Random Periodic Paths of Stochastic Periodic Semi-flows through Random Attractors, Synchronizations and Lyapunov Exponents

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Abstract

This paper discusses the existence and uniqueness of random periodic paths of stochastic periodic semi-flows. Random periodic attractors are introduced and synchronization for stochastic periodic semi-flows is proved under some conditions to find the unique random periodic path. The multiplicative ergodic theorem of stochastic periodic semi-flows is proved to characterize Lyapunov exponents. The Benzi-Parisi-Sutera-Vulpiani climate model is an example to verify the results by estimating the negative Lyapunov exponent constructed by the density function from the Fokker-Planck equation. Numerical approximations are performed with great agreement. A case of gradient systems is considered to be another example of a negative Lyapunov exponent.

Keywords: random periodic path, random attractor, Lyapunov exponent, multiplicative ergodic theorem, Fokker-Planck equation.

1 Introduction

Periodic phenomena are ubiquitous in various fields, including biology, economics, chemistry, climate dynamics and so on. The study of periodic solutions has been a cornerstone of dynamical systems theory since Poincaré's seminal work ([19]). Random periodic paths (solutions) addressed in [22] have recently emerged as a subject of significant interest. The definition of random periodic paths for stochastic periodic semi-flows was given in [14], which demonstrated the existence of stochastic periodic semi-flows generated by non-autonomous stochastic differential equations (SDEs) with additive noise. The random periodic solution

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in stochastic partial differential equations (SPDEs) was studied in [11]. Numerical approximations of random periodic paths for some SDEs are studied in [9]. The precise formulation for periodic measures was given in [12]. The existence of ergodic periodic measures for SDEs with local Lipschitz continuous property and weakly dissipative coefficients was obtained in [13].

To study the convergence for stochastic trajectories, the global attractor (a random set attracting any stochastic trajectories) plays an important role. Global attractor was discussed in [6], [7] and [20]. The periodic pullback attractors of parametric dynamical systems with cocycle property were considered in [21]. Bates et al. studied attractors of non-autonomous stochastic lattice systems ([3]). When the global attractor becomes a singleton, synchronization (all trajectories converge to a single one) happens and it was studied in [15] and [17] for random dynamical systems.

The multiplicative ergodic theorem (MET), first formulated by Oseledets in the 1960s ([18]), provides a powerful tool for analyzing the asymptotic behaviour of the trajectories of dynamical systems. A complete description of MET for random dynamical systems is given in [1].

Lyapunov exponent is a fundamental concept in the study of dynamical systems, particularly in the realm of stochastic stability analysis. The assumption in the MET is closely related to the calculation of the Lyapunov exponent. Le Jan offers a comprehensive framework for analyzing the statistical behaviour of dynamical systems governed by the products of diffeomorphisms ([16]). According to his work, the asymptotic stable property for cocycles is guaranteed by negative Lyapunov exponents.

This paper investigates the existence and uniqueness of random periodic paths for stochastic periodic semi-flows. By introducing random periodic attractors for stochastic periodic semi-flows, we extend existing synchronization results for random dynamical systems to the context of stochastic periodic semi-flows. Specifically, we show that under certain conditions, asymptotically stable stochastic periodic semi-flows possess a unique random periodic path.

In the studies of synchronizations in SDEs, we use the largest Lyapunov exponent to determine the asymptotic stability of stochastic periodic semi-flows. We prove the multiplicative ergodic theorem for stochastic periodic semi-flows to characterize the Lyapunov exponents. We find that the Lyapunov exponents satisfy the real periodic relation rather than the random periodicity. These results are of independent interest.

We intend to study the pathwise random periodic paths in SDEs with local Lipschitz continuous property and weakly dissipative coefficients. In the absence of strong dissipative property, it is difficult to find pathwise convergences for stochastic periodic semi-flows from two arbitrarily different initial points. We overcome the difficulties in specific examples which find the existence and uniqueness of random periodic paths by using random periodic attractors and Lyapunov exponents.

An example corresponding to the Benzi-Parisi-Sutera-Vulpiani (BPSV) model ([4]) is a model for the physical transition between two climates. A resonance intuitive physical explanation of transition at the same frequency as the periodic forcing was given without a rigorous analysis. An analytic estimation of the solution (density function) in the Fokker-Planck equation generated from the model is provided. It is shown that the largest Lyapunov exponent estimated by the density function is negative. Then the existence and uniqueness of random periodic paths in this model are proved. Numerical simulations have been performed with great agreement.

The other example is a case of gradient systems. We remove periodic coefficients and extend the work of Flandoli et al. in [15] to large diffusion coefficients. We show that the largest Lyapunov exponent of the stochastic periodic semi-flows in the considered SDEs is negative, which proves the asymptotic stability.

2 Preliminaries and notation

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\left(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ be a metric dynamical system, where $\theta_t : \mathbb{R}\times\Omega \to \Omega$ is a measure $\mathbb{P}$-preserving map. We consider a complete separable metric space $(\mathbb{X}, d)$, and for any $A, B \in \mathscr{B}(\mathbb{X})$, the semi-distance is defined by

$$d(A, B) = \sup\{d(x, B) : x \in A\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Taking $\Delta = \{(t, s) \in \mathbb{R}^2, s \leq t\}$, we will consider a stochastic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \to \mathbb{X}$ which satisfies for $\mathbb{P}$-a.e. ω,

$$u(t, s, \omega)\, x = u(t, r, \omega) \circ u(r, s, \omega)\, x, \text{ for all } s \leq r \leq t \text{ and } x \in \mathbb{X}, \tag{2.1}$$

and $u(t, s, \omega) : \mathbb{X} \to \mathbb{X}$ is continuous for all $(t, s) \in \Delta$. Moreover, a stochastic periodic semi-flow which can be generated by stochastic differential equations with periodic coefficients ([14]) is defined in the following:

Definition 2.1. *([9],[14]) The two-parameter random map u is called a stochastic τ-periodic semi-flow if it satisfies formula (2.1) and an additional property: there exists a constant $\tau > 0$ such that for all $(t, s) \in \Delta$,*

$$u(t + \tau, s + \tau, \omega) = u(t, s, \theta_\tau\omega), \text{ for almost all } \omega \in \Omega.$$

The random periodic paths (solutions) of stochastic periodic semi-flows are defined by:

Definition 2.2. *([14]) A random periodic path of period τ for a stochastic semi-flow u is an $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}$-measurable process $Y : \mathbb{R} \times \Omega \to \mathbb{X}$, such that for any $(t, s) \in \Delta$,*

$$u(t, s, \omega)\, Y(s, \omega) = Y(t + s, \omega), \; Y(s + \tau, \omega) = Y(s, \theta_\tau\omega) \text{ for almost all } \omega \in \Omega.$$

Random periodicity is considered as a mixture of periodicity and randomness, which differs from periodicity in deterministic situations. Feng et al. proved that if the metric dynamical system $\left(\Omega, \mathscr{F}, \mathbb{P}, (\theta_{k\tau})_{k\in\mathbb{Z}}\right)$ is ergodic, then the period τ cannot be random ([10]).

Let $(\mathscr{F}_{s,t})_{-\infty<s<t<\infty}$ be a filtration of $\mathscr{F}$ such that $\mathscr{F}_{s,t} \subseteq \mathscr{F}_{u,v}$ for $u \leq s \leq t \leq v$, and $\theta_{-r}(\mathscr{F}_{s,t}) = \mathscr{F}_{s+r,t+r}$ for all r and $(s, t) \in \Delta$. Furthermore, $\mathscr{F}_{s,t}$ and $\mathscr{F}_{u,v}$ are assumed to be independent for $s \leq t \leq u \leq v$. When $s \leq t$, the smallest σ-algebra containing all $\mathscr{F}_{s,t}$ is defined by $\mathscr{F}_t$, when $t \leq u$, the smallest σ-algebra containing all $\mathscr{F}_{t,u}$ is defined by $\mathscr{F}_{t,\infty}$, and we define $\mathscr{F}_{-\infty,\infty} = \mathscr{F}$. Moreover, $u(t, s, \omega)$ is assumed to be $\mathscr{F}_{s,t}$-measurable for every $(t, s) \in \Delta$.

Now, let us introduce a τ-periodic Markov transition probability ([13]):

$$P(\cdot,\ \cdot,\ \cdot,\ \cdot) : \Delta \times \mathbb{X} \times \mathscr{B} \to [0,1],$$

which satisfies:

(i) $P(t,s,x,\cdot)$ is a probability measure on $(\mathbb{X},\mathscr{B})$ for all $(t,s)\in\Delta$ and $x\in\mathbb{X}$;

(ii) $P(t,s,\cdot,\Gamma)$ is a $\mathscr{B}$-measurable function for all $(t,s)\in\Delta$ and $\Gamma\in\mathscr{B}$;

(iii) For all $s\le r\le t$, it satisfies

$$P(t,s,x,\Gamma)=\int_{\mathbb{R}} P(r,s,x,dy)P(t,r,y,\Gamma), x\in\mathbb{R}, \Gamma\in\mathscr{B};$$

(iv) $P\left(s,s,x,\Gamma\right)=1_{\Gamma}(x)$ for all $s\in\mathbb{R}, x\in\mathbb{X}$ and $\Gamma\in\mathscr{B}$;

(v) $P(t+\tau,s+\tau,x,\Gamma)=P(t,s,x,\Gamma),(t,s)\in\Delta,\ x\in\mathbb{X}$.

For $(t,s)\in\Delta$, the linear operator $P(s,t)$ acting on $\mathcal{B}_b(\mathbb{X})$ (the space of bounded measurable functions) is defined by

$$P(t,s)f(x)=\int_E f(y)P(t,s,x,dy),\quad f\in\mathcal{B}_b(\mathbb{X}), x\in\mathbb{X}.$$

For $(t,s)\in\Delta$, the adjoint operator $P^*(t,s)$ acting on $\mathcal{P}(\mathbb{X})$ (the space of probability measures on $(\mathbb{X},\mathscr{B})$) is defined by

$$\left(P^*(t,s)\mu\right)(\Gamma)=\int_{\mathbb{X}} P(t,s,x,\Gamma)\mu(dx),\quad \mu\in\mathcal{P}(\mathbb{X}),\ \Gamma\in\mathscr{B}.$$

The measure-valued function $\rho:\mathbb{R}\to\mathcal{P}(\mathbb{X})$ is called a τ-periodic measure of τ-periodic Markov transition probability $P(t,s,\cdot,\cdot)$ ([12]), if for any $(t,s)\in\Delta$,

$$P^*(t,s)\rho_s=\rho_t,\ \rho_{s+\tau}=\rho_s.$$

3 Random periodic paths through random periodic attractors and synchronizations

In this section, we consider the random periodic attractor associated with stochastic periodic semi-flows and give some conditions to make it a singleton. Subsequently, synchronizations of stochastic periodic semi-flows happen, and the existence and uniqueness of the random periodic paths are studied.

3.1 Random periodic attractors for stochastic periodic semi-flows

We consider a stochastic semi-flow u and let $\hat{D}$ be the collection of all nonempty bounded subsets of $\mathbb{X}$. By the work of Crauel et al. in [6], for a given $t \in \mathbb{R}$, a random set $K(t,\omega)$ is called an attracting set of u if it satisfies for all $D \in \hat{D}$,

$$\lim_{s\to-\infty} d(u(t,s,\omega)D, K(t,\omega)) = 0, \quad \text{for almost all } \omega \in \Omega.$$

We say that a stochastic semi-flow u is asymptotically compact if there exists a measurable set $\Omega_0 \subset \Omega$ with measure one such that for all $t \in \mathbb{R}$ and $\omega \in \Omega_0$, there exists a compact attracting set $K(t,\omega)$.

Definition 3.1. *([6],[7]) Let $u(t,s,\omega)$ be a stochastic semi-flow on $(\mathbb{X}, d)$ for all $(t,s) \in \Delta$, a random set $\bar{A}(t,\omega)$ is said to be a global attractor of u for every $t \in \mathbb{R}$ if it is compact and for almost all $\omega \in \Omega$, it satisfies the following conditions:*

(i) For any set $D \in \hat{D}$, $\limsup\limits_{s\to-\infty, x\in D} d(u(t,s,\omega)x, \bar{A}(t,\omega)) = 0$;

(ii) $\bar{A}$ is u-invariant, i.e. $u(t,s,\omega)\bar{A}(s,\omega) = \bar{A}(t,\omega)$, where $s \le t$.

From the Theorem 2.1 in [6], we know that for a given $t \in \mathbb{R}$ and $D \in \hat{D}$, if the stochastic semi-flow u is asymptotically compact, then the random set $A(t,D,\omega)$ of stochastic semi-flow u constructed by :

$$A(t,D,\omega) = \bigcap_{T<t} \overline{\bigcup_{s<T} u(t,s,\omega) D}, \tag{3.1}$$

can be proved to attract D. Furthermore, the random set $\bar{A}(t,\omega)$ obtained from

$$\bar{A}(t,\omega) = \overline{\bigcup_{D\in\widehat{D}} A(t,D,\omega)}. \tag{3.2}$$

is a global attractor and it is minimal.

We notice that for any $x \in A(t,D,\omega)$, there are sequences $x_n \in D$ and $s_n \to -\infty$ when $n \to \infty$ such that

$$\lim_{n\to\infty} u(t,s_n,\omega) x_n = x.$$

Given $t \in \mathbb{R}$, a global attractor $\bar{A}(t,\omega)$ is said to be a random τ-periodic attractor for a stochastic periodic semi-flow u if it satisfies $\bar{A}(t+\tau,\omega) = \bar{A}(t,\theta_\tau\omega)$. The following proposition shows that $\bar{A}(t,\omega)$ constructed by formula (3.2) is a random periodic attractor.

Proposition 3.2. *Suppose u is a stochastic periodic semi-flow with period $\tau > 0$ and the random set $A(t,D,\omega)$ is defined by formula (3.1) for every $t \in \mathbb{R}$, then $\bar{A}(t,\omega)$ constructed by formula (3.2) is a random τ-periodic attractor.*

Proof. Let $A\left(t, D, \omega\right)$ be constructed by formula (3.1) for every $t \in \mathbb{R}$, then we have

$$\begin{aligned} A\left(t+\tau, D, \omega\right) &= \bigcap_{T<t+\tau} \overline{\bigcup_{s<T} u\left(t+\tau, s, \omega\right) D} \\ &= \bigcap_{T-\tau<t} \overline{\bigcup_{s-\tau<T-\tau} u\left(t, s-\tau, \theta_\tau \omega\right) D} \\ &= \bigcap_{T<t} \overline{\bigcup_{s<T} u\left(t, s, \theta_\tau \omega\right) D} \\ &= A\left(t, D, \theta_\tau \omega\right). \end{aligned}$$

Therefore, the global attractor $\bar{A}\left(t, \omega\right)$ satisfies that $\bar{A}\left(t+\tau, \omega\right) = \bar{A}\left(t, \theta_\tau \omega\right)$ which means it is a random periodic attractor. □

3.2 Synchronization of stochastic periodic semi-flows

Given $t \in \mathbb{R}$, the global attractor $\bar{A}\left(t, \omega\right)$ attracts all compact sets (compact sets are bounded), i.e., for every compact set C and for $\mathbb{P}$-a.e. ω,

$$\limsup_{s\to-\infty,\, x\in C} d\left(u\left(t, s, \omega\right) x, \bar{A}\left(t, \omega\right)\right) = 0, \qquad t \geq s.$$

Then we define the synchronization for stochastic semi-flows below:

Definition 3.3. *Stochastic semi-flows are synchronized if the global attractor $\bar{A}$ is a singleton $\mathbb{P}$-a.e.*

Similarly to the definition in [15], asymptotically stability for stochastic semi-flows is defined.

Definition 3.4. *Suppose U is a non-empty open set, a stochastic semi-flow u is called asymptotically stable on U if there is a sequence $s_n \to -\infty$ as $n \to \infty$ such that for a given $t \in \mathbb{R}$,*

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(u\left(t, s_n, \cdot\,\right) U\right) = 0\right) > 0, \tag{3.3}$$

where

$$\operatorname{diam}\left(u\left(t, s_n, \cdot\right) U\right) = \sup_{x,\, y\in U} d\left(u\left(t, s_n, \cdot\right) x, u\left(t, s_n, \cdot\right) y\right).$$

For stochastic periodic semi-flows, we can consider the asymptotically stable property as the limit on the multiples of periods i.e. if there is a sequence $t - k_n\tau \to -\infty$, such that

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(u\left(t, t-k_n\tau, \cdot\,\right) U\right) = 0\right) > 0, \quad \tau > 0, \tag{3.4}$$

then formula (3.3) holds.

Subsequently, we consider a theorem about the synchronizations of stochastic periodic semi-flows.

Theorem 3.5. *Let u be a stochastic τ-periodic semi-flow, U is a non-empty open set. If u is asymptotically stable on U with the formula (3.4) for fixed time t, the random periodic attractor $\bar{A}(t,\omega)$ is $\mathscr{F}_t$-measurable, and $\mathbb{P}(\bar{A}(t,\omega) \subset U) > 0$, then $\bar{A}(t,\omega)$ is a singleton $\mathbb{P}$-almost surely, and synchronization of u occurs.*

Proof. Since u is asymptotically stable on U with the formula (3.4), there is a sequence $k_n \to \infty$ as $n \to \infty$, such that

$$\mathbb{P}\left(\lim_{\mathrm{n}\to\infty} \mathrm{diam}\left(u\left(t, t - k_n\tau, \omega\right) U\right) = 0\right) > 0.$$

As $\theta_{k\tau}$ is a $\mathbb{P}$-preserving map, we have

$$\mathbb{P}\left(\lim_{\mathrm{n}\to\infty} \mathrm{diam}\left(u\left(t + k_n\tau, t, \omega\right) U\right) = 0\right) > 0.$$

Noting that

$$\mathbb{P}\left(\bar{A}\left(t,\omega\right) \subset U\right) > 0,$$

$u\left(t + k_n\tau, t, \cdot\right)$ is $\mathscr{F}_{t,t+k_n\tau}$-measurable and $\bar{A}\left(t,\cdot\right)$ is $\mathscr{F}_t$-measurable. Moreover, by the independence of $\mathscr{F}_{t,t+k_n\tau}$ and $\mathscr{F}_t$, we have

$$\mathbb{P}\left(\lim_{\mathrm{n}\to\infty} \mathrm{diam}\left(u\left(t + k_n\tau, t, \omega\right) \bar{A}\left(t,\omega\right)\right) = 0\right) > 0.$$

Since $\bar{A}$ is random periodic attractor, we obtain

$$\mathbb{P}\left(\mathrm{diam}\left(\bar{A}\left(t,\omega\right)\right) = 0\right) > 0.$$

Meanwhile, for every $k > 0$,

$$\mathrm{diam}\left(\bar{A}\left(t,\omega\right)\right) = \mathrm{diam}\left(u\left(t, t - k\tau, \omega\right) \bar{A}\left(t - k\tau, \omega\right)\right), \quad \text{for almost all } \omega \in \Omega.$$

Especially, $\mathrm{diam}\left(\bar{A}\left(t - k\tau, \omega\right)\right) = 0$ implies $\mathrm{diam}\left(\bar{A}\left(t,\omega\right)\right) = 0$. Subsequently, $\bar{A}\left(t - k\tau, \omega\right) = \bar{A}\left(t, \theta_{-k\tau}\omega\right)$ implies

$$\left\{\mathrm{diam}\left(\bar{A}\left(t, \theta_{-k\tau}\omega\right)\right) = 0\right\} \subseteq \left\{\mathrm{diam}\left(\bar{A}\left(t,\omega\right)\right) = 0\right\}.$$

Furthermore,

$$\left\{\mathrm{diam}\left(\bar{A}\left(t, \theta_{-(k+1)\tau}\omega\right)\right) = 0\right\} \subseteq \left\{\mathrm{diam}\left(\bar{A}\left(t, \theta_{-k\tau}\omega\right)\right) = 0\right\} \subseteq \left\{\mathrm{diam}\left(\bar{A}\left(t,\omega\right)\right) = 0\right\},$$

then

$$\bigcap_{k\geq 0}\left\{\mathrm{diam}\left(\bar{A}\left(t, \theta_{-k\tau}\omega\right)\right) = 0\right\} \in \bigcap_{k\geq 0} \mathscr{F}_{t-k\tau}.$$

Since θ_t satisfies measure preserving property, we have

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{k\geq 0}\left\{\mathrm{diam}\left(\bar{A}\left(t, \theta_{-k\tau}\omega\right)\right) = 0\right\}\right) \\ &= \lim_{k\to\infty} \mathbb{P}\left(\mathrm{diam}\left(\bar{A}\left(t, \theta_{-k\tau}\omega\right)\right) = 0\right) \\ &= \mathbb{P}\left(\mathrm{diam}\left(\bar{A}\left(t,\omega\right)\right) = 0\right) > 0. \end{aligned}$$

And by Kolmogorov's Zero-One Law for the σ-field $\bigcap\limits_{k\geq 0} \mathscr{F}_{t-k\tau}$, we get

$$\mathbb{P}\left(\bigcap_{k\geq 0}\{\operatorname{diam}\left(\bar{A}\left(t,\theta_{-k\tau}\omega\right)\right)=0\}\right)=1,$$

so

$$\mathbb{P}\left(\operatorname{diam}\left(\bar{A}\left(t,\omega\right)\right)=0\right)=1.$$

Thus $\bar{A}$ is a singleton $\mathbb{P}$-almost surely. $\square$

Define $Y\left(t,\omega\right)=\lim\limits_{s\to-\infty}u\left(t,s,\omega\right)x$, for any $x\in C$ (C is any compact subset of $\mathbb{X}$) and given $t\in\mathbb{R}$, we have the following theorem:

Theorem 3.6. *For a given $t\in\mathbb{R}$, let u be a synchronized stochastic τ-periodic semi-flow, then $Y\left(t,\omega\right)$ is the unique random periodic path.*

Proof. By Definition 2.1, we can get that for almost every $\omega\in\Omega$ and for any $x\in C$,

$$\begin{aligned}Y\left(t+\tau,\omega\right)&=\lim_{s\to-\infty}u\left(t+\tau,s,\omega\right)x\\&=\lim_{s\to-\infty}u\left(t,s-\tau,\theta_{\tau}\omega\right)x\\&=Y\left(t,\theta_{\tau}\omega\right).\end{aligned}$$

Moreover, for any $s\leq t$, we have

$$\begin{aligned}u\left(t,s,\omega\right)Y\left(s,\omega\right)&=u\left(t,s,\omega\right)\lim_{r\to-\infty}u\left(s,r,\omega,x\right)\\&=\lim_{r\to-\infty}u\left(t,r,\omega,x\right)\\&=Y\left(t,\omega\right).\end{aligned}$$

Therefore, $Y\left(t,\omega\right)$ is the unique random periodic path by the random periodic attractor is a singleton. $\square$

4 Multiplicative ergodic theorem (MET)

We consider calculating their Lyapunov exponent to show the asymptotic stability for stochastic periodic semi-flows. In this section, we consider the multiplicative ergodic theorem (MET) for stochastic periodic semi-flows, which is an effective tool for finding the existence of Lyapunov exponents.

Note that for a stochastic τ-periodic semi-flow u, for a given $t\in\mathbb{R}$ and any $n\in\mathbb{N}$, define $u_t\left(n\tau,\omega\right):=u\left(t+n\tau,t,\omega\right)$, $\tau>0$, we have

$$\begin{aligned}u_t\left(\left(n+m\right)\tau,\omega\right)&=u\left(t+\left(n+m\right)\tau,t,\omega\right)\\&=u\left(t+\left(n+m\right)\tau,t+m\tau,\omega\right)u\left(t+m\tau,t,\omega\right)\\&=u\left(t+n\tau,t,\theta_{m\tau}\omega\right)u\left(t+m\tau,t,\omega\right)\\&=u_t\left(n\tau,\theta_{m\tau}\omega\right)u_t\left(m\tau,\omega\right),\end{aligned}\tag{4.1}$$

which means that $u_t(n\tau,\omega)$ satisfies the cocycle property. Lifting equation (4.1) to $\wedge^k \mathbb{R}^d$ $(1 \le k \le d\ )$, we obtain

$$\wedge^k u_t\left((n+m)\,\tau,\omega\right) = \wedge^k u_t\left(n\tau,\theta_{m\tau}\omega\right) \wedge^k u_t\left(m\tau,\omega\right),$$

where $\wedge^k$ is the k-fold exterior power.

4.1 Furstenberg-Kesten theorem for stochastic periodic semi-flows

For a given $t \in \mathbb{R}$ and $\tau > 0$, we define $\varphi_t(n\tau,\cdot)$, $n \in \mathbb{N}$, be a linear cocycle with random periodic property over a metric dynamical system $(\Omega,\mathscr{F},\mathbb{P},\{\theta_{n\tau}\}_{n\in\mathbb{N}})$ generated by $\hat{\varphi}$ if:

(i) $\varphi_t(n\tau,\omega) = \hat{\varphi}_t\left(\theta_{(n-1)\tau}\omega\right) \circ \cdots \circ \hat{\varphi}_t(\omega)$, where $\hat{\varphi}_t(\omega) = \varphi_t(\tau,\omega)$;

(ii) $\varphi_t(n\tau,\omega) = \varphi_t\left((n-1)\,\tau,\theta_\tau\,\omega\right) \circ \varphi_t(\tau,\omega)$;

(iii) $\varphi_{t+\tau}(n\tau,\omega) = \varphi_t((n+1)\tau,\omega) = \varphi_t(n\tau,\theta_\tau\omega)$.

In particular, for a stochastic τ-periodic semi-flow u, let $\tilde{\Omega} = \Omega \times \mathbb{X}$, $\tilde{\mathscr{F}} = \mathscr{F} \otimes \mathscr{B}(\mathbb{X})$, and $\Theta_\tau(\omega,x) = (\theta_\tau\omega, u_t(\tau,\omega,x))$. Then for $\tilde{\omega} = (\omega,x) \in \tilde{\Omega}$, we can define $\tilde{\varphi}_t(n\tau,\tilde{\omega}) = Du_t(n\tau,\omega,x)$ which satisfies the above properties over $\left(\tilde{\Omega},\tilde{\mathscr{F}},\mu,\{\Theta_{n\tau}\}_{n\in\mathbb{N}}\right)$, where $\mu \in \mathcal{P}(\Omega\times\mathbb{X})$.

Recalling the Furstenberg-Kesten theorem for random dynamical systems in [1], we directly have the following theorem of φ_t:

Theorem 4.1. *Let $\hat{\varphi}_t(\cdot) = \varphi_t(\tau,\cdot) : \Omega \to \mathbb{R}^{d\times d}$, $\varphi_t(n\tau,\omega)$, $n \in \mathbb{N}$, be a linear cocycle operator generated by $\hat{\varphi}_t(\cdot)$ for a given $t \in \mathbb{R}$ over the metric dynamical system $(\Omega,\mathscr{F},\mathbb{P},\{\theta_{n\tau}\}_{n\in\mathbb{N}})$. Then if the generator $\hat{\varphi}_t(\cdot)$ satisfies*

$$\log^+ \parallel \hat{\varphi}_t(\cdot) \parallel \in L^1(\Omega,\mathscr{F},\mathbb{P}),$$

the following properties hold:

(i) For each $k = 1,\ldots,d$, $f^k_{t,n}(\omega) = \log\left\|\wedge^k\varphi_t(n\tau,\omega)\right\|$ is subadditive and $f^{k+}_{t,1} \in L^1$;

(ii) There is a forward invariant set $\bar{\Omega} \in \mathscr{F}$ of full measure and the measurable function $\gamma^k_t : \Omega \to \mathbb{R}\cup\{-\infty\}$ such that on $\bar{\Omega}$

$$\gamma^k_t(\omega) = \lim_{n\to\infty}\frac{1}{n}\log\left\|\wedge^k\varphi_t(n\tau,\omega)\right\| \tag{4.2}$$

and

$$\gamma^k_t(\theta_\tau\omega) = \gamma^k_t(\omega), \quad \gamma^{k+m}_t(\omega) \le \gamma^k_t(\omega) + \gamma^m_t(\omega).$$

In particular, if the $\mathbb{P}$-preserving θ_τ is ergodic with respect to τ, then $\gamma^k_t(\omega) = \mathbb{E}\gamma^k_t$ is deterministic, and

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log\left\|\wedge^k\varphi_t(n\tau,\omega)\right\| = \gamma^k_t;$$

(iii) Define the measurable function $\Lambda_{t,k}$ as

$$\Lambda_{t,1}(\omega)+\Lambda_{t,2}(\omega)+\cdots+\Lambda_{t,k}(\omega)=\gamma_t^k(\omega), \quad k=1,\ldots,d,$$

and $\Lambda_{t,k}(\omega)=-\infty$ if $\gamma_t^k=-\infty$, then $\Lambda_{t,k}$ has the properties:

(a) $\Lambda_{t,k}(\omega)=\lim_{n\to\infty}\frac{1}{n}\log\delta_k\left(\varphi_t(n\tau,\omega)\right)$,
where $\delta_k\left(\varphi_t(n\tau,\omega)\right)$ are the singular values of $\varphi_t(n\tau,\omega)$;

(b) $\Lambda_{t,k}\left(\theta_\tau\omega\right)=\Lambda_{t,k}(\omega)$;

(c) $\Lambda_{t,1}(\omega)\geq\cdots\geq\Lambda_{t,d}(\omega)$.

In the case that $\mathbb{P}$-preserving θ_τ is ergodic, then $\Lambda_{t,k}(\omega)=\mathbb{E}\Lambda_{t,k}$, and

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log\delta_k\left(\varphi_t(n\tau,\cdot)\right)=\Lambda_{t,k}.$$

4.2 MET for stochastic periodic semi-flows

Theorem 4.2. *Let $\varphi_t\left(n\tau,\omega\right)$ for a given $t\in\mathbb{R}$ be a linear cocycle with*

$$\hat{\varphi}_t\left(\cdot\right)=\varphi_t\left(\tau,\cdot\right):\Omega\to\mathbb{R}^{d\times d}, \text{ and } \varphi_t\left(n\tau,\omega\right)=\hat{\varphi}_t\left(\theta_{n\tau}\omega\right)\circ\cdots\circ\hat{\varphi}_t\left(\omega\right)$$

over the metric dynamical system $(\Omega,\mathscr{F},\mathbb{P},\{\theta_{n\tau}\}_{n\in\mathbb{N}})$. If the generator $\hat{\varphi}_t\left(\omega\right)$ satisfies

$$\log^+\parallel\hat{\varphi}_t\left(\cdot\right)\parallel\in L^1\left(\Omega,\mathscr{F},\mathbb{P}\right),$$

then there is a forward invariant set $\hat{\Omega}\in\mathscr{F}$ of full measure such that for each $\omega\in\hat{\Omega}$, the followings hold:

(i) $\lim_{n\to\infty}\left(\varphi_t\left(n\tau,\omega\right)^*\varphi_t\left(n\tau,\omega\right)\right)^{\frac{1}{2n}}=\Psi_t\left(\omega\right)>0$ exists;

(ii) Let $e^{\lambda_{t,p_t(\omega)}(\omega)}<\cdots<e^{\lambda_{t,1}(\omega)}$ be the different eigenvalues of $\Psi_t\left(\omega\right)$, $\lambda_{t,p_t(\omega)}\left(\omega\right)=-\infty$ possible, and $U_{t,p_t(\omega)}\left(\omega\right),\ldots,U_{t,1}\left(\omega\right)$ be the corresponding eigenspaces with

$$d_{t,i}\left(\omega\right)=\dim\left(U_{t,i}\left(\omega\right)\right),$$

then

$$\lambda_{t,i}\left(\theta_\tau\omega\right)=\lambda_{t,i}\left(\omega\right),$$

where $i=1,\ldots p_t\left(\omega\right)$;

(iii) Putting $V_{t,p_t(\omega)+1}=\{0\}$, and let

$$V_{t,i}\left(\omega\right)=U_{t,p_t(\omega)}\left(\omega\right)\oplus\cdots\oplus U_{t,i}\left(\omega\right), \quad V_{t,p_t(\omega)}\left(\omega\right)\subset\cdots\subset V_{t,1}\left(\omega\right)=\mathbb{R}^d.$$

Then the Lyapunov exponent $\lambda_t\left(\omega,x\right)=\lim_{n\to\infty}\frac{1}{n}\log\parallel\varphi_t\left(n\tau,\omega\right)x\parallel$ exists, and

$$\lambda_t\left(\omega,x\right)=\lambda_{t,i}\left(\omega\right),$$

when $x\in V_{t,i}\left(\omega\right)\setminus V_{t,i+1}\left(\omega\right)$. This is equivalent to

$$V_{t,i}\left(\omega\right)=\left\{x\in\mathbb{R}^d:\lambda_t\left(\omega,x\right)\leq\lambda_{t,i}\left(\omega\right)\right\};$$

(iv) *For all* $x \in \mathbb{R}^d \setminus \{0\}$, $\lambda_t(\theta_\tau\omega, \hat{\varphi}_t(\omega)x) = \lambda_t(\omega, x)$, *and* $\hat{\varphi}_t(\omega) V_{t,i}(\omega) \subset V_{t,i}(\theta_\tau\omega)$;

(v) *The Lyapunov exponent satisfies that* $\lambda_{t+\tau,i}(\omega) = \lambda_{t,i}(\omega)$ *and we have* $V_{t+\tau,i}(\omega) = V_{t,i}(\omega)$, $p_{t+\tau}(\omega) = p_t(\omega)$;

(vi) *If* $\hat{\varphi}_t(\omega)$ *is inverse, then*

$$\hat{\varphi}_t(\omega) V_{t,i}(\omega) = V_{t,i}(\theta_\tau\omega) = V_{t+\tau,i}(\omega);$$

(vii) *If* $(\Omega, \mathscr{F}, \mathbb{P}, \theta_{n\tau})_{n\in\mathbb{N}}$ *is ergodic, then* $p(\cdot)$ *is constant,* $d_{t,i}(\cdot)$ *and* $\lambda_{t,i}(\cdot)$ *are constant on* $\{\omega : p(\omega) \geq i\}$, $i = 1, \ldots, d$ *as well.*

Proof. Since $\log^+ \parallel \hat{\varphi}_t(\cdot) \parallel \in L^1$, by Furstenberg-Kesten theorem, there is a forward invariant set $\hat{\Omega} \in \mathscr{F}$ of full measure, such that for all $\omega \in \hat{\Omega}$, $k = 1, \ldots, d$:

(a) $\gamma_t^k(\omega)$ which is defined in formula (4.2) exists;

(b) $\gamma_t^k(\theta_\tau\omega) = \gamma_t^k(\omega)$;

(c) $\Lambda_{t,k}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \delta_{t,k}(\varphi_t(n\tau, \omega)) = \Lambda_{t,k}(\theta_\tau\omega)$.

Then (i), (ii), (iii) can be proved directly by Theorem 3.4.1 in [1]. To prove (iv), we note that

$$\begin{aligned}
\Lambda_{t,k}(\omega) &= \lim_{n\to\infty} \frac{1}{n} \log \delta_k(\varphi_t(n\tau, \omega)), \\
\Lambda_{t+\tau,k}(\omega) &= \lim_{n\to\infty} \frac{1}{n} \log \delta_k(\varphi_{t+\tau}(n\tau, \omega)) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \delta_k(\varphi_t(n\tau, \theta_\tau\omega)) \\
&= \Lambda_{t,k}(\theta_\tau\omega) = \Lambda_{t,k}(\omega).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\lambda_{t+\tau}(\omega, x) &= \lim_{n\to\infty} \frac{1}{n} \log \parallel \varphi_{t+\tau}(n\tau, \omega) x \parallel \\
&= \lim_{n\to\infty} \frac{1}{n} \log \parallel \varphi_{t+\tau}((n-1)\tau, \theta_\tau\omega) \circ \varphi_t(\tau, \omega) x \parallel \\
&= \lambda_t(\theta_\tau\omega, \varphi_t(\tau, \omega) x).
\end{aligned} \tag{4.3}$$

Meanwhile, we note that

$$\begin{aligned}
\lambda_t(\omega, x) &= \lim_{n\to\infty} \frac{1}{n+1} \log \parallel \varphi_t((n+1)\tau, \omega) x \parallel \\
&= \lim_{n\to\infty} \frac{1}{n+1} \log \parallel \varphi_t(n\tau, \theta_\tau\omega) \circ \varphi_t(\tau, \omega) x \parallel \\
&= \lambda_t(\theta_\tau\omega, \varphi_t(\tau, \omega) x).
\end{aligned} \tag{4.4}$$

Combining equations (4.3) and (4.4), we obtain $\lambda_t(\omega, x) = \lambda_{t+\tau}(\omega, x)$. If we take $x \in V_{t,i}(\omega)$, then $\lambda_t(\omega, x) \leq \lambda_{t,i}(\omega)$. Consequently, (iv) is proved with

$$\lambda_t(\theta_\tau\omega, \hat{\varphi}_t(\omega) x) = \lambda_t(\omega, x) \leq \lambda_{t,i}(\omega) = \lambda_{t,i}(\theta_\tau\omega).$$

To prove (v), we consider that

$$\begin{aligned}\lambda_{t+\tau}(\omega,x) &= \lim_{n\to\infty}\frac{1}{n}\log\parallel \varphi_{t+\tau}(n\tau,\omega)\,x\parallel\\ &= \lim_{n\to\infty}\frac{1}{n}\log\parallel \varphi_{t}(n\tau,\theta_\tau\omega)\,x\parallel\\ &= \lambda_t(\theta_\tau\omega,x)\,,\end{aligned}$$

then

$$\lambda_{t+\tau}(\omega,x)=\lambda_t(\theta_\tau\omega,x)=\lambda_t(\omega,x)\,.$$

Noting that

$$\begin{aligned}V_{t,\,i}(\omega) &= \left\{x\in\mathbb{R}^d:\lambda_t(\omega,x)\le\lambda_{t,\,i}(\omega)\right\}\\ &= \left\{x\in\mathbb{R}^d:\lambda_{t+\tau}(\omega,x)\le\lambda_{t+\tau,\,i}(\omega)\right\}\\ &= V_{t+\tau,i}(\omega)\,,\end{aligned}$$

therefore

$$\begin{aligned}V_{t+\tau,\,i}(\omega) &= \left\{x\in\mathbb{R}^d:\lambda_{t+\tau}(\omega,x)\le\lambda_{t+\tau,\,i}(\omega)\right\}\\ &= \left\{x\in\mathbb{R}^d:\lambda_t(\theta_\tau\omega,x)\le\lambda_{t,\,i}(\theta_\tau\omega)\right\}\\ &= V_{t,\,i}(\theta_\tau\omega)\,.\end{aligned}$$

Furthermore, we get

$$V_{t,i}(\omega)=V_{t+\tau,i}(\omega)=V_{t,i}(\theta_\tau\omega)\,.$$

From (iv), we can find that

$$\hat{\varphi}_t(\omega)\,V_{t,i}(\omega)\subset V_{t,i}(\theta_\tau\omega)=V_{t+\tau,i}(\omega)\,,$$

consequently, $p_t(\omega)=p_{t+\tau}(\omega)$, and (v) is proved.

To prove (vi), we note that for any $t\in\mathbb{R}$, if taking $x\in V_{t+\tau,i}(\omega)$, then

$$\begin{aligned}\lambda_{t+\tau}\left(\omega,\hat{\varphi}_t^{-1}(\omega)\,x\right) &= \lim_{n\to\infty}\frac{1}{n}\log\parallel \varphi_t(n\tau,\omega)\circ\hat{\varphi}_t^{-1}x\parallel\\ &= \lim_{n\to\infty}\frac{1}{n}\log\parallel \varphi_t((n-1)\,\tau,\theta_\tau\omega)\circ\hat{\varphi}_t\circ\hat{\varphi}_t^{-1}x\parallel\\ &= \lim_{n\to\infty}\frac{1}{n}\log\parallel \varphi_t(n\tau,\theta_\tau\omega)\,x\parallel\\ &= \lambda_t(\theta_\tau\omega,x)\\ &\le \lambda_{t,i}(\theta_\tau\omega,x)\,.\end{aligned}$$

This implies that $\hat{\varphi}_t^{-1}V_{t+\tau,i}(\omega)\subset V_{t,i}(\theta_\tau\omega)$, and form (iv), (vi) can be proved.

Finally, (vii) can be proved directly by (vi). □

For $\log^+|Du_t(\tau,\omega,x)|\in L^1\left(\tilde{\Omega},\tilde{\mathscr{F}},\mu\right)$, the Lyapunov exponent exists and it can be considered as $\lambda_t(\omega,x)=\lim\limits_{n\to\infty}\frac{1}{n}\log|Du_t(n\tau,\omega,x)|$.

5 Applications in SDE

In this section, we show that in some examples, the largest Lyapunov exponent corresponding to stochastic periodic semi-flows is negative, then the asymptotically stable property holds, and there is a unique random periodic solution. We mainly consider the following one-dimensional SDE:

$$dX_t = b\left(t, X_t\right) dt + \sigma dW_t, \tag{5.1}$$

where W_t is one-dimensional Wiener process, $\sigma > 0$ is a constant. The state space is defined as $\mathbb{X} = \mathbb{R}$, time $t \in \mathbb{R}^+$, and $b\left(t, x\right)$ satisfies the following assumption:

Assumption 5.1.

(i) $(x-y)\left(b\left(t,x\right)-b\left(t,y\right)\right) \leq \alpha|x-y|^2$ *for all* $x, y \in \mathbb{R}$ *and for some* $\alpha > 0$*;*

(ii) $(x-y)\left(b\left(t,x\right)-b\left(t,y\right)\right) \leq L_1 - L_2|x-y|^2$ *for all* $x, y \in \mathbb{R}$*, and* $L_1, L_2 > 0$*.*

The following gives two typical examples that satisfy Assumption 5.1.

5.1 Benzi-Parisi-Sutera-Vulpiani stochastic resonance model

When $b\left(t, x\right) = x - x^3 + \varepsilon \cos t$, then equation (5.1) turns to be

$$dX_t = \left(X_t - X_t^3 + \varepsilon \cos t\right) dt + \sigma dW_t, \tag{5.2}$$

which is a stochastic resonance model introduced by Benzi et al. in [4]. It is a seminal theoretical framework for explaining the periodic occurrence of ice ages. In their work, ε was considered very small to comply with real-world issues. Small ε is assumed in this example as well.

Let $u\left(t, s, \omega\right)$ be a stochastic 2π-periodic semi-flow generated by the equation (5.2), we will show that the synchronization of $u\left(t, s, \omega\right)$ happens, and there is a unique random periodic path.

Recalling the works by Cherubini et al. in [5], the random attractor for a non-autonomous random dynamical system of (5.2) exists. Similarly, there is a random periodic attractor for stochastic periodic semi-flow. We will show the asymptotic stability of the stochastic periodic semi-flow $u\left(t, s, \omega\right)$ in (5.2) to show that the random periodic attractor is a singleton. Thus, it is important to consider the largest Lyapunov exponent, and we have the following Theorem.

Theorem 5.2. *For a given* $t \in \mathbb{R}^+$*, let* U *be an open subset of* $\mathbb{R}$*, if the largest Lyapunov exponent* $\lambda_{t,1}$ *for the stochastic periodic semi-flow* $u\left(t, s, \omega\right)$ *is negative, then the stochastic periodic semi-flow* $u\left(t, s, \omega\right)$ *is asymptotically stable on* U*, i.e. there is sequence* $s_n \to -\infty$ *for* $n \to \infty$ *such that the formula (3.3) holds.*

Proof. Le Jan in [16] proved that when the largest Lyapunov exponent for cocycle S_n is negative, then for all $\varepsilon > 0$, there is a $\delta > 0$, such that when $d\left(x, y\right) < \delta$,

$$\mathbb{P}\left(\lim_{n\to\infty} d\left(S_n\left(\omega\right)x, S_n\left(\omega\right)y\right) < \varepsilon\right) > 0, \quad x, y \in \mathbb{R}. \tag{5.3}$$

Here, for a given $t \in \mathbb{R}^+$, we consider the stochastic τ-periodic semi-flow $u(t+n\tau, t, \omega)$. Since $u(t+n\tau, t, \omega) = u(t, t-n\tau, \theta_{n\tau}\omega)$ satisfies the equation (4.1) (cocycle property in random periodic case), and the largest Lyapunov exponent $\lambda_{t,1} < 0$, then by inequality (5.3), for any $\varepsilon > 0$, there is a $\delta > 0$, such that when $d(x, y) < \delta$,

$$\mathbb{P}\left(\lim_{n\to\infty} d\left(u(t, t-n\tau, \omega)x, u(t, t-n\tau, \omega)y\right) < \varepsilon\right) > 0.$$

Let U be an open subset of $\mathbb{R}$. If $\delta \geq \operatorname{diam}(U)$, we have

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(u(t, t-n\tau, \omega)U\right) < \varepsilon\right) > 0.$$

If $\delta < \operatorname{diam}(U)$, for $x, y \in U$, we can find a sequence $\{c_m\}$ in U, such that

$$d(x = c_0, c_1) < \delta,\ d(c_1, c_2) < \delta, \cdots, d(c_{m-1}, y = c_m) < \delta,$$

then for any $\varepsilon_i > 0$,

$$\mathbb{P}\left(\lim_{n\to\infty} d\left(u(t, t-n\tau, \omega)c_i, u(t, t-n\tau, \omega)c_{i+1}\right) < \varepsilon_i\right) > 0, \quad i = 0, \ldots, m-1.$$

Since ε_i is arbitrary, taking $\varepsilon = \sum\limits_{i=0}^{m-1} \varepsilon_i$, for any $x, y \in U$, we have

$$\mathbb{P}\left(\lim_{n\to\infty} d\left(u(t, t-n\tau, \omega)x, u(t, t-n\tau, \omega)y\right) < \varepsilon\right) > 0.$$

Therefore,

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(u(t, t-n\tau, \omega)U\right) < \varepsilon\right) > 0.$$

This implies that the formula (3.4) holds. In particular, taking $s_n = t - n\tau$, we get

$$\mathbb{P}\left(\lim_{n\to\infty} \operatorname{diam}\left(u(t, s_n, \omega)U\right) < \varepsilon\right) > 0,$$

which means u is asymptotically stable on U. □

Now we will show that the largest Lyapunov exponent is negative. In fact, noting that the derivative flow $Du(t, s, \omega)$ satisfies

$$\frac{d}{dt}Du(t, s, \omega)\xi = \left(1 - 3u^2(t, s, \omega)\right) Du(t, s, \omega)\xi,$$

by Assumption 5.1 (i), we have

$$\begin{aligned} \frac{d}{dt}|Du(t, s, \omega)\xi|^2 &= 2\left(\left(1 - 3u^2(t, s, \omega)\right) Du(t, s, \omega)\xi\right) Du(t, s, \omega)\xi \\ &\leq 2\alpha |Du(t, s, \omega)\xi|^2, \end{aligned} \tag{5.4}$$

for some $\alpha > 0$. Then by Gronwall's inequality, we get

$$|Du(t, s, \omega)| \leq e^{\alpha(t-s)}.$$

For a given $t \in \mathbb{R}^+$, by MET for stochastic periodic semi-flows (Theorem 4.2), the largest Lyapunov exponent $\lambda_{t,1}$ exists and can be computed with the help of equation (5.4)

$$\lambda_{t,1} = \lim_{n\to\infty} \frac{1}{n} \int_t^{t+n\pi} \left(\left(1 - 3u^2\left(r, t, \omega, x\right)\right)\right) dr.$$

Noting that the Markov transition probability function is given by

$$P\left(t, s, x, A\right) = \mathbb{P}\left(\omega \in \Omega : u\left(t, s, \omega\right) x \in A\right), \, t \geq s.$$

It has been proved in [13] that there is an ergodic periodic measure ρ_t under Assumption 5.1. Therefore with the ergodicity, to show $\lambda_{t,1} < 0$, we need to prove that $\mathbb{E}\left(1 - 3u^2\right) < 0$ under the Markov transition probability.

By [12], we have

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\int_{\mathbb{R}} \left(\left(1 - 3u^2\left(t, s, \omega\right)\xi\right) P\left(t, s, \xi, du\right) - \left(1 - 3u^2\left(t, s, \omega\right)\xi\right) d\rho_s\left(u\right)\right)\right) ds = 0.$$

Thus we will prove

$$\int_0^{2\pi} \int_{\mathbb{R}} \left(1 - 3u^2\right) d\rho_s\left(u\right) ds < 0, \tag{5.5}$$

to guarantee the largest Lyapunov exponent is negative.

Moreover, from the work of [13], ρ_s has the density function $q\left(s, x\right)$, and $q(s, x)$ satisfies the Fokker-Planck equation

$$\frac{\partial q\left(s, x\right)}{\partial s} = -\frac{\partial}{\partial x}\left(\left(x - x^3 + \varepsilon \cos s\right) q(s, x)\right) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} q(s, x). \tag{5.6}$$

Noting that

$$\int_{\mathbb{R}} \left(1 - 3u^2\right) d\rho_s\left(u\right) = \int_{\mathbb{R}} \left(1 - 3u^2\right) q\left(s, u\right) du,$$

and we take $x = u$, then proving the inequality (5.5) is equivalent to prove that for $s \in [0, 2\pi]$,

$$\int_{\mathbb{R}} \left(1 - 3x^2\right) q\left(s, x\right) dx < 0.$$

Firstly, noting that when $x \in \left(-\infty, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, \infty\right)$, $\left(1 - 3x^2\right) < 0$, then for large $a \gg \frac{\sqrt{3}}{3}$,

$$\int_{\mathbb{R}} \left(1 - 3x^2\right) q\left(s, x\right) dx < \int_{-a}^{a} \left(1 - 3x^2\right) q\left(s, x\right) dx.$$

Thus it is enough to prove that on a bounded interval $[-a, a]$ ($a \gg \frac{\sqrt{3}}{3}$),

$$\int_{-a}^{a} \left(1 - 3x^2\right) q\left(s, x\right) dx < 0. \tag{5.7}$$

Inspired by the proof of averaging method in [2], the estimation of the inequality (5.7) is presented in the following Theorem.

Theorem 5.3. *Let $\hat{q}(x) = e^{-\frac{2}{\sigma^2}\left(-\frac{1}{2}x^2+\frac{1}{4}x^4\right)}$, and $q(s,x)$ be the density function satisfying the Fokker-Planck equation (5.6), then there exists a bounded function $f(s,x)$ such that*

$$\lim_{\varepsilon\to 0}(q - \hat{q} - \varepsilon\hat{q}f) = 0,$$

and the inequality (5.7) holds for all $s \in [0, 2\pi]$ and for all $\sigma > 0$ on a bounded interval $[-a, a]$, where $a \gg \frac{\sqrt{3}}{3}$.

Proof. Firstly, we prove the existence of the solution $f(s,x)$ of the partial differential equation:

$$\frac{\partial f}{\partial s} = \left(x - x^3\right)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \cos s \cdot \frac{2}{\sigma^2}\left(x - x^3\right). \tag{5.8}$$

This PDE can be rewritten as

$$-\frac{d\bar{f}}{ds} = A\bar{f} + h,$$

where

$$\bar{f} = \begin{pmatrix} f \\ \frac{\partial f}{\partial x} \\ \left(x - x^3\right) f \\ \left(1 - 3x^3\right) f \end{pmatrix}, \ A = \begin{pmatrix} 0 & -\frac{\sigma^2}{2}\frac{d}{dx} & -\frac{d}{dx} & 1 \\ 0 & -\frac{d}{ds} & 0 & 0 \\ 0 & 0 & -\frac{d}{ds} & 0 \\ 0 & 0 & 0 & -\frac{d}{ds} \end{pmatrix}, \ h = \begin{pmatrix} -\cos s \cdot \frac{2}{\sigma^2}\left(x - x^3\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let B be a linear operator:

$$B\,v = -\frac{dv}{ds}.$$

According to [8], for $\hat{\lambda} \in \hat{\rho}(B)$, where $\hat{\rho}(B)$ is the resolvent set of B, if

$$\left(\hat{\lambda}I - B\right)^{-1} = R\left(\hat{\lambda}, B\right)$$

exists, then it satisfies

$$R\left(\hat{\lambda}, B\right) v(s) = \int_0^s e^{-\hat{\lambda}(s-r)} v(r)\, ds$$

and

$$\left\| R\left(\hat{\lambda}, B\right)\right\|_{L^p} \leq \frac{1}{\hat{\lambda}}, \quad \hat{\lambda} \in \mathbb{R},\ \hat{\lambda} > 0. \tag{5.9}$$

Now we consider the equation

$$B\,v = A\,v + h,$$

and we will find a bounded sequence $v_n \in L^p$ satisfying

$$-B_n\,v_n + A\,v_n + h = 0, \quad n \in \mathbb{N}^+, \tag{5.10}$$

where $B_n = n^2 R(n, B)$ is chosen by Proposition 3.1 in [8] and it satisfies $\lim_{n\to\infty} \| B_n - B \|_{L^p} = 0$. In fact, the equation (5.10) can be written as

$$(A - n)\,v_n = n^2 \int_0^s e^{-n(s-r)} v_n dr - h - n v_n.$$

Taking $(n-A)^{-1} = R(n,A)$ on both sides, we get

$$(-I + nR(n,A))\, v_n = n^2 e^{-ns} \int_0^s e^{nr} R(n,A)\, v_n dr - R(n,A)\, h. \tag{5.11}$$

Let

$$w_n = \int_0^s e^{nr} R(n,A)\, v_n dr, \quad w_n' = e^{ns} R(n,A)\, v_n,$$

substituting into equation (5.11), we have

$$(-I + nR(n,A))\, w_n' = e^{ns} R(n,A)\, w_n - R^2(n,A)\, e^{ns} h,$$

as well as

$$(-I + nR(n,A))\, w_n = \int_0^s e^{n^2 R(n,A)(s-r)} R^2(n,A)\, e^{-nr} h\, dr.$$

Let σ be a finite positive constant, $x \in [-a,\, a]$, then by the definition of A, we obtain

$$nI - A = \begin{pmatrix} nI & nI + \frac{d}{dx}\frac{\sigma^2}{2} & nI + \frac{d}{dx} & (n-1)\, I \\ 0 & nI + \frac{d}{dt} & 0 & 0 \\ 0 & 0 & nI + \frac{d}{ds} & 0 \\ 0 & 0 & 0 & nI + \frac{d}{ds} \end{pmatrix},$$

and its reverse $(nI - A)^{-1}$ is

$$-\frac{1}{n}\left(nI + \frac{d}{ds}\right)^{-1} \begin{pmatrix} -\left(nI + \frac{\sigma^2}{2}\frac{d}{dx}\right) & \left(nI + \frac{\sigma^2}{2}\frac{d}{dx}\right) & \left(nI + \frac{d}{ds}\right) & (n-1)\, I \\ 0 & \left(nI + \frac{d}{ds}\right)^{-1} & 0 & 0 \\ 0 & 0 & \left(nI + \frac{d}{ds}\right)^{-1} & 0 \\ 0 & 0 & 0 & \left(nI + \frac{d}{ds}\right)^{-1} \end{pmatrix}.$$

By the inequality (5.9), we get $\|R(n,B)\|_{L^p} < 1/n$. Therefore

$$\|nR(n,A)\|_{L^p} < \infty, \quad \left\|(-I + nR(n,A))^{-1}\right\|_{L^p} < \infty.$$

Define

$$M = \sup_{k\in\mathbb{N}, n>0} \left\| n^k R(n,A)^k \right\|_{L^p} < \infty,$$

then we have

$$\begin{aligned} & \left\| e^{n^2 R(n,A)(s-r)} R^2(n,A) \right\|_{L^p} \\ = & \left\| \sum_{k=0}^{\infty} \frac{1}{k!} \left(n^2 R(n,A)(s-r)\right)^k R^2(n,A) \right\|_{L^p} \\ = & \left\| \sum_{k=0}^{\infty} \frac{1}{k!} \left(n^{2k} R^{k+2}(n,A)(s-r)^k\right) \right\|_{L^p} \\ \le & M \sum_{k=0}^{\infty} \frac{n^{2k}(s-r)^k}{k! n^{k+2}} = \frac{M e^{n(s-r)}}{n^2}. \end{aligned}$$

Thus

$$\|(-I+nR(n,A))\,v_n\|_{L^p} \le M\left(\int_0^s \|h\|_{L^p}\,dr + \frac{1}{n}\|h\|_{L^p}\right) \le MK\,\|h\|_{L^p}\,,$$

where K is some finite positive constant. Furthermore, we get

$$||v_n||_{L^p} \le \bar{M}||h||_{L^p} \tag{5.12}$$

for some finite positive constant $\bar{M}$. Thus, we showed a bounded sequence v_n in equation (5.10).

Now we rewrite equation (5.10) as

$$-B_n\left(v_n-\bar{f}\right)+A\left(v_n-\bar{f}\right)-B_n\bar{f}+B\bar{f}=0, \tag{5.13}$$

and define

$$\bar{h}=B\bar{f}-B_n\bar{f}, \quad V=v_n-\bar{f}.$$

By inequality (5.12) and equation (5.13), we have

$$\|V\|_{L^p}=\left\|v_n-\bar{f}\right\|_{L^p} \le \bar{M}\left\|\bar{h}\right\|_{L^p}=\bar{M}\left\|B\bar{f}-B_n\bar{f}\right\|_{L^p}, \tag{5.14}$$

which means $\lim\limits_{n\to\infty}\left\|v_n-\bar{f}\right\|_{L^p}=0$. Therefore, we proved the existence of the solution $f(s,x)$ in PDE (5.8).

We come to the estimation of the density function $q(s,x)$. Initially, we find $\hat{q}$ satisfies equations:

$$\frac{\partial\hat{q}}{\partial s}=-\left(1-3x^2\right)\hat{q}-\left(x-x^3\right)\frac{\partial\hat{q}}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2\hat{q}}{\partial x^2}, \tag{5.15}$$

and

$$\frac{\partial\hat{q}}{\partial x}=\frac{2}{\sigma^2}\left(x-x^3\right)\hat{q}. \tag{5.16}$$

Multiplying equation (5.8) by $\hat{q}$ on both sides, and then substituting equation (5.16) into equation (5.8), we have

$$\hat{q}\frac{\partial f}{\partial s}=-\left(x-x^3\right)\hat{q}\frac{\partial f}{\partial x}+\sigma^2\frac{\partial\hat{q}}{\partial x}\frac{\partial f}{\partial x}+\frac{\sigma^2}{2}\hat{q}\frac{\partial^2 f}{\partial x^2}+\cos s\cdot\frac{\partial\hat{q}}{\partial x}. \tag{5.17}$$

From equations (5.15) and (5.17), we obtain $g(s,x)=\hat{q}+\varepsilon\hat{q}f$ satisfying the equation:

$$\frac{\partial g}{\partial s}=-\left(1-3x^2\right)g-\left(x-x^3\right)\frac{\partial g}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}+\varepsilon\cos s\frac{\partial\hat{q}}{\partial x}.$$

Combined with equation (5.6), the equation about $\tilde{q}=q-g$ is

$$\frac{\partial\tilde{q}}{\partial s}=-\left(1-3x^2\right)\tilde{q}-\left(x-x^3\right)\frac{\partial\tilde{q}}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2\tilde{q}}{\partial x^2}-\varepsilon\cos s\frac{\partial q}{\partial x}+\varepsilon\cos s\frac{\partial\hat{q}}{\partial x}. \tag{5.18}$$

Meanwhile, we note that

$$\varepsilon\cos s\frac{\partial g}{\partial x}=\varepsilon\cos s\frac{\partial\hat{q}}{\partial x}+\varepsilon^2\cos s\frac{\partial\hat{q}}{\partial x}f+\varepsilon^2\cos s\frac{\partial f}{\partial x}\hat{q}, \tag{5.19}$$

and

$$\varepsilon \cos s \frac{\partial q}{\partial x} = \varepsilon \cos s \frac{\partial \tilde{q}}{\partial x} + \varepsilon \cos s \frac{\partial g}{\partial x}. \tag{5.20}$$

Substituting equations (5.19) and (5.20) into equation (5.18), we get

$$\begin{aligned} \frac{\partial \tilde{q}}{\partial s} =& -\left(1-3x^2\right)\tilde{q} - \left(x-x^3\right)\frac{\partial \tilde{q}}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial x^2} \\ & - \varepsilon \cos s \frac{\partial \tilde{q}}{\partial x} - \varepsilon^2 \cos s \frac{\partial \hat{q}}{\partial x} f - \varepsilon^2 \cos s \frac{\partial f}{\partial x}\hat{q}. \end{aligned} \tag{5.21}$$

When $x \in [-a, a]$, by inequalities (5.12) and (5.14), the solution f of equation (5.8) is bounded, then we have

$$G(s,x) = \cos s \frac{\partial \hat{q}}{\partial x} f + \cos s \frac{\partial f}{\partial x}\hat{q} \le C, \; C \text{ is some positive constant.}$$

Equation (5.21) can be written as

$$-L(\tilde{q}) = \varepsilon^2 G(s,x),$$

where L is an operator satisfying

$$L(\tilde{q}) = \frac{\partial \tilde{q}}{\partial s} + \left(1-3x^2\right)\tilde{q} - \left(x-x^3\right)\frac{\partial \tilde{q}}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial x^2} + \varepsilon \cos s \frac{\partial \tilde{q}}{\partial x}.$$

Therefore, we have

$$\left|\frac{q-g}{\varepsilon^2}\right| \le \left|\hat{C}\right|, \quad x \in [-a,a], \, s \in [0, 2\pi],$$

where $\hat{C}$ is some finite positive constant. Thus $\lim_{\varepsilon \to 0} (q - \hat{q} - \varepsilon \hat{q} f) = 0$.

Finally, for very small ε to prove

$$\int_{[-a,a]} \left(1-3x^2\right) q dx < 0,$$

it is enough to prove

$$\int_{[-a,a]} \left(1-3x^2\right) \hat{q} dx + \varepsilon \int_{[-a,a]} \left(1-3x^2\right) \hat{q} f dx < 0.$$

Flandoli et al. have shown that

$$\int_{[-a,a]} \left(1-3x^2\right) \hat{q} dx < 0$$

in [15], and since f is bounded, then we have

$$\int_{[-a,a]} \left(1-3x^2\right) \hat{q} f dx < 0.$$

Therefore inequality (5.7) holds, and the theorem has been completely proved. □

Thus by Theorem 5.2, we have $u(t,s,\omega)$ is asymptotically stable in any open set U. For the random periodic attractor $\bar{A}$, it is straightforward to show that it is swift transitive and contracting on large sets in a similar way to the proof of Proposition 3.10 in [15], and from Lemma 2.10 and Theorem 2.14 in [15], $\mathbb{P}\left(\bar{A} \subset U\right) > 0$. Therefore by Theorem 3.5, u is synchronized, and by Theorem 3.6, there is a unique random periodic path.

We perform some numerical simulations in Matlab to verify the above results. The density function $q(s,x)$ is firstly calculated to approach the largest Lyapunov exponent. Calculating the density function $q(s,x)$ with infinity x is not realistic. The numerical results show that $|x| < 100$ is enough for the calculation because the result is nearly invariant when choosing x from -200 to 200.

In fact, we take s from 0 to 2π, $\sigma = 1$, x from -100 to 100 and -200 to 200 separately. We also choose some different values of small ε. Then we solve the Fokker-Planck equation (5.6) numerically, and substitute the numerical solution $q_{1,num}(s,x)$ (x choosing from -100 to 100) and $q_{2,num}(s,x)$ (x choosing from -200 to 200) into

$$\hat{\lambda}_1(s) = \int_{-100}^{100} \left(1 - 3x^2\right) q_{1,num}(s,x)\, dx, \quad \hat{\lambda}_2(s) = \int_{-200}^{200} \left(1 - 3x^2\right) q_{2,num}(s,x)\, dx.$$

According to inequality (5.7), it is enough to verify $\hat{\lambda}_1 = \max\limits_{s\in[0,2\pi]} \left(\hat{\lambda}_1(s)\right)$ and $\hat{\lambda}_2 = \max\limits_{s\in[0,2\pi]} \left(\hat{\lambda}_2(s)\right)$ are negative with respect to different ε. Numerical results in Table 1 show that$\hat{\lambda}_1$ and $\hat{\lambda}_2$ are negative for small values of ε and they are much similar.

Table 1: Numerical results of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

ε	0.01	0.05	0.1	0.2	0.5
$\hat{\lambda}_1$	-35.5678	-35.5479	-35.5478	-35.5407	-35.5379
$\hat{\lambda}_2$	-35.5498	-35.5315	-35.5287	-35.5207	-35.5179

The numerical results showed great agreement with our analytical results for small ε, which means that there is a unique random periodic path in the equation (5.2). As an open question, one can take large values of ε to see if the largest Lyapunov exponent is negative or not.

5.2 A case of gradient systems

From the above example, we have shown the largest Lyapunov exponent is negative with a small periodic coefficient, for the large periodic coefficient, it may be more difficult to calculate the largest Lyapunov exponent. In this part, we consider a case of gradient systems where the periodic function is removed:

$$dX_t = -\nabla V(X_t)\, dt + \sigma dW_t, \quad \sigma > 0. \tag{5.22}$$

Flandoli et al. in [15] considered equation (5.22) with

$$V(x) \geq C_0 \log|x|, \quad ||D^2 V(x)|| \leq C_0 |x|^N, \tag{5.23}$$

for all $x \geq R_0$ and some constants $C_0,\ R_0 \geq 1,\ N \geq 0$. They proved the largest Lyapunov exponent $\lambda_1 < 0$ for small $\sigma > 0$. We consider an example of equation (5.22) with $V(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4$ satisfying condition (5.23) for arbitrary $\sigma > 0$:

$$dX_t = \left(X_t + X_t^2 - X_t^3\right) dt + \sigma dW_t, \tag{5.24}$$

We will prove the largest Lyapunov exponent is negative in this example for all $\sigma > 0$.

According to [15], the density function of equation (5.24) is

$$q(x) = \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2}V(x)}, \quad Z_\sigma = \int_{-\infty}^{\infty} e^{-\frac{2}{\sigma^2}V(x)} dx, \quad V(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4.$$

The largest Lyapunov exponent is

$$\lambda_1(x) = \int_{-\infty}^{\infty} \left(1 + 2x - 3x^2\right) \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2}V(x)} dx.$$

One the one hand, when $1 + 2x - 3x^2 \leq 0$, i.e., $x \leq -\frac{1}{3}$ or $x \geq 1$, we take a small δ ($0 < \delta < \frac{1}{4}$ in this paper), then

$$\begin{aligned} \lambda_1(x) &= \int_{-\infty}^{\infty} \left(1 + 2x - 3x^2\right) \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2}V(x)} dx \\ &\leq \int_{-\frac{7}{12}}^{-\frac{1}{3}-\delta} \left(1 + 2x - 3x^2\right) \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2}V(x)} dx \\ &+ \int_{-\frac{1}{3}-\delta}^{1+\delta} \left(1 + 2x - 3x^2\right) \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2}V(x)} dx \\ &+ \int_{1+\delta}^{1.63} \left(1 + 2x - 3x^2\right) \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2}V(x)} dx, \end{aligned} \tag{5.25}$$

where $0 < \frac{1}{Z_\sigma} < 1$ can be proved by

$$Z_\sigma = \int_{-\infty}^{\infty} e^{-\frac{2}{\sigma^2}V(x)} dx \geq \int_{-0.5}^{0.5} e^{-\frac{2}{\sigma^2}V(x)} dx > \int_{-0.5}^{0.5} e^{-\frac{V(0)}{\sigma^2}} = 1.$$

Noting that in inequality (5.25),

$$\begin{aligned} &\int_{-\frac{7}{12}}^{-\frac{1}{3}-\delta} \left(1 + 2x - 3x^2\right) e^{-\frac{2}{\sigma^2}V(x)} dx \\ \leq &\int_{-\frac{7}{12}}^{-\frac{1}{3}-\delta} \left(1 + 2x - 3x^2\right) e^{-\frac{2}{\sigma^2}V\left(-\frac{1}{3}-\delta\right)} dx \\ = &\left(-\frac{9}{64} + \delta^3 + 2\delta^2\right) e^{-\frac{2}{\sigma^2}V\left(-\frac{1}{3}-\delta\right)}, \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} &\int_{-\frac{1}{3}-\delta}^{1+\delta} \left(1 + 2x - 3x^2\right) e^{-\frac{2}{\sigma^2}V(x)} dx \\ \leq &\int_{-\frac{1}{3}}^{1} \left(1 + 2x - 3x^2\right) e^{-\frac{2}{\sigma^2}V(x)} dx = \frac{32}{27} e^{-\frac{2}{\sigma^2}V(1)}, \end{aligned} \tag{5.27}$$

as well as

$$
\begin{aligned}
&\int_{1+\delta}^{1.63} \left(1+2x-3x^2\right) e^{-\frac{2}{\sigma^2}V(x)} dx \\
&\leq \int_{1+\delta}^{1.63} \left(1+2x-3x^2\right) e^{-\frac{2}{\sigma^2}V(1+\delta)} dx \\
&= \left(-1.043847+\delta^3+2\delta^2\right) e^{-\frac{2}{\sigma^2}V(1+\delta)}.
\end{aligned} \tag{5.28}
$$

We consider $\sigma^2 < 1.756$ and $\delta = \frac{1}{5}$, then

$$
-\frac{2}{\sigma^2} V\left(1+\delta\right) = \frac{972}{625\sigma^2}, \quad -\frac{2}{\sigma^2} V\left(-\frac{1}{3}-\delta\right) = \frac{7232}{50626\sigma^2}, \quad -\frac{2}{\sigma^2} V\left(1\right) = \frac{7}{6\sigma^2}. \tag{5.29}
$$

Combining equations (5.26), (5.27), (5.28) and (5.29), we have

$$
\left(-1.043847+\delta^3+2\delta^2\right) e^{-\frac{2}{\sigma^2}V(1+\delta)} + \left(-\frac{9}{64}+\delta^3+2\delta^2\right) e^{-\frac{2}{\sigma^2}V\left(-\frac{1}{3}-\delta\right)} + \frac{32}{27} e^{-\frac{2}{\sigma^2}V(1)} < 0.
$$

By inequality (5.25) and (5.26), (5.27), (5.28), we proved $\lambda_1 < 0$. Therefore, when $\sigma^2 < 1.756$, $\lambda_1 < 0$.

On the other hand, when $x \leq \frac{2-\sqrt{22}}{3}$ or $x \geq \frac{2+\sqrt{22}}{3}$, $-2V\left(x\right) \leq 0$, and $1+2x-3x^2 < 0$, then $\lambda_1 < 0$. Meanwhile, we have

$$
\begin{aligned}
\lambda_1 &\leq \int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} \left(1+2x-3x^2\right) e^{-\frac{2}{\sigma^2}V(x)} dx \\
&= \int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} e^{-\frac{2}{\sigma^2}V(x)} dx + 2\int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} x e^{-\frac{2}{\sigma^2}V(x)} dx - 3\int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} x^2 e^{-\frac{2}{\sigma^2}V(x)} dx.
\end{aligned}
$$

When $x = \frac{1+\sqrt{5}}{2}$, $-2V\left(x\right)$ takes the maximum value $\frac{13}{12} + \frac{5\sqrt{5}}{12} < 2.1$. Then we obtain

$$
\begin{aligned}
&\int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} e^{-\frac{2}{\sigma^2}V(x)} dx \leq \frac{2\sqrt{22}}{3} e^{\frac{2.1}{\sigma^2}}, \\
&2\int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} x e^{-\frac{2}{\sigma^2}V(x)} dx < 2\int_{0}^{\frac{2+\sqrt{22}}{3}} x e^{-\frac{2}{\sigma^2}V(x)} dx = \frac{26+4\sqrt{22}}{9} e^{\frac{2.1}{\sigma^2}}, \\
&3\int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} x^2 e^{-\frac{2}{\sigma^2}V(x)} dx \geq \int_{\frac{2-\sqrt{22}}{3}}^{\frac{2+\sqrt{22}}{3}} x^2 dx = \frac{68\sqrt{22}}{27}.
\end{aligned}
$$

Thus

$$
\lambda_1 < \frac{1}{Z_\sigma} \frac{2\sqrt{22}}{3} e^{\frac{2.1}{\sigma^2}} + \frac{1}{Z_\sigma} \frac{26+\frac{1}{Z_\sigma}4\sqrt{22}}{9} e^{\frac{2.1}{\sigma^2}} - \frac{1}{Z_\sigma} \frac{68\sqrt{22}}{27}.
$$

With some calculations, we can prove $\lambda_1 < 0$ for $\sigma^2 > 1.58$. Combining the previous result $\sigma^2 < 1.756$, we conclude that the Lyapunov exponent $\lambda_1 < 0$ for arbitrary $\sigma > 0$.

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Conflict of interest statement

The authors declare that there are no conflicts of interest, we do not have any possible conflicts of interest.

References

[1] L. Arnold. *Random Dynamical Systems*. Springer-Verlag, Berlin Heidelberg New York, 2003.

[2] V.I. Arnold, V.V. Kozlov, and A.I. Neishtadt. *Mathematical Aspects of Classical and Celestial Mechanics*. Springer, New York, 3rd edition, 2006.

[3] P. W. Bates, K. Lu, and B. Wang. Attractors of non-autonomous stochastic lattice systems in weighted spaces. *Physica D*, 289:32–50, 2014.

[4] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani. Stochastic resonance in climatic change. *Tellus*, 34:10–16, 1982.

[5] A. M. Cherubini, J. Lamb, M. Rasmussen, and Y. Sato. A random dynamical systems perspective on stochastic resonance. *Nonlinearity*, 30:2835–2853, 2017.

[6] H. Crauel, A. Debussche, and F. Flandoli. Random attractor. *Journal of Dynamics and Differential Equations*, 9:307–341, 1997.

[7] H. Crauel and F. Flandoli. Attractor for random dynamical systems. *Probability Theory and Related Fields*, 100:365–393, 1994.

[8] G. Da Prato and E. Sinestrari. Differential operators with non dense domain. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie 4*, 14:285–344, 1987.

[9] C. Feng, Y. Liu, and H. Zhao. Numerical approximation of random periodic solutions of stochastic differential equations. *Zeitschrift für angewandte Mathematik und Physik*, 68:119, 2017.

[10] C. Feng, Y. Liu, and H. Zhao. Periodic measures and Wasserstein distance for analyzing periodicity of time series datasets. *Communications in Nonlinear Science and Numerical Simulation*, 120:107166, 2023.

[11] C. Feng and H. Zhao. Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding. *Journal of Functional Analysis*, 262:4377–4422, 2012.

[12] C. Feng and H. Zhao. Random periodic processes, periodic measures and ergodicity. *Journal of Differential Equations*, 269:7382–7428, 2020.

[13] C. Feng, H. Zhao, and J. Zhong. Existence of geometric ergodic periodic measures of stochastic differential equations. *Journal of Differential Equations*, 359:67–106, 2023.

[14] C. Feng, H. Zhao, and B. Zhou. Pathwise random periodic solutions of stochastic differential equations. *Journal of Differential Equations*, 251:119–149, 2011.

[15] F. Flandoli, B. Gess, and M. Scheutzow. Synchronization by noise. *Probability Theory and Related Fields*, 168:511–556, 2017.

[16] Y. Le Jan. Équilibre statistique pour les produits de difféomorphismes aléatoires indépendants. *Annales de l'I.H.P. Probabilités et statistiques*, 23:111–120, 1987.

[17] J. Newman. Necessary and sufficient conditions for stable synchronization in random dynamical systems. *Ergodic Theory and Dynamical Systems*, 38:1857–1875, 2018.

[18] V. I. Oseledets. A multiplicative ergodic theorem, Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[19] H. Poincare. Mémoire sur les courbes définies par une Équation différentielle. *J. Math. Pures Appli.*, 3:375–442, 1881. Additional parts published in Vol. 3 (1882), 251–296; Vol. 4 (1885), 167–244; Vol. 4 (1886), 151–217.

[20] B. Schmalfuß. Backward cocycle and attractors of stochastic differential equations. In *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*, 1992.

[21] B. Wang. Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems. *Journal of Differential Equations*, 253:1544–1583, 2012.

[22] H. Zhao and Z. H. Zheng. Random periodic solutions of random dynamical systems. *Journal of Differential Equations*, 246:2020–2038, 2009.