infty$ as $x \to \infty$ that contradicts $\mathbb{D}(f, \chi(n)n^{it}; x) < \infty$. Also, for $1/\log x \ll |t| \leqslant x$, by (2), we have

$$2\mathbb{D}(f, \chi(n)n^{it}; x) \geqslant \mathbb{D}(1, \chi^2(n)n^{i2t}; x) = \mathbb{D}(1, \chi_0(n)n^{i2t}; x) = \log(1 + |2t|\log x) + O(1) \qquad (11)$$

and the right-hand side of (11) tends to ∞ as $x \to \infty$. This contradicts $\mathbb{D}(f, \chi(n)n^{it}; x) < \infty$, unless $|t| \ll 1/\log x$ as $x \to \infty$ which implies $t = 0$. Moreover, according to the extension process, the size of (4) can be bounded by the upper bound of $\delta(k)$, for which

$$|\mathcal{J}(k)| \leqslant \frac{1}{2}k. \qquad \square$$

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