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Gradual convergence for Langevin dynamics on a degenerate potential

Gerardo Barrera [a],*, Conrado da-Costa [b], Milton Jara [c]

[a] Center for Mathematical Analysis, Geometry and Dynamical Systems, Mathematics Department, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001, Lisboa, Portugal

[b] Department of Mathematical Sciences, Durham University, Upper Mountjoy Campus, Durham DH1 3LE, United Kingdom

[c] Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, Brazil

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ABSTRACT

In this paper, we study an ordinary differential equation with a degenerate global attractor at the origin, to which we add a white noise with a small parameter that regulates its intensity. Under general conditions, for any fixed intensity, as time tends to infinity, the solution of this stochastic dynamics converges exponentially fast in total variation distance to a unique equilibrium distribution. We suitably accelerate the random dynamics and show that the preceding convergence is gradual, that is, the function that associates to each fixed $t \geq 0$ the total variation distance between the accelerated random dynamics at time t and its equilibrium distribution converges, as the noise intensity tends to zero, to a decreasing function with values in $(0,1)$. Moreover, we prove that this limit function for each fixed $t \geq 0$ corresponds to the total variation distance between the marginal, at time t, of a stochastic differential equation that comes down from infinity and its corresponding equilibrium distribution. This completes the classification of all possible behaviors of the total variation distance between the time marginal of the aforementioned stochastic dynamics and its invariant measure for one dimensional well-behaved convex potentials. In addition, there is no cut-off phenomenon for this one-parameter family of random processes and asymptotics of the mixing times are derived.

1. Introduction

The study of random dynamical systems and their convergence to equilibrium is one of the most studied subject in probability theory and mathematical physics with a vast literature such as stochastic control [1], slow–fast systems [2], small noise limit [3–6], small noise asymptotics for invariant densities [4,7–9], sharp estimates on transit and exit times [10], couplings and quantitative contraction rates for Langevin dynamics [11–13], convergence to equilibrium in Fokker–Planck equations [14–16], random attractors for stochastic dissipative systems [17], numerical computations of geometric ergodicity [18,19], multi-scale analysis, ergodicity and exponential loss of memory of the initial condition [20–22], regularity for Lyapunov exponents [23], metastability and large deviations [24,25], optimal transport [26], etc.

The goal of this paper is the study of the *convergence to equilibrium* in the so-called *zero-noise limit* for a family of stochastic small random perturbations of a given one-dimensional dynamical system. We consider an ordinary differential equation with a degenerate (non-hyperbolic) global attractor at the origin. Under appropriate conditions on the dynamics, as time increases, for any initial condition the solution of this differential equation tends to the origin polynomially fast. We then consider a perturbation of the deterministic dynamics by a Brownian motion of small intensity. This random dynamics possesses a unique invariant probability

* Corresponding author.
E-mail addresses: gerardo.barrera.vargas@tecnico.ulisboa.pt (G. Barrera), conrado.da-costa@durham.ac.uk (C. da-Costa), mjara@impa.br (M. Jara).

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measure and for any initial condition, the solution converges in the total variation distance to such invariant probability measure as times increases. We prove that the convergence occurs gradually, that is, when the strength of the noise (ϵ) tends to zero, with a suitable scaling of time ($a_\epsilon, \epsilon > 0$), the function that associates to each fixed $t \geq 0$ the total variation distance between the marginal of the random dynamics at time $a_\epsilon t$ and its equilibrium tends, as $\epsilon \to 0$, to a *decreasing* function with values in the open interval $(0, 1)$, see also Definition 1.5 below. This fact, with the help of Proposition 1.9 implies no cut-off phenomenon in the context of random processes.

1.1. The degenerate Langevin dynamics

In this subsection, we specify the *degenerate Langevin dynamics* that we consider in this paper. Here we say that a Langevin dynamics is degenerate when its vector field possesses a degenerate fixed (critical) point.

The Langevin dynamics was introduced by P. Langevin in 1908 in his seminal article [27]. It is perhaps one of the most popular models in molecular systems. For details on its history and phenomenological treatment, we refer to [28,29] and the references therein.

Let $\epsilon \in (0, 1]$ be the parameter that controls the intensity of the noise and let $X^\epsilon(x) := \left(X_t^\epsilon(x), t \geq 0\right)$ be the unique strong solution of the one-dimensional Stochastic Differential Equation (for short SDE)

$$\begin{cases} \mathrm{d}X_t^\epsilon = -V'(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}B_t \quad \text{for} \quad t \geq 0, \\ X_0^\epsilon = x, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$ is a deterministic initial condition, $B := \left(B_t, t \geq 0\right)$ is a one-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $V : \mathbb{R} \to [0, \infty)$ is a given function that will be referred to as *the potential*. In order to avoid technicalities and since we want to be able to use Itô's formula, we assume the following conditions for V.

Hypothesis 1.1 (*Regularity*). We assume that the potential V is a twice continuously differentiable, convex and even function with $V(0) = 0$.

Since for the dynamics (1.1) we only consider from the potential V its derivative V', the value of V at 0 is not crucial and it is only imposed in Hypothesis 1.1 to fix a unique potential once given its derivative. Moreover, since V is even and differentiable, we have $V'(0) = 0$. We recall that 0 is a degenerate fixed point when $V''(0) = 0$. In what follows, we assume the following local behavior at 0.

Hypothesis 1.2 (*Local Behavior at the Origin*). There exist positive constants C_0 and α such that

$$\lim_{\lambda \to 0} \sup_{|z| \leq 1} \left| \frac{V'(\lambda z)}{\lambda^{1+\alpha}} - C_0 |z|^{1+\alpha} \mathrm{sgn}(z) \right| = 0, \quad \text{where} \quad \mathrm{sgn}(z) := z\,|z|^{-1} \mathbb{1}_{\{z \neq 0\}}. \tag{1.2}$$

Remark 1.3. An intuition for Hypothesis 1.2 is to think of it as a generalization of the behavior of the monomial potential $V_0 : \mathbb{R} \to [0, \infty)$ given by $V_0(x) := |x|^{2+\alpha}$ with $\alpha > 0$ to smooth potentials $V : \mathbb{R} \to [0, \infty)$ with leading behavior at the origin given by $C_0|x|^{2+\alpha}$ and suitable $C_0 > 0$ as described in (1.2). That is, $V(x) = C_0|x|^{2+\alpha} + \mathrm{o}(|x|^{2+\alpha})$ as $x \to 0$. For instance, for the case $V(x) = |x|^{2+\alpha}$, $x \in \mathbb{R}$, we have $\frac{V'(\lambda z)}{\lambda^{1+\alpha}} = (2+\alpha)|z|^{1+\alpha}\mathrm{sgn}(z)$ for $z \in \mathbb{R}$ and $\lambda \neq 0$, and hence Condition (1.2) is satisfied with $C_0 := 2+\alpha$.

Roughly speaking, the local behavior of the potential at the origin captured in (1.2) controls the convergence to equilibrium in (1.1). In fact, the convex potential V drives the trajectories of (1.1) to the origin and the convergence to equilibrium depends on the intensity of the noise and the strength of drift determined by the potential. See Section 2.1 for further heuristics.

We also point out that Hypothesis 1.2 is equivalent to

$$\lim_{\lambda \to 0} \sup_{|z| \leq K} \left| \frac{V'(\lambda z)}{\lambda^{1+\alpha}} - C_0 |z|^{1+\alpha} \mathrm{sgn}(z) \right| = 0 \quad \text{for any} \quad K > 0.$$

Finally, in order to control the growth of V' around infinity and to ensure that (1.1) has a unique invariant probability measure, we assume the following growth condition.

Hypothesis 1.4 (*Growth at Infinity*). There exist $c_0, R_0 \in (0, \infty)$, and $\beta \in (-1, \infty)$ such that

$$V'(z) \geq c_0 z^{1+\beta} \quad \text{for all} \quad z \geq R_0.$$

An interesting example, which satisfies Hypotheses 1.1, 1.2 and 1.4, is the one-well potential $V(z) = |z|^{2+\alpha}$, $z \in \mathbb{R}$ for some $\alpha > 0$. When $\alpha = 0$, we have that (1.1) corresponds to the Ornstein–Uhlenbeck process which exhibits profile cut-off for $x \neq 0$ and it does not when $x = 0$, for further details see [30,31]. For $\alpha > 0$ we have $V'(0) = V''(0) = 0$ and hence Theorem 2.1 in [31] cannot be applied. In fact, in the degenerate case, for any initial condition the convergence to equilibrium is gradual, in the sense of Definition 1.5 below, which implies no cut-off. This is in stark contrast with the Ornstein–Uhlenbeck process and it is natural from the dynamical point of view, since the fixed point changes from hyperbolic to non-hyperbolic (degenerate). For instance, the qualitative behavior of hyperbolic systems and degenerate systems are very different, the former are structurally stable whereas the latter are not. In the hyperbolic attracting case, it is shown in Theorem 2.2 in [31] that profile cut-off phenomenon holds true and the proof relies on the Hartman–Grobman theorem, which breaks down at degenerate points. In the present degenerate setting, we introduce a time space scaling and obtain gradual convergence to equilibrium, see Theorem 1.7 below. Moreover, there is a qualitative change of behavior in the model: the cut-off phenomenon is not present in this setting, see Corollary 1.10 below. We also give asymptotics for the mixing times in (1.14) in Corollary 1.10.

Another example which underlies our motivation to study this type of model is the so-called Ginzburg–Landau potential. More precisely, for a given $\eta \in \mathbb{R}$ the *Ginzburg–Landau potential* $V_\eta : \mathbb{R} \to \mathbb{R}$ is defined by

$$V_\eta(x) := \cosh(x) - \frac{1}{2}\eta x^2 = 1 + \frac{(1-\eta)x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad \text{for any} \quad x \in \mathbb{R}.$$

Note that $V_\eta'(0) = 0$ for all $\eta \in \mathbb{R}$. Moreover, for $\eta \leq 1$, we have that V_η is convex and $V_\eta''(x) \geq (1-\eta)$ for all $x \in \mathbb{R}$ with $V_\eta''(0) = 1-\eta$. For $\eta < 1$, V_η is coercive and hence Theorem 2.1 in [31] applies, yielding the cut-off phenomenon (abrupt convergence) to equilibrium for (1.1). For $\eta > 1$, V_η is no longer convex and has the classical double-well shape used in models which exhibit metastability, see [24,25,32–34]. We point out that metastable models do not exhibit cut-off phenomenon, see [35,36]. Finally, at the critical value $\eta = 1$, up to a translation, the potential V_1 satisfies Hypotheses 1.1, 1.2 and 1.4. Therefore, Theorem 1.7 below yields gradual convergence to equilibrium for (1.1) when the potential is V_1.

With the present result, we improve the classification of the behaviors of the total variation distance between the marginal X_t^ε given in (1.1) and its invariant measure for one dimensional smooth convex potentials V. More precisely, provided the convex potential V satisfies regularity conditions, such as it grows at infinity and is well-behaved at the origin, then we can classify the convergence to equilibrium as follows.

(1) *Cut-off phenomenon*. This occurs when the fixed point of the deterministic dynamics associated to (1.1) is hyperbolic, i.e., $V''(0) > 0$, see Theorem 2.1 in [31].
(2) *Gradual convergence to equilibrium*. This occurs when the fixed point of the deterministic dynamics associated to (1.1) is degenerate, i.e., $V''(0) = 0$, see Theorem 1.7 below.

When the potential V is not convex and possesses finitely many (hyperbolic) stable equilibria, it is well-known that *metastability phenomenon* occurs, see [24,25,32–34].

By Hypothesis 1.1 the SDE (1.1) has a unique strong solution, see Theorem 3.5 in [37, p. 58] or Theorem 10.2.2 in [38, p. 255]. Hence, $X^\varepsilon(x)$ is a well-defined stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, Lemma 2.1 below yields that (1.1) is exponentially ergodic in total variation distance with a unique invariant probability measure μ^ε given by

$$\mu^\varepsilon(\mathrm{d}z) = \frac{e^{-\frac{2}{\varepsilon}V(z)}}{C_\varepsilon}\mathrm{d}z \qquad \text{with} \qquad C_\varepsilon := \int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}V(y)}\,\mathrm{d}y.$$

1.2. Results

In this section, before we state the main results of the paper, we recall the definition of the total variation distance and fix some conventions.

In the sequel, we adopt the convention that $\mathrm{sgn}(0)\infty = 0$ and since $\varepsilon \in (0,1]$, for simplicity we write $\varepsilon \to 0$ instead of $\varepsilon \to 0^+$. We point out that for any $x \in \mathbb{R}$ and $t > 0$, the marginal $X_t^\varepsilon(x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Then we measure the distance between the law of $X_t^\varepsilon(x)$ and its limiting distribution μ^ε by the total variation distance, defined by

$$\mathrm{d}_{\mathrm{TV}}(\nu_1, \nu_2) := \sup_{F \in \mathcal{F}} |\nu_1(F) - \nu_2(F)|$$

for any ν_1 and ν_2 probability measures in the same measurable space $(\Omega, \mathcal{F})$. For convenience, we do not distinguish a random variable X_1 and its law $\mathbb{P}_{X_1}$ as an argument of d_{TV}. In other words, for random variables X_1 and X_2 and probability measure μ we write $\mathrm{d}_{\mathrm{TV}}(X_1, X_2)$ in place of $\mathrm{d}_{\mathrm{TV}}(\mathbb{P}_{X_1}, \mathbb{P}_{X_2})$ and write $\mathrm{d}_{\mathrm{TV}}(X_1, \mu)$ instead of $\mathrm{d}_{\mathrm{TV}}(\mathbb{P}_{X_1}, \mu)$. For further details on the total variation distance, we refer to [20, Ch. 2] or [39, Sec. 3.3].

To state our main result, we introduce the following definition.

Definition 1.5 (*Gradual Convergence*)**.** For each $\varepsilon \in (0,1]$, let $X^\varepsilon = (X_t^\varepsilon, t \geq 0)$ be a stochastic process with unique invariant probability measure μ^ε, and fix a deterministic $a_\varepsilon > 0$. We say that the family of stochastic processes $(X^\varepsilon, \varepsilon \in (0,1])$ *exhibits gradual convergence to equilibrium at scale* $(a_\varepsilon, \varepsilon > 0)$ *with respect to the total variation distance as* $\varepsilon \to 0$, when the map $d_\varepsilon : (0,\infty) \to [0,1]$ defined by

$$d_\varepsilon(t) := \mathrm{d}_{\mathrm{TV}}(X_{a_\varepsilon t}^\varepsilon, \mu^\varepsilon), \quad t \geq 0,$$

converges as ε tends to zero to a function $d_0 : (0,\infty) \to (0,1)$ in the sense that for almost all $t > 0$

$$\lim_{\varepsilon \to 0} d_\varepsilon(t) = d_0(t).$$

We say that the family of stochastic processes $(X^\varepsilon, \varepsilon \in (0,1])$ *exhibits gradual convergence to equilibrium with respect to the total variation distance* or simply *exhibits gradual convergence* when there is a scale $(a_\varepsilon, \varepsilon > 0)$ for which the family of stochastic processes $(X^\varepsilon, \varepsilon \in (0,1])$ *exhibits gradual convergence to equilibrium at scale* $(a_\varepsilon, \varepsilon > 0)$ *with respect to the total variation distance as* $\varepsilon \to 0$.

Remark 1.6. We point out that gradual convergence and cut-off do not form a dichotomy. Indeed, in principle, one could have a process with a unique invariant measure that converges after scaling to a profile function f such that

$$f(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ \frac{1}{2} & \text{if } t \in (1,2], \\ 0 & \text{if } t > 2. \end{cases} \tag{1.3}$$

The above profile function is not compatible with gradual convergence, where we require $f(t) \in (0,1)$ for all $t > 0$ nor it is compatible with cut-off, as the function does not drop abruptly to 0, since reaches a plateau with value $\frac{1}{2}$ for $t \in (1,2]$. One example for this would be to consider two decks of cards, one red, one black. Assume that the shuffling for the black deck of cards converges to equilibrium after scaling at the deterministic time 1 and that the shuffling for the red one, converges to equilibrium at the deterministic time 2. If we select the deck of cards according to the outcome of a fair coin toss, red deck if the coin lands "heads" and the black deck if the coin lands "tails". This process will exhibit convergence to a profile (1.3) which is not gradual not cut-off. We point out that discontinuous profile functions may arise in which one still retains gradual convergence, see for instance [40].

The main result of this paper whose proof is given in Section 2 is the following.

Theorem 1.7. *Assume that Hypotheses 1.1, 1.2 and 1.4 hold true. For $\varepsilon \in (0,1]$ and $x \in \mathbb{R}$, let $X^{\varepsilon}(x)$ be the unique strong solution of (1.1) and denote by μ^{ε} its unique invariant probability measure. Define the scaling parameter*

$$a_{\varepsilon} := \varepsilon^{-\frac{\alpha}{2+\alpha}}, \quad \text{where } \alpha > 0 \text{ is given in Hypothesis 1.2} \tag{1.4}$$

Then for any $t > 0$ it follows that

$$\lim_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_{t a_{\varepsilon}}(x), \mu^{\varepsilon} \right) = \mathrm{d}_{\mathrm{TV}} \left(Y_t(\mathrm{sgn}(x)\infty), \nu \right) \in (0,1), \tag{1.5}$$

where $Y_t(\mathrm{sgn}(x)\infty) := \lim_{r\to\infty} Y_t(\mathrm{sgn}(x)r)$ and for $r > 0$, $(Y_t(\mathrm{sgn}(x)r), t \geq 0)$ is the strong solution of the SDE

$$\begin{cases} \mathrm{d}Y_t = -C_0 |Y_t|^{1+\alpha} \mathrm{sgn}(Y_t)\mathrm{d}t + \mathrm{d}W_t \quad \text{for} \quad t > 0, \\ Y_0 = \mathrm{sgn}(x) r, \end{cases} \tag{1.6}$$

$(W_t, t \geq 0)$ is a standard Brownian motion, ν is the unique invariant probability measure for (1.6), and the constant C_0 is defined in Hypothesis 1.2. Moreover, the map

$$t \mapsto \mathrm{d}_{\mathrm{TV}} \left(Y_t(\mathrm{sgn}(x)\infty), \nu \right) \text{ is continuous and strictly decreasing.} \tag{1.7}$$

The computation of the scaling (1.4) is given in Section 2.3.

We point out that $Y_t(\mathrm{sgn}(x)\infty)$ comes down from infinity, that is, $Y_t(\mathrm{sgn}(x)\infty) \in \mathbb{R}$ for any $t > 0$. It is not surprising that an equation in the form of (1.6) should "come down from infinity" and should admit a continuous Markovian extension. Since we did not find a reference with a full proof of this result, we have devoted Appendix A to explain this in detail. The continuous Markovian extension of the SDE (1.6) to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is done in detail using basic ODE/probabilistic techniques in Appendix A and here we only outline the main steps. First, based on a monotonic comparison, which follows from the synchronous coupling, and uniform second moment bounds for $x \in \mathbb{R}$, the SDE (1.6) can be extended to $\overline{\mathbb{R}}$, see Appendix A.1. Then because $\pm\infty$ are entrance boundaries for the dynamics in $\mathbb{R}$, the extended family $(Y(x) := (Y_t(x), t \geq 0), x \in \overline{\mathbb{R}})$ is Markovian, see Appendix A.2. The rigorous definition of (1.6) is given in Proposition 2.5 below. Moreover, Theorem 1.7 actually provides the essentially unique scale $(a_{\varepsilon}, \varepsilon \in (0,1])$ that captures the convergence to equilibrium. In fact, since by the Chapman–Kolmogorov equation, the map $t \mapsto \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_t(x), \mu^{\varepsilon} \right)$ is non-increasing, any sequence $(t_{\varepsilon}, \varepsilon \in (0,1])$ for which

$$\lim_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_{t_{\varepsilon}}(x), \mu^{\varepsilon} \right) = \mathrm{d}_{\mathrm{TV}} \left(Y_t(\mathrm{sgn}(x)\infty), \nu \right)$$

must satisfy

$$\lim_{\varepsilon \to 0} \frac{t_{\varepsilon}}{a_{\varepsilon}} = t. \tag{1.8}$$

Indeed, assume that $\limsup_{\varepsilon\to 0} \frac{t_{\varepsilon}}{a_{\varepsilon}} > t + \delta$ for some $\delta > 0$, then by (1.5) and (1.7)

$$\liminf_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_{t_{\varepsilon}}(x), \mu^{\varepsilon} \right) \leq \liminf_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_{(t+\delta)a_{\varepsilon}}(x), \mu^{\varepsilon} \right) = \mathrm{d}_{\mathrm{TV}} \left(Y_{t+\delta}(\mathrm{sgn}(x)\infty), \nu \right) < \mathrm{d}_{\mathrm{TV}} \left(Y_t(\mathrm{sgn}(x)\infty), \nu \right).$$

Similarly, if $\liminf_{\varepsilon\to 0} \frac{t_{\varepsilon}}{a_{\varepsilon}} < t - \delta$ for some $\delta \in (0,t)$, then

$$\liminf_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_{t_{\varepsilon}}(x), \mu^{\varepsilon} \right) \geq \liminf_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}} \left(X^{\varepsilon}_{(t-\delta)a_{\varepsilon}}(x), \mu^{\varepsilon} \right) = \mathrm{d}_{\mathrm{TV}} \left(Y_{t-\delta}(\mathrm{sgn}(x)\infty), \nu \right) > \mathrm{d}_{\mathrm{TV}} \left(Y_t(\mathrm{sgn}(x)\infty), \nu \right).$$

We now fix a (rough) notion of abrupt convergence to equilibrium, the cut-off phenomenon. For further details, see Definition 1.8 in [41] and also [35].

Definition 1.8 (*Cut-off Phenomenon*)**.** For each $\varepsilon \in (0,1]$, let $X^{\varepsilon} = (X^{\varepsilon}_t, t \geq 0)$ be a stochastic process with unique invariant probability measure μ^{ε} and deterministic $t_{\varepsilon} > 0$. We say that the one-parameter family of stochastic processes $(X^{\varepsilon}, \varepsilon \in (0,1])$ *exhibits cut-off with respect to the total variation distance at scale* $(t_{\varepsilon}, \varepsilon \in (0,1])$ *when* $t_{\varepsilon} \to \infty$ *as* $\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0} \mathrm{d}_{\mathrm{TV}}\left(X^{\varepsilon}_{\delta t_{\varepsilon}}, \mu^{\varepsilon}\right) = \begin{cases} 1 & \text{for} \quad \delta \in (0,1), \\ 0 & \text{for} \quad \delta \in (1,\infty). \end{cases} \tag{1.9}$$

We say that the family of stochastic processes $(X^{\varepsilon}, \varepsilon \in (0,1])$ *exhibits cut-off with respect to the total variation distance* or simply *exhibits cut-off* if there is a scale $(t_{\varepsilon}, \varepsilon \in (0,1])$ such that $t_{\varepsilon} \to \infty$ and (1.9) holds true.

Similar arguments leading to (1.8) allows one to obtain that the scale in the cut-off phenomenon is essentially unique, see also [42].

Roughly speaking, one generally expects that a one-parameter family of well-mixing stochastic processes will exhibit abrupt convergence of the marginals to the equilibrium distribution as a function of the parameter. This is known in the literature as the *cut-off phenomenon* introduced by [43] in the context of card shuffling. Actually, the notion of cut-off applies to a wide range of random models. In the discrete setting, the cut-off phenomenon has been proved for many different models such as Markovian shuffling cards and random transpositions [43–48], random walks on the hypercube [49,50], birth and death chains [35,36], sparse Markov chains [51], Glauber dynamics [52], SSEP dynamics [53], SEP in the circle [54,55], random walks in random regular graphs [56], Ornstein–Uhlenbeck processes [30,57–59], mean field zero-range process [60], averaging processes [61], sampling chains and processes [62–64], star transpositions [65], etc. There are relatively few examples of Markov processes, taking values in continuous state-spaces for which the cut-off phenomenon has been studied, such processes include linear and nonlinear SDEs driven by small Lévy noise [31,41,57,66–70], Dyson–Ornstein–Uhlenbeck process [71], the biased adjacent walk on the simplex [72], Brownian motion on families of compact Riemannian manifolds [73].

No cut-off: A classical example of a Markov dynamics that does not exhibit cut-off phenomenon is the random walk on the circle $\mathbb{Z}_n$, see Example 18.5 in [50, Ch.18, p.253] or [74, Thm. 2.2.1, p.55]. Numerical results yields that the cut-off phenomenon does not occur for the entropy in the sense of information theory, see [75]. It has been also proved that the "insect Markov chain" does not have cut-off, see [76]. Moreover, it has been showed the absence of cut-off for several classes of trees, including spherically symmetric trees, Galton–Watson trees of a fixed height, and sequences of random trees converging to the Brownian CRT, see [77,78]. More recently, it is shown that the TASEP in the coexistence line does not have cut-off, see [79], and that there is no cut-off for sparse chains, see Corollary 4 in [80].

By the Chapman–Kolmogorov equation, for any $x \in \mathbb{R}$ and $\varepsilon \in (0,1]$ it follows that the map

$$t \mapsto d^{\varepsilon}_t(x) := \mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_t(x), \mu^{\varepsilon}) \quad \text{is non-increasing.} \tag{1.10}$$

By ergodicity, see Lemma 2.1 below, $\lim_{t\to\infty} d^{\varepsilon}_t(x) = 0$. This allows us to define for any $x \in \mathbb{R}$, $\varepsilon \in (0,1)$, and $\eta \in (0,1)$, the η-mixing time for the process $(X^{\varepsilon}_t(x), t \geq 0)$ by

$$\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta) := \inf\{t \geq 0 : d^{\varepsilon}_t(x) \leq \eta\}.$$

That is, one seeks the time required by the process $X^{\varepsilon}(x)$ for the total variation distance to its invariant measure μ^{ε} to be equal to or smaller than a prescribed error η.

The phenomenon of cut-off can be detected with the help of the notion of mixing times, see for instance [56,81] or Chapter 18 of [50]. Following Equation (18.3) in [50], the cut-off phenomenon is equivalent to the following relation between mixing times

$$\lim_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{\tau^{\varepsilon,x}_{\mathrm{mix}}(1-\eta)} = 1 \quad \text{for all} \quad \eta \in (0,1).$$

Due to its natural relevance, it has been extensively studied in many stochastic models, see for instance [50,52,81–88] and the references therein.

The following proposition provides a rather general technique to prove no cut-off in ergodic systems. For convenience we keep the notation and parameters that are proper to our model, but we emphasize that the result can be seen as a method for proving no-cut-off valid for any stochastic process satisfying the hypotheses in Proposition 1.9 below. In rough terms, it states that if there exists a non-trivial behavior for a suitable scale, then there is no cut-off for any scale.

Proposition 1.9 (*No Cut-Off and Mixing Times Asymptotics*)**.** *Let $x \in \mathbb{R}$ be given and assume that the scale $(a_{\varepsilon} = a_{\varepsilon}(x), \varepsilon \in (0,1])$ satisfies*

(i) $\lim_{\varepsilon\to 0} a_{\varepsilon} = \infty$.
(ii) For any $t > 0$

$$0 < \liminf_{\varepsilon\to 0} d^{\varepsilon}_{a_{\varepsilon}t}(x) \leq \limsup_{\varepsilon\to 0} d^{\varepsilon}_{a_{\varepsilon}t}(x) < 1. \tag{1.11}$$

Then there is no cut-off for the family $(X^{\varepsilon}(x), \varepsilon \in (0,1])$ as ε tends to zero. In addition, if the following limit exists

$$\lim_{\varepsilon\to 0} d^{\varepsilon}_{a_{\varepsilon}t}(x) = G_x(t) \in (0,1) \tag{1.12}$$

and the map $t \mapsto G_x(t)$ is continuous and strictly decreasing then, for any $\eta \in (0,1)$,

$$\lim_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_{\varepsilon}} = \inf\{t > 0 : G_x(t) \leq \eta\}. \tag{1.13}$$

The following result, which establishes the gradual convergence to equilibrium of the family defined in (1.1) is a consequence of Theorem 1.7 and Proposition 1.9.

Corollary 1.10 (*No Cut-Off Phenomenon*). *With the assumptions and notations of Theorem 1.7, for any $x \in \mathbb{R}$, the family of processes $(X^{\varepsilon}(x), \varepsilon \in (0,1])$ does not exhibit cut-off as $\varepsilon \to 0$. In addition,*

$$\lim_{\varepsilon \to 0} \frac{\tau^{\varepsilon,x}_{\mathsf{mix}}(\eta)}{a_{\varepsilon}} = \inf\{t \geq 0 : \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), \nu\right) \leq \eta\}. \tag{1.14}$$

Proof. By Theorem 1.7 and Proposition 1.9 we obtain Corollary 1.10. Indeed, by (1.4) and (1.5) we obtain that the scale function $(a_{\varepsilon}, \varepsilon \in (0,1])$ given in Theorem 1.7 satisfies conditions (i) and (ii) of Proposition 1.9 with

$$G_x(t) = \lim_{\varepsilon \to 0} d^{\varepsilon}_{a_{\varepsilon} t}(x) = \lim_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_t(x), \mu^{\varepsilon}) = \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), \nu\right) \in (0,1). \tag{1.15}$$

Therefore, the family of processes $(X^{\varepsilon}(x), \varepsilon \in (0,1])$ does not exhibit cut-off as $\varepsilon \to 0$. To obtain (1.14) it suffices to combine (1.13) with (1.15) and to note, by (1.7), that the map $t \mapsto G_x(t)$ is continuous and strictly decreasing. □

Structure of the paper. In Section 2 we explain the proof of Theorem 1.7 and in Section 3 we complete the outline of the proof of Theorem 1.7. The Appendix is divided in four sections. Appendix A proves that the process defined in (1.6) admits a continuous Markovian extension to $\overline{\mathbb{R}}$. Appendix B is devoted to the proof of uniform bounds for the entrance on compact sets, a crucial estimate to control the coupling rate of the process with the equilibrium measure. In Appendix C we give the proofs of the convergence of the invariant measures for the processes $X^{\varepsilon}(x)$ after suitable scaling. Appendix D contains results of technical nature that we collect to make the presentation more self-contained.

2. Proof of Theorem 1.7

This section is divided in five parts. Firstly, we give an heuristic argument for (1.5). Secondly, we examine, for fixed $\varepsilon \in (0,1]$, convergence to the unique invariant measure of $X^{\varepsilon}_t(x)$ as $t \to \infty$. Thirdly, we perform a scale analysis to deduce $(a_{\varepsilon}, \varepsilon \in (0,1])$. Fourthly, we introduce a localization argument which allows us to simplify the potential V under analysis. Finally, we state the key results used in the proof of (1.5).

2.1. Heuristics

Assume that V satisfies Hypotheses 1.1, 1.2 and 1.4. Let $\varphi(x) := \left(\varphi_t(x), t \geq 0\right)$ be the solution of the Ordinary Differential Equation (for short ODE)

$$\begin{cases} \mathrm{d}\varphi_t = -V'(\varphi_t)\mathrm{d}t \quad \text{for} \quad t \geq 0, \\ \varphi_0 = x. \end{cases} \tag{2.1}$$

The intuitive reason to consider (2.1) is that, with high probability, at early stages of the random evolution (1.1), the process stays close to the deterministic evolution (2.1). By the contracting nature of the random evolution (1.1), which follows from the convexity of V (Hypothesis 1.1) and its growth condition at infinity (Hypothesis 1.4), for every $x \neq 0$ the noise gets dissipated and the process is driven to zero, falling back into the stream of the deterministic evolution. For this reason, (2.1) is actually a good approximation of (1.1) for a long period of time and for large times, what matters is the behavior of V at zero. Moreover, Hypothesis 1.2 ensures the drift in (2.1) can be approximated at the origin, $V'(z) \sim C_0 |z|^{1+\alpha}\,\mathrm{sgn}(z)$, and the properly rescaled process should converge to $Y(\mathrm{sgn}(x)\infty)$ as $\varepsilon \to 0$, where $Y(x) = \left(Y_t(x), t \geq 0\right)$ is the unique strong solution of the following SDE

$$\begin{cases} \mathrm{d}Y_t = -C_0|Y_t|^{1+\alpha}\mathrm{sgn}(Y_t)\mathrm{d}t + \mathrm{d}W_t \quad \text{for} \quad t \geq 0, \\ Y_0 = x. \end{cases} \tag{2.2}$$

The exact scale and validity of the replacement of (1.1) by (2.2) is not immediate and is explained in the remainder this section.

2.2. The invariant probability measure

By the next lemma, (1.1) admits a unique invariant probability measure μ^{ε}.

Lemma 2.1 (*Exponential Ergodicity*). *Assume V satisfies Hypotheses 1.1 and 1.4. Let $\varepsilon \in (0,1]$ be fixed and for each $x \in \mathbb{R}$ let $X^{\varepsilon}(x)$ be the unique strong solution of (1.1). Then there exists a unique probability measure μ^{ε} such that for any $c > 0$ there are positive constants $C_1 = C_1(c,\varepsilon)$ and $C_2 = C_2(c,\varepsilon)$ for which*

$$\mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_t(x), \mu^{\varepsilon}) \leq C_1 e^{-C_2 t}\left(e^{c|x|} + \int_{\mathbb{R}} e^{c|y|}\mu^{\varepsilon}(\mathrm{d}y)\right) \quad \textit{for all} \quad x \in \mathbb{R},\ t \geq 0. \tag{2.3}$$

Furthermore, μ^{ε} is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and its density $\rho^{\varepsilon} : \mathbb{R} \to (0,\infty)$ is given by

$$\rho^{\varepsilon}(x) = \frac{e^{-\frac{2}{\varepsilon}V(x)}}{C_{\varepsilon}} \qquad \textit{with} \qquad C_{\varepsilon} := \int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}V(y)}\,\mathrm{d}y. \tag{2.4}$$

The proof of Lemma 2.1 is given in Appendix D.

2.3. Scale analysis

In this section we clarify the scaling factor in (1.4) and the need to consider $Y_t(\mathrm{sgn}(x)\infty) := \lim_{r\to\infty} Y_t(\mathrm{sgn}(x)r)$ given in Theorem 1.7. Define, for $t \geq 0$,

$$\mathcal{X}_t^{\varepsilon,x} := \frac{X^{\varepsilon}_{a_\varepsilon t}(x)}{b_\varepsilon} \tag{2.5}$$

and let us determine the time and space scaling parameters $a_\varepsilon > 0$ and $b_\varepsilon > 0$. By Itô's formula the stochastic process $(\mathcal{X}_t^{\varepsilon,x}, t \geq 0)$ has the same law as $(Y_t^{\varepsilon}(xb_\varepsilon^{-1}), t \geq 0)$, where $Y^{\varepsilon}(y) := (Y_t^{\varepsilon}(y), t \geq 0)$ is the unique strong solution of the following SDE

$$\begin{cases} \mathrm{d}Y_t^{\varepsilon} = -\frac{a_\varepsilon}{b_\varepsilon} V'(b_\varepsilon Y_t^{\varepsilon})\mathrm{d}t + \frac{\sqrt{\varepsilon a_\varepsilon}}{b_\varepsilon}\mathrm{d}B_t \quad \text{for} \quad t \geq 0, \\ Y_0^{\varepsilon} = y. \end{cases} \tag{2.6}$$

In what follows, we shall refer to the process $Y^{\varepsilon}(y)$ as the rescaled process, since it arises from the rescaling of the process X^{ε}. Note that, by Hypothesis 1.2, if $b_\varepsilon \to 0$ as $\varepsilon \to 0$ then for any $z \in \mathbb{R}$

$$\frac{a_\varepsilon}{b_\varepsilon} V'(b_\varepsilon z) \sim C_0 \frac{a_\varepsilon}{b_\varepsilon} |b_\varepsilon z|^{1+\alpha} \mathrm{sgn}(z) = C_0 a_\varepsilon b_\varepsilon^{\alpha} |z|^{1+\alpha} \mathrm{sgn}(z). \tag{2.7}$$

Therefore, to obtain a non-trivial limit for $\mathcal{X}_t^{\varepsilon,x}$ we remove the scaling factors of (2.6) by defining the pair a_ε and b_ε to be the solution of the system

$$\begin{cases} \frac{\sqrt{\varepsilon a_\varepsilon}}{b_\varepsilon} = 1, \\ a_\varepsilon b_\varepsilon^{\alpha} = 1. \end{cases} \tag{2.8}$$

The solution of (2.8) is given by

$$a_\varepsilon = \varepsilon^{-\frac{\alpha}{2+\alpha}} \qquad \text{and} \qquad b_\varepsilon = \varepsilon^{\frac{1}{2+\alpha}}. \tag{2.9}$$

Condition (2.8) sets the scale analysis to a fixed magnitude of the noise ($\varepsilon a_\varepsilon = b_\varepsilon^2$) and a constant strength of the velocity field at the origin ($a_\varepsilon b_\varepsilon^{\alpha} = 1$). By (2.7) and (2.8) the dynamics of (2.6) converges to the dynamics (2.2) on compact intervals as $\varepsilon \to 0$. However, for any initial condition $x \neq 0$ of (1.1), the family of processes we consider after scaling, $(\mathcal{X}_t^{\varepsilon,x}, t \geq 0)$, have initial condition xb_ε^{-1}, which diverges as $\varepsilon \to 0$. Therefore, the zero-noise limit of (2.6) requires a rigorous analysis at infinity.

2.4. Coupling near the origin

In this section we show that the problem in Theorem 1.7 is local. That is, we prove that one may replace V' in (1.1) with the derivative of a suitable function $\widetilde{V}$ that is well behaved at a neighborhood of the origin and satisfies mild growth conditions. More precisely, let x be the initial condition of (1.1). In the sequel, we consider a convex potential $\widetilde{V} = \widetilde{V}_x$ that satisfies

$$\widetilde{V}(z) = V(z) \quad \text{for any } z \text{ with} \quad |z| \leq L, \tag{2.10}$$

where $L > 0$ is such that $L^2 \geq 1 + |x|^2$. Additionally, we assume the following growth condition.

Hypothesis 2.2 (*Polynomial Growth at Infinity*)**.** There exist positive constants c, C and R such that

$$\widetilde{V}'(z) \geq cz^{1+\alpha} \quad \text{for} \quad z \geq R \tag{G1}$$

and such that

$$|\widetilde{V}'(z)| \leq Ce^{z^2} \quad \text{for} \quad |z| \geq R, \tag{G2}$$

where $\alpha > 0$ is given in Hypothesis 1.2.

Furthermore, note that $\widetilde{V}$ satisfies Hypotheses 1.1 and 1.2. The existence of $\widetilde{V}$ is guaranteed by Lemma D.1 in Appendix D.

For each $\varepsilon \in (0,1]$ and $x \in \mathbb{R}$ we consider the unique strong solution $\widetilde{X}^{\varepsilon}(x) := (\widetilde{X}_t^{\varepsilon}(x))_{t\geq 0}$ of the SDE

$$\begin{cases} \mathrm{d}\widetilde{X}_t^{\varepsilon} = -\widetilde{V}'(\widetilde{X}_t^{\varepsilon})\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t \quad \text{for} \quad t \geq 0, \\ \widetilde{X}_0^{\varepsilon} = x. \end{cases} \tag{2.11}$$

Since $\widetilde{V}$ is a convex function, Theorem 3.5 in [37, p. 58] yields that the SDE (2.11) has a unique strong solution. Furthermore, Lemma 2.1 implies that (2.11) possesses a unique invariant probability measure $\widetilde{\mu}^{\varepsilon}$. Recall that μ^{ε} is the unique invariant probability measure for (1.1) and for any $t \geq 0$ let

$$d_t^{\varepsilon}(x) := \mathrm{d}_{\mathrm{TV}}\left(X_t^{\varepsilon}(x), \mu^{\varepsilon}\right) \qquad \text{and} \qquad \widetilde{d}_t^{\varepsilon}(x) := \mathrm{d}_{\mathrm{TV}}(\widetilde{X}_t^{\varepsilon}(x), \widetilde{\mu}^{\varepsilon}). \tag{2.12}$$

The next lemma yields that $d^{\varepsilon}_{a_\varepsilon t}(x)$ and $\widetilde{d}^{\varepsilon}_{a_\varepsilon t}(x)$ are asymptotically equivalent in the following precise sense.

Lemma 2.3 (*Localization and Replacement of Potentials*)**.** *For all $x \in \mathbb{R}$ and $t \geq 0$ it follows that*

$$\lim_{\varepsilon\to 0} |d^{\varepsilon}_{a_\varepsilon t}(x) - \widetilde{d}^{\varepsilon}_{a_\varepsilon t}(x)| = 0,$$

where $(a_\varepsilon, \varepsilon \in [0,1))$ is defined in (1.4).

The proof of Lemma 2.3 is given in Section 3.2. By Lemma 2.3 it is enough to show Theorem 1.7 under Hypotheses 1.1, 1.2 and 2.2.

2.5. Bound via limit replacements

From this point onwards, we assume that V satisfies Hypotheses 1.1, 1.2 and 2.2.

Due to the scale invariance of the total variation distance ($\mathrm{d}_{\mathrm{TV}}(cX, cY) = \mathrm{d}_{\mathrm{TV}}(X, Y)$ for any $c \neq 0$ and any pair of random variables X and Y, see for instance Lemma A.1 in [70]) the distance $d^{\varepsilon}_{a_\varepsilon t}(x)$ in (2.12) can be expressed in terms of Y^{ε}_t and ν^{ε}, a "scalar multiple" of μ^{ε}. For convenience, we denote by X^{ε}_{∞} a random variable with the law μ^{ε} and by Y_{∞} a random variable with law ν which is the unique invariant probability measure for (1.6). With this notation, we have that ν^{ε} is the law of $Y^{\varepsilon}_{\infty} := b^{-1}_{\varepsilon} X^{\varepsilon}_{\infty}$ and therefore

$$d^{\varepsilon}_{a_\varepsilon t}(x) = \mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_{a_\varepsilon t}(x), X^{\varepsilon}_{\infty}) = \mathrm{d}_{\mathrm{TV}}\left(b^{-1}_{\varepsilon} X^{\varepsilon}_{a_\varepsilon t}(x), b^{-1}_{\varepsilon} X^{\varepsilon}_{\infty}\right) = \mathrm{d}_{\mathrm{TV}}\left(\mathcal{X}^{\varepsilon,x}_t, Y^{\varepsilon}_{\infty}\right), \tag{2.13}$$

where $\mathcal{X}^{\varepsilon,x}_t$ is given in (2.5). Now, by the triangle inequality we have

$$d^{\varepsilon}_{a_\varepsilon t}(x) \leq \mathrm{d}_{\mathrm{TV}}\left(\mathcal{X}^{\varepsilon,x}_t, Y_t(\mathrm{sgn}(x)\infty)\right) + \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_{\infty}\right) + \mathrm{d}_{\mathrm{TV}}\left(Y_{\infty}, Y^{\varepsilon}_{\infty}\right). \tag{2.14}$$

We stress that $Y_t(\mathrm{sgn}(x)\infty)$ is well-defined as we show in Proposition 2.5 below. Informally, the idea is that the drift dominates the noise and is strong enough to ensure that the process comes down from infinity. The triangle inequality also implies that

$$\mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_{\infty}\right) \leq \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), \mathcal{X}^{\varepsilon,x}_t\right) + d^{\varepsilon}_{a_\varepsilon t}(x) + \mathrm{d}_{\mathrm{TV}}\left(Y^{\varepsilon}_{\infty}, Y_{\infty}\right). \tag{2.15}$$

Combining (2.14) and (2.15) we obtain the following *key estimate* that we state as a lemma.

Lemma 2.4 (*Decoupling Inequality*). *Assume Hypotheses 1.1, 1.2 and 2.2 hold true. Then for any $x \in \mathbb{R}$, $\varepsilon > 0$ and $t \geq 0$ it follows that*

$$|d^{\varepsilon}_{a_\varepsilon t}(x) - \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_{\infty}\right)| \leq \mathrm{d}_{\mathrm{TV}}\left(\mathcal{X}^{\varepsilon,x}_t, Y_t(\mathrm{sgn}(x)\infty)\right) + \mathrm{d}_{\mathrm{TV}}\left(Y^{\varepsilon}_{\infty}, Y_{\infty}\right). \tag{2.16}$$

The following proposition states that the right-hand side of (2.16) tends to zero as $\varepsilon \to 0$.

Proposition 2.5. *Assume Hypotheses 1.1, 1.2 and 2.2 hold true. Then the following holds true:*

(1) Continuous Markovian extension: The real valued process defined by (2.2) admits a continuous Markovian extension to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

(2) Convergence for fixed marginal: For all $x \in \mathbb{R}$ and $t > 0$ it follows that

$$\lim_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), \mathcal{X}^{\varepsilon,x}_t\right) = 0, \tag{2.17}$$

where $(\mathcal{X}^{\varepsilon,x}_t, t \geq 0)$ is defined in (2.5).

(3) Convergence of invariant measures: Let Y_{∞} and Y^{ε}_{∞} denote random variables distributed according to the unique invariant distributions of the dynamics given by (2.2) and (2.6), respectively. The following limit holds true

$$\lim_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}}\left(Y_{\infty}, Y^{\varepsilon}_{\infty}\right) = 0. \tag{2.18}$$

The proof of Proposition 2.5 is given in Section 3.3. To complete the proof of (1.5) we rely on the following proposition.

Proposition 2.6. *For all $t > 0$ and $x \in \mathbb{R}$*

$$0 < \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_{\infty}\right) < 1. \tag{2.19}$$

The proof of Proposition 2.6 is given in Section 3.4.

Now, we are ready to prove Theorem 1.7, which is a consequence of what we have already stated up to here.

Proof of Theorem 1.7. Inequality (2.16) with the help of (2.17), (2.18) implies that

$$\lim_{\varepsilon \to 0} d^{\varepsilon}_{a_\varepsilon t}(x) = \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_{\infty}\right).$$

In addition, (2.19) implies (1.5). The proof of (1.7) is given in Lemma D.4 in Appendix D. □

3. Proofs: details

In this section, we give the proof of Proposition 1.9 and complete the proofs of the statements in Section 2. To be more precise, in Section 3.1 we prove Proposition 1.9, the proof of Lemma 2.3 is given in Section 3.2, the proof of Proposition 2.5 is given in Section 3.3 and the proof of Proposition 2.6 is given in Section 3.4.

3.1. Proof of Proposition 1.9

To prove that there is no cut-off phenomenon, by Definition 1.8, we need to show that for any scale $(t_\varepsilon, \varepsilon \in [0,1])$ with $\lim_{\varepsilon\to 0} t_\varepsilon = \infty$ the condition (1.9) does not hold. Let $(t_\varepsilon, \varepsilon \in (0,1])$ be such that $\lim_{\varepsilon\to 0} t_\varepsilon = \infty$. First, we assume that

$$\limsup_{\varepsilon\to 0} \frac{t_\varepsilon}{a_\varepsilon} < \infty, \tag{3.1}$$

that is, there are constants $C_1 > 0$ and $\varepsilon_0 \in (0,1]$ such that $t_\varepsilon \le C_1 a_\varepsilon$ for any $\varepsilon \in (0,\varepsilon_0]$. By (1.10), for any $\delta > 0$ and $\varepsilon \in (0,\varepsilon_0]$ we have $d^\varepsilon_{\delta C_1 a_\varepsilon}(x) \le d^\varepsilon_{\delta t_\varepsilon}(x)$. Therefore, by (1.11), for $\delta > 1$ we have

$$0 < \liminf_{\varepsilon\to 0} d^\varepsilon_{\delta C_1 a_\varepsilon}(x) \le \liminf_{\varepsilon\to 0} d^\varepsilon_{\delta t_\varepsilon}(x).$$

Hence, condition (1.9) fails at the scale $(t_\varepsilon, \varepsilon \in (0,1])$ for the family $(X^\varepsilon(x), \varepsilon \in (0,1])$.

If (3.1) fails, then there exists a sequence $(\varepsilon_k, k \in \mathbb{N})$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{t_{\varepsilon_k}}{a_{\varepsilon_k}} = \infty.$$

In particular, there exists $k_0 \in \mathbb{N}$ such that $a_{\varepsilon_k} \le t_{\varepsilon_k}$ for all $k \ge k_0$. By (1.10) and (1.11), for $0 < \delta < 1$ we have

$$\limsup_{k\to\infty} d^{\varepsilon_k}_{\delta t_{\varepsilon_k}}(x) \le \limsup_{k\to\infty} d^{\varepsilon_k}_{\delta a_{\varepsilon_k}}(x) \le \limsup_{\varepsilon\to 0} d^\varepsilon_{\delta a_\varepsilon}(x) < 1.$$

Hence, by Definition 1.8 there is no cut-off at the scale $(t_\varepsilon, \varepsilon \in (0,1])$ for the family $(X^\varepsilon(x), \varepsilon \in (0,1])$. Since $(t_\varepsilon, \varepsilon \in (0,1])$ is any function with $\lim_{\varepsilon\to 0} t_\varepsilon = \infty$, it follows that there is no cut-off phenomenon for the family $(X^\varepsilon(x), \varepsilon \in (0,1])$.

We now prove (1.13). By assumption (1.12) for any $t > 0$ we have

$$\lim_{\varepsilon\to 0} d^\varepsilon_{a_\varepsilon t}(x) = G_x(t) \in (0,1). \tag{3.2}$$

Moreover, the map $t \mapsto G_x(t)$ is continuous and strictly decreasing. Now, for each $\eta \in (0,1)$ we define $H_x(\eta) := \inf\{t \ge 0 : G_x(t) \le \eta\}$. To prove (1.13) we show that

$$\limsup_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_\varepsilon} \le H_x(\eta) \quad \text{and} \tag{3.3}$$

$$\liminf_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_\varepsilon} \ge H_x(\eta). \tag{3.4}$$

To prove (3.3), let $\gamma^* \in (0,\eta)$ be fixed and let $t^* := t^*(\eta - \gamma^*, x) > 0$ be such that $G_x(t^*) = \eta - \gamma^*$, and $G_x(t) > \eta - \gamma^*$ for all $t < t^*$. By (3.2) there is $\varepsilon^* := \varepsilon^*(\eta, \gamma^*, x) > 0$ such that

$$-\gamma^* < \mathrm{d}_{\mathrm{TV}}\left(X^\varepsilon_{t^* a_\varepsilon}(x), \mu^\varepsilon\right) - G_x(t^*) < \gamma^* \quad \text{for all} \quad \varepsilon \in (0,\varepsilon^*),$$

which implies that $\mathrm{d}_{\mathrm{TV}}\left(X^\varepsilon_{t^* a_\varepsilon}(x), \mu^\varepsilon\right) < \eta$ for all $\varepsilon \in (0,\varepsilon^*)$. Therefore, $\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta) \le t^* a_\varepsilon$ for all $\varepsilon \in (0,\varepsilon^*)$, which yields

$$\limsup_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_\varepsilon} \le t^* = t^*(\eta - \gamma^*, x).$$

Since $\gamma^* \in (0,\eta)$ is arbitrary and $t \mapsto G_x(t)$ is continuous and strictly decreasing we obtain that

$$\limsup_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_\varepsilon} \le \lim_{\gamma^*\to 0} t^*(\eta - \gamma^*, x) = H_x(\eta).$$

To prove (3.4), let $\gamma_* \in (0, 1-\eta)$ be fixed and let $t_* = t_*(\eta + \gamma_*, x) > 0$ be such that $G_x(t_*) = \eta + \gamma_*$ and $G_x(t_*) > \eta + \gamma_*$ for all $t < t_*$. By (3.2) there is $\varepsilon_* := \varepsilon_*(\eta, \gamma_*, x) > 0$ such that

$$\mathrm{d}_{\mathrm{TV}}\left(X^\varepsilon_{t_* a_\varepsilon}(x), \mu^\varepsilon\right) > \eta \quad \text{for all} \quad \varepsilon \in (0,\varepsilon_*).$$

Therefore, $\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta) \ge t_* a_\varepsilon$ for all $\varepsilon \in (0,\varepsilon_*)$, which implies

$$\liminf_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_\varepsilon} \ge t_* = t_*(\eta + \gamma_*, x).$$

Therefore

$$\liminf_{\varepsilon\to 0} \frac{\tau^{\varepsilon,x}_{\mathrm{mix}}(\eta)}{a_\varepsilon} \ge \lim_{\gamma^*\to 0} t_*(\eta + \gamma_*, x) = H_x(\eta).$$

This completes the proof of (1.13). □

3.2. Proof of Lemma 2.3

By the triangle inequality for the total variation distance we have that

$$d^\varepsilon_{a_\varepsilon t}(x) \le \mathrm{d}_{\mathrm{TV}}(X^\varepsilon_{a_\varepsilon t}(x), \widetilde{X}^\varepsilon_{a_\varepsilon t}(x)) + \widetilde{d}^\varepsilon_{a_\varepsilon t}(x) + \mathrm{d}_{\mathrm{TV}}(\widetilde{\mu}^\varepsilon, \mu^\varepsilon). \tag{3.5}$$

Similarly,

$$\widetilde{d}^{\varepsilon}_{a_\varepsilon t}(x) \le \mathrm{d}_{\mathrm{TV}}(\widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x), X^{\varepsilon}_{a_\varepsilon t}(x)) + d^{\varepsilon}_{a_\varepsilon t}(x) + \mathrm{d}_{\mathrm{TV}}(\mu^{\varepsilon}, \widetilde{\mu}^{\varepsilon}). \tag{3.6}$$

By (3.5) and (3.6) we obtain

$$\left| d^{\varepsilon}_{a_\varepsilon t}(x) - \widetilde{d}^{\varepsilon}_{a_\varepsilon t}(x) \right| \le \mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_{a_\varepsilon t}(x), \widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x)) + \mathrm{d}_{\mathrm{TV}}(\mu^{\varepsilon}, \widetilde{\mu}^{\varepsilon}) \quad \text{for all} \quad t \ge 0. \tag{3.7}$$

By Lemma 3.1 below and Eq. (C.2) from Lemma C.1 in Appendix C, we deduce that the right-hand side of (3.7) tends to zero as $\varepsilon \to 0$ and thereby conclude the proof of Lemma 2.3. $\square$

Lemma 3.1 (*Convergence of the Drift-Modified Process Close to the Origin*)**.** *For any $x \in \mathbb{R}$ and $t \ge 0$ the following limit holds*

$$\lim_{\varepsilon \to 0} \mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_{a_\varepsilon t}(x), \widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x)) = 0,$$

where $(a_\varepsilon, \varepsilon \in [0,1))$ is defined in (1.4).

Proof. The proof follows the steps given in the proof of Proposition 4.1, item (ii), of [31]. Recall the definition of $\widetilde{V}$ given in (2.10). In particular, note that $L = L_x$ is chosen such that $L^2 \ge |x|^2 + 1$. Let $\varepsilon \in (0,1]$ be fixed. The variational formulation of the total variation distance yields $\mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_{a_\varepsilon t}(x), \widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x)) \le \mathbb{Q}(X^{\varepsilon}_{a_\varepsilon t}(x) \ne \widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x))$ for any coupling $\mathbb{Q}$ of the random variables $X^{\varepsilon}_{a_\varepsilon t}(x)$ and $\widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x)$. Moreover, as $|x| < L$ for the synchronous coupling $\mathbb{P}$ (where processes are driven by the same noise), we have $\widetilde{X}^{\varepsilon}_s(x) = X^{\varepsilon}_s(x)$ for $0 \le s < \widetilde{\tau}^{\varepsilon}(x)$, where $\widetilde{\tau}^{\varepsilon}(x) := \inf\{s \ge 0 : |\widetilde{X}^{\varepsilon}_s(x)| > L\}$. Therefore,

$$\mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_{a_\varepsilon t}(x), \widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x)) \le \mathbb{P}(\widetilde{\tau}^{\varepsilon}(x) \le a_\varepsilon t) \quad \text{for any} \quad t \ge 0. \tag{3.8}$$

Note that

$$\mathbb{P}\left(\widetilde{\tau}^{\varepsilon}(x) \ge a_\varepsilon t\right) = \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{X}^{\varepsilon}_s(x)| \le L \right). \tag{3.9}$$

Since $\widetilde{V}$ is a smooth, convex, and even function, Itô's formula yields $\mathbb{P}$-almost surely that

$$\begin{aligned} |\widetilde{X}^{\varepsilon}_t(x)|^2 &= |x|^2 - 2\int_0^t \widetilde{X}^{\varepsilon}_s(x) \widetilde{V}'(\widetilde{X}^{\varepsilon}_s(x))\mathrm{d}s + \varepsilon t + \widetilde{M}^{\varepsilon}_t(x) \\ &\le |x|^2 + \varepsilon t + \widetilde{M}^{\varepsilon}_t(x) \quad \text{for all} \quad t \ge 0, \end{aligned} \tag{3.10}$$

where $\widetilde{M}^{\varepsilon}_t(x) := 2\sqrt{\varepsilon}\int_0^t \widetilde{X}^{\varepsilon}_s(x)\mathrm{d}B_s$, $t \ge 0$. By a localization procedure, it follows that

$$\mathbb{E}[|\widetilde{X}^{\varepsilon}_t(x)|^2] \le |x|^2 + \varepsilon t \quad \text{for all} \quad t \ge 0 \tag{3.11}$$

and hence $(\widetilde{M}^{\varepsilon}_t(x), t \ge 0)$ is a true martingale. By (2.9) we have $\varepsilon a_\varepsilon = \varepsilon^{\frac{2}{2+\alpha}}$, which tends to zero as $\varepsilon \to 0$. Then for any $t > 0$ fixed there exists $\varepsilon_0 = \varepsilon_0(t,\alpha) > 0$ such that $1 - \varepsilon a_\varepsilon t > 1/2$ for all $\varepsilon \in (0, \varepsilon_0)$. By (3.10) for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{X}^{\varepsilon}_s(x)| \ge L \right) &= \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{X}^{\varepsilon}_s(x)|^2 \ge L^2 \right) \le \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{M}^{\varepsilon}_s(x)| \ge L^2 - |x|^2 - \varepsilon a_\varepsilon t \right) \\ &\le \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{M}^{\varepsilon}_s(x)| \ge 1/2 \right) = \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{M}^{\varepsilon}_s(x)|^2 \ge 1/4 \right), \end{aligned}$$

where for the last inequality we used that $|x|^2 + 1 < L^2$ and $1 - \varepsilon a_\varepsilon t > 1/2$. Now, by Doob's submartingale inequality, Itô's isometry and (3.11) we have for all $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{X}^{\varepsilon}_s(x)| \ge L \right) &\le \mathbb{P}\left(\sup_{0 \le s \le a_\varepsilon t} |\widetilde{M}^{\varepsilon}_s(x)|^2 \ge 1/4 \right) \le 4\mathbb{E}[|\widetilde{M}^{\varepsilon}_{a_\varepsilon t}(x)|^2] \\ &= 16\varepsilon \int_0^{a_\varepsilon t} \mathbb{E}[|\widetilde{X}^{\varepsilon}_s(x)|^2]\mathrm{d}s \le 16|x|^2 \varepsilon a_\varepsilon t + 8\varepsilon^2 a_\varepsilon^2 t^2. \end{aligned} \tag{3.12}$$

By (3.8), (3.9) and (3.12) we deduce

$$\mathrm{d}_{\mathrm{TV}}(X^{\varepsilon}_{a_\varepsilon t}(x), \widetilde{X}^{\varepsilon}_{a_\varepsilon t}(x)) \le 16|x|^2 \varepsilon a_\varepsilon t + 8\varepsilon^2 a_\varepsilon^2 t^2 \quad \text{for any} \quad t \ge 0,$$

which implies the statement as $\varepsilon \to 0$. $\square$

3.3. Proof of Proposition 2.5

The proof of Proposition 2.5 is divided in three parts, one for each claim of the proposition. To ease the exposition, we only give here the main steps of the proofs and postpone the technical details to Appendix.

3.3.1. Continuous Markovian extension

The continuous Markovian extension of the SDE (2.2) is done in three steps. Their proofs are given in detail in Appendix A and here we only outline the main steps. First, based on a monotonic coupling and uniform moment bounds for $x \in \mathbb{R}$, SDE (2.2) can be extended to $\overline{\mathbb{R}}$, see Appendix A.1. Second, because $\pm\infty$ are exit boundaries for the dynamics in $\mathbb{R}$, the extended family $(Y(x), x \in \overline{\mathbb{R}})$ is Markovian, see Appendix A.2. Finally, in Appendix A.3 we show that the extension is continuous in the sense that

$$\lim_{x\to\pm\infty} \mathrm{d}_{\mathrm{TV}}\left(Y_t(x), Y_t(\pm\infty)\right) = 0. \tag{3.13}$$

3.3.2. Convergence for fixed marginal

In this section we show the limit (2.17). For simplicity, we consider only the case when the initial condition x in (1.1) is positive, the case when x is negative can be treated by an analogous argument, while the case $x = 0$ is easier as no scaling of the initial condition is required and (2.17) follows from the uniform convergence of the velocity fields, see (3.16) below. To ease notation and clarify the limit procedures, we denote by F_0, F_ε the velocity fields of (2.2) and (2.6), respectively. That is, for any $\varepsilon \in [0,1]$ and $z \in \mathbb{R}$ we define

$$F_0(z) := -C_0\,|z|^{1+\alpha}\,\mathrm{sgn}(z) \qquad \text{and} \qquad F_\varepsilon(z) := -\frac{a_\varepsilon}{b_\varepsilon} V'(b_\varepsilon z) = -\frac{V'(b_\varepsilon z)}{b_\varepsilon^{1+\alpha}}. \tag{3.14}$$

To ease notation, denote by $Y^0(x)$ the solution of (2.2). With this, $(Y^\varepsilon(x), \varepsilon \in [0,1])$ solves

$$\begin{cases} \mathrm{d}Y_t = F_\varepsilon(Y_t)\mathrm{d}t + \mathrm{d}B_t \quad \text{for} \quad t \geq 0, \\ Y_0 = x. \end{cases} \tag{3.15}$$

In what follows, we consider uniform bounds for $Y^\varepsilon(x)$ with $\varepsilon \in [0,1]$ and we will take the limit of such processes as $\varepsilon \to 0$. First, since $b_\varepsilon \to 0$ as $\varepsilon \to 0$, Hypothesis 1.2 yields for all $K > 0$

$$\lim_{\varepsilon\to 0} \sup_{|z|\leq K} \left|F_\varepsilon(z) - F_0(z)\right| = 0. \tag{3.16}$$

Also, by Proposition A.3 it follows that, almost surely, for any $\varepsilon \in [0,1]$, the limit

$$Y_t^\varepsilon(\infty) := \lim_{x\to\infty} Y_t^\varepsilon(x)$$

exists and is finite for $t > 0$. Next, by Lemma A.5 for all $t > 0$ and $\varepsilon \in [0,1]$,

$$\lim_{x\to\infty} \mathrm{d}_{\mathrm{TV}}\left(Y_t^\varepsilon(x), Y_t^\varepsilon(\infty)\right) = 0. \tag{3.17}$$

Now, we fix $\eta > 0$. By the uniform behavior at infinity, see Proposition B.1, it follows that for any $a > 0$, there are $b > 0$ and $\delta \in (0,t)$ such that

$$\sup_{\varepsilon\in[0,1]} \mathbb{P}\left(Y_\delta^\varepsilon(\infty) \notin [a,b]\right) \leq \eta/8. \tag{3.18}$$

By (3.17), we may choose $a > 0$ large enough so that

$$\sup_{x\geq a} \mathrm{d}_{\mathrm{TV}}\big(Y_t^0(x), Y_t^0(\infty)\big) \leq \eta/4. \tag{3.19}$$

Now, given a, b and $\delta \in (0,t)$ we claim that there is $\varepsilon_0 = \varepsilon(\eta) > 0$ for which

$$\sup_{0\leq\varepsilon\leq\varepsilon_0} \sup_{x\in[a,b]} \mathrm{d}_{\mathrm{TV}}\big(Y_{t-\delta}^0(x), Y_{t-\delta}^\varepsilon(x)\big) \leq \eta/4. \tag{3.20}$$

The proof of (3.20) is given in Appendix B.2.

Now, let $x_\varepsilon := x b_\varepsilon^{-1}$ and define μ_δ^ε to be the synchronous coupling (both SDEs are driven with the same noise) of $Y_\delta^0(\infty)$ and $Y_\delta^\varepsilon(x_\varepsilon)$. We write $\mu_\delta^\varepsilon(A,B) := \mathbb{P}(Y_\delta^0(\infty) \in A, Y_\delta^\varepsilon(x_\varepsilon) \in B)$ for any A, B Borelian subsets of $\mathbb{R}$. We may choose $a > 0$ for which (3.19) holds, then we choose $b > a$ and $\delta \in (0,t)$ such that (3.18) and (3.20) also hold true. With these choices, it follows that for any $\varepsilon \in [0,1]$

$$\mu_\delta^\varepsilon(\mathbb{R}^2 \setminus [a,b]^2) \leq \mathbb{P}\big(Y_\delta^0(\infty) \notin [a,b]\big) + \mathbb{P}\big(Y_\delta^\varepsilon(\infty) \notin [a,b]\big) \leq \eta/4.$$

The disintegration inequality, see Proposition D.2, and the triangle inequality for the total variation distance imply that for each $x > 0$ and $t > 0$ there is $\varepsilon_0 > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$ we choose $a > 0$, $b > a$ and $\delta \in (0,t)$ for which (3.18), (3.19), and (3.20) hold true and therefore

$$\begin{aligned}
\mathrm{d}_{\mathrm{TV}}\left(Y_t^0(\infty), Y_t^\varepsilon(x_\varepsilon)\right) &\leq \int_{\mathbb{R}^2} \mathrm{d}_{\mathrm{TV}}\left(Y_{t-\delta}^0(x), Y_{t-\delta}^\varepsilon(y)\right) \mu_\delta^\varepsilon(\mathrm{d}x,\mathrm{d}y) \\
&\leq \mu_\delta^\varepsilon(\mathbb{R}^2 \setminus [a,b]^2) + \int_{[a,b]^2} \mathrm{d}_{\mathrm{TV}}\left(Y_{t-\delta}^0(x), Y_{t-\delta}^\varepsilon(y)\right) \mu_\delta^\varepsilon(\mathrm{d}x,\mathrm{d}y) \\
&\leq \qquad \eta/4 \qquad + \int_{[a,b]^2} \mathrm{d}_{\mathrm{TV}}\left(Y_{t-\delta}^0(x), Y_{t-\delta}^0(\infty)\right) \mu_\delta^\varepsilon(\mathrm{d}x,\mathrm{d}y) \\
&\quad + \int_{[a,b]^2} \mathrm{d}_{\mathrm{TV}}\left(Y_{t-\delta}^0(\infty), Y_{t-\delta}^0(y)\right) \mu_\delta^\varepsilon(\mathrm{d}x,\mathrm{d}y) + \int_{[a,b]^2} \mathrm{d}_{\mathrm{TV}}\left(Y_{t-\delta}^0(y), Y_{t-\delta}^\varepsilon(y)\right) \mu_\delta^\varepsilon(\mathrm{d}x,\mathrm{d}y) \\
&\leq \eta/4 + \eta/4 + \eta/4 + \eta/4 = \eta.
\end{aligned}$$

Recall (2.5) and observe that $\mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), \mathcal{X}_t^{\varepsilon,x}\right) = \mathrm{d}_{\mathrm{TV}}\left(Y_t^0(\infty), Y_t^{\varepsilon}(x_\varepsilon)\right)$. Since $\eta > 0$ is arbitrary, the proof of (2.17) is complete. □

3.3.3. Convergence of invariant measures

Recall the notation introduced above (2.13), that is, $Y_\infty \stackrel{d}{=} \nu$ and $\widetilde{X}^{\varepsilon}_\infty \stackrel{d}{=} \widetilde{\mu}^{\varepsilon}$, where $\stackrel{d}{=}$ denotes equality in the distribution sense. By Lemma 2.1 it follows that

$$\nu(\mathrm{d}z) = C^{-1}\exp\big(-2V_0(z)\big)\mathrm{d}z, \tag{3.21}$$

where C is a normalization constant, $V_0(z) := (2+\alpha)^{-1}C_0|z|^{2+\alpha}$ with α and C_0 defined in Hypothesis 1.2. Similarly, $\widetilde{\mu}^{\varepsilon}$ is the density of $\widetilde{X}^{\varepsilon}_\infty$ and it is given by

$$\widetilde{\mu}^{\varepsilon}(\mathrm{d}z) = \widetilde{C}_\varepsilon^{-1}\exp\left(-2\frac{\widetilde{V}(z)}{\varepsilon}\right)\mathrm{d}z.$$

By the change of variable theorem, with $(b_\varepsilon, \varepsilon\in[0,1))$ as defined in (2.9), the density of $Y^{\varepsilon}_\infty = \frac{\widetilde{X}^{\varepsilon}_\infty}{b_\varepsilon}$ is given by

$$b_\varepsilon\widetilde{C}_\varepsilon^{-1}\exp\left(-2\frac{\widetilde{V}(b_\varepsilon z)}{\varepsilon}\right)\mathrm{d}z. \tag{3.22}$$

By (3.21), (3.22), and Scheffé's lemma ([39, Lemma 3.3.2, p.95]), to conclude the proof of (2.18), it suffices to show

$$\lim_{\varepsilon\to 0}\frac{b_\varepsilon}{\widetilde{C}_\varepsilon}e^{-2\frac{\widetilde{V}(b_\varepsilon z)}{\varepsilon}} = \frac{1}{C}e^{-2V_0(z)} \quad \text{for any} \quad z\in\mathbb{R}. \tag{3.23}$$

The proof of (3.23) is given in Lemmas C.2 and C.3 in Appendix C.

3.4. Strict inequalities for the rescaled process

In this section we show

$$0 < \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_\infty\right) < 1 \quad \text{for any} \quad t>0.$$

First, we prove the upper bound and then we show the lower bound.

The upper bound. We first note that for any $t>0$, $x\in\mathbb{R}$, the marginal $Y_t(x)$ has full support in $\mathbb{R}$, see Proposition D.3 in Appendix D for a proof. By Proposition 2.5 the family $(Y(x), x\in\overline{\mathbb{R}})$ is Markovian, and hence, by semigroup property, $Y_t(\infty)$ is equal in law to $Y_{t/2}(Y_{t/2}(\infty))$ for any $t>0$. Since $\mathbb{P}(Y_{t/2}(\infty)\in\mathbb{R}) = 1$ it follows by our previous discussion that $Y_t(\infty)$ with law ν_t possesses a continuous density $\rho_t : \mathbb{R}\to(0,\infty)$, that is $\nu_t(\mathrm{d}x) = \rho_t(x)\mathrm{d}x$. Furthermore, the invariant distribution of Y corresponding to the random variable Y_∞ has explicit density function $\rho : \mathbb{R}\to(0,\infty)$, which is given in (3.21). To conclude that $\mathrm{d}_{\mathrm{TV}}(Y_t(\infty), Y_\infty) < 1$ we note that

$$\mathrm{d}_{\mathrm{TV}}(Y_t(\infty), Y_\infty) = 1 - \int_{\mathbb{R}}\min\{\rho_t(z), \rho(z)\}\,\mathrm{d}z < 1. \tag{3.24}$$

The lower bound: injective evolution map. To prove the lower bound, we first define the evolution map on the space of measures. Let $\mathcal{P}$ be the space of probability measures on $\mathbb{R}$ that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and let $C_b(\mathbb{R})$ be the set of bounded continuous functions on $\mathbb{R}$. For each $\mu\in\mathcal{P}$ and $t\geq 0$ let $\varphi = \varphi(\mu, t)$ be the measure such that for every $f\in C_b(\mathbb{R})$

$$\int f(x)\mathrm{d}\varphi(x) := \int\mathbb{E}[f(Y_t(x))]\mathrm{d}\mu(x).$$

By Proposition D.3 we have that $\varphi(\mu,t)\in\mathcal{P}$ for all $\mu\in\mathcal{P}$ and $t>0$. For fixed time $t>0$, the evolution map is injective in the sense that

$$\text{if} \quad \mu,\mu'\in\mathcal{P} \quad \text{and} \quad \mu\neq\mu' \quad \text{then for all} \quad t\geq 0 \quad \varphi(\mu,t)\neq\varphi(\mu',t). \tag{3.25}$$

Moreover, since the dynamics is uniquely ergodic, see Lemma 2.1, for all $t>0$ the map $\mu\mapsto\varphi(\mu,t)$ admits a unique fixed point, that is, there is a unique $\nu\in\mathcal{P}$ such that

$$\varphi(\nu,t) = \nu \quad \text{for any} \quad t\geq 0. \tag{3.26}$$

Recall that we denote the law of Y_∞ by ν. By Propositions 2.5 and D.3 it follows that for all $\delta>0$ the law of $Y_\delta(\mathrm{sgn}(x)\infty)$ denoted by μ^∞_δ belongs to $\mathcal{P}$. By the Markov property of the extended process, see Proposition 2.5, we have for any $\delta\in(0,t)$

$$\mu^\infty_t = \varphi(\mu^\infty_\delta, t-\delta). \tag{3.27}$$

Let $t>0$ be fixed. We observe that there exists $\delta\in(0,t)$ such that $\mu^\infty_\delta\neq\nu$. By (3.25), (3.26) and (3.27) it follows that $\mu^\infty_t = \varphi(\mu^\infty_\delta, t-\delta)\neq\varphi(\nu, t-\delta) = \nu$ and hence

$$0 < \mathrm{d}_{\mathrm{TV}}\left(\mu^\infty_t, \nu\right) = \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), Y_\infty\right).$$

Ethical approval

Not applicable.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The continuous Markovian extension: details

In this section we prove that the SDEs defined in (2.2) and (2.6), or equivalently in (3.15), with state space $\mathbb{R}$ may be extended to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Furthermore, we show that this extension is Markovian and that the family of transition kernels associated to it is continuous with respect to the initial condition, in the sense of (3.13).

The main reason for this appendix is to provide a full proof of the continuous Markovian extension of SDE (3.15). The methods we employ here are of a probabilistic and path-wise nature offering an alternative to the classical analytical techniques of generators and resolvents presented in [89, p.366 ff], [90],[91, Chap. 17], [92]. More specifically, we apply martingale convergence methods and L^2-bounds, which can be found in [93, Chap. 5] together with standard methods for ODEs and SDEs which can be found in [94, Thm. 1] and [95, Thm. 1.1].

For the extension, we consider $\overline{\mathbb{R}}$ endowed with the Borel σ-algebra associated to the metric $d_\infty : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}_+$ defined by

$$d_\infty(x_1, x_2) := \left|\arctan(x_1) - \arctan(x_2)\right|,$$

where $\arctan : \overline{\mathbb{R}} \to \mathbb{R}$ is the continuous function defined by

$$\arctan(v) := \begin{cases} -\pi/2 & \text{for } v = -\infty, \\ \int_0^v \frac{1}{1+u^2}\,\mathrm{d}u & \text{for } v \in \mathbb{R}, \\ \pi/2 & \text{for } v = \infty. \end{cases}$$

Let $Y^\varepsilon(x) = (Y_t^\varepsilon(x), t \geq 0)$ be the unique strong solution of (3.15). For $x \in \mathbb{R}$ let P_x^ε be the law induced by $Y^\varepsilon(x)$ on the space of real valued continuous functions $(C, \mathcal{C})$ and let $\overline{P}_x^\varepsilon$ be its law on the space of $\overline{\mathbb{R}}$-valued continuous functions $(\overline{C}, \overline{\mathcal{C}})$. To complete the extension we define $\overline{P}_x^\varepsilon$ for $x \in \{-\infty, \infty\}$ as the law on $(\overline{C}, \overline{\mathcal{C}})$ induced by $Y^\varepsilon(\infty)$ and $Y^\varepsilon(-\infty)$ where for all $t \geq 0$

$$Y_t^\varepsilon(\infty) := \lim_{x\to\infty} Y_t^\varepsilon(x) \qquad \text{and} \qquad Y_t^\varepsilon(-\infty) := \lim_{x\to-\infty} Y_t^\varepsilon(x). \tag{A.1}$$

The above extension is well-defined since, by the comparison lemma for SDEs, see [95, Thm. 1.1],

$$x \leq x' \Rightarrow \mathbb{P}(Y_t^\varepsilon(x) \leq Y_t^\varepsilon(x') \quad \forall t \geq 0) = 1. \tag{A.2}$$

This section is divided into three subsections. In Appendix A.1 we prove trajectory properties of the above extension. In Appendix A.2 we prove that the extension is Markovian. Finally, in Appendix A.3 we prove that the extension is continuous with respect to the initial condition.

A.1. Coming down from infinity

We next explain when a solution to an SDE *comes down from infinity*. This is based on entrance conditions at the boundary for ODEs. In fact, as we shall see in Lemma A.2, for all $t > 0$, $\varepsilon \in [0,1]$, the family $(Y_t^{\varepsilon}(x), x \in \mathbb{R})$ satisfies a uniform L^2 bound and so are a.s. finite for all positive times. This section is organized as follows: First, we prove an entrance condition for ODEs. Then we show the uniform bounds in L^2. Finally, we define what is meant by the integral form of the solution when the initial condition is $\pm\infty$. We include an explanation of these standard techniques for completeness and to prepare for specific results we will need.

A.1.1. Entrance condition for ODEs

Let

$$\mathfrak{L} := \left\{ G : \mathbb{R} \to \mathbb{R} |\ G \text{ is locally Lipschitz and } -\infty < \int_R^{\infty} \frac{1}{G(u)} \mathrm{d}u < 0 \text{ for some } R > 0 \right\} \tag{A.3}$$

be the space of velocity fields in which we are interested in.

Lemma A.1 (*Descent from Infinity*). *Given any fixed $G \in \mathfrak{L}$ and any $x \in \mathbb{R}$, let $\psi(x) := (\psi_t(x), t \geq 0)$ be the unique solution of the differential equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t = G(\psi_t) & \text{for} \quad t \geq 0, \\ \psi_0 = x. \end{cases}$$

Then for all $t > 0$, the limit $\psi_t(\infty) := \lim_{x\to\infty} \psi_t(x)$ is well-defined and finite.

Proof. By the comparison lemma for ODEs we have

$$\forall\, t \geq 0,\ x_1, x_2 \in \mathbb{R},\ x_1 \leq x_2 \Rightarrow \psi_t(x_1) \leq \psi_t(x_2). \tag{A.4}$$

Therefore, the limit $\psi_t(\infty) := \lim_{x\to\infty} \psi_t(x)$ is well-defined but may be infinite. In the sequel, we show that $\psi_t(\infty) < \infty$ for any $t > 0$. Since G is locally Lipschitz and satisfies (A.3), $G(x) < 0$ for all $x \geq R$. Fix $T > 0$ and let $L := \psi_T(R)$. By uniqueness of solutions, the map $t \mapsto \psi_t(R)$ is decreasing and for all $x \geq L$ $G(x) < 0$. Let $F_{L,R} : [L, R] \to [0, T]$ be such that $F_{L,R}(\psi_t(R)) = t$ for all $t \in [0, T]$ and note that $F'_{L,R}(u) = 1/G(u)$. Since $F_{L,R}(R) = 0$ and $F_{L,R}(L) = T$, we obtain

$$-T = \int_L^R F'_{L,R}(u)\mathrm{d}u = \int_L^R \frac{1}{G(u)}\mathrm{d}u. \tag{A.5}$$

Now let $F_L : [L, \infty] \to [F_L(\infty), 0]$ be given by $F_L(x) := \int_L^x \frac{1}{G(u)}\mathrm{d}u$ for $x \in [L, \infty)$, and set $F_L(\infty) := \lim_{x\to\infty} F_L(x)$. By (A.3) and (A.5), $F_L(\infty) \in (-\infty, 0)$. By (A.4), for any $x \in [R, \infty)$, $t \leq T$, we have $\psi_t(x) \geq L$ and $\frac{\mathrm{d}}{\mathrm{d}t} F_L(\psi_t(x)) = 1$. We thereby, conclude that

$$F_L(\psi_t(x)) = t + F_L(x), \quad t \in [0, T]. \tag{A.6}$$

Again by (A.4) and the continuity of F_L, we can take the limit $x \to \infty$ in (A.6) to obtain

$$F_L(\psi_t(\infty)) = t + F_L(\infty), \quad t \in [0, T]. \tag{A.7}$$

Therefore $\psi_t(\infty) < \infty$ for any $t > 0$. $\square$

We may now define the *extended ODE*. By Lemma A.1 $\psi(\infty) := (\psi_t(\infty), t \geq 0)$ solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t = G(\psi_t) & \text{for} \quad t > 0, \\ \psi_0 = \infty, \end{cases} \tag{A.8}$$

in the sense that $\psi_0(\infty) = \infty$, and for any $t_0 > 0$, the following integral relation holds

$$\psi_t(\infty) = \psi_{t_0}(\infty) + \int_{t_0}^t G(\psi_s(\infty))\mathrm{d}s \quad \text{for all} \quad t \geq t_0. \tag{A.9}$$

Eq. (A.9) is a consequence of the Fundamental Theorem of Calculus. Indeed, by (A.7), for any $s \geq t_0$, $\psi_s(\infty) := F_L^{-1}(s + F_L(\infty)) \in \mathbb{R}$ and taking derivatives on both sides of (A.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi_s(\infty) = \frac{1}{F'_L(\psi_s(\infty))} = G(\psi_s(\infty)). \quad \square$$

A.1.2. Uniform L^2 bounds

In what follows, we prove a second moment bound for any fixed $t > 0$ for $(Y_t^{\varepsilon}(x), x \in \mathbb{R})$.

Lemma A.2 (L^2-*Bound*). *If $\varepsilon \in [0,1]$, then the process $Y^{\varepsilon}(x)$ defined by (3.15) satisfies for any $t > 0$,*

$$\sup_{x\in\mathbb{R}} \mathbb{E}[|Y_t^{\varepsilon}(x)|^2] < \infty. \tag{A.10}$$

Proof. Fix $\varepsilon \in [0,1]$ and let $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ be given by $G_\varepsilon(y) := yF_\varepsilon(y)$ for all $y \in \mathbb{R}$ with F_ε defined in (3.14). Recall that V satisfies Hypotheses 1.1, 1.2 and 2.2. Since V' is an odd function, it follows that $G_\varepsilon(y) = G_\varepsilon(|y|) \le 0$ for all $y \in \mathbb{R}$. By Hypothesis 1.2 and Hypothesis 2.2 (Condition (G1)) there is $c_* > 0$ for which

$$G_\varepsilon(y) \le -c_*|y|^{2+\alpha} \quad \text{for all} \quad y \in \mathbb{R}. \tag{A.11}$$

By Itô's formula, for $t > 0$, we have

$$|Y_t^\varepsilon(x)|^2 = x^2 + 2\int_0^t G_\varepsilon(Y_s^\varepsilon(x))\mathrm{d}s + t + M_t^x, \tag{A.12}$$

where $M^x = (M_t^x, t \ge 0)$ is a local martingale given by

$$M_t^x = 2\int_0^t Y_s^\varepsilon(x)\mathrm{d}B_s. \tag{A.13}$$

Since $G_\varepsilon(y) \le 0$ for all $y \in \mathbb{R}$, a localization argument yields that, $\mathbb{E}[|Y_t^\varepsilon(x)|^2] \le x^2 + t$ for any $t \ge 0$. As a consequence, we have that M^x is a true mean-zero martingale. Now, if we take expectation on both sides of equality (A.12), apply Fubini's theorem and use (A.11) we obtain for all $t > 0$

$$\mathbb{E}[|Y_t^\varepsilon(x)|^2] = x^2 + 2\int_0^t \mathbb{E}[G_\varepsilon\left(Y_t^\varepsilon(x)\right)]\mathrm{d}s + t \le x^2 - 2c_*\int_0^t \mathbb{E}[|Y_s^\varepsilon(x)|^{2+\alpha}]\mathrm{d}s + t. \tag{A.14}$$

By Jensen's inequality we obtain

$$\mathbb{E}[|Y_t^\varepsilon(x)|^{2+\alpha}] \ge (\mathbb{E}[|Y_t^\varepsilon(x)|^2])^{1+\alpha/2} \quad \text{for all} \quad t \ge 0. \tag{A.15}$$

By (A.14) and (A.15) if we denote $\psi_t^\varepsilon(x) := \mathbb{E}[|Y_t^\varepsilon(x)|^2]$ and let $\widetilde{G}(y) := -2c_*|y|^{1+\alpha/2} + 1$ for all $y \in \mathbb{R}$ then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^\varepsilon(x) \le \widetilde{G}(\psi_t^\varepsilon(x)) \quad \text{for} \quad t \ge 0.$$

Now, we let $(\widetilde{\psi}_t(x), t \ge 0)$ be the solution of (A.8) for $G = \widetilde{G}$ and with initial condition $\widetilde{\psi}_0(x) = \psi_0^\varepsilon(x) = x^2$. Observe that $\widetilde{G} \in \mathfrak{L}$, where $\mathfrak{L}$ is defined in (A.3). To conclude (A.10), we rely on monotonicity and Lemma A.1. Indeed, for any $x \in \mathbb{R}$ and $t > 0$

$$\mathbb{E}\left[|Y_t^\varepsilon(x)|^2\right] \le \widetilde{\psi}_t(x) \le \lim_{z\to\infty} \widetilde{\psi}_t(z) = \widetilde{\psi}_t(\infty) < \infty. \quad \square \tag{A.16}$$

A.1.3. Integral expression

Now, we examine the integral form of the limit process.

Proposition A.3 (*Integral Form*). *For any fixed $\varepsilon \in [0,1]$, let F_ε be as defined in (3.14). Then, the limit process $Y_t^\varepsilon(\mathrm{sgn}(x)\infty) := \lim_{r\to\infty} Y^\varepsilon(\mathrm{sgn}(x)\cdot r)$ solves*

$$\begin{cases} \mathrm{d}Y_t = F_\varepsilon(Y_t)\mathrm{d}t + \mathrm{d}B_t & \text{for} \quad t > 0, \\ Y_0 = \mathrm{sgn}(x)\infty, \end{cases}$$

in the sense that almost surely $\lim_{t\to 0} Y_t = \mathrm{sgn}(x)\infty$ and for any $0 < t_0 < t$

$$Y_t = Y_{t_0} + \int_{t_0}^t F_\varepsilon(Y_s)\,\mathrm{d}s + B_t - B_{t_0}. \tag{A.17}$$

Proof. Assume without loss of generality that $x > 0$. By (A.2) $Y_t^\varepsilon(r)$ increases with r and therefore the limit $Y_t^\varepsilon(\infty)$ exists. By (A.10) it follows that $\mathbb{P}(Y_t^\varepsilon(\infty) < \infty) = 1$ for any $t > 0$. Given $T > t_0 > 0$, we claim that, for every $\delta > 0$

$$\lim_{r\to\infty} \mathbb{P}\Big(\sup_{t\in[t_0,T]} \left|F_\varepsilon(Y_t^\varepsilon(\infty)) - F_\varepsilon(Y_t^\varepsilon(r))\right| > \delta\Big) = 0. \tag{A.18}$$

The proof of (A.18) is postponed to Lemma A.4 below. Now, note that, almost surely

$$Y_t^\varepsilon(r) = Y_{t_0}^\varepsilon(r) + \int_{t_0}^t F_\varepsilon(Y_s^\varepsilon(r))\,\mathrm{d}s + B_t - B_{t_0}.$$

By (A.18) we may take the limit inside the above integral and therefore

$$\mathbb{P}\Big(Y_t^\varepsilon(\infty) = Y_{t_0}^\varepsilon(\infty) + \int_{t_0}^t F_\varepsilon(Y_s^\varepsilon(\infty))\,\mathrm{d}s + B_t - B_{t_0} \quad \forall\, t > t_0\Big) = 1. \quad \square$$

Lemma A.4. *For any $T > t_0 > 0$ and $\delta > 0$ the equality in (A.18) holds true.*

Proof. We first note that (A.18) is a consequence of

$$\lim_{A\to\infty} \mathbb{P}\left(\sup_{t\in[t_0,T]} \left|Y_t^\varepsilon(\infty)\right| > A\right) = 0 \tag{A.19}$$

and

$$\forall\delta > 0 \quad \limsup_{r\to\infty} \mathbb{P}\left(\sup_{t\in[t_0,T]} \left|Y_t^\varepsilon(\infty) - Y_t^\varepsilon(r)\right| > \delta\right) = 0. \tag{A.20}$$

Indeed, as F_ε is locally Lipschitz, for any $\delta > 0$ and $A > 0$ there is $\delta' = \delta'(\delta, A, \varepsilon) > 0$ for which

$$\mathbb{P}\Big(\sup_{t\in[t_0,T]} \big|F_\varepsilon(Y_t^\varepsilon(\infty)) - F_\varepsilon(Y_t^\varepsilon(r))\big| > \delta\Big) \le \mathbb{P}\left(\sup_{t\in[t_0,T]} \big|Y_t^\varepsilon(\infty) - Y_t^\varepsilon(r)\big| > \delta'\right) + \mathbb{P}\left(\sup_{t\in[t_0,T]} \big|Y_t^\varepsilon(\infty)\big| > A\right).$$

By monotonicity and Lemma A.2, $\lim_{r\to\infty} Y_t^\varepsilon(r) = Y_t^\varepsilon(\infty) \in \mathbb{R}$, for any $t > 0$. The pointwise limit, does not guarantee (A.19) and (A.20). In order to obtain the above uniform bounds we will show that the family $(Y^\varepsilon(r), r \ge 0)$ is tight in the space of continuous paths C. Tightness in C and pointwise convergence imply uniform convergence of the family and the bounds (A.19) and (A.20). By Aldous' tightness criterion, see [96, Thm. 16.10, p.178] or [97, Thm. 4.1.3, p.51] we only need to show that

$$\forall\, t \in [t_0, T] \quad \lim_{A\to\infty} \sup_{r\in\mathbb{R}} \mathbb{P}\left(\big|Y_t^\varepsilon(r)\big| > A\right) = 0, \tag{A.21}$$

and that

$$\forall \eta > 0 \quad \lim_{\delta\to 0} \sup_{r\in\mathbb{R}} \mathbb{P}\left(\sup_{|t-s|<\delta} \big|Y_t^\varepsilon(r) - Y_s^\varepsilon(r)\big| > \eta\right) = 0. \tag{A.22}$$

Proof of (A.21). By Lemma A.2 we have that

$$C_t^\varepsilon := \sup_{r\in\mathbb{R}} \mathbb{E}[\big|Y_t^\varepsilon(r)\big|^2] < \infty. \tag{A.23}$$

Therefore, by Chebyshev's inequality, for any $t \in [t_0, T]$ it follows that

$$\sup_{r\in\mathbb{R}} \mathbb{P}\left(\big|Y_t^\varepsilon(r)\big| > A\right) \le \sup_{r\in\mathbb{R}} \frac{\mathbb{E}[\big|Y_t^\varepsilon(r)\big|^2]}{A^2} \le \frac{C_t^\varepsilon}{A^2} \to 0 \quad \text{as} \quad A \to \infty.$$

Proof of (A.22). We first write $Y_t^\varepsilon(r) - Y_s^\varepsilon(r) = \int_s^t F_\varepsilon(Y_u^\varepsilon(r))\,\mathrm{d}u + B_t - B_s$. By the triangle inequality and the continuity of Brownian motion, to verify (A.22) it suffices to prove that for any $\eta > 0$

$$\lim_{\delta\to 0} \sup_{r\in\mathbb{R}} \mathbb{P}\Big(\sup_{|t-s|\le\delta} \int_s^t F_\varepsilon(Y_u^\varepsilon(r))\,\mathrm{d}u > \eta\Big) = 0.$$

Fix $K > 0$, $\eta > 0$, and let $A_K^r := \{\sup_{u\in[t_0,T]} |Y_u^\varepsilon(r)| > K\}$. Now note that there is $\delta = \delta(K, \eta, \varepsilon)$ such that $\delta \sup_{|y|\le K} \big|F_\varepsilon(y)\big| \le \eta$ and therefore for any $K > 0$, $\eta > 0$

$$\lim_{\delta\to 0} \sup_{r\in\mathbb{R}} \mathbb{P}\Big(\sup_{|t-s|\le\delta} \int_s^t F_\varepsilon(Y_u^\varepsilon(r))\,\mathrm{d}u > \eta\Big) \le \sup_{r\in\mathbb{R}} \mathbb{P}\left(A_K^r\right).$$

Since the left-hand side of the above inequality does not depend on K, we obtain that

$$\lim_{\delta\to 0} \sup_{r\in\mathbb{R}} \mathbb{P}\Big(\sup_{|t-s|\le\delta} \int_s^t F_\varepsilon(Y_u^\varepsilon(r))\,\mathrm{d}u > \eta\Big) \le \lim_{K\to\infty} \sup_{r\in\mathbb{R}} \mathbb{P}(A_K^r).$$

Hence, to obtain (A.22) it is enough to prove that $\lim_{K\to\infty} \sup_{r\in\mathbb{R}} \mathbb{P}(A_K^r) = 0$. By (A.12) and (A.13), since $G_\varepsilon(y) = yF_\varepsilon(y) \le 0$ for all $y \in \mathbb{R}$ we have that

$$\sup_{t\in[t_0,T]} |Y_t^\varepsilon(r)|^2 \le |Y_{t_0}^\varepsilon(r)|^2 + T + \sup_{t\in[t_0,T]} \left|\int_{t_0}^t 2Y_s^\varepsilon(r)\mathrm{d}B_s\right|.$$

The estimate on A_K^r then becomes

$$\mathbb{P}\left(A_K^r\right) \le \mathbb{P}\left(|Y_{t_0}^\varepsilon(r)|^2 + T > K^2/2\right) + \mathbb{P}\Big(\sup_{t\in[t_0,T]} \left|\int_{t_0}^t 2Y_s^\varepsilon(r)\mathrm{d}B_s\right| > K^2/2\Big).$$

First, note that for $K^2/2 > 2T$ and by (A.23) we have

$$\mathbb{P}\left(|Y_{t_0}^\varepsilon(r)|^2 + T > K^2/2\right) \le \mathbb{P}\left(|Y_{t_0}^\varepsilon(r)|^2 > K^2/4\right) \le 4\frac{C_{t_0}^\varepsilon}{K^2}.$$

To conclude, note that by Doob's submartingale inequality, Itô's isometry and (A.23) it follows that

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[t_0,T]} \left|\int_{t_0}^t 2Y_s^\varepsilon(r)\mathrm{d}B_s\right| > K^2/2\Big) &\le \frac{\int_{t_0}^T 4\mathbb{E}[|Y_s^\varepsilon(r)|^2]\mathrm{d}s}{K^4/4} \le \frac{\int_{t_0}^T 16C_s^\varepsilon\,\mathrm{d}s}{K^4} \\ &\le \frac{16T \sup_{s\in[t_0,T]} C_s^\varepsilon}{K^4} \to 0 \quad \text{as} \quad K \to \infty,\end{aligned}$$

where for the last passage we note that by (A.16), $s \mapsto C_s^\varepsilon$ is bounded by a continuous function in $(0, \infty)$. □

A.2. Markov property of the extended family

To prove that the extended family obtained by (A.1) is Markovian, one needs to verify the conditions stated in Theorem 5.16 in [93, Ch 2, p.78]. These are the (i) compatible initial values, (ii) the measurability of the transition laws, and (iii) Markov property.

Note that for all $x \in \mathbb{R}$, $\varepsilon > 0$, and $t > 0$, $\mathbb{P}(Y_t^\varepsilon(x) \in \{\pm\infty\}) = 0$ so conditions (i)–(iii) are satisfied for all finite initial values. It only remains to see that (i)–(iii) is satisfied for initial values $x \in \{-\infty, \infty\}$. We only consider $x = \infty$, the case $x = -\infty$ is analogous.

For (i), note that

$$\mathbb{P}(Y_0^\varepsilon(\infty)=\infty)=\lim_{R\to\infty}\mathbb{P}(Y_0^\varepsilon(\infty)>R)=\lim_{R\to\infty}\lim_{x\to\infty}\mathbb{P}(Y_0^\varepsilon(x)>R)=1.$$

For (ii), Proposition A.3 implies that for any $t_0>0$ and $T>0$, the process $Y^\varepsilon(n)$ converges uniformly on the interval $[t_0,T]$ to $Y^\varepsilon(\infty)$ as $n\to\infty$. Therefore by the monotonicity in (A.2) and the continuity of probability, for all $k\in\mathbb{N}$, $t_1<\cdots<t_k$, and $a_1,\ldots,a_k\in\mathbb{R}$ one has

$$\liminf_{n\to\infty}\mathbb{P}(Y_{t_1}^\varepsilon(n)>a_1,\ldots,Y_{t_k}^\varepsilon(n)>a_k)=\mathbb{P}(Y_{t_1}^\varepsilon(\infty)>a_1,\ldots,Y_{t_k}^\varepsilon(\infty)>a_k).$$

The measurability follows by extending the above using Dynkin's π-λ theorem.

Condition (iii) is a consequence of (A.17) in Proposition A.3. Indeed, for any fixed $s>0$, if we let $(W_t:=B_{t+s}-B_s,t\geq 0)$ we have that almost surely for any $t>0$

$$Y_{t+s}^\varepsilon(\infty)=Y_s^\varepsilon(\infty)+\int_0^t F_\varepsilon(Y_{u+s}^\varepsilon(\infty))\,\mathrm{d}u+W_t.$$

Now, if we let $\mathbb{Y}_t:=Y_{t+s}^\varepsilon(\infty)$, it follows that $(\mathbb{Y}_t,t\geq 0)$ solves the SDE (3.15) with initial condition $Y_s^\varepsilon(\infty)$. Furthermore, by Theorem 3.5 in [37, p. 58], Eq. (3.15) is well-posed in $\mathbb{R}$ and, by Lemma A.2, $Y_s^\varepsilon(\infty)\in\mathbb{R}$ almost surely for any $s>0$. Therefore, with the help of Theorem 9.1 in [37, p. 86] we conclude that

$$\mathbb{P}(Y_{t+s}^\varepsilon(\infty)\in A|Y_s^\varepsilon(\infty)=y)=\mathbb{P}(Y_t^\varepsilon(y)\in A),$$

which yields (iii) and concludes that the extended family is Markovian.

A.3. Continuity at infinity

Let $\varepsilon\in[0,1]$ be fixed. In this section we prove that the map $x\mapsto Y_t^\varepsilon(x)$ is continuous with respect to the total variation distance for any $t>0$. We first note that the map above is continuous in $\mathbb{R}$, see Theorem 1.3 in [98] and Theorem 1.1 in [99] for a proof. Therefore, it only remains to verify the continuity at infinity. This is the content of the following lemma.

Lemma A.5 (*Continuity in Total Variation at Infinity*). *For any $\varepsilon\in[0,1]$ and any $t>0$ it follows that*

$$\lim_{x\to\infty}\mathrm{d}_{\mathrm{TV}}\left(Y_t^\varepsilon(x),Y_t^\varepsilon(\infty)\right)=0\qquad\textit{and}\qquad\lim_{x\to-\infty}\mathrm{d}_{\mathrm{TV}}\left(Y_t^\varepsilon(x),Y_t^\varepsilon(-\infty)\right)=0.$$

Proof. We only prove the case for which $x\to+\infty$. The case when $x\to-\infty$ follows from the symmetry of V^ε. Let $\mu_t^{\varepsilon,x}$ be the measure in $\mathbb{R}^2$ defined by

$$\mu_t^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)=\mathbb{P}(Y_t^\varepsilon(x)\in\mathrm{d}z_1,Y_t^\varepsilon(\infty)\in\mathrm{d}z_2).$$

For $s\in(0,t)$ we define $f:\mathbb{R}^2\to[0,1]$ by $f(z_1,z_2):=\mathrm{d}_{\mathrm{TV}}(Y_{t-s}^\varepsilon(z_1),Y_{t-s}^\varepsilon(z_2))$. By the Markovian property of the extended family $\left(Y_\cdot^\varepsilon(x),x\in\overline{\mathbb{R}}\right)$ and Proposition D.2 for any $K>0$ we have that

$$\begin{aligned}\mathrm{d}_{\mathrm{TV}}\left(Y_t^\varepsilon(x),Y_t^\varepsilon(\infty)\right)&\leq\int_{\mathbb{R}^2}f(z_1,z_2)\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)\\&\leq\int_{|z_1|,|z_2|\leq K}f(z_1,z_2)\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)+\mathbb{P}(|Y_t^\varepsilon(x)|>K)+\mathbb{P}(|Y_t^\varepsilon(\infty)|>K).\end{aligned}$$

By Lemma A.2 and Chebyshev's inequality it follows that

$$\limsup_{K\to\infty}\limsup_{x\to\infty}\left(\mathbb{P}(|Y_t^\varepsilon(x)|>K)+\mathbb{P}(|Y_t^\varepsilon(\infty)|>K)\right)=0.$$

It suffices to show that for any $K>0$

$$\limsup_{x\to\infty}\int_{|z_1|,|z_2|\leq K}f(z_1,z_2)\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)=0.\tag{A.24}$$

We now define for any $\delta>0$

$$\omega_{f,K}(\delta):=\max\{f(z_1,z_2):\,|z_1|,|z_2|\leq K,|z_1-z_2|\leq\delta\}.$$

Since f is continuous and $f(z,z)=0$, it follows that

$$\omega_{f,K}(\delta)<\infty\qquad\text{and}\qquad\lim_{\delta\to 0}\omega_{f,K}(\delta)=0.\tag{A.25}$$

Given $\delta>0$, consider the following split of the integral in (A.24),

$$\begin{aligned}\int_{|z_1|,|z_2|\leq K}f(z_1,z_2)\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)&=\int_{|z_1|,|z_2|\leq K}f(z_1,z_2)\,\mathbb{1}_{|z_1-z_2|\leq\delta}\,\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)+\int_{|z_1|,|z_2|\leq K}f(z_1,z_2)\,\mathbb{1}_{|z_1-z_2|>\delta}\,\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2)\\&\leq\omega_{f,K}(\delta)+\eta(x,\delta),\end{aligned}$$

where $\eta(x,\delta):=\mu_s^{\varepsilon,x}(|z_1-z_2|>\delta)=\mathbb{P}(|Y_s^\varepsilon(x)-Y_s^\varepsilon(\infty)|>\delta)$. By (A.1),

$$\lim_{x\to\infty} \eta(x,\delta) = 0 \quad \text{for any } \delta > 0.$$

To conclude the proof of Lemma A.5 we note that, by (A.25)

$$\limsup_{x\to\infty} \int_{|z_1|,|z_2|\le K} f(z_1,z_2)\mu_s^{\varepsilon,x}(\mathrm{d}z_1,\mathrm{d}z_2) \le \inf_{\delta>0} \omega_{f,K}(\delta) = 0. \qquad \square$$

Appendix B. Uniform bounds

In this section we prove the bounds (3.18) and (3.20).

B.1. Uniform entrance in a compact

The bound (3.18) is a consequence of the following proposition.

Proposition B.1. *For any $\eta > 0$ and $a > 0$ there are $b > 0$ and $\delta \in (0,\eta)$ such that*

$$\sup_{\varepsilon\in[0,1]} \mathbb{P}\left(Y_\delta^\varepsilon(\infty) \notin [a,b]\right) \le \eta.$$

The proof of Proposition B.1 is based on the two following statements, whose proofs are given afterwards.

$$\text{For any } a > 0, \quad \lim_{\delta\to 0} \sup_{\varepsilon\in[0,1]} \mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| \le a) = 0. \tag{B.1}$$

$$\text{For any } \delta > 0, \quad \lim_{b\to\infty} \sup_{\varepsilon\in[0,1]} \mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| > b) = 0. \tag{B.2}$$

Proof of Proposition B.1. Given $\eta > 0$ and $a > 0$, by (B.1) there is $\delta \in (0,\eta)$ such that $\mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| < a) < \eta/2$ for any $\varepsilon \in [0,1]$. Next, by (B.2), we choose $b > 0$ such that $\mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| > b) < \eta/2$ for any $\varepsilon \in [0,1]$. With this, we conclude that for every $\varepsilon \in [0,1]$

$$\mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| \notin [a,b]) = \mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| > b) + \mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| < a) < \eta. \qquad \square$$

In what follows we prove (B.1) and (B.2).

Proof of (B.1). Fix any $a > 0$. For $D > 0$, let $\Omega(\delta,D) := \{\sup_{t\le\delta} |B_t| \le D\}$. Fix $x := 2(a+D)$ and choose $K := 2x$. Now let $\sigma := \tau(a) \wedge \tau(K)$ where $\tau(v) = \tau(a,x,\varepsilon) := \inf\{t > 0 : Y_t^\varepsilon(x) = v\}$ for $v \in \mathbb{R}$. Note that $Y_\delta^\varepsilon(\infty) \ge Y_\delta^\varepsilon(x)$, and that almost surely

$$Y_{\delta\wedge\sigma}^\varepsilon(x) = x - \int_0^{\delta\wedge\sigma} \left|F_\varepsilon(Y_s^\varepsilon(x))\right| \mathrm{d}s + B_{\delta\wedge\sigma}. \tag{B.3}$$

By Hypothesis 1.2 and 3.14 it follows that

$$C(K) := \sup_{\varepsilon\in[0,1]} \sup_{|y|\le K} \left|F_\varepsilon(y)\right| = \sup_{\varepsilon\in[0,1]} \sup_{|y|\le K} \left|\frac{V'(b_\varepsilon y)}{b_\varepsilon^{1+\alpha}}\right| < \infty. \tag{B.4}$$

Given $D > 0$, there is $\eta > 0$ such that $\delta \in (0,\eta)$, implies $\delta < \sigma$ on $\Omega(\delta,D)$. Indeed, by (B.4) there is $C = C(D) > 0$ which allows (B.3) to be bounded by

$$Y_{\delta\wedge\sigma}^\varepsilon(x) \ge x - C(\delta\wedge\sigma) - D \ge x - D - C\delta \qquad \text{and} \qquad Y_{\delta\wedge\sigma}^\varepsilon(x) \le x + D < K.$$

Since $x - D > 2a$, for any $\delta \in (0, C^{-1}a)$ it follows that $Y_{\delta\wedge\sigma}^\varepsilon(x) \in (a,K)$. In conclusion, on the event $\Omega(\delta,D)$, we have that $Y_\delta^\varepsilon(\infty) \ge Y_\delta^\varepsilon(x) = Y_{\delta\wedge\sigma}^\varepsilon(x) > a$. Since $\mathbb{P}(\Omega(\delta,D)) \to 1$ as $\delta \to 0$, the proof of (B.1) is complete. $\square$

Proof of (B.2). We start the proof with uniform L^2 bounds, that is, we prove that for any $t > 0$

$$\sup_{\varepsilon\in[0,1]} \mathbb{E}[|Y_t^\varepsilon(x)|^2] < \infty.$$

For any $x \in \mathbb{R}$ and $t \ge 0$, inequality (A.16) yields

$$\mathbb{E}\left[|Y_t^\varepsilon(x)|^2\right] \le \widetilde{\psi}_t(x),$$

where $(\widetilde{\psi}_t(x), t \ge 0)$ is the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = \widetilde{G}(\psi_t(x)), \\ \psi_0(x) = x \end{cases}$$

with $\widetilde{G}(y) := -2c_*|y|^{1+\alpha/2} + 1$ for all $y \in \mathbb{R}$ for some $c_* > 0$. The monotone convergence theorem with the help of Lemma A.1 implies

$$\mathbb{E}\left[|Y_t^\varepsilon(\infty)|^2\right] \le \widetilde{\psi}_t(\infty) < \infty. \tag{B.5}$$

To conclude (B.2) note that (B.5) yields,

$$\sup_{\varepsilon\in[0,1]} \mathbb{P}(\left|Y_\delta^\varepsilon(\infty)\right| > b) \le \frac{\widetilde{\psi}_t(\infty)}{b^2} \to 0 \quad \text{as} \quad b \to \infty. \qquad \square$$

B.2. Uniform convergence in total variation distance

The bound (3.20) is a consequence of the following proposition.

Proposition B.2. *For any $t>0$, $a>0$, $b>a$ and $\eta>0$ there is $\varepsilon_0>0$ for which*

$$\sup_{0\le\varepsilon<\varepsilon_0}\sup_{x\in[a,b]} \mathrm{d}_{\mathrm{TV}}\left(Y_t^0(x),Y_t^\varepsilon(x)\right)<\eta. \tag{B.6}$$

Proof. By Theorem 5.1 in [100], we have

$$\left|\mathrm{d}_{\mathrm{TV}}\left(Y_t^0(x),Y_t^\varepsilon(x)\right)\right|^2\le 2\int_0^t \mathbb{E}\left[\left|F_0(Y_s^0(x))-F_\varepsilon(Y_s^\varepsilon(x))\right|^2\right]\mathrm{d}s.$$

To conclude (B.6) we show that

$$\lim_{\varepsilon_0\to0}\sup_{0\le\varepsilon<\varepsilon_0}\sup_{x\in[a,b]}\sup_{s\in[0,t]}\mathbb{E}\left[\left|F_0(Y_s^0(x))-F_\varepsilon(Y_s^\varepsilon(x))\right|^2\right]=0. \tag{B.7}$$

First, we define the event

$$A_{M,\varepsilon}=A_{M,\varepsilon}(x,t):=\left\{\sup_{s\in[0,t]}|Y_s^0(x)|\vee\left|Y_s^\varepsilon(x)\right|\le M\right\}. \tag{B.8}$$

Now, for any $M>0$, we may write the expectation term in (B.7) as

$$\mathbb{E}\left[\left|F_0(Y_s^0(x))-F_\varepsilon(Y_s^\varepsilon(x))\right|^2\mathbb{1}_{A_{M,\varepsilon}}\right]+\mathbb{E}\left[\left|F_0(Y_s^0(x))-F_\varepsilon(Y_s^\varepsilon(x))\right|^2\mathbb{1}_{A^c_{M,\varepsilon}}\right].$$

Since $M>0$ is arbitrary, to prove (B.7) it suffices to show that for any $M>0$

$$\lim_{\varepsilon_0\to0}\sup_{0\le\varepsilon<\varepsilon_0}\sup_{x\in[a,b]}\sup_{s\in[0,t]}\mathbb{E}\left[\left|F_0(Y_s^0(x))-F_\varepsilon(Y_s^\varepsilon(x))\right|^2\mathbb{1}_{A_{M,\varepsilon}}\right]=0 \tag{B.9}$$

and that

$$\lim_{M\to\infty}\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]}\mathbb{E}\left[\left|F_0(Y_s^0(x))-F_\varepsilon(Y_s^\varepsilon(x))\right|^2\mathbb{1}_{A^c_{M,\varepsilon}}\right]=0.\quad\square \tag{B.10}$$

Proof of (B.9) Let $\Delta_s^\varepsilon(x):=Y_s^\varepsilon(x)-Y_s^0(x)$. We recall that for each $\varepsilon\in[0,1]$ the process $Y^\varepsilon(x)=(Y_t^\varepsilon(x),t\ge0)$ solves (3.15). Moreover, the processes $(Y^\varepsilon(x),\varepsilon\in[0,1])$ are coupled with the same noise, i.e., there is $(B_t,t\ge0)$ a Brownian motion under $\mathbb{P}$ such that for all $\varepsilon\in[0,1]$ we have almost surely that

$$Y_t^\varepsilon(x)=x+\int_0^t F_\varepsilon(Y_s^\varepsilon(x))\mathrm{d}s+B_t\quad\text{ for all }\quad t>0. \tag{B.11}$$

Now, we may use (B.11) to write

$$\Delta_s^\varepsilon(x)=Y_s^\varepsilon(x)-Y_s^0(x)=\int_0^s[F_\varepsilon(Y_u^\varepsilon(x))-F_0(Y_u^0(x))]\,\mathrm{d}u.$$

By the mean value theorem we have

$$\begin{aligned}\Delta_s^\varepsilon(x)&=\int_0^s[F_\varepsilon(Y_u^\varepsilon(x))-F_\varepsilon(Y_u^0(x))-\left(F_0(Y_u^0(x))-F_\varepsilon(Y_u^0(x))\right)]\,\mathrm{d}u\\&=\int_0^s[F_\varepsilon'(\Theta_u^\varepsilon)\Delta_u^\varepsilon(x)-\left(F_0(Y_u^0(x))-F_\varepsilon(Y_u^0(x))\right)]\,\mathrm{d}u,\end{aligned}$$

where $\Theta_u^\varepsilon\in\left(Y_u^\varepsilon(x)\wedge Y_u^0(x),Y_u^\varepsilon(x)\vee Y_u^0(x)\right)$ for all $u\ge0$. By the convexity of V, it follows that $F_\varepsilon'(\Theta_u^\varepsilon)\le0$. By the chain rule and the fact that $|x|\le1+x^2$ for all $x\in\mathbb{R}$ we have that

$$\begin{aligned}|\Delta_s^\varepsilon(x)|^2&=|\Delta_0^\varepsilon(x)|^2+\int_0^s 2\Delta_u^\varepsilon(x)\mathrm{d}\Delta_u^\varepsilon(x)\\&=2\int_0^s[F_\varepsilon'(\Theta_u^\varepsilon)|\Delta_u^\varepsilon(x)|^2-\Delta_u^\varepsilon(x)\left(F_0(Y_u^0(x))-F_\varepsilon(Y_u^0(x))\right)]\,\mathrm{d}u\\&\le2\int_0^s(1+|\Delta_u^\varepsilon(x)|^2)\left|F_0(Y_u^0(x))-F_\varepsilon(Y_u^0(x))\right|\,\mathrm{d}u.\end{aligned}$$

If we let $\psi_s^\varepsilon(x):=\sup_{u\in[0,s]}|\Delta_s^\varepsilon(x)|^2\mathbb{1}_{A_{M,\varepsilon}}$ and $K(M,\varepsilon):=\sup_{|z|\le M}\left|F_0(z)-F_\varepsilon(z)\right|$ we obtain that $\psi_s^\varepsilon(x)\le2K(M,\varepsilon)\int_0^s(1+\psi_u^\varepsilon(x))\mathrm{d}u$ for any $x\in\mathbb{R}$. Now, for any fixed $M>0$, by Hypothesis 1.2 we have $K(M,\varepsilon)\to0$ as $\varepsilon\to0$. This implies that for any $\eta>0$ and $M>0$ there is ε_0 such that $\sup_{\varepsilon\in[0,\varepsilon_0]}\sup_{x\in\mathbb{R}}\psi_t^\varepsilon(x)\le\eta$. This completes the proof of (B.9). $\square$

Proof of (B.10) Since $(r_1+r_2)^2\le2\left(r_1^2+r_2^2\right)$ for any $r_1,r_2\in\mathbb{R}$, the expectation in (B.10) can be bounded by

$$2\mathbb{E}\left[\left|F_0(Y_s^0(x))\right|^4\mathbb{1}_{A^c_{M,\varepsilon}(x,s)}\right]+2\mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^4\mathbb{1}_{A^c_{M,\varepsilon}(x,s)}\right],$$

it remains to show that

$$\lim_{M\to\infty}\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]}\mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^4\mathbb{1}_{A^c_{M,\varepsilon}(x,s)}\right]=0.$$

By Cauchy–Schwarz inequality, we have

$$\mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^4 \mathbb{1}_{A^c_{M,\varepsilon}}\right] \leq \left(\mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^8\right]\right)^{1/2} \cdot \left(\mathbb{P}\left(A^c_{M,\varepsilon}\right)\right)^{1/2}. \tag{B.12}$$

We now claim that

$$\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]} \mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^8\right] < \infty, \tag{B.13}$$

and that, recall (B.8), for any fixed $t > 0$

$$\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]} \mathbb{P}(A^c_{M,\varepsilon}(x,t)) \to 0 \quad \text{as} \quad M \to \infty. \tag{B.14}$$

From (B.12), (B.13) and (B.14) we conclude (B.10). It remains to prove (B.13) and (B.14). □

Proof of (B.13) We first note that Hypothesis 1.2 and Hypothesis 2.2(Condition (G2)) imply that there is $\tilde{c} > 0$ such that $\left|V'(z)\right| \leq \tilde{c}|z|^{1+\alpha}\exp(z^2)$ for any $z \in \mathbb{R}$. Therefore

$$\mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^8\right] = \mathbb{E}\left[\left|V'(b_\varepsilon Y_s^\varepsilon(x))/b_\varepsilon^{1+\alpha}\right|^8\right] \leq \tilde{c}^8\mathbb{E}\left[\left|Y_s^\varepsilon(x)\right|^{8(1+\alpha)}\left|1+\exp(8b_\varepsilon\left|Y_s^\varepsilon(x)\right|^2)\right|\right].$$

Now, since $b_\varepsilon \to 0$ as $\varepsilon \to 0$ and $|z|^{1+\alpha} \leq \exp(z^2)$ for all $z \in \mathbb{R}$, there is $\tilde{C} > 0$ for which

$$\mathbb{E}\left[\left|F_\varepsilon(Y_s^\varepsilon(x))\right|^8\right] \leq \tilde{C}\mathbb{E}\left[\exp(\left|Y_s^\varepsilon(x)\right|^2)\right].$$

To conclude, we now show that

$$\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]} \mathbb{E}\left[\exp(\left|Y_s^\varepsilon(x)\right|^2)\right] = C(t,a,b) < \infty. \tag{B.15}$$

Indeed, by Itô's formula for $H(z) = \exp(z^2)$, $z \in \mathbb{R}$, we have that

$$H(Y_s^\varepsilon(x)) = H(x) + \int_0^s H(Y_u^\varepsilon(x))\big(2Y_u^\varepsilon(x)F_\varepsilon(Y_u^\varepsilon(x)) + 2(Y_u^\varepsilon(x))^2 + 1\big)\mathrm{d}u + M_s,$$

where $(M_s, s \geq 0)$ is a local martingale. Recall that $G_\varepsilon(z) = zF_\varepsilon(z)$ for all $z \in \mathbb{R}$. By (A.11) we deduce that

$$\sup_{\varepsilon\in[0,1]}\sup_{z\in\mathbb{R}}\big(2zF_\varepsilon(z) + 2z^2 + 1\big) = C < \infty.$$

Therefore, if we let $\tau_K = \tau(K,\varepsilon,x) := \inf\{s > 0 : \left|Y_s^\varepsilon(x)\right| > K\}$ we obtain

$$\mathbb{E}\left[H(Y^\varepsilon_{s\wedge\tau_K}(x))\right] \leq H(x) + C\int_0^s \mathbb{E}\left[H(Y^\varepsilon_{u\wedge\tau_K}(x))\right]\mathrm{d}u. \tag{B.16}$$

Now, by Grönwall's inequality we obtain for $x \in [a,b]$ and $s \in [0,t]$ that

$$\mathbb{E}\left[H(Y^\varepsilon_{s\wedge\tau_K}(x))\right] \leq H(x)\exp(Cs) \leq \big(H(a)+H(b)\big)\exp(Ct).$$

Since the constant C in (B.16) does not depend on ε, Fatou's lemma implies

$$\begin{aligned}\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]} \mathbb{E}\left[\exp(\left|Y_s^\varepsilon(x)\right|^2)\right] &= \sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]} \mathbb{E}\left[\liminf_{K\to\infty} H\left(Y^\varepsilon_{s\wedge\tau_K}(x)\right)\right] \\ &\leq \sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]} \liminf_{K\to\infty}\mathbb{E}\left[H(Y^\varepsilon_{s\wedge\tau_K}(x))\right] \\ &\leq \big(H(a)+H(b)\big)\exp(Ct),\end{aligned} \tag{B.17}$$

which yields (B.15). This completes the proof of (B.13).

Proof of (B.14) Since $zF_\varepsilon(z) = G_\varepsilon(z) \leq 0$ for all $z \in \mathbb{R}$, it follows from (A.12) that

$$\sup_{s\in[0,t]}|Y_s^\varepsilon(x)|^2 \leq x^2 + t + \sup_{s\in[0,t]}\left|M_s^{\varepsilon,x}\right|,$$

where $M^{\varepsilon,x} = (M_s^{\varepsilon,x}, s \geq 0)$ is the local martingale given by (A.13). Therefore, for any $x \in [a,b]$ and any M such that $M > a^2+b^2+t$

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,t]}|Y_s^\varepsilon(x)|^2 > M\right) &\leq \mathbb{P}\left(x^2+t+\sup_{s\in[0,t]}\left|M_s^{\varepsilon,x}\right| > M\right) = \mathbb{P}\left(\sup_{s\in[0,t]}|M_s^{\varepsilon,x}|^2 > (M-x^2-t)^2\right) \\ &\leq \frac{\mathbb{E}\left[|M_t^{\varepsilon,x}|^2\right]}{(M-x^2-t)^2} = \frac{C_{\varepsilon,t,x}}{(M-x^2-t)^2},\end{aligned}$$

where the second inequality follows from Doob's L^2 submartingale inequality and by Itô's isometry, $C_{\varepsilon,t,x} := 4\int_0^t \mathbb{E}\left[|Y_s^\varepsilon(x)|^2\right]\mathrm{d}s$. Now, since $z^2 \leq \exp(z^2)$ for all $z \in \mathbb{R}$, by (B.17) it follows that

$$\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]}\sup_{s\in[0,t]} C_{\varepsilon,t,x} = C'(t,a,b) < \infty.$$

Now, recall the definition in (B.8). Since $x^2 \leq a^2+b^2$ for any $x \in [a,b]$, it follows that for any M such that $M > a^2+b^2+t$

$$\sup_{\varepsilon\in[0,1]}\sup_{x\in[a,b]} \mathbb{P}(A^c_{M,\varepsilon}(x,t)) \leq \frac{C'(t,a,b)}{(M-a^2-b^2-t)^2}$$

and so, we obtain (B.14) and thereby conclude the proof of Proposition B.2. □

Appendix C. Convergence of invariant measures: details

Recall the notation introduced above (2.13), that is, $Y_\infty \stackrel{d}{=} \nu$, $X^\varepsilon_\infty \stackrel{d}{=} \mu^\varepsilon$ and $\widetilde{X}^\varepsilon_\infty \stackrel{d}{=} \widetilde{\mu}^\varepsilon$. By Lemma 2.1 it follows that

$$\nu(\mathrm{d}z) = C^{-1}\exp\big(-2V_0(z)\big)\mathrm{d}z,$$

where C is a normalization constant, $V_0(z) := (2+\alpha)^{-1}C_0|z|^{2+\alpha}$ with α and C_0 defined in Hypothesis 1.2. Similarly, μ^ε and $\widetilde{\mu}^\varepsilon$ is the density of X^ε_∞ and $\widetilde{X}^\varepsilon_\infty$ given by, and they are given by

$$\mu^\varepsilon(\mathrm{d}z) = C_\varepsilon^{-1}\exp\left(-2\frac{V(z)}{\varepsilon}\right)\mathrm{d}z \qquad \text{and} \qquad \widetilde{\mu}^\varepsilon(\mathrm{d}z) = \widetilde{C}_\varepsilon^{-1}\exp\left(-2\frac{\widetilde{V}(z)}{\varepsilon}\right)\mathrm{d}z.$$

respectively. For $(b_\varepsilon, \varepsilon\in[0,1))$ as defined in (2.9), the same argument leading to (3.22) yields that the density of $\frac{X^\varepsilon_\infty}{b_\varepsilon}$ and $Y^\varepsilon_\infty = \frac{\widetilde{X}^\varepsilon_\infty}{b_\varepsilon}$ is given by

$$b_\varepsilon C_\varepsilon^{-1}\exp\left(-2\frac{V(b_\varepsilon z)}{\varepsilon}\right)\mathrm{d}z \qquad \text{and} \qquad b_\varepsilon \widetilde{C}_\varepsilon^{-1}\exp\left(-2\frac{\widetilde{V}(b_\varepsilon z)}{\varepsilon}\right)\mathrm{d}z.$$

Lemma C.1 (*Asymptotic Coupling of the Invariant Measures*). *For each $\varepsilon > 0$, let X^ε_∞ and $\widetilde{X}^\varepsilon_\infty$ be the random variables whose distributions are the invariant measures of the SDE given by (1.1) and (2.11), respectively. Assume that the potential V of (1.1) satisfies Hypotheses 1.1, 1.2, and 1.4 and assume that the potential $\widetilde{V}$ of (2.11) satisfies Hypotheses 1.1, 1.2, and 2.2. Under those assumptions, it follows that*

$$\lim_{\varepsilon\to 0}\mathrm{d}_{\mathrm{TV}}\left(\frac{X^\varepsilon_\infty}{b_\varepsilon}, Y_\infty\right) = 0 \qquad \textit{and} \qquad \lim_{\varepsilon\to 0}\mathrm{d}_{\mathrm{TV}}\left(\frac{\widetilde{X}^\varepsilon_\infty}{b_\varepsilon}, Y_\infty\right) = 0, \tag{C.1}$$

where $(b_\varepsilon, \varepsilon\in[0,1))$ is defined in (2.9). In particular,

$$\lim_{\varepsilon\to 0}\mathrm{d}_{\mathrm{TV}}(X^\varepsilon_\infty, \widetilde{X}^\varepsilon_\infty) = \lim_{\varepsilon\to 0}\mathrm{d}_{\mathrm{TV}}(\mu^\varepsilon, \widetilde{\mu}^\varepsilon) = 0. \tag{C.2}$$

The following two lemmas will be instrumental for the proof of Lemma C.1.

Lemma C.2 (*Uniform Convergence of the Potentials*). *Under the same hypotheses of Lemma C.1, for any $K > 0$ it follows that*

$$\lim_{\varepsilon\to 0}\sup_{|z|\le K}\left|\frac{V(b_\varepsilon z)}{\varepsilon} - V_0(z)\right| = 0 \tag{C.3}$$

and

$$\lim_{\varepsilon\to 0}\sup_{|z|\le K}\left|\frac{\widetilde{V}(b_\varepsilon z)}{\varepsilon} - V_0(z)\right| = 0. \tag{C.4}$$

where $V_0(z) = (2+\alpha)^{-1}C_0|z|^{2+\alpha}$ for any $z\in\mathbb{R}$ with α and C_0 defined in Hypothesis 1.2. □

Proof. In the sequel, we show (C.3). Let $K > 0$ and $\eta > 0$ be fixed and define $\widetilde{\eta} := \eta K^{-1} > 0$. By (2.9), $b_\varepsilon \to 0$, as $\varepsilon \to 0$. Now, by Hypothesis 1.2 there exists $\varepsilon_0 = \varepsilon_0(K, \widetilde{\eta}) > 0$ such that for any $|z| \le K$ and $\varepsilon < \varepsilon_0$,

$$C_0|z|^{1+\alpha}\mathrm{sgn}(z) - \widetilde{\eta} < \frac{V'(b_\varepsilon z)}{b_\varepsilon^{1+\alpha}} < C_0|z|^{1+\alpha}\mathrm{sgn}(z) + \widetilde{\eta}.$$

If we integrate each term from 0 to x in the above inequality, use Hypothesis 1.1 and note that, by (2.9), $b_\varepsilon^{2+\alpha} = \varepsilon$ we obtain that for any $|z| \le K$ and $\varepsilon < \varepsilon_0$

$$\sup_{|z|\le K}\left|\frac{V(b_\varepsilon z)}{\varepsilon} - V_0(z)\right| \le \widetilde{\eta}K = \eta.$$

Since $\eta > 0$ is arbitrary, the proof of (C.3) is complete.

By construction, we stress that $\widetilde{V}$ also satisfies Hypotheses 1.1 and 1.2. Hence the proof of (C.4) is analogous. □

Lemma C.3 (*Convergence of the Normalizing Constants*). *Under the same hypotheses of Lemma C.1 it follows that*

$$\lim_{\varepsilon\to 0} b_\varepsilon C_\varepsilon^{-1} = C \tag{C.5}$$

and

$$\lim_{\varepsilon\to 0} b_\varepsilon \widetilde{C}_\varepsilon^{-1} = C. \tag{C.6}$$

Proof. By the change of variables $z \mapsto b_\varepsilon z$ we obtain that $\frac{C_\varepsilon}{b_\varepsilon} = \int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z$. Now, by Lemma C.2 and Fatou's lemma we have

$$C = \int_{\mathbb{R}} e^{-2V_0(z)}\mathrm{d}z \le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z = \liminf_{\varepsilon\to 0}\frac{C_\varepsilon}{b_\varepsilon}.$$

Similarly, for $\widetilde{V}$ we obtain

$$C \le \liminf_{\varepsilon\to 0}\frac{\widetilde{C}_\varepsilon}{b_\varepsilon}. \tag{C.7}$$

We next show that $\limsup_{\varepsilon\to 0}\frac{C_\varepsilon}{b_\varepsilon}\le C$. Note first that

$$\limsup_{\varepsilon\to 0}\frac{C_\varepsilon}{b_\varepsilon}\le \lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|\le K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z+\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z.$$

By Lemma C.2, the dominated convergence theorem, and the monotone convergence theorem we obtain that

$$\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|\le K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z=\lim_{K\to\infty}\int_{|z|\le K} e^{-2V_0(z)}\mathrm{d}z=C.$$

Similarly, for $\widetilde{V}$ we obtain

$$\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|\le K} e^{-\frac{2}{\varepsilon}\widetilde{V}(b_\varepsilon z)}\mathrm{d}z=C. \tag{C.8}$$

It remains to show that

$$\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z=0 \tag{C.9}$$

and

$$\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-\frac{2}{\varepsilon}\widetilde{V}(b_\varepsilon z)}\mathrm{d}z=0. \tag{C.10}$$

In the sequel, we give the proof of (C.9), which we divide in two cases, depending on whether $\beta\ge\alpha$ or $\beta<\alpha$, where α and β are defined in Hypotheses 1.2 and 1.4, respectively. Note first that by Hypothesis 1.2 for any $\delta>0$ there is $c_0(\delta)>0$ such that for any z with $|z|\le\delta$

$$\left|V'(z)\right|\ge c_0(\delta)\,|z|^{1+\alpha}. \tag{C.11}$$

Assume that $\beta\ge\alpha$. By Hypotheses 1.1 and 1.4 there is an $R>1$ and $c>0$ such that for any z with $|z|\ge R_0$,

$$\left|V'(z)\right|\ge c\,|z|^{1+\beta}\ge c\,|z|^{1+\alpha}. \tag{C.12}$$

By (C.11) and (C.12) there is a $c_1(\delta)>0$ such that for any $z\in\mathbb{R}$, $\left|V'(z)\right|\ge c_1(\delta)\,|z|^{1+\alpha}$. Since $V'(z)=V'(|z|)\mathrm{sgn}(z)$ and $V(0)=0$, if we compute the integral from 0 to z of both sides of (C.12) we obtain that there is a $c(\delta)>0$ such that $V(z)\ge c(\delta)\,|z|^{2+\alpha}$ for any $z\in\mathbb{R}$. The preceding inequality implies that $-V(b_\varepsilon z)\le -c(\delta)b_\varepsilon^{2+\alpha}|z|^{2+\alpha}$ which together with $b_\varepsilon^{2+\alpha}=\varepsilon$ yields that

$$\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z\le \lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-2c(\delta)|z|^{2+\alpha}}\mathrm{d}z=0. \tag{C.13}$$

This completes the case $\beta\ge\alpha$.

Now, we assume that $-1<\beta<\alpha$. Since V satisfies Hypothesis 1.4, we may now take $R_0\ge[(2+\alpha)(2+\beta)^{-1}]^{\frac{1}{\alpha-\beta}}$ and let $\kappa_0(\delta):=\min\{V'(z)|z|^{-1-\alpha}:\delta\le z\le R_0\}$, where $\delta>0$ is given in (C.11). Note that $\kappa_0(\delta)>0$ and that $k_1(\delta):=\min\{c_0(\delta),\kappa_0(\delta)\}>0$ is such that $V'(z)\ge k_1(\delta)z^{1+\alpha}$ for $z\in[0,R_0]$. Now, by Hypothesis 1.4 there is $c>0$ such that $V'(z)\ge cz^{1+\beta}$ for any $z\ge R_0$ and therefore, for $\widetilde{c}=\widetilde{c}(\delta):=\min\{k_1(\delta),c\}>0$

$$V'(z)\ge\begin{cases}\widetilde{c}\,z^{1+\alpha} & \text{for}\quad z\in[0,R_0],\\ \widetilde{c}\,z^{1+\beta} & \text{for}\quad z>R_0.\end{cases}$$

As $V(0)=0$, integrating from 0 to z in the both sides of the above inequality we obtain

$$V(z)\ge\begin{cases}\widetilde{c}\,z^{2+\alpha}(2+\alpha)^{-1} & \text{for}\quad z\in[0,R_0],\\ \frac{\widetilde{c}\,z^{2+\beta}}{(2+\beta)}+\widetilde{c}\left[\frac{R_0^{2+\alpha}}{2+\alpha}-\frac{R_0^{2+\beta}}{(2+\beta)}\right] & \text{for}\quad z>R_0.\end{cases}$$

Since $R_0\ge[(2+\alpha)(2+\beta)^{-1}]^{\frac{1}{\alpha-\beta}}$, it follows that $\frac{R_0^{2+\alpha}}{2+\alpha}-\frac{R_0^{2+\beta}}{2+\beta}\ge 0$, and because V is an even function we deduce the existence of $\kappa=\kappa(\delta)>0$ such that

$$V(z)\ge\begin{cases}\kappa|z|^{2+\alpha} & \text{for}\quad |z|\le R_0,\\ \kappa|z|^{2+\beta} & \text{for}\quad |z|\ge R_0.\end{cases}$$

Since $b_\varepsilon^{2+\alpha}=\varepsilon$ and $b_\varepsilon^{2+\beta}/\varepsilon=b_\varepsilon^{\beta-\alpha}=b_\varepsilon^{-|\beta-\alpha|}\to\infty$ as $\varepsilon\to 0$, the dominated convergence theorem yields

$$\begin{aligned}\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\mathrm{d}z&=\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>K} e^{-\frac{2}{\varepsilon}V(b_\varepsilon z)}\left(\mathbb{1}_{|b_\varepsilon z|\le R_0}+\mathbb{1}_{|b_\varepsilon z|>R_0}\right)\mathrm{d}z\\&\le\lim_{K\to\infty}\limsup_{\varepsilon\to 0}\left(\int_{|z|>K} e^{-2\kappa|z|^{2+\alpha}}\mathrm{d}z+\int_{|z|>K} e^{-2\kappa b_\varepsilon^{\beta-\alpha}|z|^{2+\beta}}\mathrm{d}z\right)\\&=\lim_{K\to\infty}\int_{|z|>K} e^{-2\kappa|z|^{2+\alpha}}\mathrm{d}z=0.\end{aligned} \tag{C.14}$$

This completes the case $-1<\beta<\alpha$.

Combining (C.13) and (C.14) we obtain (C.9). This finishes the proof of (C.5).

In the sequel, we stress that (C.6) is just a consequence from above case $\beta \geq \alpha$. Indeed, by (C.7) and (C.8), it is enough to show (C.10). Since $\widetilde{V}$ satisfies Hypothesis 1.4 with $\beta = \alpha$, the proof is already covered in (C.13).

The proof of Lemma C.3 is finished. □

Proof of Lemma C.1. By (3.21), (3.22) and Scheffé's lemma ([39, Lemma 3.3.2, p.95]), to obtain (C.1), it suffices to note that Lemma C.2 and Lemma C.3 imply that

$$\lim_{\varepsilon\to 0} \frac{b_\varepsilon}{C_\varepsilon} e^{-2\frac{V(b_\varepsilon z)}{\varepsilon}} = \frac{1}{C} e^{-2V_0(z)} \quad \text{for any} \quad z \in \mathbb{R}$$

and

$$\lim_{\varepsilon\to 0} \frac{b_\varepsilon}{\widetilde{C}_\varepsilon} e^{-2\frac{\widetilde{V}(b_\varepsilon z)}{\varepsilon}} = \frac{1}{C} e^{-2V_0(z)} \quad \text{for any} \quad z \in \mathbb{R}. \quad \square$$

Appendix D. Complements

In this section we include, for completeness of the exposition, a few results that have been used throughout the text with a brief explanation.

Proof of Lemma 2.1. We apply Theorem 3.3.4 of [20, Ch. 3, p.91]. By Hypothesis 1.4 for all $|z| \geq R$ we have $-V'(z)\frac{z}{|z|^{1+\kappa}} \leq -c|z|^{\rho-\kappa}$ for any $\kappa \in (0, \rho)$, and therefore

$$\lim_{|z|\to\infty} \left(-V'(z)z|z|^{-1-\kappa}\right) = -\infty < 0.$$

Hence, the field $-V'$ satisfies the drift condition eq. (3.3.4) in [20, Ch. 3, p.86]. By Theorem 3.3.4 of [20] we have the existence and uniqueness of the invariant measure μ^ε. In addition, for any $c > 0$ there are $C_1 = C_1(c,\kappa,\varepsilon) > 0$ and $C_2 = C_2(c,\kappa,\varepsilon) > 0$ such that

$$\mathrm{d}_{\mathrm{TV}}(X_t^\varepsilon(x), X_t^\varepsilon(y)) \leq C_1 e^{-C_2 t}(e^{c|x|} + e^{c|y|}) \quad \text{for any} \quad x, y \in \mathbb{R},\ t \geq 0.$$

By Hypothesis 1.4, we have $\int_{\mathbb{R}} e^{c|z|}\mu^\varepsilon(\mathrm{d}z) < \infty$. Therefore, Theorem 3.3.4 in [20] yields (2.3). Moreover, formula (2.4) follows from Proposition 4.2 in [101, p. 110]. □

Let C^2 represent the set of twice continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$. □

Lemma D.1 (*Existence of a Regular Potential*). *Assume that V satisfies Hypothesis 1.1 and Hypothesis 1.2 with $\alpha > 0$. For each $M > 0$, there exist an even C^2 convex function $V_M = V_{M,\alpha} : \mathbb{R} \to [0,\infty)$ and positive constants $c = c_{M,\alpha}$, $C = C_{M,\alpha}$ and $R = R_{M,\alpha}$ such that*

$$V_M(z) = V(z) \quad \textit{for} \quad |z| \leq M \tag{D.1}$$

and

$$V_M'(z) \geq cz^{1+\alpha} \qquad \textit{and} \qquad |V_M'(z)| \leq Ce^{z^2} \quad \textit{for all} \quad z \geq R. \tag{D.2}$$

In particular, the potential V_M satisfies Hypotheses 1.1, 1.2 and 2.2.

Proof. The proof follows by a standard mollifier procedure. We mimic the lines given in Proposition 4.10 of [31]. Let $g : \mathbb{R} \to [0,1]$ be an increasing C^∞-function such that $g(u) = 0$ for $u \leq 1/2$, $g(u) = 1$ for $u \geq 1$, and $g(u) \in (0,1)$ for all $u \in (1/2, 1)$. Let $M > 0$ be fixed and define

$$G_M(u) = \left(1 - g\left(\frac{u^2}{2M^2}\right)\right) V''(u) + g\left(\frac{u^2}{2M^2}\right)|u|^\alpha \quad \text{for all} \quad u \in \mathbb{R}.$$

Observe that $G_M(u) = V''(u)$ for all $|u| \leq M$, and $G_M(u) = |u|^\alpha$ for all $|u| \geq \sqrt{2}M$. We note that G_M is a non-negative continuous function and then we set $H_M(u) := \int_0^u G_M(y)\mathrm{d}y$ for all $u \in \mathbb{R}$. Finally we define $V_M(z) := \int_0^{|z|} H_M(u)\mathrm{d}u$ for all z. Since G_M is an even function, it follows that H_M is odd and V_M is again even. Now, since $V_M(0) = V(0) = 0$, $V_M'(0) = V'(0) = 0$ and $V_M''(z) = V''(z)$ for $z \leq M$ it follows that V_M satisfies (D.1). Moreover, since there is $C > 0$ for which $|u|^\alpha \leq C\exp(u^2)$ for all $u \in \mathbb{R}$ it follows that V_M satisfies (D.2). □

Proposition D.2 (*Disintegration Inequality*). *Suppose that $\{X(x) = (X_t(x), t \geq 0), x \in S\}$ and $Y = \{Y(y) = (Y_t(y), t \geq 0), y \in S\}$ are Markov families on the measurable space $(S, \mathcal{S})$ and defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for all $r, s > 0$, $a, b \in S$, the following disintegration inequality for the total variation distance holds:*

$$\mathrm{d}_{\mathrm{TV}}\left(X_{r+s}(a), Y_{r+s}(b)\right) \leq \int_{S^2} \mathrm{d}_{\mathrm{TV}}\left(X_s(x), Y_s(y)\right) \mathbb{P}(X_r(a) \in \mathrm{d}x, Y_r(b) \in \mathrm{d}y).$$

Proof. Write $t = r + s$. Since the families $\{X(x), x \in S\}$ and $\{Y(y), y \in S\}$ are Markovian and are defined on the same probability space, for any $a \in S$ and $B \in \mathcal{S}$ we have that

$$\mathbb{P}(X_t(a) \in B) = \int_S \mathbb{P}(X_s(x) \in B)\mathbb{P}(X_r(a) \in \mathrm{d}x) = \int_{S^2} \mathbb{P}(X_s(x) \in B)\mathbb{P}(X_r(a) \in \mathrm{d}x, Y_r(a) \in \mathrm{d}y). \tag{D.3}$$

Similarly, we have that

$$\mathbb{P}(Y_t(a) \in B) = \int_{S^2} \mathbb{P}(Y_s(y) \in B)\mathbb{P}(X_r(a) \in \mathrm{d}x, Y_r(a) \in \mathrm{d}y). \tag{D.4}$$

Therefore, from the definition of total variation distance, together with (D.3) and (D.4) we obtain that

$$\begin{aligned}
\mathrm{d}_{\mathrm{TV}}\big(X_t(a), Y_t(b)\big) &= \sup_{B \in \mathcal{S}} \left|\mathbb{P}\big(X_t(a) \in B\big) - \mathbb{P}\big(Y_t(b) \in B\big)\right| \\
&= \sup_{B \in \mathcal{S}} \left|\int_{S^2} \Big(\mathbb{P}\big(X_s(x) \in B\big) - \mathbb{P}\big(Y_s(y) \in B\big)\Big)\mathbb{P}\big(X_r(a) \in \mathrm{d}x, Y_r(b) \in \mathrm{d}y\big)\right| \quad \square \\
&\leq \int_{S^2} \mathrm{d}_{\mathrm{TV}}\big(X_s(x), Y_s(y)\big)\mathbb{P}\big(X_r(a) \in \mathrm{d}x, Y_r(b) \in \mathrm{d}y\big).
\end{aligned}$$

Proposition D.3 (*Support Theorem for Diffusions*). *For any $x \in \mathbb{R}$ and $\varepsilon \in [0,1]$ let $Y^\varepsilon(x) = (Y_t^\varepsilon(x), t \geq 0)$ be the solution of (3.15). For each fixed $t > 0$, the law of $Y_t^\varepsilon(x)$ is absolutely continuous with respect to the Lebesgue measure and it has full support on $\mathbb{R}$.*

Proof. Fix $\varepsilon \in [0,1]$. Now, write for simplicity $Y_t = Y_t^\varepsilon(x)$, $F = F_\varepsilon$ and note that, almost surely, for every $t \geq 0$

$$Y_t = x + \int_0^t F(Y_s)\,\mathrm{d}s + B_t. \tag{D.5}$$

The proof is done in two steps. On the first step, following the ideas in [102], we prove that for any $t > 0$ the law of Y_t denoted by μ_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Let ρ_t represent a density of μ_t, i.e. for any $a, b \in \mathbb{R}$ with $a < b$

$$\mathbb{P}(Y_t \in [a,b]) = \mu_t([a,b]) = \int_a^b \rho_t(z)\,\mathrm{d}z.$$

On the second step, we prove, with the help of the maximum principle in [103], that $\rho_{t+s}(z) > 0$ for all $z \in \mathbb{R}$. Since $t > 0$ and $s > 0$ are arbitrary, this completes the proof that $\mu_t(\mathrm{d}z) = \rho_t(z)\mathrm{d}z$ with $\rho_t(z) > 0$ for all $t > 0$, i.e. the law of Y_t has full support.

We remark that a standard localization argument is not straightforward with the methods in [102]. Indeed, as the authors themselves say

> "Our result might be deduced from [Aronson-1968] by a localization argument, however, we did not succeed in this direction".

Step 1. We adapt to our case the proof of Theorem 2.1 in [102]. This means that $\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}B_t$ with $b(z) = F(z)$ and $\sigma(z) = 1$ for all $z \in \mathbb{R}$. Since b is not bounded by a linear function we cannot apply Theorem 2.1 in [102] directly. However, the field F is convex and drives the trajectories towards the origin which allows us to obtain L^2 bounds and replicate the main steps in the proof. Moreover, the noise term is simpler and this allows us to ignore the auxiliary function f_δ defined in Lemma 1.2 in [102].

Now, for $\delta \in (0,t)$, consider the random variable $Z_\delta := Y_{t-\delta} + B_t - B_{t-\delta}$. Note that for any $b \in \mathbb{R}$

$$\left|\mathbb{E}[\exp(\mathrm{i}bZ_\delta)|\mathcal{F}_{t-\delta}]\right| = \left|\exp(\mathrm{i}bY_{t-\delta} - \delta b^2/2)\right| = \exp(-\delta b^2/2), \tag{D.6}$$

where $(\mathcal{F}_t, t \geq 0)$ is the natural filtration of the Brownian motion $B = (B_t, t \geq 0)$ and i is the unit imaginary. By (D.5) it follows that

$$Y_t - Z_\delta = \int_{t-\delta}^t F(Y_s)\,\mathrm{d}s.$$

By (B.13), there is $C = C_t$ such that $\sup_{s \in [0,t]} \mathbb{E}[|F(Y_s)|^2] \leq C^2$ and therefore by Jensen's and Cauchy–Schwarz's inequalities

$$\left(\mathbb{E}[|Y_t - Z_\delta|]\right)^2 \leq \mathbb{E}[|Y_t - Z_\delta|^2] = \mathbb{E}\left[\left(\int_0^\delta F(Y_{t-\delta+s})\,\mathrm{d}s\right)^2\right] \leq \delta \int_0^\delta \mathbb{E}[|F(Y_{t-\delta+s})|^2] \leq C^2\delta^2. \tag{D.7}$$

Let μ_t be the law of Y_t and let $\widehat{\mu}_t$ be the characteristic function of μ_t defined by $\widehat{\mu}_t(b) := \mathbb{E}[\exp(\mathrm{i}bY_t)]$. We note that for any $\delta \in (0,t)$ and $b \in \mathbb{R}$, by (D.6) and (D.7), we have

$$\left|\widehat{\mu}_t(b)\right| = \left|\mathbb{E}[\exp(\mathrm{i}bY_t)]\right| \leq \left|\mathbb{E}[\exp(\mathrm{i}bZ_\delta)]\right| + |b|\,\mathbb{E}[|Y_t - Z_\delta|] \leq \exp(-\delta b^2/2) + C\,|b|\,\delta. \tag{D.8}$$

Let $R_t > 0$ be such that $(\log|b|)^2/b^2 < t$ when $|b| > R_t$. For each b with $|b| \geq R_t$ we choose $\delta_b := (\log|b|)^2/b^2$ and so the bound in (D.8) implies that

$$\left|\widehat{\mu}_t(b)\right| \leq \exp(-\delta_b b^2/2) + C\,|b|\,\delta_b = \exp(-(\log|b|)^2/2) + C(\log|b|)^2/b.$$

Since $\left|\widehat{\mu}_t(b)\right| \leq 1$ for all $b \in \mathbb{R}$ it follows that $\int_{-\infty}^\infty \left|\widehat{\mu}_t(b)\right|^2 \mathrm{d}b < \infty$ and so, by Lemma 1.1 in [102] it follows that μ_t has density in $\mathbb{R}$.

Step 2. Note that ρ_{t+s} is the solution of

$$Lu + F'u = 0 \tag{D.9}$$

at time s with initial condition ρ_t where $Lu := \partial_t u - \frac{1}{2}(\partial_x)^2 u + F\partial_x u$. By Theorem 3 in [103], $(\rho_{t+h}, h \geq 0)$ is a non-negative solution of (D.9) and therefore either $\rho_{t+s}(z) > 0$ for all $z \in \mathbb{R}$ or $\rho_{t+s}(z) = 0$ for all $z \in \mathbb{R}$, and since $\int_{\mathbb{R}} \rho_{t+s}(z)\mathrm{d}z = 1$ it follows that $\rho_{t+s}(z) > 0$ for all $z \in \mathbb{R}$. This concludes the proof. □

We conclude this section with the following result.

Lemma D.4 (*Monotonicity and Continuity*). *For all $x \in \mathbb{R}$ the function $t \mapsto G_t(x)$ defined in (3.2) is continuous and strictly decreasing in t.*

Proof. Let $x \in \mathbb{R}$ be fixed. By the triangle inequality for all $t > 0$ and $s > 0$ we have

$$\left|G_x(t) - G_x(s)\right| \leq \mathrm{d}_{\mathrm{TV}}(Y_t(\mathrm{sgn}(x)\infty), Y_s(\mathrm{sgn}(x)\infty)).$$

Then it is enough to show that the right-hand side of the preceding inequality tends to zero as $t \to s$. For short, we write $Y_u = Y_u(\mathrm{sgn}(x)\infty)$, $u \geq 0$. By Proposition D.3 it follows that for every $t > 0$, the law of Y_t is absolutely continuous with respect to the Lebesgue measure and has a full support density $\rho_t(y)$. Moreover, $(\rho_t(y))_{t\geq 0}$ solves the so-called Fokker–Planck equation

$$\partial_t \rho_t(y) = \frac{1}{2}\partial_y^2 \rho_t - \partial_y(F_0(y)\rho_t(y)),$$

where F_0 is as defined in (3.14), see for instance [23, Section 2.2]. Then $\lim_{t\to s} \rho_t(y) = \rho_s(y)$ for all $y \in \mathbb{R}$ and therefore by Scheffé's lemma, see [39, Lemma 3.3.2, p.95], we have $\lim_{t\to s} \mathrm{d}_{\mathrm{TV}}(Y_t, Y_s) = 0$. This completes the proof that $t \mapsto G_t(x)$ is continuous.

We now turn to the proof that $G_t(x)$ is strictly decreasing in t. Recall that $G_x(t) = \mathrm{d}_{\mathrm{TV}}\left(Y_t(\mathrm{sgn}(x)\infty), \nu\right)$ for $t \geq 0$. Let $x \in \mathbb{R}$ and $t > 0$ be fixed. By (3.24) we have $G_x(t) < 1$. For short let $\theta_{x,t} := G_x(t)$ and denote the law of $Y_t(\mathrm{sgn}(x)\infty)$ by $\mu_{x,t}$. Let $(P_s)_{s\geq 0}$ be the semigroup associated to the Markov process $(Y_s(z), s \geq 0, z \in \mathbb{R})$ and note the invariance $P_s(\nu) = \nu$, $s \geq 0$. Since $\theta_{x,t}$ is the total variation distance between $Y_t(\mathrm{sgn}(x)\infty)$ and ν, there exists a coupling between $\mu_{x,t}$ and ν such that $\mu_{x,t} = (1-\theta_{x,t})\nu + \theta_{x,t}\eta_{x,t}$, where $\eta_{x,t}$ is a probability measure on $\mathbb{R}$. By the semigroup property we have for any $s > 0$

$$\begin{aligned}\mathrm{d}_{\mathrm{TV}}(\mu_{x,t+s}, \nu) &= \mathrm{d}_{\mathrm{TV}}(P_s(\mu_{x,t}), \nu) = \mathrm{d}_{\mathrm{TV}}((1-\theta_{x,t})P_s(\nu) + \theta_{x,t}P_s(\eta_{x,t}), \nu)\\ &= \mathrm{d}_{\mathrm{TV}}((1-\theta_{x,t})\nu + \theta_{x,t}P_s(\eta_{x,t}), \nu) = \theta_{x,t}\mathrm{d}_{\mathrm{TV}}(P_s(\eta_{x,t}), \nu).\end{aligned}$$

Now, we claim that $\mathrm{d}_{\mathrm{TV}}(P_s(\eta_{x,t}), \nu) < 1$. Indeed, by disintegration we have

$$\mathrm{d}_{\mathrm{TV}}(P_s(\eta_{x,t}), \nu) \leq \int_{\mathbb{R}} \mathrm{d}_{\mathrm{TV}}(P_s(z), \nu)\eta_{x,t}(\mathrm{d}z) = \int_{\mathbb{R}} G_z(s)\eta_{x,t}(\mathrm{d}z).$$

Hence, $\mathrm{d}_{\mathrm{TV}}(P_s(\eta_{x,t}), \nu) = 1$ if and only if $G_z(s) = 1$ for z-almost surely with respect to the measure $\eta_{x,t}$. This yields a contradiction with (3.24) and hence the proof that the function G_x is strictly decreasing is finished. □

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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