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ORIGINAL ARTICLE

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Viscoelastic bidispersive convection with a Kelvin–Voigt fluid

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Abstract We develop a theory for thermal convection in a double porosity material of Brinkman–Forchheimer type when there is a single temperature. The saturating fluid is one of Kelvin–Voigt type, and the equation for the temperature is one due to C.I. Christov. It is shown that the global nonlinear stability threshold coincides with the linear stability one. A thoroughly analytical discussion of both linear instability analysis and global nonlinear energy stability is provided. Numerical results show that the relative permeability and Brinkman viscosity between the macro and micro pores are key parameters which play a dominant role in determining the critical Rayleigh number for the onset of convective motions.

Keywords Bidispersive porous media · Brinkman–Forchheimer convection · Linear and nonlinear stability · Kelvin–Voigt equations · Christov heat law

1 Introduction

There has been a significant research activity on flows in porous materials which possess a double porosity structure. These materials have relatively large pores, known as macro pores, but additionally, the solid skeleton has cracks which give rise to a microporosity structure.

The increase in the research activity is driven by the many applications such as to landslides, see e.g. Borja et al. [1], Scotto di Santolo and Evangelista [2]; to chemical engineering issues, see Enterria et al. [3], Huang et al. [4], Ly et al. [5]; to self heating/ignition in piles of coal, see Hooman and Maas [6]; and to oil reservoir recovery, see Olusola et al. [7]. In addition, there are many applications in areas associated with water resources, for example, in soil drainage and dealing with storm runoff, see e.g. Haws et al. [8]; and accessibility of clean drinking water from an aquifer, Love et al. [9], Simmons et al. [10], Ghasemizadeh et al. [11], Fretwell et al. [12]. A particularly novel use is that of Professor Marina Bergen Jensen who has developed large scale use of a double porosity filter for polluted groundwater, see Jensen et al. [13], University of Copenhagen report [14]. A further area where double porosity materials are important is in renewable energy and the creation of electricity and desalinized water by means of a solar pond, see e.g. Kumaravel et al. [15], Tawalbeh et al. [16], Yuvaperiyasamy et al. [17]. We also mention that double porosity is believed to

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be important in convection mechanisms in magma in a volcano, see e.g. Bagdassarov and Fradkov [18], De Campos et al. [19], De Campos et al. [20]. For the last class of problem the fluid is likely to be non-Newtonian, and so we here allow the saturating fluid to be a viscoelastic one of Kelvin–Voigt type, see Oskolkov [21], Sukacheva [22]. Viscoelastic flows in bidisperse porous media are the subject of much modern research, see e.g. De et al. [23], Ibezim et al. [24].

Theories of fluid flow in double porosity materials, also known as bidisperse materials, were developed by Nield and Kuznetsov [25,26] and by Nield [27], see also Nield and Bejan [28]. The fundamental work of Nield and Kuznetsov [25] produced a theory capable of encompassing thermal convection in a bidispersive porous material. Thermal effects are very important because they may produce thermal stresses which in turn induce cracking in the solid skeleton, leading to the production of micro pores, see Gelet et al. [29], Kim and Hosseini [30]. See also Rees et al. [31], where a hot fluid is injected into a cold porous skeleton. Indeed, the parameters of the equations describing porous media flows often have to take into account the fact that there are different length scales operating within them, see e.g. Wang et al. [32], Ogden et al. [33]. Porosities in two different length scales represent one of the many facets of fluid flow in porous media.

The theory of Nield and Kuznetsov [25] introduces independent fields of velocity, pressure and temperature U_i^f, p^f and T^f, in the macro pores, and U_i^p, p^p and T^p, in the micro pores. For many applications it is reasonable to adopt a single temperature T, in both the macro and micro pores and this approach has been successfully embraced by Capone et al. [34], Capone et al. [35], Capone et al. [36], Franchi et al. [37,38], Gentile and Straughan [39], Straughan and Barletta [40] and Straughan [41,42]. However, the articles just listed largely employ a Darcy theory for both the macro and micro phases, whereas the original theory of Nield and Kuznetsov [25] used a Brinkman theory. There are many instances where a Brinkman theory is desirable, see e.g. Barletta et al. [43], Nield [44], Rees [45]. It must also be noted that Fried and Gurtin [46] argue that when flow dimensions are small, then length scale effects are paramount; this strongly suggests the use of the Brinkman theory. However, recent works on flow in micro channels suggest that the pressure drop against flow rate is not adequately represented by a linear relationship, see e.g. Christov et al. [47], Wang and Christov [48], Christov [49], Wang et al. [50]. Hence, we here additionally employ the Forchheimer theory both in the macro and micro pores.

Our main concern in this paper is to investigate in depth the classical Rayleigh-Bénard convection problem in a horizontal layer uniformly heated from below for a general theory governing a bidisperse Brinkman–Forchheimer porous medium, in the presence of a single temperature T.

We underline that, to achieve our goal, we employ a Boussinesq approximation in the buoyancy force terms in both the macro and micro phase equations. It is worth to remember that the Boussinesq approximation, allowing the gravity effects to involve linear functions of the temperature, is discussed at length in the articles by Barletta [51,52], Nield and Barletta [53], and general multi-constituent porous media flows as models in continuum mechanics are presented in Allen [54].

The outline of the paper is as follows. The mathematical formulation of the convection problem is set up in Sect. 2: more precisely, in the first subsection, we present the boundary value problem for the general bidisperse Brinkman–Forchheimer porous material, but allowing for different essential coefficients in the macro and micro pores. Then, in the second subsection, after determining the basic steady conduction solution, in whose stability we are interested, we establish the boundary value problem for the dimensionless perturbation system. In Sect. 3, as a first step, working with its linearized version and aimed to determine the linear instability boundary, we establish the strong form of the so called *principle of exchange of stabilities*, i.e. convection can occur only through steady motions, and consequently the linear and nonlinear stability thresholds are shown to be the same. Next, we perform a detailed linear stability analysis towards the determination of the critical Rayleigh number. It is *a priori* evident that the nonlinear Forchheimer terms do not affect this analysis. In Sect. 4, we introduce the energy method to investigate also the global stability of the conduction solution. Finally, in Sect. 5 we complete the analytical discussion with numerical developments, which enhance the key role of some relevant parameters on the critical Rayleigh number.

2 Bénard convection in a bidisperse Brinkman–Forchheimer porous medium

2.1 Governing equations

To begin, we remember that the macro and micro porosities are usually denoted by ϕ and ϵ, so that the expressions $\epsilon(1-\phi)$ and $\epsilon_1 = (1-\epsilon)(1-\phi)$ stand for the fractions of the volume occupied by the micro pores and the solid matrix, respectively.

As usual in bidispersive contexts, the fluid velocities involved in the macro and micro pores are interpreted as the pore-averaged velocities, herein denoted by U_i^f and U_i^p, which are related to the actual velocities V_i^f and V_i^p, through the relations $U_i^f = \phi V_i^f$ and $U_i^p = \epsilon(1-\phi)V_i^p$.

The continuity equations reduce to the solenoidality constraints for U_i^f and U_i^p. The momentum equations follow the prescription of Nield and Kuznetsov [25], but we allow the fluid to be a viscoelastic one of Kelvin–Voigt order zero type, see e.g. Oskolkov [21], Oskolkov and Shadiev [55], Sukacheva [22,56], Sviridyuk and Shipilov [57], Sviridyuk and Kazak [58]. A fluid of Kelvin–Voigt order zero is also known as a Navier–Stokes–Voigt fluid, or an Oskolkov fluid, and in addition to the inertia acceleration term includes a term involving the Laplacian of the acceleration. The momentum equations also allow for Brinkman–Forchheimer porous effects, and employ a Boussinesq approximation in their buoyancy terms and include interaction transfer terms. For the energy balance equation for T, we modify the appropriate equation employed by Gentile and Straughan [39] and follow Christov [59] who suggests a modification to the classical heat equation which involves higher gradients of temperature and heat flux, arguing that such effects are likely to be of importance in microfluid situations. Such an equation is also employed by Kaya and Celebi [60] and Kaya [61] who analyse non-isothermal flow in a fluid which is governed by a system of Navier–Stokes–Voigt equations which contains not only a Kelvin–Voigt term for the velocity field, but also one for the temperature field.

Standard indicial notation is used throughout, in compliance with the Einstein summation convention, with the subscript $,i$ denoting $\partial/\partial x_i$ and Δ being the 3D Laplacian operator.

In the interests of clarity we briefly review the ideas behind a Navier–Stokes–Voigt fluid. For an incompressible viscous fluid with velocity v_i and pressure p, the Navier–Stokes–Voigt equations are, with zero body force,

$$\begin{aligned} &v_{i,t} + v_j v_{i,j} - \lambda\Delta v_{i,t} = -\frac{1}{\rho}\, p_{,i} + \nu\Delta v_i, \\ &v_{i,i} = 0. \end{aligned} \tag{1}$$

This system was investigated with respect to existence and regularity of the solution by Oskolkov [62]. As Oskolkov and Shadiev [63] point out, Ladyzhenskaya [64] suggested this modification of the Navier–Stokes equations due to the fact that the term $-\lambda\Delta v_{i,t}$ would have good regularization properties. One may ask how does this relate to the momentum balance equation

$$\rho\dot{v}_i = \sigma_{ij,j}, \tag{2}$$

where σ_{ij} is the Cauchy stress and the dot denotes the material derivative. One theory is that the Cauchy stress could be given by the constitutive equation

$$\sigma_{ij} = -p\delta_{ij} + 2\mu d_{ij} + 2\lambda d_{ij,t}. \tag{3}$$

However, as Damázio et al. [65] point out, $d_{ij,t}$ is not an objective derivative. Pavlovskii [66] gives convincing arguments to support employing a similar equation when modelling the flow of water containing a weak solution of polymer. He suggests instead of (3) the relation

$$\sigma_{ij} = -p\delta_{ij} + 2\mu d_{ij} + 2\lambda\frac{Dd_{ij}}{Dt}, \tag{4}$$

i.e. replace the partial time derivative of d_{ij} by the material derivative. However, this derivative is still not objective. Earlier, Beard and Walters [67] argued one could model the physically important case of "*an elastico-viscous liquid that is mobile and not highly elastic*" with a constitutive theory like

$$\sigma_{ij} = -p\delta_{ij} + 2\mu d_{ij} + 2\lambda\frac{\mathfrak{D}d_{ij}}{\mathfrak{D}t}, \tag{5}$$

where $\mathfrak{D}/\mathfrak{D}t$ is an objective derivative, and they chose an Oldroyd derivative. The model presented by Beard and Walters [67] is now known as a Walters' B fluid. It is pertinent to draw attention to the fact that the question of objective derivatives and their compatibility with continuum thermodynamics has recently been investigated in detail in the fluid mechanics scenario and in the case of fluid saturated porous media by Morro [68,69] and by Giorgi and Morro [70]. Straughan [71] has investigated the use of a theory like (5) in various hydrodynamic stability problems and has shown that the choice of objective derivative plays an important role. However, Straughan [72] investigates such theories for flow in porous media and for linear instability analyses often (3) is adequate. Indeed, if the co-rotational objective derivative is employed then (3) may well be sufficient for a nonlinear energy stability analysis.

In this article we employ essentially (3) for flow in a bidisperse porous medium. However, we incorporate nonlinear effects via a traditionally accepted route of including Forchheimer terms. Hence, the relevant equations for our model read as follows

$$\begin{aligned}
\rho_0 c_a^f U_{i,t}^f - \hat{\kappa}_f \Delta U_{i,t}^f =& \tilde{\mu}^f \Delta U_i^f - \tilde{F}_1 |\mathbf{U}^f| U_i^f - \frac{\mu}{K_f} U_i^f \\
& - \zeta (U_i^f - U_i^p) - p_{,i}^f + \rho_0 \alpha g T k_i \,, \\
U_{i,i}^f =& 0 \,, \\
\rho_0 c_a^p U_{i,t}^p - \hat{\kappa}_p \Delta U_{i,t}^p =& \tilde{\mu}^p \Delta U_i^p - \tilde{F}_2 |\mathbf{U}^p| U_i^p - \frac{\mu}{K_p} U_i^p \\
& - \zeta (U_i^p - U_i^f) - p_{,i}^p + \rho_0 \alpha g T k_i \,, \\
U_{i,i}^p =& 0 \,, \\
(\rho c)_m T_{,t} - \hat{\zeta} \Delta T_{,t} + (\rho c)_f & \left[U_i^f + (\rho c)_r U_i^p \right] T_{,i} = \kappa_m \Delta T \,,
\end{aligned} \tag{6}$$

where μ and ζ are the dynamic viscosity and coefficient for the momentum transfer between the macro and micro phases; $\tilde{\mu}^f$, $\tilde{\mu}^p$, K_f, K_p, $\tilde{F}_1$, $\tilde{F}_2$, p^f and p^p are the Brinkman viscosities, the permeabilities, the Forchheimer coefficients and the pressures in the macro and micro pores, respectively; g is the gravity constant, $\mathbf{k} = (0, 0, 1)$, ρ_0 and α are the reference density and the coefficient of thermal expansion in the fluid, arising from the Boussinesq approximation see Franchi et al. [73], Barletta [52]. Additionally, $(\rho c)_m = [\epsilon_1 (\rho c)_s + \phi (\rho c)_f + \epsilon (1-\phi)(\rho c)_p]$, the terms $(\rho c)_s$, $(\rho c)_f$ and $(\rho c)_p$ standing for the *heat capacities* of the solid skeleton and of the fluid in the macro and micro pores and $(\rho c)_r = \dfrac{(\rho c)_p}{(\rho c)_f}$, being for the relative *heat capacity*. We observe that, under the hypothesis $(\rho c)_r = 1$, we have $(\rho c)_m = \epsilon_1 (\rho c)_s + [\phi + \epsilon(1-\phi)](\rho c)_f$, see e.g. Franchi et al. [37]. Moreover, if κ_s, κ_f and κ_p denote the thermal conductivities of the solid and of the fluid in the macro and micro pores, let $\kappa_m = [\epsilon_1 \kappa_s + \phi \kappa_f + \epsilon(1-\phi)\kappa_p]$. The terms in the momentum equations having coefficient $\hat{\kappa}_f$, $\hat{\kappa}_p$ are Kelvin–Voigt terms and $\hat{\kappa}_f$, $\hat{\kappa}_p$ are the Kelvin–Voigt coefficients. Likewise, $\hat{\zeta}$ in the temperature equation is a Kelvin-Voigt coefficient for the temperature field. The inertia terms contain coefficients c_a^f, c_a^p which are acceleration coefficients for the macro phase and the micro phase. This coefficients are discussed in section 1.5 of Nield and Bejan [28] in the single porosity case and by Straughan [41] for the bidisperse situation. The Forchheimer coefficients F_1, F_2 are discussed in section 1.5 of Nield and Bejan [28] who indicate that they have form

$$F_1 = \frac{\rho_0 c_F^f}{\sqrt{K}}, \qquad F_2 = \frac{\rho_0 c_F^p}{\sqrt{K}},$$

where c_F^f, c_F^p are drag coefficients for the macro and micro phases. The effect of the Forchheimer term on convection in a single porosity situation is analysed in depth by Rees [74,75]. Equations (6) hold in the horizontal layer $\Omega = \mathbb{R}^2 \times \{0 < z < d\}$, for all $t > 0$, and are subject to the boundary conditions

$$\begin{aligned}
& U_i^f = U_i^p = 0 \,, \qquad \text{on } z = 0\,,\ d\,; \\
& T = T_L \,, \quad \text{on } z = 0\,; \quad T = T_U \,, \quad \text{on } z = d \,,
\end{aligned} \tag{7}$$

where T_L and T_U are constant, with $T_L > T_U$.

2.2 Basic steady conduction solution and dimensionless perturbation equations

The steady conduction solution of system (6)–(7) in whose stability we are interested herein is one with

$$\bar{U}_i^f = \bar{U}_i^p = 0\,, \quad \bar{T} = -\beta z + T_L\,, \tag{8}$$

where $\beta = \dfrac{(T_L - T_U)}{d} > 0$ is the adverse temperature gradient.

As a consequence, from $(6)_1$ and $(6)_3$, the steady pressures $\bar{p}^f$ and $\bar{p}^p$ have form

$$\bar{p}^f(z) = \bar{p}^p(z) = -\rho_0 g\alpha\beta\frac{z^2}{2} + \rho_0 g\alpha T_L z\,,$$

selecting both pressure scales to vanish at $z = 0$.

Following the standard perturbation technique, we introduce the perturbation functions u_i^f, u_i^p, θ, π^f and π^p, such that

$$\begin{aligned} &U_i^f = \bar{U}_i^f + u_i^f\,, \quad U_i^p = \bar{U}_i^p + u_i^p\,, \quad T = \bar{T} + \theta\,, \\ &p^f(z) = \bar{p}^f(z) + \pi^f\,, \quad p^p(z) = \bar{p}^p(z) + \pi^p\,, \end{aligned}$$

to be replaced in system (6) to yield the nonlinear perturbation system.

Up to this point, as in [25], we have envisaged different heat capacities for the fluid in the macro and micro pores. However, for our model we have considered only one temperature field, and hence it turns out to be highly probable that $(\rho c)_r = 1$. Therefore, we focus on this simplified background. Instead, the possibility of different heat conductivities is preserved.

The nonlinear perturbation system therefore has form

$$\begin{aligned} &\rho_0 c_a^f u_{i,t}^f - \hat{\kappa}_f \Delta u_{i,t}^f = \tilde{\mu}^f \Delta u_i^f - \tilde{F}_1 |\mathbf{u}^f| u_i^f - \frac{\mu}{K_f} u_i^f \\ &\qquad\qquad - \zeta(u_i^f - u_i^p) - \pi_{,i}^f + \rho_0 \alpha g \theta k_i\,, \\ &u_{i,i}^f = 0\,, \\ &\rho_0 c_a^p u_{i,t}^p - \hat{\kappa}_p \Delta u_{i,t}^p = \tilde{\mu}^p \Delta u_i^p - \tilde{F}_2 |\mathbf{u}^p| u_i^p - \frac{\mu}{K_p} u_i^p \\ &\qquad\qquad - \zeta(u_i^p - u_i^f) - \pi_{,i}^p + \rho_0 \alpha g \theta k_i\,, \\ &u_{i,i}^p = 0\,, \\ &(\rho c)_m \theta_{,t} - \hat{\xi} \Delta\theta_{,t} + (\rho c)_f (u_i^f + u_i^p)\theta_{,i} = \beta(\rho c)_f (w^f + w^p) + \kappa_m \Delta\theta\,, \end{aligned} \tag{9}$$

where $w^f = u_3^f$ and $w^p = u_3^p$. The associated boundary conditions become

$$u_i^f = u_i^p = 0\,, \quad \theta = 0\,, \qquad \text{on} \quad z = 0,\, d\,, \tag{10}$$

and $(u_i^f, u_i^p, \theta, \pi^f, \pi^p)$ are assumed to have an (x, y)-dependence consistent with one that has a repetitive shape tiling the plane, such as typical hexagonal convection cell forms found in real life, see Chandrasekhar [76, pages 43–52].

To better highlight the role of the different involved effects within the model, we proceed with a convenient dimensionless process, according to the following scalings

$$\begin{aligned} &x = x^* d\,, \quad t = t^* \tau\,, \quad \tau = \frac{(\rho c)_m d^2}{\kappa_m}\,, \\ &u_i^f = u_i^{f*} U\,, \quad u_i^p = u_i^{p*} U\,, \quad U = \frac{\kappa_m}{(\rho c)_f d}\,, \\ &\pi^f = \pi^{f*} P\,, \quad \pi^p = \pi^{p*} P\,, \quad P = \frac{\mu d U}{K_f}\,, \end{aligned}$$

$$\theta = \theta^* T^\sharp , \quad T^\sharp = \frac{\beta U (\rho c)_f d^2}{\kappa_m} ,$$

and non-dimensional inertia and Kelvin–Voigt coefficients,

$$J_1 = \frac{\rho_0 c_a^f K_f \kappa_m}{\mu (\rho c)_m d^2} , \qquad J_2 = \frac{\rho_0 c_p^f K_f \kappa_m}{\mu (\rho c)_m d^2} , \qquad J_3 = \frac{\hat{\zeta}}{(\rho c)_m d^2} ,$$
$$\lambda_f = \frac{\hat{\kappa}_f K_f \kappa_m}{\mu (\rho c)_m d^4} , \qquad \lambda_p = \frac{\hat{\kappa}_p K_f \kappa_m}{\mu (\rho c)_m d^4} .$$

As a consequence, the following non-dimensional parameters $\lambda, \tilde{\mu}_r, \xi, K_r, f_1$ and f_2, representing the Brinkman coefficient in the macro pores, the relative Brinkman viscosity, the transfer coefficient, the relative permeability and the Forchheimer coefficients in the macro and micro pores, arise and are defined as

$$\begin{aligned} &\lambda = \frac{\tilde{\mu}_f K_f}{\mu d^2} , \quad \tilde{\mu}_r = \frac{\tilde{\mu}_p}{\tilde{\mu}_f} , \quad \xi = \frac{\zeta K_f}{\mu} , \\ &K_r = \frac{K_f}{K_p} , \quad f_1 = \frac{K_f \tilde{F}_1 U}{\mu} , \quad f_2 = \frac{K_f \tilde{F}_2 U}{\mu} . \end{aligned} \tag{11}$$

Finally, the Rayleigh number for our theory is given by

$$R = \frac{\rho_0 \alpha g \beta d^2 (\rho c)_f K_f}{\mu \kappa_m} . \tag{12}$$

With these scalings and dropping all *-superscripts, the dimensionless form of (9) becomes

$$\begin{aligned} &J_1 u^f_{i,t} - \lambda_f \Delta u^f_{i,t} = \lambda \Delta u^f_i - f_1 |\mathbf{u}^f| u^f_i - u^f_i - \xi (u^f_i - u^p_i) - \pi^f_{,i} + R\theta k_i , \\ &u^f_{i,i} = 0 , \\ &J_2 u^p_{i,t} - \lambda_p \Delta u^p_{i,t} = \lambda \tilde{\mu}_r \Delta u^p_i - f_2 |\mathbf{u}^p| u^p_i - K_r u^p_i - \xi (u^p_i - u^f_i) - \pi^p_{,i} + R\theta k_i , \\ &u^p_{i,i} = 0 , \\ &\theta_{,t} - J_3 \Delta \theta_{,t} + (u^f_i + u^p_i) \theta_{,i} = w^f + w^p + \Delta\theta . \end{aligned} \tag{13}$$

Its domain is now the horizontal layer $\mathbb{R}^2 \times \{0 < z < 1\}$, with $t > 0$. Obviously, the boundary conditions are

$$u^f_i = u^p_i = 0 , \qquad \theta = 0 , \qquad \text{on} \quad z = 0 , 1 \tag{14}$$

together with the requirement that the perturbations field satisfies a plane tiling periodicity in x and y. In what follows, we preferably work on the period cell $V = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times (0, 1)$, $t > 0$, where a, such that $a^2 = a_x^2 + a_y^2$, is the wavenumber.

3 Exchange of stabilities and linear instability analysis

To find the linear instability threshold, we firstly linearize (13), i.e. we discard the Forchheimer terms and the nonlinear convective term, and we look for perturbations accounting for a separate dependence on t of form $e^{\sigma t}$, $\sigma = \sigma_r + i\sigma_i$ playing the role of the complex growth rate. This yields the system

$$\begin{aligned} &J_1 \sigma u^f_i - \sigma \lambda_f \Delta u^f_i = \lambda \Delta u^f_i - u^f_i - \xi (u^f_i - u^p_i) - \pi^f_{,i} + R\theta k_i , \\ &u^f_{i,i} = 0 , \\ &J_2 \sigma u^p_i - \sigma \lambda_p \Delta u^p_i = \lambda \tilde{\mu}_r \Delta u^p_i - K_r u^p_i - \xi (u^p_i - u^f_i) - \pi^p_{,i} + R\theta k_i , \\ &u^p_{i,i} = 0 , \\ &\sigma \theta - \sigma J_3 \Delta \theta = w^f + w^p + \Delta\theta . \end{aligned} \tag{15}$$

The boundary conditions are (14) together with the periodicity in (x, y).

We need to prove that σ is real, which means that our model satisfies the principle of exchange of stabilities in its strong form. To this aim, without any misunderstanding, let now $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and the norm on the complex Hilbert space $L^2(V)$ and use the notation * for the complex conjugate of a perturbation field. Multiply $(15)_1$ by $u_i^{f\,\star}$, $(15)_3$ by $u_i^{p\,\star}$ and $(15)_5$ by $\theta^\star$ and then integrate over V the resulting equations.

After use of standard identities, integration by parts and taking into account the solenoidality and boundary conditions, this procedure leads to

$$\begin{aligned}
&J_1\sigma\|\mathbf{u}^f\|^2+\sigma\lambda_f\|\nabla\mathbf{u}^f\|^2\\
&\quad-R(\theta,w^{f\,\star})=-\lambda\|\nabla\mathbf{u}^f\|^2-(1+\xi)\|\mathbf{u}^f\|^2+\xi(u_i^p,u_i^{f\,\star})\,,\\
&J_2\sigma\|\mathbf{u}^p\|^2+\sigma\lambda_p\|\nabla\mathbf{u}^p\|^2\\
&\quad-R(\theta,w^{p\,\star})=-\lambda\tilde{\mu}_r\|\nabla\mathbf{u}^p\|^2-(K_r+\xi)\|\mathbf{u}^p\|^2+\xi(u_i^f,u_i^{p\,\star})\,,\\
&\sigma\|\theta\|^2+\sigma J_3\|\nabla\theta\|^2=-\|\nabla\theta\|^2+(w^f,\theta^\star)+(w^p,\theta^\star)\,.
\end{aligned}\tag{16}$$

Next, upon forming a suitable combination among $(16)_1$, $(16)_2$ and $R(16)_3$ and rearranging, we obtain

$$\begin{aligned}
&R\sigma\|\theta\|^2+R\sigma J_3\|\nabla\theta\|^2+J_1\sigma\|\mathbf{u}^f\|^2+\sigma\lambda_f\|\nabla\mathbf{u}^f\|^2\\
&\quad+J_2\sigma\|\mathbf{u}^p\|^2+\sigma\lambda_p\|\nabla\mathbf{u}^p\|^2\\
&=-R\|\nabla\theta\|^2-\lambda(\|\nabla\mathbf{u}^f\|^2+\tilde{\mu}_r\|\nabla\mathbf{u}^p\|^2)\\
&\quad-(1+\xi)\|\mathbf{u}^f\|^2-(K_r+\xi)\|\mathbf{u}^p\|^2+\xi[(u_i^p,u_i^{f\,\star})+(u_i^f,u_i^{p\,\star})]\\
&\quad+R[(\theta,w^{f\,\star})+(w^f,\theta^\star)+(\theta,w^{p\,\star})+(w^p,\theta^\star)]\,.
\end{aligned}\tag{17}$$

The right hand side of (17) is real, and hence taking the imaginary part of (17), it follows that

$$\begin{aligned}
&\sigma_i\Big\{R\|\theta\|^2+RJ_3\|\nabla\theta\|^2+J_1\|\mathbf{u}^f\|^2\\
&\quad+\lambda_f\|\nabla\mathbf{u}^f\|^2+J_2\|\mathbf{u}^p\|^2+\lambda_p\|\nabla\mathbf{u}^p\|^2\Big\}=0\,.
\end{aligned}$$

This yields $\sigma_i=0$, namely oscillatory convection can not arise and exchange of stabilities is thus proven. As a consequence, the marginal states are characterized by $\sigma=0$ and hence the linear instability boundary is found by writing $\sigma=0$ in (15), since convection occurs for steady motions. This is very important since it means linear instability theory has completely captured the physics of the onset of thermal convection. Therefore, to find the critical Rayleigh number, we remove the π^f and π^p terms, by applying the double curl operator of $(15)_1$ and $(15)_3$ and then retain only the third components of the resulting equations. This leaves one to solve the stationary system

$$\begin{aligned}
&-\lambda\Delta^2w^f+(1+\xi)\Delta w^f-\xi\Delta w^p-R\Delta^\star\theta=0\,,\\
&-\lambda\tilde{\mu}_r\Delta^2w^p+(K_r+\xi)\Delta w^p-\xi\Delta w^f-R\Delta^\star\theta=0\,,\\
&\ \Delta\theta+w^f+w^p=0\,,
\end{aligned}\tag{18}$$

where Δ^* is the horizontal Laplacian, i.e. $\Delta^*=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$. The boundary conditions are

$$w^f=w^p=\theta=0\,,\qquad \text{on}\quad z=0,\,1,$$

under the periodicity assumption in x, y of periods $\dfrac{2\pi}{a_x}$ and $\dfrac{2\pi}{a_y}$, respectively.

One now employs the Fourier normal modes technique in (18) writing $w^f=W^f(z)h(x,y)$, with similar forms for w^p and θ, where $h(x,y)$, is a suitable planform which tiles the plane and satisfies the condition

$\Delta^\star h = -a^2 h$, for a wavenumber a, see Chandrasekhar [76, pages 43–52]. Upon using these forms for w^f, w^p and θ in (18), one has to solve the eigenvalue problem

$$\begin{aligned}
&\lambda(D^2-a^2)^2W^f-(1+\xi)(D^2-a^2)W^f+\xi(D^2-a^2)W^p=Ra^2\Theta\,,\\
&\lambda\tilde{\mu}_r(D^2-a^2)^2W^p-(K_r+\xi)(D^2-a^2)W^p+\xi(D^2-a^2)W^f=Ra^2\Theta\,,\\
&(D^2-a^2)\Theta+W^f+W^p=0\,,
\end{aligned}\tag{19}$$

where $D=d/dz$, and the boundary conditions are

$$W^f=W^p=\Theta=0\,,\qquad \text{on}\quad z=0\,,\,1\,.\tag{20}$$

One requires two extra boundary conditions for W^f and W^p. These depend on what sort of surface one has. If the surfaces $z=0$ and $z=1$ are fixed, then it is necessary to add

$$DW^f=DW^p=0\qquad \text{on}\quad z=0\,,\,1\tag{21}$$

whereas, when the surfaces are stress-free, one adds

$$D^2W^f=D^2W^p=0\qquad \text{on}\quad z=0\,,\,1\tag{22}$$

together with an analogous condition on Θ.

We here employ (22) which allows an analytical solution of the eigenvalue problem. Since we seek an instability threshold one selects $W^f(z)=\hat{W}^f\sin(n\pi z)$, $n\in\mathbb{N}$, with constant amplitude $\hat{W}^f$, and analogous representations for $W^p(z)$ and $\Theta(z)$.

Thus, (19) leads to

$$\begin{aligned}
&\lambda\Lambda_n^2\hat{W}^f+(1+\xi)\Lambda_n\hat{W}^f-\xi\Lambda_n\hat{W}^p=Ra^2\hat{\Theta}\,,\\
&\lambda\tilde{\mu}_r\Lambda_n^2\hat{W}^p+(K_r+\xi)\Lambda_n\hat{W}^p-\xi\Lambda_n\hat{W}^f=Ra^2\hat{\Theta}\,,\\
&\Lambda_n\hat{\Theta}=\hat{W}^f+\hat{W}^p\,,
\end{aligned}$$

where $\Lambda_n=n^2\pi^2+a^2$.

Then, one reduces the calculation of the Rayleigh number to

$$R(n^2,a^2)=\frac{\Lambda_n^2}{a^2}\frac{\left[\lambda^2\tilde{\mu}_r\Lambda_n^2+\lambda\Lambda_n\left(\tilde{\mu}_r(1+\xi)+K_r+\xi\right)+K_r+\xi+\xi K_r\right]}{(4\xi+1+K_r+\lambda(1+\tilde{\mu}_r)\Lambda_n)}\,.\tag{23}$$

To find the critical Rayleigh number for the onset of thermal instabilities in our bidispersive porous setting, one must minimize R^2 in n^2 and a^2. A direct calculation shows $\partial R/\partial n^2\geq 0$ and so we may take $n^2=1$. The critical Rayleigh number calculation then reduces to the minimization of $R=R(a^2)$ in a^2, where

$$R(a^2)=\frac{\Lambda^2}{a^2}\frac{\left[\lambda^2\tilde{\mu}_r\Lambda^2+\lambda\Lambda\left(\tilde{\mu}_r(1+\xi)+K_r+\xi\right)+K_r+\xi+\xi K_r\right]}{(4\xi+1+K_r+\lambda(1+\tilde{\mu}_r)\Lambda)}\tag{24}$$

with $\Lambda=\pi^2+a^2$.

Remark 1 The mathematical analysis of section 3 can be shown to also hold with the boundary conditions of Celli and Kuznetsov [77], who model rough boundary conditions for flows in an incompressible fluid. On the macroscopic scale, where the macro and micro pores touch a (smooth) rigid boundary, the actual boundary of the pores will always possess a (random) roughness. Thus, one could argue that in porous media flow the boundary conditions of Celli and Kuznetsov [77] should be usefully employed. To use them in the case of equations (13), the boundary conditions (14) are replaced by

$$\begin{aligned}
&w^f=w^p=\theta=0\,, && \text{on } z=0\,,1\,,\\
&\frac{\partial u_j^f}{\partial z}=-\frac{\alpha}{\sqrt{K_f}}u_j^f\,,\quad \frac{\partial u_j^p}{\partial z}=-\frac{\alpha}{\sqrt{K_p}}u_j^p && \text{on } z=0\,,\quad j=1,2\,,\\
&\frac{\partial u_j^f}{\partial z}=\frac{\alpha}{\sqrt{K_f}}u_j^f\,,\quad \frac{\partial u_j^p}{\partial z}=\frac{\alpha}{\sqrt{K_p}}u_j^p && \text{on } z=1\,,\quad j=1,2\,.
\end{aligned}$$

The coefficient α in the above conditions depends on the fluid, the porous material, and the bounding interface, cf. Beavers and Joseph [78], Saffman [79], Payne and Straughan [80].

Of course the expressions (23) and (24) for the Rayleigh number will be different and require numerical calculations.

Remark 2 If we consider thermosolutal convection when the fluid is heated from below and the layer is also salted from below then exchange of stabilities will not hold. In that case one will obtain oscillatory convection for certain parameter ranges. The Kelvin–Voigt coefficients for the macro and micro phases as well as the equivalent one for the temperature field will play a major role and lead to substantial lowering of the critical Rayleigh number threshold for the onset of convective motion in the fluid saturated layer.

4 Global nonlinear stability

The threshold found by minimizing (24) yields an instability curve. However, it yields no information on nonlinear stability. For completeness of results we perform a fully nonlinear stability analysis. From the fully nonlinear perturbation system (13) we multiply $(13)_1$ by u_i^f, we multiply $(13)_3$ by u_i^p and we multiply $(13)_5$ by θ and then integrate the resulting equations over the period cell V. After some integration by parts and use of our boundary conditions, we arrive at the equations

$$\begin{aligned} \frac{d}{dt}\Big(\frac{J_1}{2}\|\mathbf{u}^f\|^2 + \frac{\lambda_f}{2}\|\nabla \mathbf{u}^f\|^2\Big) &= -\lambda\|\nabla \mathbf{u}^f\|^2 - \|\mathbf{u}^f\|^2 - f_1\|\mathbf{u}^f\|_3^3 \\ &\quad - \xi(u_i^f, u_i^f - u_i^p) + R(\theta, w^f)\,, \\ \frac{d}{dt}\Big(\frac{J_2}{2}\|\mathbf{u}^p\|^2 + \frac{\lambda_p}{2}\|\nabla \mathbf{u}^p\|^2\Big) &= -\lambda\tilde{\mu}_r\|\nabla \mathbf{u}^p\|^2 - K_r\|\mathbf{u}^p\|^2 - f_2\|\mathbf{u}^p\|_3^3 \\ &\quad + \xi(u_i^p, u_i^f - u_i^p) + R(\theta, w^p)\,, \\ \frac{d}{dt}\Big(\frac{1}{2}\|\theta\|^2 + \frac{J_3}{2}\|\nabla\theta\|^2\Big) &= [(w^f,\theta) + (w^p,\theta)] - \|\nabla\theta\|^2\,, \end{aligned} \tag{25}$$

where $\|\cdot\|_3$ denotes the norm on $L^3(V)$.

Define now $\hat{\theta} = R^{1/2}\theta$ and then we replace θ by $\hat{\theta}$ in equations (25). We then add the three resulting equations to obtain

$$\frac{dE}{dt} = R^{1/2}I - \mathcal{D} - E_3\,, \tag{26}$$

where

$$E = \frac{J_1}{2}\|\mathbf{u}^f\|^2 + \frac{\lambda_f}{2}\|\nabla\mathbf{u}^f\|^2 + \frac{J_2}{2}\|\mathbf{u}^p\|^2 + \frac{\lambda_p}{2}\|\nabla\mathbf{u}^p\|^2 + \frac{1}{2}\|\hat{\theta}\|^2 + \frac{J_3}{2}\|\nabla\hat{\theta}\|^2, \tag{27}$$

and

$$I = 2[(w^f,\hat{\theta}) + (w^p,\hat{\theta})]\,, \tag{28}$$

and

$$\begin{aligned} \mathcal{D} = \|\nabla\hat{\theta}\|^2 + \lambda(\|\nabla\mathbf{u}^f\|^2 + \tilde{\mu}_r\|\nabla\mathbf{u}^p\|^2) \\ + \|\mathbf{u}^f\|^2 + K_r\|\mathbf{u}^p\|^2 + \xi\|\mathbf{u}^f - \mathbf{u}^p\|^2\,. \end{aligned} \tag{29}$$

with additionally

$$E_3 = f_1\|\mathbf{u}^f\|_3^3 + f_2\|\mathbf{u}^p\|_3^3. \tag{30}$$

Define now the threshold R_E as

$$\frac{1}{R_E} = \max_{\mathcal{H}}\frac{I}{\mathcal{D}}\,, \tag{31}$$

where $\mathcal{H}$ is the space of admissible perturbations, i.e. consisting of $H^1(V)$-divergence free functions u_i^f and u_i^p and $H^1(V)$ functions $\hat{\theta}$, satisfying the homogeneous boundary conditions (14) on $z = 0, 1$.

From (26) one may obtain

$$\frac{dE}{dt} \leq -\mathcal{D}\left(1 - \frac{R^{1/2}}{R_E}\right) - E_3 \,. \tag{32}$$

For $R^{1/2} < R_E$, we put $\kappa = (1 - R^{1/2}/R_E) > 0$; thus integration of (32), after use of the Poincaré inequality, allows us to find the estimate for a constant $b > 0$

$$E(t) + \int_0^t e^{-b\kappa(t-s)} E_3(s)\,ds \leq E(0)e^{-b\kappa t} \,. \tag{33}$$

Consequently, $\|\mathbf{u}^f\|$, $\|\nabla\mathbf{u}^f\|$, $\|\mathbf{u}^p\|$, $\|\nabla\mathbf{u}^p\|$, $\|\hat{\theta}\|$, $\|\nabla\hat{\theta}\|$ decay to zero exponentially in time. We conclude that the condition $R^{1/2} < R_E$ guarantees the global (i.e. for all initial data) nonlinear stability of the steady conduction state under investigation.

Finally, in order to determine the critical Rayleigh number R_E, by solving the variational problem (31), we need to derive the Euler-Lagrange equations. Let now $\eta^f(\mathbf{x})$ and $\eta^p(\mathbf{x})$ be the associated Lagrange multipliers. One may show that the related Euler-Lagrange equations have form

$$\begin{aligned}
&R_E\hat{\theta}k_i + \lambda\Delta u_i^f - u_i^f - \xi(u_i^f - u_i^p) = \eta^f_{,i}\,,\\
&R_E\hat{\theta}k_i + \lambda\tilde{\mu}_r\Delta u_i^p - K_r u_i^p - \xi(u_i^p - u_i^f) = \eta^p_{,i}\,,\\
&\Delta\hat{\theta} + R_E w^f + R_E w^p = 0\,.
\end{aligned} \tag{34}$$

Upon inspection bearing in mind the definition of $\hat{\theta}$, we see that equations (34) are just equivalent to perturbation equations (13) without the time derivative terms and the nonlinear convective term. This completes the picture: the nonlinear critical Rayleigh threshold is just the same as the one of the linear instability analysis. Hence, the key physics of the onset of thermal convection is fully captured by the linear instability theory.

5 Numerical results and conclusions

In this section we report our critical values of the Rayleigh number and wavenumber which are found by numerically minimizing $R(a^2)$ in a^2 in equation (24).

Before reporting on numerical results we observe that the parameters λ, K_r and ξ are not present in I in (28) whereas they appear linearly in the dissipative term $\mathcal{D}$. Since R_E^{-1} is defined by max $I/\mathcal{D}$ we expect the parameters λ, K_r and ξ to each lead to increased values of the Rayleigh number R, i.e. we expect each to have a stabilizing influence on the onset of convection. This is indeed what we witness numerically.

Furthermore, from (24) we may observe some asymptotic results, all of which we have verified numerically. As $\lambda \to \infty$ we see from (24) that $R/\lambda \to \Lambda^3/2a^2$ and so $a^2_{crit} = \pi^2/2$ and $R/\lambda \to 27\pi^4/8$, where a_{crit} denotes the critical value of a for instability. When $\lambda = 0$ then

$$R \to \frac{\Lambda^2}{a^2}\,\frac{(K_r + \xi + \xi K_r)}{(4\xi + 1 + K_r)}$$

and so

$$R \to 4\pi^2\,\frac{(K_r + \xi + \xi K_r)}{(4\xi + 1 + K_r)}$$

with $a^2_{crit} = \pi^2$, as found in Gentile and Straughan [39]. For $\lambda << 1$ we may differentiate (24) with respect to a^2 and then perform an asymptotic analysis, expanding a^2 in powers of λ. In this case we find

$$a^2 = \pi^2 - X_1 + O(\lambda^2),$$

Table 1 Critical Rayleigh and wave numbers for various values of K_r. Here $\lambda = 18.471$, $\xi = 1.3578 \times 10^{-1}$.

Ra	a^2	K_r	$\tilde{\mu}_r$
389.01	9.60	10	10^{-6}
752.41	9.36	20	10^{-6}
1096.80	9.15	30	10^{-6}
4166.51	7.49	150	10^{-6}
7110.07	6.29	400	10^{-6}
9524.49	5.56	1000	10^{-6}
402.30	9.29	10	10^{-3}
764.89	9.21	20	10^{-3}
1108.52	9.05	30	10^{-3}
4122.50	7.48	150	10^{-3}
7112.34	6.28	400	10^{-3}
9525.10	5.56	1000	10^{-3}
514.85	7.75	10	10^{-2}
872.79	8.23	20	10^{-2}
1210.74	8.35	30	10^{-2}
4175.74	7.38	150	10^{-2}
7132.66	6.27	400	10^{-2}
9530.52	5.56	1000	10^{-2}

where

$$X_1 = \frac{-2\pi^4(1 + 4\xi + 4\xi K_r + 8\xi^2 + K_r^2)}{(K_r + \xi + \xi K_r)(1 + 4\xi + K_r)}.$$

As λ varies from 0 to ∞ we expect to see a^2 vary from π^2 to $\pi^2/2$ for all values of K_r and ξ, and this is what we see numerically.

Numerical results are given in Tables 1, 2. We have computed results for various combinations of λ, K_r and ξ, and the tables displayed represent only a few of our computations.

In their study of convection in a bidisperse material with only Darcy theory in the micropores and no inertia terms Gentile and Straughan [81] report values of the relative permeability in the range $K_r = 25, \ldots, 2322$. They employ water as the saturating fluid and this has a dynamic viscosity of 8.9×10^{-4} Pa s at 25°C. However, a viscoelastic fluid usually has a dynamic viscosity much higher. For example, from the website Engineeringtoolbox.com the oils SAE 10, 20, 30, 40, 50 have dynamic viscosities of 0.079, 0.170, 0.310, 0.430, 0.630 Pa s at 20°C. Thus, we calculate values of the parameters based upon such an oil. The macropermeability may be calculated from the Carmen - Kozeny relation, Chen [82], Nield [83], and this is

$$K_f = \frac{d_f^2}{172.8} \frac{\phi^3}{(1-\phi)^2}.$$

We suppose the porous media is based on 5 mm glass beads and so $d_f = 5 \times 10^{-3}$m, and we take the porosity value $\phi = 0.972$ of the open cell rigid foam of Givler and Altobelli [84]. This yields a value of K_f as

$$K_f = 1.6946 \times 10^{-4} \text{ m}^2. \tag{35}$$

Givler and Altobelli [84] deduce from experiments that the relation between μ and $\tilde{\mu}_f$ lies in the range $\mu = 5.1\tilde{\mu}_f$ to $\mu = 10.9\tilde{\mu}_f$. Then from (11) we calculate λ as $\lambda = \tilde{\mu}_f K_f/\mu d^2$, and for a layer of 1 cm depth we find

$$\lambda = 18.471. \tag{36}$$

The interaction coefficient ζ is determined by Hooman et al. [85] as 63.3 Ps s m^{-2} and from (11) we calculate $\xi = \zeta K_f/\mu$ and then for an SAE 10 oil we determine

$$\xi = 0.13578. \tag{37}$$

Table 2 Critical Rayleigh and wave numbers for various values of K_r. Here $\lambda = 18.471$, $\xi = 1.3578 \times 10^{-1}$

Ra	a^2	K_r	$\tilde{\mu}_r$
1456.31	5.58	10	10^{-1}
1775.54	5.98	20	10^{-1}
2073.28	6.24	30	10^{-1}
3952.74	6.75	110	10^{-1}
4660.31	6.69	150	10^{-1}
7325.95	6.10	400	10^{-1}
9583.50	5.53	1000	10^{-1}
6192.78	4.98	10	1
6297.92	5.02	20	1
6399.22	5.06	30	1
7383.40	5.31	150	1
8637.99	5.41	400	1
10012.71	5.32	1000	1
1.10852×10^4	4.94	10	10
1.10889×10^4	4.95	20	10
1.10925×10^4	4.95	30	10
1.11344×10^4	4.95	150	10
1.12120×10^4	4.97	400	10
1.13589×10^4	4.99	1000	10

Thus, equipped with the values of λ and ξ from (36) and (37) we minimize R in (24). The values of R are calculated for a range of relative permeabilities K_r. To use (24) we also need a value for $\tilde{\mu}_r = \tilde{\mu}_p/\tilde{\mu}_f$. There are two schools of thought on the relative Brinkman viscosity values. One is that $\tilde{\mu}_p$ will be much smaller than $\tilde{\mu}_f$ because Brinkman theory will likely be more valid in the macro pores. However, Fried and Gurtin [46] argue that in a microfluidic situation higher velocity gradients will be important. This suggests the Brinkman term in the micropores should be important as compared to that in the macropores. Therefore, we compute critical Rayleigh and wave numbers for various values of $\tilde{\mu}_r$.

In Tables 1, 2, $\lambda = 18.471$, $\xi = 0.13578$. We allow K_r to vary from 10 to 1000. Table 1 shows Rayleigh and wavenumber values for $\tilde{\mu}_r = 10^{-6}, 10^{-3}, 10^{-2}$, whereas in Table 2 $\tilde{\mu}_r = 0.1, 1, 10$.

For the small values of $\tilde{\mu}_r$ in Table 1 there is a large variation in both critical Rayleigh number and critical wavenumber. In Table 1 the critical Rayleigh number always increases with increasing K_r as predicted. However, the critical wavenumber decreases in each case. Since the wavenumber is inversely proportional to the aspect ratio of the convection cell (width to depth) this means that increasing K_r leads to wider convection cells. Thus, increasing K_r leads to a greater stability before convection commences and the cells are wider once it does. The pattern of behaviour of the Rayleigh number is the same in Table 2, but the increase is much less as K_r increases, as $\tilde{\mu}_r$ likewise increases. We have also computed with $\tilde{\mu}_r = 100$ and 1000. In the former case R increases monotonically from 1.20740×10^4 when $K_r = 10$ to 1.20782×10^4 when $K_r = 1000$. The equivalent wavenumber squared increases very slowly from 4.94 50 4.95 over the same range. For the second case where $\tilde{\mu}_r = 1000$ we find $R = 1.21831 \times 10^4$ over the whole range and $a^2 = 4.95$. However, in Table 2 the wavenumber increases very slowly with increasing K_r when $\tilde{\mu}_r = 10$. But, in the cases $\tilde{\mu}_r = 0.1, 1$ the wavenumber squared increases in value from that when $K_r = 10$ to a maximum when $K_r = 110$ and thereafter decreases again when $\tilde{\mu}_r = 0.1$. There is similar behaviour when $\tilde{\mu}_r = 1$, but the maximum is achieved when $K_r = 400$.

We believe the results presented here show that there is a need for the model for bidisperse convection with Brinkman effects and inertia effects in the macro and micro pores, especially when considering viscoelastic fluids. It would be very useful to have experimental values of the relation between the dynamic viscosity μ and the micropore Brinkman viscosity $\tilde{\mu}_p$, in the vein of the results of Altobelli [84]. If theoretical estimates for this relationship were available this would also be very useful, in the same way as Rees [86,87] calculates theoretically parameter values for a monodisperse porous model under a local non-thermal equilibrium frame.

Summarizing, the theory of a bidisperse porous medium, accounting for Brinkman-Forchheimer effects and inertia and Kelvin–Voigt effects in both the macro and micro pores, is employed to generalize the classical Bénard convection problem in a horizontal layer, uniformly heated from below. The role of the relative parameters $\tilde{\mu}_r$, K_r, ξ and λ is highlighted and play a strategic role in determining when thermal convective motion may occur. We have shown the key result that the linear instability threshold is just the same as the global nonlinear stability one. This means that linear instability theory captures fully the physics of the onset of thermal convection and subcritical instabilities do not arise.

Interestingly, the result of coincidence of the linear instability boundary and the global nonlinear stability boundary holds even in the case where one, at the micro level, argues for rough boundaries of a porous medium: in this case, one introduces novel Saffman type boundary conditions for the vertical gradients of the horizontal components of the velocities, as in the recent work of Celli and Kuznetsov [77].

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Declarations

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Author contributions The authors have equally contributed to each part of this paper. They conceived and thoroughly discussed the main ideas, mathematical models and results of this paper by mutual consent. All of them carried out in detail the proofs and calculations. All authors gave final approval for publication and agree to be held accountable for the work performed therein.

Data Availability Any sources of data are indicated in the text.

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