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ORIGINAL PAPER

3d Farey graph, lambda lengths and SL_2-tilings

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Abstract
We explore a three-dimensional counterpart of the Farey tessellation and its relations to Penner's lambda lengths and SL_2-tilings. In particular, we prove a three-dimensional version of the Ptolemy relation, and generalise results of Short to classify tame SL_2-tilings over Eisenstein integers in terms of pairs of paths in the 3D Farey graph.

Keywords Farey graph · Ford circle · SL_2-tiling · lambda length

Mathematics Subject Classification 05E15 · 05B99 · 51F15 · 11A55 · 13F60

1 Introduction and main results

We study geometric aspects of the Farey graph over the Eisenstein integers and its realisation in hyperbolic three-dimensional space as the 1-skeleton of the union of the symmetry planes (including points at the absolute) of the reflection group of the regular ideal hyperbolic tetrahedron. Our first main goal is to generalise relations between Penner's λ-lengths and $SL_2(\mathbb{Z})$-tilings and to prove a three-dimensional version of the Ptolemy relation. Secondly, we classify tame SL_2-tilings over Eisenstein integers in terms of pairs of paths in the 3D Farey graph.

The classical notion of the Farey graph, together with its close relatives such as circle packings, continued fractions, Conway-Coxeter friezes and SL_2-tilings, is a subject of large and ever growing literature. Overviews of different aspects of the theory can be found in [12,

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20, 29]. Moreover, the Farey graph also appears in the study of discrete group of symmetries of the hyperbolic plane $\mathbb{H}^2$, which has been studied exhaustively. It is then natural to ask which features of this theory can be generalised or extended to higher dimensions.

There are many natural generalisations of the above mentioned classical notions based on the substitution of the ring of integers $\mathbb{Z}$ with other rings. In 1887 in his fundamental work [15], A. Hurwitz initiated a systematic study of the continued fractions over $\mathbb{C}$ and over various subrings in $\mathbb{C}$ (see also [14]). In 1951, Cassels, Ledermann and Mahler made the first steps in the Farey graphs for Gaussian and Eisenstein numbers in [5]. A little later in his paper [27], Schmidt introduced a counterpart of the Farey graph for all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$, where d is square-free, see also paper [34] by Vulakh for further generalisations (recall that $d = 1, 3$ correspond to the Gaussian and Eisenstein numbers considered in [5]). We would also like to mention an essentially different three-dimensional approach to the Farey graph developed by Beaver and Garrity [2], based on multidimensional Farey addition that does not appear in the theory of complex continued fractions (see [28] for further details). Furthermore, various other objects related to the Farey graph have also been extended beyond the classical setting. In particular, a three-dimensional analogue of Ford circles for $d = 1$ was originally introduced by Ford [11]; for other fields see e.g. [23, 26]. In [33], Stange studied various circle packings arising from Bianchi groups $PSL_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K. Coxeter [8] considered examples of friezes with quadratic irrational entries, Holm and Jorgensen [13] used p-angulations of polygons to classify friezes with quiddity row consisting of positive integer multiples of $2\cos(\pi/p)$.

In this paper, we consider a 3-dimensional analogue of the Farey graph arising from a tessellation of hyperbolic space $\mathbb{H}^3$ by regular hyperbolic ideal simplices (used in place of a tessellation of $\mathbb{H}^2$ by ideal triangles). We call it the *tetrahedral graph* $\mathcal{T}$. The graph $\mathcal{T}$ inherits many good properties of the classical Farey graph $\mathcal{F}$ (see Sect. 3 for details and essential definitions). In particular, the vertices of $\mathcal{T}$ are precisely points of $\widehat{\mathbb{Q}}(\sigma) = \mathbb{Q}(\sigma) \cup \{\infty\}$, where $\sigma = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, the group of symmetries of $\mathcal{T}$ is the Bianchi group $Bi(3)$, and the edges of $\mathcal{T}$ can be described, similarly to the ones of the Farey graph, via determinants: two irreducible fractions p/q and $r/s \in \widehat{\mathbb{Q}}(\sigma)$ are joined by an edge if and only if $|ps - rq| = 1$ (see Sect. 3.2). Furthermore, as for the Farey graph, faces of $\mathcal{T}$ can be described via Farey addition (see Sect. 3.3).

Another property inherited by the tetrahedral graph is the relation with λ-*lengths*. Given two points $x, y \in \partial\mathbb{H}^d$ and a choice of horospheres h_x, h_y centred at x and y, Penner [24] introduced the notion of λ-length λ_{xy} between x and y as $\lambda_{xy} = e^{d/2}$, where d is the signed distance between h_x and h_y. Penner also showed that for an ideal quadrilateral $xyzt$, the corresponding λ-lengths satisfy the Ptolemy relation

$$\lambda_{xz}\lambda_{yt} = \lambda_{xy}\lambda_{zt} + \lambda_{yz}\lambda_{xt}.$$

Given two irreducible fractions $p/q, r/s \in \widehat{\mathbb{Q}}(\sigma)$, we can also define the *det-length* $l(p/q, r/s)$ as the absolute value of the determinant $l(p/q, r/s) = |ps - rq|$. We then choose a distinguished set of horospheres at points of $\widehat{\mathbb{Q}}(\sigma)$ (see Sect. 4.2; the horospheres are represented by *Ford spheres* [26]) and show that λ-lengths computed with respect to these horospheres coincide with det-lengths.

Theorem 4.12 *Let $X, Y \in \widehat{\mathbb{Q}}(\sigma)$ be two irreducible fractions. Let the standard horosphere be chosen at every point of $\widehat{\mathbb{Q}}(\sigma)$. Then $\lambda_{XY} = l_{XY}$.*

To prove Theorem 4.12, we first show that λ-lengths between vertices of $\mathcal{T}$ satisfy an analogue of the Ptolemy relation:

Theorem 4.7 *Let $A_1A_2A_3A_4$ be a fundamental tetrahedron of $\mathcal{T}$ with vertices in $\widehat{\mathbb{Q}}(\sigma)$, and choose any $X \in \widehat{\mathbb{Q}}(\sigma)$ distinct from A_i. Let $\lambda_i = \lambda_{XA_i}$ be the λ-length of XA_i, $i = 1, \ldots, 4$. Then*

$$\sum_{i=1}^{4} \lambda_i^4 = \sum_{1 \le i < j \le 4} \lambda_i^2 \lambda_j^2.$$

We prove Theorem 4.7 in two ways, namely by a direct computation and as a corollary of the Soddy-Gosset theorem relating the radii of five mutually tangent spheres in $\mathbb{R}^3$. We then show (in Theorem 4.8) that det-lengths satisfy the same the Ptolemy relation, which eventually implies Theorem 4.12.

Theorem 4.7 is a particular case of Theorem 4.18 which we state next; the latter can be considered as a 3-dimensional counterpart of the Ptolemy relation which can be applied to any five points in $\widehat{\mathbb{C}}$:

Theorem 4.18 *Let $X_1, \ldots, X_5 \in \widehat{\mathbb{C}} = \partial \mathbb{H}^3$ be 5 distinct points. Suppose that there are horospheres chosen at these points. Let $\lambda_{ij} = \lambda_{X_i X_j}$. Then*

$$\sum_{(ij)(klm) \in S_5} \lambda_{ij}^4 \lambda_{kl}^2 \lambda_{lm}^2 \lambda_{mk}^2 = \sum_{(ijklm) \in S_5} \lambda_{ij}^2 \lambda_{jk}^2 \lambda_{kl}^2 \lambda_{lm}^2 \lambda_{mi}^2,$$

where $(ij)(klm)$ and $(ijklm)$ denote cycle decompositions of permutations in the symmetric group S_5.

Next, we apply $\mathcal{T}$ to generalise results of Short [30] to classify $SL_2(\mathbb{Z}[\sigma])$-tilings. A *path* (v_i) in $\mathcal{T}$ is a (bi-infinite) sequence of vertices of $\mathcal{T}$ such that v_i and v_{i+1} are connected by an edge of $\mathcal{T}$. We normalise the paths by requiring that the expressions $v_i = p_i/q_i$ satisfy the condition $p_i q_{i+1} - p_{i+1} q_i = 1$. We then prove the following result, which is a direct generalisation of the result of [30].

Theorem 5.20 *Given two normalised paths $v_i = p_i/q_i$ and $u_j = r_j/s_j$, the map $(u_i, v_j) \mapsto m_{ij} = p_i s_j - q_i r_j$ provides a bijection between equivalence classes of tame $SL_2(\mathbb{Z}[\sigma])$-tilings and pairs of paths in $\mathcal{T}$ considered up to the simultaneous action of $SL_2(\mathbb{Z}[\sigma])$ on both paths.*

Here, we say that two $SL_2(\mathbb{Z}[\sigma])$-tilings are equivalent if one is obtained from the other by multiplication of even rows by σ^k and of odd rows by σ^{-k}, together with multiplication by σ^l (resp. σ^{-l}) of even (resp. odd) columns.

In fact, the bijection given in Theorem 5.20 can be refined to enumeration of individual tilings rather than equivalence classes. This is achieved by using $\mathcal{T}$*-angle sequences* of paths we introduce in Sect. 6.

Given a path in $\mathcal{T}$, we construct a sequence of numbers from $\mathbb{Z}[\sigma]$ called a $\mathcal{T}$-angle sequence of the path (also known as a quiddity sequence for friezes or as an itinerary in [30]). We then use Theorem 5.20 to provide a geometric interpretation of the classification of SL_2-tilings obtained in [3].

In conclusion, we note that in the classical two-dimensional setting there are also beautiful connections between the combinatorics of SL_2-tilings, triangulations of polygons or more generally apeirogons, cluster algebras, as well as the representation theory of quivers. It would be interesting to see to what extent it is possible to incorporate these ideas into the three-dimensional setting. One example of connections between cluster algebras and triangulated three-dimensional hyperbolic manifolds can be found in [22].

We would also like to note that since the first version of this paper was released several related works have appeared. In [31], Farey complexes over arbitrary rings are studied and applied to describe tame SL_2-tilings. In [18], a complex analog of Penner's λ-lengths is used to provide a counterpart of the Ptolemy relation for skew ideal quadrilaterals in $\mathbb{H}^3$ (this result can also be proved in a way similar to the one described in Remark 4.20). In particular, in [18, Theorem 2] a complex version of Theorem 4.12 is proved, which implies our Theorem 4.12 after taking the absolute values.

The paper is organised as follows. In Sect. 2, we recall basic properties of the classical Farey graph $\mathcal{F}$. In Sect. 3, we introduce the tetrahedral graph $\mathcal{T}$ and describe its properties. Section 4 is devoted to establishing various relations on λ-lengths, first between the points of $\widehat{\mathbb{Q}}(\sigma)$ and then generally in $\mathbb{H}^3$. In Sect. 5, we show that SL_2-tilings over $\mathbb{Z}[\sigma]$ are classified by pairs of paths in $\mathcal{T}$. In Sect. 6, we describe paths in $\mathcal{T}$ in terms of sequences of numbers from $\mathbb{Z}[\sigma]$ ($\mathcal{T}$-angles of the paths), we then enumerate SL_2-tilings over $\mathbb{Z}[\sigma]$ by pairs of infinite sequences (together with an element of $SL_2(\mathbb{Z}[\sigma])$). Finally, Sect. 7 is devoted to various remarks, including the connection between paths in $\mathcal{T}$ and continued fractions over $\mathbb{Z}[\sigma]$, discussion of some properties of $\mathcal{T}$-angles, and generalisation of the results to other imaginary quadratic fields. We also give a geometric argument to reprove a recent result of Cuntz and Holm [10] concerning friezes over algebraic numbers.

2 Preliminaries: the Farey graph and its symmetries

In this section we recall some classic notions and definitions. In particular we recall the definitions of the Farey graph, Farey addition, Ford circles, and the Farey tessellation of the hyperbolic plane. We also discuss the group of symmetries of the Farey tessellation.

Notation. Throughout the paper we denote $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We also use the notations $\widehat{\mathbb{R}}$, $\widehat{\mathbb{Q}}$, $\widehat{\mathbb{Z}}$, $\widehat{\mathbb{Z}}[\sigma]$ for the sets $\mathbb{R} \cup \{\infty\}$, $\mathbb{Q} \cup \{\infty\}$, $\mathbb{Z} \cup \{\infty\}$, and $\mathbb{Z}[\sigma] \cup \{\infty\}$ respectively.

2.1 The Farey graph

Definition 2.1 The *Farey graph* $\mathcal{F}$ is an infinite graph whose vertices are $\widehat{\mathbb{Q}}$; two vertices $u, v \in \widehat{\mathbb{Q}}$ with irreducible fractions p/q and r/s respectively are connected by an edge if and only if $|ps - rq| = 1$.

It is convenient to visualise the Farey graph $\mathcal{F}$ by drawing it in the upper half-plane model of the hyperbolic plane $\mathbb{H}^2$. Namely, we identify $\mathbb{H}^2$ with $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \subset \widehat{\mathbb{C}}$; its boundary is therefore identified with $\widehat{\mathbb{R}}$. Then the vertices of the graph $\mathcal{F}$ are naturally identified with rational points of the absolute; for each edge uv in the graph $\mathcal{F}$ we draw a hyperbolic line connecting u and v (i.e. a semicircle with endpoints at u and v), see Fig. 1.

Observe that the geodesic semicircles representing any two distinct edges of the graph do not intersect each other in $\mathbb{H}^2$, however they might have a common vertex on the absolute. Moreover, the obtained diagram is a tessellation of $\mathbb{H}^2$ by ideal triangles (i.e. geodesic triangles with vertices at the absolute); we call it the *Farey tessellation*.

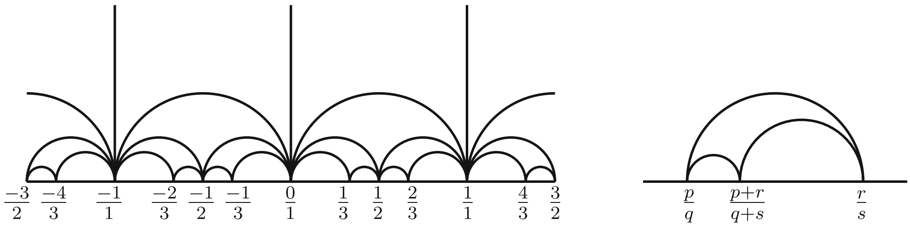

Fig. 1 The Farey graph

2.2 Farey addition in $\widehat{\mathbb{Q}}$ and ideal triangles in the tessellation

Given an edge connecting p/q with r/s in the Farey graph (we assume $q > 0, s \geq 0$), the third vertex of the triangle lying right below this edge is given by *Farey addition*, i.e.

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s},$$

see Fig. 1, right. One can check that the third vertex of the triangle lying right above the edge can be written by $\frac{p}{q} \ominus \frac{r}{s} = \frac{p-r}{q-s}$, and that in terms of hyperbolic geometry the two triangles can be obtained from each other by applying a reflection with respect the hyperbolic line connecting p/q with r/s. Note that $u \ominus v = v \ominus u$.

2.3 Symmetry group of $\mathcal{F}$

Recall that the group $PSL_2(\mathbb{Z})$ naturally acts on the upper half-plane, here for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we consider the mapping $z \to \frac{az+b}{cz+d}$. This action preserves the value $|ps - rq|$ for every two irreducible fractions $p/q, r/s \in \widehat{\mathbb{Q}}$. Therefore, every element of $PSL_2(\mathbb{Z})$ takes the Farey graph to itself. Thus, we can identify $PSL_2(\mathbb{Z})$ with a subgroup of symmetries of $\mathcal{F}$.

In fact, $PSL_2(\mathbb{Z})$ is the group of orientation-preserving symmetries of $\mathcal{F}$. This group is an index 2 subgroup of the group of all symmetries of $\mathcal{F}$, the latter is generated by reflections with respect to the the sides of the ideal triangle with vertices $0, 1, \infty$. The group $PSL_2(\mathbb{Z})$ acts transitively on the vertices of $\mathcal{F}$, on the edges of $\mathcal{F}$, and on the triangles in the tessellation.

2.4 Ford circles

Consider the Euclidean circle of (Euclidean) radius $1/2$ tangent to the real axis at the point 0 (in terms of hyperbolic geometry it represents a horocycle). The action of $PSL_2(\mathbb{Z})$ takes this circle to infinitely many circles tangent to the real axis at each rational point, see Fig. 2. All such circles are called *Ford circles*. Observe that for every pair of rational points connected by an edge in the Farey graph the corresponding Ford circles are tangent (while all other circles in the family are disjoint).

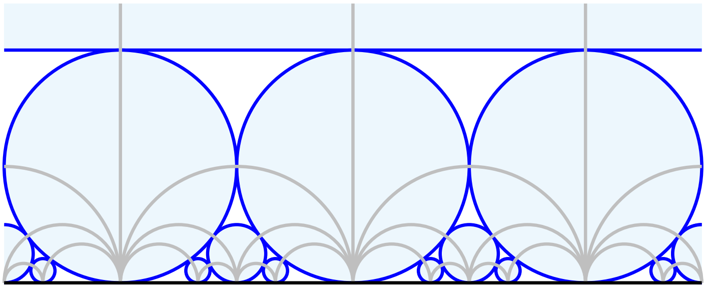

Fig. 2 Ford circles

3 3D Farey graph: tetrahedral graph $\mathcal{T}$

Recall that we denote $\sigma = e^{i\pi/3} = \frac{1+i\sqrt{3}}{2}$. Let $\mathbb{Q}(\sigma)$ be the field of fractions of the ring $\mathbb{Z}[\sigma]$.

3.1 Tetrahedral graph $\mathcal{T}$ and its symmetries

Consider a regular ideal tetrahedron $T \subset \mathbb{H}^3$ with vertices $(0, 1, \sigma, \infty)$. Let H be the group generated by reflections with respect to the faces of T.

The group H acts discretely on $\mathbb{H}^3$ and T is a fundamental domain for the action. Throughout the paper we call all images of T under the action of H *fundamental tetrahedra*.

Definition 3.1 (Tetrahedral graph, tetrahedral tiling) Denote by $\mathcal{T}$ the tiling of $\mathbb{H}^3$ by tetrahedra hT, $h \in H$, and denote by $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$, the sets of all vertices, edges, faces, and tetrahedra themselves of the tetrahedra in the tiling respectively.

The following proposition is due to Bianchi [4].

Proposition 3.2 *The set of vertices of $\mathcal{T}$ is $\widehat{\mathbb{Q}}(\sigma)$. The symmetry group of $\mathcal{T}$ is the Bianchi group $Bi(3) = PGL_2(\mathbb{Z}[\sigma]) \rtimes \langle \tau \rangle$, where τ is complex conjugation. The group $PGL_2(\mathbb{Z}[\sigma])$ of orientation-preserving symmetries of $\mathcal{T}$ acts transitively on $\mathcal{T}_0$ and $\mathcal{T}_1$.*

3.2 Edges in $\mathcal{T}$ and det-length

First, we recall the notion of irreducible fractions.

Definition 3.3 (Irreducible fractions in $\widehat{\mathbb{Q}}(\sigma)$) A *fraction* is an expression $\frac{p}{q}$, $p, q \in \mathbb{Z}[\sigma]$. A fraction $\frac{p}{q}$ is *irreducible* if for any $k \in \mathbb{Z}[\sigma]$ such that $p = kp'$, $q = kq'$ with $p', q' \in \mathbb{Z}[\sigma]$ one has $|k| = 1$.

Note that the ring $\mathbb{Z}[\sigma]$ has six units, i.e. six powers of σ. Thus, if the fraction $p/q \in \widehat{\mathbb{Q}}(\sigma)$ is irreducible, then the other five irreducible fractions $p\sigma^i/q\sigma^i$ for $i = 1, \ldots, 5$ represent the same element of $\widehat{\mathbb{Q}}(\sigma)$.

Remark 3.4 It is an easy observation that $PGL_2(\mathbb{Z}[\sigma])$ takes irreducible fractions to irreducible fractions.

Definition 3.5 (Det-length) Given two irreducible fractions $z_i = \frac{p_i}{q_i}$, $p_i, q_i \in \mathbb{Z}[\sigma], i = 1, 2$, define *det-length* $l(z_1, z_2)$ by

$$l(z_1, z_2) = \left|\det\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}\right|.$$

Remark 3.6 Assume that $z_1, z_2 \in \mathbb{Q}$. Then the value of det-length $l(z_1, z_2)$ coincides with the integer sine of the integer lattice angle with vertex at the origin and edges passing through $(z_1, 1)$ and $(z_2, 1)$ respectively, which is equal to the index of the sublattice generated by integer points of these two edges in the integer lattice (see, e.g., [16]).

Although the following proposition is well known, we give a proof for completeness.

Proposition 3.7 *The group $PGL_2(\mathbb{Z}[\sigma])$ preserves the det-length l.*

Proof Let an element of $PGL_2(\mathbb{Z}[\sigma])$ be represented by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}[\sigma])$. Let p/q and m/n be two irreducible fractions, $p, q, m, n \in \mathbb{Z}[\sigma]$. Then

$$l(M\begin{pmatrix} p \\ q \end{pmatrix}, M\begin{pmatrix} m \\ n \end{pmatrix}) = \left|\det\begin{pmatrix} ap+bq & am+bn \\ cp+dq & cm+dn \end{pmatrix}\right| =$$

$$\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \left|\det\begin{pmatrix} p & m \\ q & n \end{pmatrix}\right| = 1 \cdot l(\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} m \\ n \end{pmatrix}),$$

which shows that the det-length is preserved. $\square$

Now, we give a complete description of the edges in $\mathcal{T}$.

Proposition 3.8 *Let $\frac{p_i}{q_i}$, $p_i, q_i \in \mathbb{Z}[\sigma]$, $i = 1, 2$, be two irreducible fractions. Then the line connecting $\frac{p_1}{q_1}$ to $\frac{p_2}{q_2}$ is an edge of some tetrahedron hT, $h \in H$, if and only if $l(\frac{p_1}{q_1}, \frac{p_2}{q_2}) = 1$.*

Proof First, suppose that $p_1/q_1 = \infty$. As p_1/q_1 is irreducible, we have $q_1 = 0$ and $p_1 = \sigma^k$ for $k \in \{0, 1, \ldots, 5\}$. So,

$$\left|\det\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & p_2 \\ 0 & q_2 \end{pmatrix}\right| = |q_2|,$$

which implies that the determinant is a unit if and only if $p_2/q_2 \in \mathbb{Z}[\sigma]$. On the other hand, one can see directly from the tiling of $\mathbb{H}^3$ by ideal tetrahedra that the points in $\widehat{\mathbb{Q}}(\sigma)$ connected to ∞ by an edge of some tetrahedron hT, $h \in H$, are precisely ones lying in $\mathbb{Z}[\sigma]$.

Next, consider arbitrary irreducible fractions p_1/q_1 and p_2/q_2. According to Proposition 3.2, there exists an element $f \in PGL_2(\mathbb{Z}[\sigma])$ taking $p_1/q_1 \in \widehat{\mathbb{Q}}(\sigma)$ to ∞. In view of Remark 3.4, the consideration above implies that the statement holds for $f(p_1/q_1)$ and $f(p_2/q_2)$. Since f preserves the det-length (Proposition 3.7), we conclude that the statement holds for p_1/q_1 and p_2/q_2 as well. $\square$

Remark 3.9 Consider the hyperbolic plane Π in $\mathbb{H}^3$ containing the line $\mathrm{Im}(z) = 0$, and let $\mathcal{F}$ be the classical Farey graph lying in Π. Propositions 3.7 and 3.8 imply that $\mathcal{T}$ can also be constructed as the orbit of $\mathcal{F}$ under the action of the Bianchi group $Bi(3)$ (this follows closely the construction of *Schmidt arrangement* for any imaginary quadratic field, see [33]). Based on that, the description of faces and fundamental tetrahedra of $\mathcal{T}$ (up to the action of the Bianchi group) can be deduced from [27]. In the rest of this section we give an explicit description of faces and fundamental tetrahedra of $\mathcal{T}$ in terms of *Farey addition*.

3.3 Faces in $\mathcal{T}$ and symmetric Farey addition

3.3.1 Symmetric Farey addition in $\mathcal{T}$

As we have already mentioned in Sect. 2.2, for the irreducible adjacent vertices $\frac{p}{q}, \frac{r}{s} \in \widehat{\mathbb{Q}}$ of the classical Farey graph (where we assume $q > 0, s \geq 0$) the third point adjacent to both (and lying between them) is given by Farey addition $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$. The other vertex of the Farey graph adjacent to both $\frac{p}{q}$ and $\frac{r}{s}$ is given by $\frac{p}{q} \ominus \frac{r}{s} = \frac{p-r}{q-s}$.

Define the *symmetric Farey sum* of $\frac{p}{q}, \frac{r}{s} \in \widehat{\mathbb{Q}}$ as

$$\frac{p}{q} \oslash \frac{r}{s} = \left\{\frac{p+r}{q+s}, \frac{p-r}{q-s}\right\}.$$

The symmetric Farey sum provides the set of all vertices that form all fundamental triangles with the given edge joining $\frac{p}{q}$ and $\frac{r}{s}$.

The symmetric Farey sum has a natural generalisation to $\widehat{\mathbb{Q}}(\sigma)$.

Definition 3.10 Let p/q and r/s be irreducible fractions in $\widehat{\mathbb{Q}}(\sigma)$. The *symmetric Farey sum* of p/q and r/s is the following set

$$\frac{p}{q} \oslash \frac{r}{s} = \left\{\frac{p+\sigma^i r}{q+\sigma^i s} \in \widehat{\mathbb{Q}}(\sigma) \mid i = 0, 1, 2, 3, 4, 5\right\}.$$

Remark 3.11 Note that the resulting set does not depend on the choice of irreducible fractions representing p/q and r/s. The definition may look asymmetric (the second summand multiplied by σ^i while the first is not), however, it is symmetric in $\widehat{\mathbb{Q}}(\sigma)$.

3.3.2 Faces in $\mathcal{T}$

By the construction, all the faces in $\mathcal{T}$ are triangles. Below, we describe them in terms of symmetric Farey summation.

Proposition 3.12 *Three points $\alpha, \beta, \gamma \in \mathbb{Q}(\sigma)$ are the vertices of a triangle in the graph $\mathcal{T}$ if and only if $l(\alpha, \beta) = 1$ and*

$$\gamma \in \alpha \oslash \beta.$$

Remark 3.13 There are precisely 6 triangles adjacent to any edge of $\mathcal{T}$.

Proof Without loss of generality (keeping in mind Proposition 3.2) we set $\alpha = \frac{1}{0}$ and $\beta = \frac{0}{1}$. Then let $\gamma = \frac{p}{q}$.

By Proposition 3.8 we have $l(\alpha, \gamma) = 1 = l(\beta, \gamma)$, and therefore $q = \sigma^i$ and $p = \sigma^j$ for some i, j. It is clear that the sets $\{\frac{\sigma^j}{\sigma^i} \mid i, j = 0, \ldots, 5\}$ and $\alpha \oslash \beta$ coincide. □

3.4 Fundamental tetrahedra in $\mathcal{T}$

We now describe quadruples of vertices of the fundamental tetrahedra in $\mathcal{T}$.

Proposition 3.14 *Four points $\alpha, \beta, \gamma, \delta \in \widehat{\mathbb{Q}}(\sigma)$ are the vertices of one fundamental tetrahedron in $\mathcal{T}$ if and only if they are of the form*

$$\frac{p}{q}, \quad \frac{r}{s}, \quad \frac{p+\sigma^i r}{q+\sigma^i s}, \quad \frac{p+\sigma^{i+1} r}{q+\sigma^{i+1} s}$$

for some p, q, r, s satisfying $l(\frac{p}{q}, \frac{r}{s}) = 1$, where $i \in \mathbb{Z}$ is considered modulo 6.

Proof First, we check that the two fundamental tetrahedra containing the triangle $\alpha = \frac{1}{0}$, $\beta = \frac{0}{1}, \gamma = \frac{1}{1}$ are exactly of the required shape. Indeed, the condition of Proposition 3.8 for the edges (α, δ) and (β, δ) imply $\delta = \frac{\sigma^i}{\sigma^j}$. The condition for (γ, δ) implies that $|\sigma^i - \sigma^j| = 1$, which is satisfied if and only if $i = j \pm 1$ (modulo 6). So, either $\delta = \frac{\sigma}{1}$ or $\delta = \frac{1}{\sigma}$. This means that $(\alpha, \beta, \gamma, \delta)$ coincides with either $(\frac{1}{0}, \frac{0}{1}, \frac{1+0\cdot\sigma^0}{0+1\cdot\sigma^0}, \frac{1+0\cdot\sigma^{-1}}{0+1\cdot\sigma^{-1}})$ or $(\frac{1}{0}, \frac{0}{1}, \frac{1+0\cdot\sigma^0}{0+1\cdot\sigma^0}, \frac{1+0\cdot\sigma}{0+1\cdot\sigma})$.

By Proposition 3.12 any triangle in $\mathcal{T}$ can be written as $(\frac{p}{q}, \frac{r}{s}, \frac{p+\sigma^i r}{q+\sigma^i s})$. This triangle can be obtained from $(\frac{1}{0}, \frac{0}{1}, \frac{1}{1})$ by applying a transformation $\begin{pmatrix} p & \sigma^i r \\ q & \sigma^i s \end{pmatrix} \in PGL_2(\mathbb{Z}[\sigma])$. This map takes $\frac{\sigma}{1}$ and $\frac{1}{\sigma}$ to $\frac{p+\sigma^{i+1} r}{q+\sigma^{i+1} s}$ and $\frac{p+\sigma^{i-1} r}{q+\sigma^{i-1} s}$ respectively. So, we get the required statement (up to swapping γ and δ). □

Proposition 3.14 can be reformulated in the following way.

Corollary 3.15 *Four points $\alpha, \beta, \gamma, \delta \in \widehat{\mathbb{Q}}(\sigma)$ are the vertices of one fundamental tetrahedron in $\mathcal{T}$ if and only if they are of the form*

$$\frac{p}{q}, \quad \frac{r}{s}, \quad \frac{p+r}{q+s}, \quad \textit{and} \quad \frac{p+\sigma r}{q+\sigma s},$$

where $|ps - qr| = 1$.

4 Relations on λ-lengths

In this section we derive some relations on λ-lengths which can be considered as 3-dimensional analogues of the Ptolemy relation. We start by recalling Penner's definition of λ-lengths and one of the proofs of the hyperbolic version of the Ptolemy relation. Then we continue by using similar ideas in a 3-dimensional setting.

4.1 λ-lengths and the Ptolemy relation

The following definition was introduced by Penner [24].

Definition 4.1 (λ-length) Given two points $A, B \in \partial\mathbb{H}^n$ together with the choice of horoballs h_A and h_B centred at A and B respectively, the λ-*length* λ_{AB} of the segment AB is defined as $\exp(d/2)$, where d is the (signed) distance between the horospheres h_A and h_B. Here $d = 0$ when the horoballs h_A and h_B are tangent and $d < 0$ when the horoballs intersect non-trivially (in the latter case d is defined as the negative distance $d(a, b)$ between the intersection points $a = AB \cap h_A$ and $b = AB \cap h_B$ of the line AB with the horoballs).

The computation in the next example is a particular case of [25, Corollary 4.2].

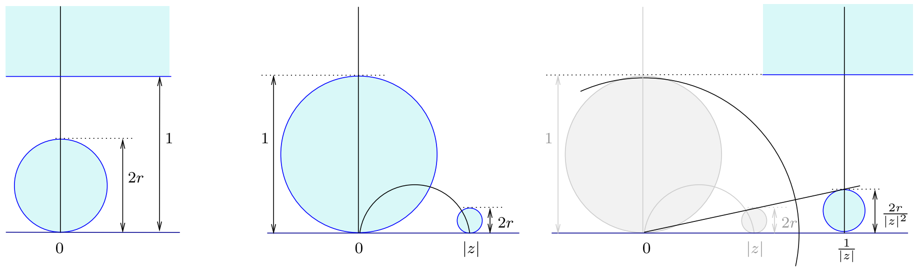

Fig. 3 To Example 4.2: (a) on the left, (b) in the middle, and inversion with respect to the unit circle on the right

Example 4.2 (Computing λ-lengths) Consider the upper halfplane model of the hyperbolic plane $\mathbb{H}^2$ with complex coordinate z.

(a) Let $A = \infty$, $B = 0$, let h_A be given by equation $\mathrm{Im}(z) = 1$ (horizontal line) and h_B be given by the (Euclidean) circle of radius r tangent to the absolute $\mathrm{Im}(z) = 0$, see Fig. 3, left. Then

$$d = d(i, 2ri) = \ln \frac{1}{2r},$$

and thus

$$\lambda_{AB} = \exp(d/2) = \frac{1}{\sqrt{2r}}.$$

(b) Let $A = 0$, $B = z$, let h_A be given by Euclidean circle of radius $1/2$ and h_B be given by Euclidean circle of radius r, then $\lambda_{AB} = \frac{|z|}{\sqrt{2r}}$. This is easy to check using inversion with respect to the unit circle centred at the origin and applying the result of (a), see Fig. 3 middle and right.

(c) The result of (b) implies the Ptolemy relation for ideal quadrilaterals in $\mathbb{H}^2$ (see Example 4.3 below).

Example 4.3 (The Ptolemy relation) It was shown by Penner in [24] that λ-lengths satisfy the Ptolemy relation:

Given an ideal quadrilateral $ABCD$ (i.e. $A, B, C, D \subset \partial\mathbb{H}^2$) *with any choice of horocycles at points A, B, C, D, the λ-lengths of sides and diagonals of $ABCD$ satisfy*

$$\lambda_{AC}\lambda_{BD} = \lambda_{AB}\lambda_{CD} + \lambda_{BC}\lambda_{AD}. \tag{4.1}$$

We prove this relation in three steps:

a: *Reducing to the quadrilateral* $(0, 1, \infty, x) \subset \mathbb{H}^2$. By using the isometry group of $\mathbb{H}^2$ we can assume that a quadrilateral $ABCD$ has vertices $0, 1, \infty, x$, we will denote it by $(0, 1, \infty, x) \subset \mathbb{H}^2$.

b: *Choosing the horocycles.* Notice that a choice of horocycles does not affect validity of (4.1). Indeed, changing a horocycle at one vertex $Z \in \{A, B, C, D\}$ changes the length d of every edge incident to Z by the same number γ, and hence the corresponding λ-lengths are multiplied by the same number $e^{\gamma/2}$, which preserves (4.1) as the relation is homogeneous. In particular, we may assume that three of the horocycles are mutually tangent.

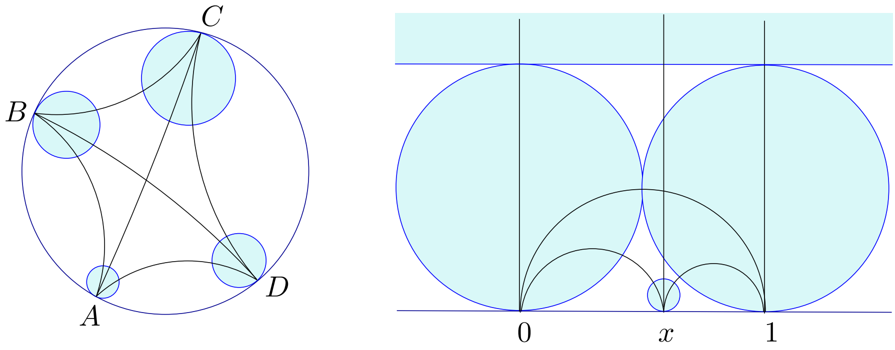

Fig. 4 The Ptolemy Theorem and one of its proofs

c: *Proof for quadrilateral* $(0, 1, \infty, x) \subset \mathbb{H}^2$, *with three mutually tangent horocycles.* It is now sufficient to show (4.1) for the ideal quadrilateral $(0, 1, \infty, x) \subset \mathbb{H}^2$, with $x \in \mathbb{R}$, $0 < x < 1$, and the horocycles chosen as follows (see also Fig. 4, right). Let $\mathrm{Im}(z) = 1$ be the horocycle at ∞, $|z - i/2| = 1/2$ and $|z - 1 - i/2| = 1/2$ be the horocycles at 0 and 1 respectively, and let $|z - x - ir| = r$ be the horocycle at x. Then three of the horocycles are mutually tangent, so that $\lambda_{0,\infty} = \lambda_{1,\infty} = \lambda_{0,1} = 1$. We can also compute using the result of Example 4.2 (b):

$$\lambda_{0,x} = \frac{x}{\sqrt{2r}}, \qquad \lambda_{1,x} = \frac{1-x}{\sqrt{2r}}, \qquad \lambda_{\infty,x} = \frac{1}{\sqrt{2r}},$$

so that we get $\lambda_{0,x} + \lambda_{x,1} = \lambda_{x,\infty}$, which is exactly the Ptolemy relation for the quadrilateral $(0, 1, \infty, x)$.

Remark 4.4 (λ-lengths and det-lengths) One can observe that det-lengths $l(u, v)$ between points $u, v \in \widehat{\mathbb{Q}}$ (see Definition 3.5) satisfy the Ptolemy relation (e.g. by combining results of [6] and [21]). Computing λ-length between points of $\widehat{\mathbb{Q}}$ with respect to Ford circles shows that both λ-lengths and det-lengths of the sides of fundamental triangles of the Farey tessellation are equal to 1. Applying the Ptolemy relation iteratively we conclude that

$$\lambda_{u,v} = l(u, v)$$

for every $u, v \in \widehat{\mathbb{Q}}$.

4.2 The Ptolemy relation in $\mathcal{T}$

Definition 4.5 (Standard horospheres at $\mathcal{T}_0$: counterpart of Ford circles) Consider the fundamental tetrahedron $T = (0, 1, \sigma, \infty)$ of $\mathcal{T}$, choose four pairwise tangent horospheres at its vertices (i.e. horospheres represented by three balls of radii $1/2$ and a horizontal plane at the height 1). Notice that any isometry of $\mathbb{H}^3$ preserving T takes the four horospheres to the same four horospheres. Due to the action of $PGL_2(\mathbb{Z}[\sigma])$, we can pick a horosphere at each point of $\mathcal{T}_0 = \widehat{\mathbb{Q}}(\sigma)$ (so that at every fundamental tetrahedron the four horospheres are mutually tangent). This choice of horospheres at $\widehat{\mathbb{Q}}(\sigma)$ will be called *standard*.

Remark 4.6 The definition above is equivalent to the definition of *Ford spheres* given by Northshield [23].

In this section, we assume by default the standard choice of horospheres at $\mathcal{T}_0$. This assumption simplifies the proofs without affecting the statements, since the relations considered below are all homogeneous. The only exception here will be Theorem 4.12, as the standard choice of horospheres is essential there.

Theorem 4.7 *Let $A_1A_2A_3A_4$ be a fundamental tetrahedron with vertices in $\widehat{\mathbb{Q}}(\sigma)$, and choose any $X \in \widehat{\mathbb{Q}}(\sigma)$ distinct from A_i. Let $\lambda_i = \lambda_{XA_i}$ be the λ-length of XA_i, $i = 1, \dots, 4$. Then*

$$\sum_{i=1}^{4} \lambda_i^4 = \sum_{1 \le i < j \le 4} \lambda_i^2 \lambda_j^2. \tag{4.2}$$

Proof The setting of the theorem is illustrated in Fig. 5, left. Applying an isometry, we may assume that $A_1A_2A_3A_4 = (0, 1, \sigma, \infty)$ and that $X = z \in \widehat{\mathbb{Q}}(\sigma)$. Let the horosphere at z be represented by a Euclidean sphere of radius $r/2$. Then from Example 4.2(b) applied to the plane through points z, A_i, ∞ we get that

$$\lambda_i = \frac{|z - A_i|}{\sqrt{2r}},\ i = 1, 2, 3, \qquad \lambda_4 = \frac{1}{\sqrt{2r}}.$$

At this point, it is sufficient to check that the values $a_i = |z - A_i|^2$, $i = 1, 2, 3$ and $a_4 = 1$ satisfy the equation

$$\sum_{i=1}^{4} a_i^2 = \sum_{1 \le i < j \le 4} a_i a_j.$$

This is a straightforward computation after the following substitutions are implemented: $a_1 = z\overline{z}$, $a_2 = (z - 1)(\overline{z} - 1)$, and $a_3 = (z - \sigma)(\overline{z} - \overline{\sigma})$. □

4.3 λ-lengths and Soddy-Gosset theorem

It was noted to the authors by Arthur Baragar and Ian Whitehead that Eq. 4.2 can be considered as a corollary of the *Soddy-Gosset Theorem* relating the radii of five mutually tangent spheres in Euclidean space (see e.g. [17]). Given $n + 2$ mutually tangent spheres in $\mathbb{R}^n$ of radii r_i, $i = 1, \dots, n + 2$, denote $k_i = 1/r_i$. Then the Soddy-Gosset theorem asserts that

$$\left(\sum_{i=1}^{n+2} k_i\right)^2 = n \sum_{i=1}^{n+2} k_i^2. \tag{4.3}$$

The plane version of this theorem (4 mutually tangent circles in $\mathbb{E}^2$) is called the *Descartes Circle Theorem*. We will use the version with $n = 3$ and five spheres, one of which is of infinite radius, and hence has $k_5 = 0$. Equation 4.3 is then reduced to

$$\sum_{1 \le i < j \le 4} k_i k_j = \sum_{i=1}^{4} k_i^2, \tag{4.4}$$

cf. Equation 4.2.

Now, consider the configuration of points A_1, A_2, A_3, A_4 and X as in Theorem 4.7: here, $A_1A_2A_3A_4$ is a fundamental tetrahedron of $\mathcal{T}$ (or any other regular ideal tetrahedron, i.e. an image of the fundamental tetrahedron under any isometry of $\mathbb{H}^3$) and X is any point on the absolute of $\mathbb{H}^3$. After applying an isometry we may assume that $X = \infty$ (we use the upper halfspace model with coordinates $\{(z, t) \mid z \in \mathbb{C}, t \in \mathbb{R}_+\}$).

Next, we choose four mutually tangent horospheres centred at points A_1, A_2, A_3, A_4 (this is possible as $A_1A_2A_3A_4$ is a regular ideal tetrahedron). The four horospheres together with the plane representing the absolute play the role of the five mutually tangent spheres in the Soddy-Gosset theorem. Let r_i be the radius of the horosphere centred at A_i. Applying the Soddy-Gosset theorem as in Eq. 4.4, we get

$$\sum_{1\le i<j\le 4} \frac{1}{r_i}\frac{1}{r_j} = \sum_{i=1}^{4} \frac{1}{r_i^2}. \tag{4.5}$$

We also choose the horosphere h_X centred at X as the plane given by $t = 1$ and denote by λ_i the λ-lengths of A_iX. In view of the computation in Example 4.2(a) we have

$$\lambda_i = \frac{1}{\sqrt{2r_i}},$$

so, replacing $1/r_i$ by $2\lambda_i^2$ in Eq. 4.5 we get Eq. 4.2.

4.4 Det-lengths and λ-lengths

In this section, we show that det-lengths between points of $\widehat{\mathbb{Q}}(\sigma)$ coincide with λ-lengths with respect to standard horospheres.

We start by showing that det-lengths satisfy the counterpart of Relation 4.2.

Theorem 4.8 *For points $A_1A_2A_3A_4$ and X as in Theorem 4.7, define $l_i := l_{XA_i}$. Then*

$$\sum_{i=1}^{4} l_i^4 = \sum_{1\le i<j\le 4} l_i^2 l_j^2.$$

Proof As in the proof of Theorem 4.7, we assume that $A_1A_2A_3A_4 = (0, 1, \sigma, \infty)$ and that $X = z \in \mathbb{Q}(\sigma)$. We can write $z = \frac{p}{q} = \frac{p_1+p_2\sigma}{q_1+q_2\sigma}$, $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, and hence obtain

$$l_1 = l_{0,z} = |p|, \qquad l_2 = l_{1,z} = |p-q|, \qquad l_3 = l_{\sigma,z} = |p-q\sigma|, \qquad l_4 = l_{\infty,z} = |q|.$$

Taking into account that $|p|^2 = p\overline{p} = p_1^2 + p_2^2 + p_1p_2$ (and that there are similar expressions for $|q|^2$, $|p-q|^2$ and $|p-\sigma q|^2$), one can easily check the identity claimed in the theorem. $\square$

Remark 4.9 Theorem 4.8 is a counterpart of the 2-dimensional statement mentioned in Remark 4.4.

Remark 4.10 Since the equations above are homogeneous, Theorem 4.8 is not affected when the fraction for z is not irreducible. Indeed, when we change the horosphere (resp. multiplying the numerator and denominator by the same factor), each of the summands in the sum is multiplied by the same factor.

Remark 4.11 The relation in Theorems 4.7 is quadratic with respect to λ_4^2. It has two positive roots, these roots have the following geometrical meaning. Let A_4' be the point obtained from A_4 by the reflection with respect to the plane $A_1A_2A_3$ (i.e. $A_1A_2A_3A_4'$ is the fundamental tetrahedron adjacent to $A_1A_2A_3A_4$ along $A_1A_2A_3$). We assume that A_4' lies in the same

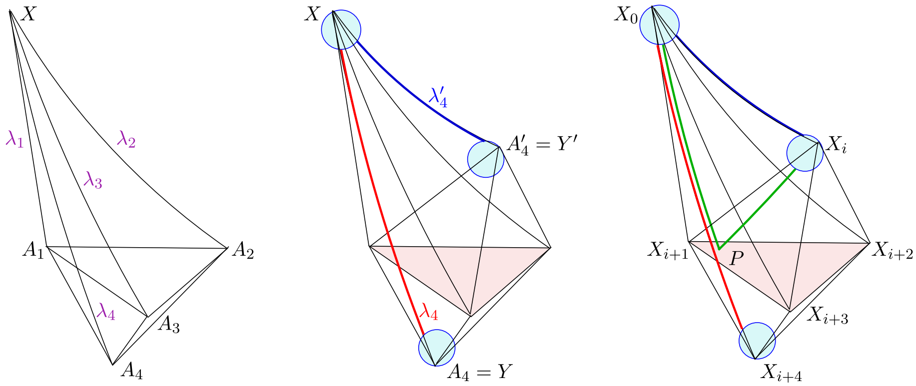

Fig. 5 Notation for Theorems 4.7, 4.12, and 4.16

half-space with respect to $A_1A_2A_3$ as X while A_4 lies in the other half-space, see Fig. 5, middle.

Denote $\lambda_4' = \lambda_{XA_4'}$ the corresponding λ-length. Then λ_4 and λ_4' are the two roots of the quadratic equation in Theorem 4.7 (as in Theorem 4.7, we do not make any assumptions on the position of point X).

Remark 4.11 gives rise to the following theorem.

Theorem 4.12 *Let $X, Y \in \widehat{\mathbb{Q}}(\sigma)$ be two irreducible fractions. Let the standard horosphere be chosen at every point of $\widehat{\mathbb{Q}}(\sigma)$. Then $\lambda_{XY} = l_{XY}$.*

Proof We consider two cases: either the geodesic XY lies in a plane containing a face of a fundamental tetrahedron of $\mathcal{T}$ or it does not.

First, suppose that XY lies in a plane Π containing a face of a fundamental tetrahedron of $\mathcal{T}$. The tessellation $\mathcal{T}$ induces a Farey triangulation on Π. Denote by S the set of triangles whose interior is intersected by the geodesic XY, let $N = |S|$. We proceed by induction on N. If $N = 0$, then XY is an edge of some triangle (and edge of $\mathcal{T}$) and $l_{XY} = 1 = \lambda_{XY}$ (by Proposition 3.8 and since the horospheres are mutually tangent). Otherwise, $N \geq 2$.

Let $T_Y \in S$ be the triangle with vertex Y. Denote by Y_1 and Y_2 the other two vertices of T_Y. Then the set of triangles intersected by geodesics XY_1 and XY_2 are subsets of S not containing T_Y, so their det-lengths coincide with λ-lengths by the induction assumption.

Consider the quadrilateral with vertices X, Y_1, Y_2, Y. As we have seen above, for all its edges the det-lengths coincide with λ-lengths, as well as for Y_1Y_2. Then both λ_{XY} and l_{XY} satisfy the same Ptolemy relation (see Remark 4.9), and thus coincide as well.

Now, suppose that XY does not lie in the plane of a face of any fundamental tetrahedron of $\mathcal{T}$, the proof is very similar to the two-dimensional case. Denote by S the set of (closed) fundamental tetrahedra intersected by XY in an interior point of $\mathbb{H}^3$, let $N = |S|$. Again, we proceed by induction on N.

Let $T_Y \in S$ be the fundamental tetrahedron with vertex Y. Denote by Y_1, Y_2, Y_3 the other three vertices of T_Y. For every $i = 1, 2, 3$ either the set of tetrahedra intersected by the geodesic XY_i is a subset of S not containing T_Y, or XY_i belongs to the plane of a face of some fundamental tetrahedron. By the induction assumption (and the two-dimensional case), the det-lengths of all XY_i coincide with their λ-lengths.

Consider the triangular bipyramid with vertices X, Y and Y_i. For all its edges the det-lengths coincide with λ-lengths. By Theorems 4.7 and 4.8, the values λ_{XY} and l_{XY} satisfy the same quadratic equation, and we are left to show that it should be the same root.

Let T'_Y be the fundamental tetrahedron adjacent to T_Y along the face $Y_1Y_2Y_3$, and let Y' be its fourth vertex. From Remark 4.11 we know that $\lambda_{XY'}$ and $l_{XY'}$ are also the roots of the same quadratic equation as λ_{XY} and l_{XY}. By the inductive assumption, $\lambda_{XY'} = l_{XY'}$ (as XY' intersect a smaller number of tetrahedra than N). This implies that each of λ_{XY} and l_{XY} is the *other* root of the same quadratic equation, which implies $\lambda_{XY} = l_{XY}$. □

Remark 4.13 Theorem 4.12 implies that the set of all λ-lengths of arcs with ends at $\widehat{\mathbb{Q}}(\sigma)$ coincides with the set of absolute values of all elements of $\mathbb{Z}[\sigma]$.

Remark 4.14 There is another, purely computational, approach to Theorem 4.12. Given an irreducible fraction $p/q \in \widehat{\mathbb{Q}}(\sigma)$, the radius of the standard horosphere at p/q is equal to $1/2|q|^2$ (see e.g. [26] by Rieger). Using Example 4.2(a), one can see that the λ-length between p/q and $\infty = 1/0$ is equal to the corresponding det-length. Now, every point of $\widehat{\mathbb{Q}}(\sigma)$ can be taken to $\sigma^k/0$ by an element of $PSL_2(\mathbb{Z}[\sigma])$, and as both λ-length and det-length are invariant, the result follows.

Remark 4.15 It was pointed out to the authors by the anonymous referee that the proof mentioned in Remark 4.14 can be easily extended to provide a more general statement. Namely, given any two pairs of complex numbers (p_1, q_1) and (p_2, q_2) and horospheres H_1, H_2 of Euclidean radii $1/2|q_1|^2$ and $1/2|q_2|^2$ centered at points $X_1 = p_1/q_1$ and $X_2 = p_2/q_2$ respectively, the lambda length $\lambda_{X_1X_2}$ computed with respect to the horospheres H_1 and H_2 is equal to $|p_1q_2 - p_2q_1|$. The proof is similar to the one of Proposition 6.2 from [32] by Springborn. Here, the counterpart of Rieger's result is a $\mathbb{C}$-version of the PGL_2-equivariance of the corresponding horospheres, see e.g. [32, Proposition 5.1].

4.5 Linear relation

Consider a geodesic γ connecting two points X and Y in $\widehat{\mathbb{Q}}(\sigma)$. Consider all fundamental tetrahedra crossed by γ. Denote $X = X_0$, let X_1, X_2, X_3 be the other three vertices of the first tetrahedron crossed by γ, and let X_i, $i > 3$, be the vertices added on the way when tetrahedra are attached one by one along γ.

Theorem 4.16 *Let $b_i = \lambda^2_{X_0X_i}$. Then*

(a) $b_i + b_{i+4} = b_{i+1} + b_{i+2} + b_{i+3}$;
(b) $b_i \in \mathbb{Z}$;
(c) $b_i < b_{i+4}$.

Proof In view of Theorem 4.7, the values b_i and b_{i+4} are the two roots of the following quadratic equation

$$x^2 - x(b_{i+1} + b_{i+2} + b_{i+3}) + b^2_{i+1} + b^2_{i+2} + b^2_{i+3} + b_{i+2}b_{i+3} + b_{i+1}b_{i+3} + b_{i+1}b_{i+2} = 0,$$

which implies $b_i + b_{i+4} = b_{i+1} + b_{i+2} + b_{i+3}$, which settles part (a).

Applying equation (a) iteratively we see that $b_i \in \mathbb{Z}$ for all i, as $b_1 = b_2 = b_3 = 1$ and $b_4 = 3$ (where the latter follows from Theorem 4.7), which proves (b).

To prove (c), notice that X_0 is separated from X_{i+4} by the plane $X_{i+1}X_{i+2}X_{i+3}$, and X_i is the reflection image of X_{i+4} with respect to $X_{i+1}X_{i+2}X_{i+3}$. To compare $\lambda_{X_0X_{i+4}}$ with $\lambda_{X_0X_i}$

notice that the distance from the horosphere at X_0 to the horosphere at X_{i+4} is equal to the sum of the distance from the horosphere at X_0 to the point $P = X_{i+1}X_{i+2}X_{i+3} \cap X_0X_{i+4}$ and the distance from P to the horosphere at X_i, see Fig. 5, right. The latter sum is larger than the distance from the horosphere at X_0 to the horosphere at X_i, which implies $b_{i+4} > b_i$. □

Remark 4.17 Equation (a) of Theorem 4.16 is a counterpart of the following relation in the classical Farey graph:

$$b_i + b_{i+3} = 2(b_{i+1} + b_{i+2}),$$

where $b_i = \lambda^2_{X_0X_i}$ and triangles $X_iX_{i+1}X_{i+2}$ and $X_{i+1}X_{i+2}X_{i+3}$ are fundamental ones.

4.6 3D Ptolemy relation: general formula

In this section we generalise relation (4.2) to any pair of adjacent ideal tetrahedra in $\mathbb{H}^3$.

Theorem 4.18 *Let $X_1, \dots, X_5 \in \widehat{\mathbb{C}} = \partial\mathbb{H}^3$ be 5 distinct points. Suppose that there are horospheres chosen at these points. Let $\lambda_{ij} = \lambda_{X_iX_j}$. Then*

$$\sum_{(ij)(klm)\in S_5} \lambda^4_{ij}\lambda^2_{kl}\lambda^2_{lm}\lambda^2_{mk} = \sum_{(ijklm)\in S_5} \lambda^2_{ij}\lambda^2_{jk}\lambda^2_{kl}\lambda^2_{lm}\lambda^2_{mi},$$

where $(ij)(klm)$ and $(ijklm)$ denote cycle decompositions of permutations in the symmetric group S_5.

Proof First, we apply the action of $PSL_2(\mathbb{C})$ on $\mathbb{H}^3$ to map X_1, X_2, X_3 to $0, 1, \infty$ (this does not change any of the λ_{ij}). Suppose that X_4 and X_5 are mapped to points $z, w \in \mathbb{C}$. Notice that the choice of horospheres does not affect whether the equation is true or not (since the equation is homogeneous). So, we can take three tangent horospheres at $0, 1, \infty$ and the two other horospheres tangent to the horosphere at ∞. This makes five of the λ-lengths equal to 1. The other five we can compute using the formula from Example 4.2, and check that the obtained expressions satisfy the equation, we omit the details. □

Remark 4.19 It was noted to the authors by Ivan Izmestiev that Theorem 4.18, as well as similar formulae relating λ-lengths between $n + 2$ points at the boundary of n-dimensional hyperbolic space for any $n \geq 4$, can be obtained as follows. Given an isotropic vector u in the hyperboloid model of $\mathbb{H}^n$, every horosphere centred at u can be written as $h_{u,c} = \{x \in \mathbb{H}^n \mid \langle x, cu\rangle = -1/\sqrt{2}\}$ for some $c \in \mathbb{R}_+$. Then for two points $u, v \in \partial\mathbb{H}^n$ the λ-length $\lambda_{u,v}$ with respect to the horospheres $h_{u,a}$ and $h_{v,b}$ can be written as $\lambda_{u,v} = \sqrt{-\langle au, bv\rangle}$, see [25, Lemma 4.1]. Now, the formula for $n + 2$ points $\{u_i\}$ can be obtained by expanding the determinant of the matrix $(\langle c_iu_i, c_ju_j\rangle)$, which vanishes since the vectors are linearly dependent.

Remark 4.20 As it was observed by the anonymous referee, one can avoid using the non-conformal model of $\mathbb{H}^n$ in the proof sketched in Remark 4.19 by considering the Euclidean distance matrix instead (whose determinant also vanishes), and then applying [25, Corollary 4.2] to convert Euclidean distances to λ-lengths to obtain the required relation.

Remark 4.21 Suppose we are given a triangulated cusped hyperbolic 3-manifold, where all tetrahedra in the triangulation are ideal. Consider two tetrahedra in the triangulation sharing

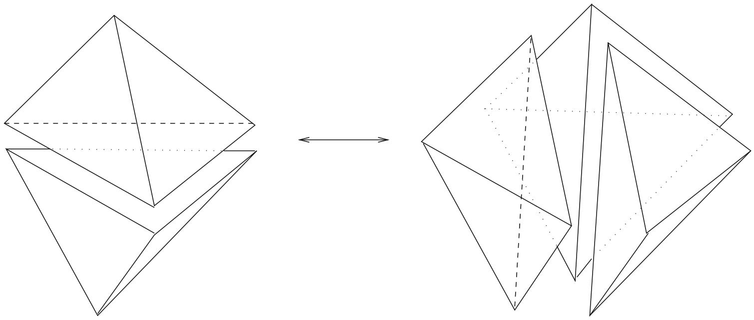

Fig. 6 2–3 move

a facet. Assuming the union of the two tetrahedra is a convex bipyramid, one can change triangulation by applying a 2–3 *move*, see Fig. 6. Theorem 4.18 describes the relations between λ-lengths of the arcs of the initial and the resulting triangulations.

5 SL_2-tilings over $\mathbb{Z}[\sigma]$

Taking their origin in Conway-Coxeter frieze patterns [6, 7], SL_2-tilings were introduced in [1] and became a topic of a rapidly growing field of studies connecting combinatorics, geometry, cluster algebras and many related domains, see the review by Morier-Genoud [19] concerning the connections. It was shown by Short [30] that all SL_2-tilings over $\mathbb{Z}$ can be classified in terms of pairs of paths on the Farey graph $\mathcal{F}$.

In this section, we provide a classification of SL_2-tilings with entries in $\mathbb{Z}[\sigma]$ in terms of pairs of paths in $\mathcal{T}$, generalising the results of [30].

5.1 Normalised paths

Definition 5.1 We say that $(v_i)_{i=k}^n$ for $k \in \{-\infty, 0\}$ and $n \in \mathbb{Z}_+ \cup \{+\infty\}$ is a *path* on $\mathcal{T}$ if $v_i v_{i+1}$ is an edge of $\mathcal{T}$ for every i satisfying $k \le i < n$.

We will abuse notation by writing (v_i) if the path is bi-infinite.

Definition 5.2 A path $(v_i)_{i=k}^n$ represented by irreducible fractions $v_i = p_i/q_i$ for all i is called *normalised* if

$$\det \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = 1$$

for all admissible i.

Remark 5.3 Given an irreducible fraction p_0/q_0 representing $v_0 \in \widehat{\mathbb{Q}}(\sigma)$ (so that we fix one of the six fractions of the form $\sigma^k p_0/\sigma^k q_0$), there is a unique fraction p_1/q_1 representing $v_1 \in \widehat{\mathbb{Q}}(\sigma)$ and satisfying the condition for a normalised path given in Definition 5.2. Similarly each p_i/q_i in the normalised path (v_i) can be uniquely reconstructed from p_0/q_0. Therefore, for every nontrivial path there are precisely six different normalisations (obtained from one of them by multiplying both p_i and q_i by $\sigma^{(-1)^i \cdot k}$).

5.2 SL_2-tilings

Definition 5.4 (SL_2-tiling)
Let $M = (m_{ij})$ be a bi-infinite matrix, $i, j \in \mathbb{Z}$, $m_{ij} \in R$, where R is an integral domain. M is an SL_2-*tiling* if

$$\begin{pmatrix} m_{i,j} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} \in SL_2(R)$$

for any $i, j \in \mathbb{Z}$.
An SL_2-tiling M is *tame* if

$$\det \begin{pmatrix} m_{i,j} & m_{i,j+1} & m_{i,j+2} \\ m_{i+1,j} & m_{i+1,j+1} & m_{i+1,j+2} \\ m_{i+2,j} & m_{i+2,j+1} & m_{i+2,j+2} \end{pmatrix} = 0$$

for any $i, j \in \mathbb{Z}$.

Given two bi-infinite normalised paths (u_i) and (v_j), where $u_i = p_i/q_i$, $v_j = r_j/s_j$, consider the numbers

$$m_{ij} = \det \begin{pmatrix} p_i & r_j \\ q_i & s_j \end{pmatrix}. \tag{5.1}$$

Proposition 5.5 ([30]) *If $R = \mathbb{Z}$, the bi-infinite matrix (m_{ij}) is a tame SL_2-tiling. Moreover, (5.1) provides a bijection between tame SL_2-tilings (modulo multiplication of all entries by -1) and pairs of bi-infinite normalised paths in the Farey graph $\mathcal{F}$ (considered up to the simultaneous action of $SL_2(\mathbb{Z})$).*

5.3 SL_2-tilings over $\mathbb{Z}[\sigma]$ and pairs of paths in $\mathcal{T}$

The goal of this section is to prove a counterpart of Proposition 5.5 based on the paths in $\mathcal{T}$ (see Theorems 5.18 and 5.20).

Definition 5.6 Consider two bi-infinite normalised paths (p_i/q_i) and (r_j/s_j) in $\mathcal{T}$. The *scalar product of two paths* is a matrix $(m_{i,j})$ where

$$m_{i,j} = p_i r_j + q_i s_j$$

for all i and j. In this case we write $(m_{i,j}) = (p_i/q_i) \cdot (r_j/s_j)$.

Remark 5.7 In terms of matrices, the scalar product of two paths can be understood as a matrix multiplication:

$$(m_{i,j}) = \begin{pmatrix} \dots & \dots \\ p_{i-1} & q_{i-1} \\ p_i & q_i \\ p_{i+1} & q_{i+1} \\ \dots & \dots. \end{pmatrix} \begin{pmatrix} \dots & r_{j-1} & r_j & r_{j+1} & \dots \\ \dots & s_{j-1} & s_j & s_{j+1} & \dots \end{pmatrix}.$$

Proposition 5.8 *The scalar product $(m_{i,j})$ of two bi-infinite normalised paths (p_i/q_i) and (r_j/s_j) in $\mathcal{T}$ forms a tame $SL_2(\mathbb{Z}[\sigma])$-tiling.*

$$
\begin{array}{c|ccccccc}
 & \cdots & \frac{0}{-\overline{\sigma}} & \frac{\sigma}{\overline{\sigma}} & \frac{-1}{1} & \frac{-1}{0} & \frac{0}{-1} & \cdots \\
\hline
\vdots & \ddots & & & \vdots & & & \iddots \\
\frac{1}{0} & & 0 & \sigma & -1 & -1 & 0 & \\
\frac{0}{1} & & -\overline{\sigma} & \overline{\sigma} & 1 & 0 & -1 & \\
\frac{-1}{-1} & \cdots & \overline{\sigma} & -1 & 0 & 1 & 1 & \cdots \\
\frac{\overline{\sigma}}{-\sigma} & & 1 & 0 & -1 & -\overline{\sigma} & \sigma & \\
\frac{\overline{\sigma}}{0} & & 0 & 1 & -\overline{\sigma} & -\overline{\sigma} & 0 & \\
\vdots & \iddots & & & \vdots & & & \ddots
\end{array}
$$

Fig. 7 $SL_2(\mathbb{Z}[\sigma])$-tiling as a scalar product of two normalised paths in $\mathcal{T}$. The shaded region is the initial block, all shifted blocks are obtained from that one by multiplication by powers of σ (see Example 5.10)

Proof A direct calculation shows that the scalar product forms an $SL_2(\mathbb{Z}[\sigma])$-tiling (cf. the proof of Proposition 3.7). The tameness follows from the fact that $(m_{i,j})$ is a product of two (infinite) matrices of rank two. □

Remark 5.9 Note that tameness is equivalent to the requirement that every row of the matrix $(m_{i,j})$ is a linear combination of two adjacent rows, say $(m_{0,j})$ and $(m_{1,j})$.

Example 5.10 Consider two periodic paths in $\mathcal{T}$: $(u_i) = (\dots, \infty, 0, 1, \sigma, \infty, \dots)$, and $(v_j) = (\dots, 0, \sigma^2, -1, \infty, 0, \dots)$. Normalise the paths as follows:

$$
u_0 = \frac{1}{0},\ u_1 = \frac{0}{1},\ u_2 = \frac{-1}{-1},\ u_3 = \frac{\overline{\sigma}}{-\sigma},\ \text{and}\ \ u_{i+4} = \frac{\sigma^{(-1)^{i+1}}}{\overline{\sigma}^{(-1)^{i+1}}} u_i;
$$

$$
v_0 = \frac{0}{-\overline{\sigma}},\ v_1 = \frac{\sigma}{\overline{\sigma}},\ v_2 = \frac{-1}{1},\ v_3 = \frac{-1}{0},\ \text{and}\ \ v_{i+4} = \frac{\sigma^{(-1)^{i}}}{\overline{\sigma}^{(-1)^{i}}} v_i.
$$

The initial four values and the recursive formulae define the (bi-infinite) normalised paths uniquely. By taking the scalar product, we obtain a tiling shown in Fig. 7. The tiling has a block structure with the initial block of size 4×4 shown in Fig. 7, with

$$
m_{i+4k,j+4l} = \sigma^{l-k} m_{i,j}.
$$

Note that we get a counterpart of an antiperiodic SL_2-tiling considered in [21, Section 3.3]. The reason for this is the periodicity of the paths (u_i) and (v_j).

Proposition 5.11 *Any tame $SL_2(\mathbb{Z}[\sigma])$-tiling is a scalar product of two normalised paths.*

Proof Let (m_{ij}) be a tame $SL_2(\mathbb{Z}[\sigma])$-tiling. First, we construct the path (r_j/s_j). Set $(p_0/q_0, p_1/q_1) = (1/0, 0/1)$. Then we can set $r_j = m_{0,j}$, $s_j = m_{1,j}$. This would imply that the condition $m_{i,j} = p_i r_j + q_i s_j$ is satisfied for $i = 0$ and all j. Notice that since

$$\det \begin{pmatrix} m_{0,j} & m_{0,j+1} \\ m_{1,j} & m_{1,j+1} \end{pmatrix} = 1,$$

each fraction $r_j/s_j = m_{0,j}/m_{1,j}$ is irreducible, and by the same reason the path (r_j/s_j) is normalised.

Now, we use the path (r_j/s_j) and the values $(p_0/q_0, p_1/q_1) = (1/0, 0/1)$ to construct (p_i/q_i) for all i. We know from tameness of $(m_{i,j})$ that every row of $(m_{i,j})$ is a linear combination of the rows $(m_{0,j})$ and $(m_{1,j})$. Denote by α_i and β_i the coefficients of this linear combination for the i-th row, i.e.

$$m_{i,j} = \alpha_i m_{0,j} + \beta_i m_{1,j}.$$

The values α_i and β_i are uniquely defined from

$$\begin{pmatrix} \alpha_i & \beta_i \end{pmatrix} \begin{pmatrix} m_{0,0} & m_{0,1} \\ m_{1,0} & m_{1,1} \end{pmatrix} = \begin{pmatrix} m_{i,0} & m_{i,1} \end{pmatrix}, \tag{5.2}$$

moreover, $\alpha_i, \beta_i \in \mathbb{Z}[\sigma]$ as $\det \begin{pmatrix} m_{0,0} & m_{0,1} \\ m_{1,0} & m_{1,1} \end{pmatrix} = 1$.

We set $p_i = \alpha_i$ and $q_i = \beta_i$. As $r_j = m_{0,j}$ and $s_j = m_{1,j}$, this implies that $m_{i,j} = p_i m_{0,j} + q_i m_{1,j} = p_i r_j + q_i s_j$ for all i, j.

We are left to check that the constructed path (p_i/q_i) is normalised. Notice that (5.2) implies that α_i/β_i is an irreducible fraction. Also, notice that

$$\begin{pmatrix} \alpha_i & \beta_i \\ \alpha_{i+1} & \beta_{i+1} \end{pmatrix} \begin{pmatrix} m_{0,0} & m_{0,1} \\ m_{1,0} & m_{1,1} \end{pmatrix} = \begin{pmatrix} m_{i,0} & m_{i,1} \\ m_{i+1,0} & m_{i+1,1} \end{pmatrix}$$

implies that $\det \begin{pmatrix} \alpha_i & \beta_i \\ \alpha_{i+1} & \beta_{i+1} \end{pmatrix} = 1$, and hence, the path is normalised. □

Remark 5.12 Notice that once the initial values $(p_0/q_0, p_1/q_1) = (1/0, 0/1)$ are fixed, the sequences (p_i/q_i) and (s_j/r_j) are uniquely determined by $(m_{i,j})$.

Remark 5.13 An easy computation shows that the action of $SL_2(\mathbb{Z}[\sigma])$ on paths takes normalised paths to normalised ones (cf. the proof of Proposition 3.7).

Lemma 5.14 *Let (p_i/q_i) and (r_j/s_j) be two normalised paths. Let $A \in SL_2(\mathbb{Z}[\sigma])$ and consider*

$$\begin{pmatrix} p_i' \\ q_i' \end{pmatrix} = A \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad \begin{pmatrix} r_j' \\ s_j' \end{pmatrix} = (A^T)^{-1} \begin{pmatrix} r_j \\ s_j \end{pmatrix}.$$

Let M and M' be $SL_2(\mathbb{Z}[\sigma])$-tilings given by

$$M = ((p_i/q_i) \cdot (r_j/s_j)) \quad \text{and} \quad M' = ((p_i'/q_i') \cdot (r_j'/s_j')).$$

Then M and M' coincide.

Proof This is a direct computation:

$$(m'_{ij}) = \begin{pmatrix} \cdots & \cdots \\ p'_{i-1} & q'_{i-1} \\ p'_i & q'_i \\ p'_{i+1} & q'_{i+1} \\ \cdots & \cdots. \end{pmatrix} \begin{pmatrix} \cdots & r'_{j-1} & r'_j & r'_{j+1} & \cdots \\ \cdots & s'_{j-1} & s'_j & s'_{j+1} & \cdots \end{pmatrix} =$$

$$\begin{pmatrix} \cdots & \cdots \\ p_{i-1} & q_{i-1} \\ p_i & q_i \\ p_{i+1} & q_{i+1} \\ \cdots & \cdots. \end{pmatrix} A^T (A^T)^{-1} \begin{pmatrix} \cdots & r_{j-1} & r_j & r_{j+1} & \cdots \\ \cdots & s_{j-1} & s_j & s_{j+1} & \cdots \end{pmatrix} = (m_{ij}).$$

□

Remark 5.15 Notice that different normalisations of the paths may lead to different $SL_2(\mathbb{Z}[\sigma])$-tilings. More precisely, if p_i and q_i are multiplied by $\sigma^{(-1)^i k}$ (cf. Remark 5.3), then the i-th row of $(m_{i,j})$ is multiplied by $\sigma^{(-1)^i k}$. Similarly, changing the normalisation of (r_j/s_j) affects columns of $(m_{i,j})$. This leads to 36 tame $SL_2(\mathbb{Z}[\sigma])$-tilings, generically 18 of them are distinct tilings (as a simultaneous multiplication of columns and rows by -1 preserves the tiling).

Definition 5.16 We call $SL_2(\mathbb{Z}[\sigma])$-tilings obtained from different normalisations of the same pair of paths *equivalent*.

Lemma 5.17 *Let* $((p_i/q_i), (r_j/s_j))$ *and* $((p'_i/q'_i), (r'_j/s'_j))$ *be two pairs of normalised paths and* $M = (p_i/q_i) \cdot (r_j/s_j)$, $M' = (p'_i/q'_i) \cdot (r'_j/s'_j)$ *be their scalar products.*

If M *is equivalent to* M' *then there exists a matrix* $A \in SL_2(\mathbb{Z}[\sigma])$ *and normalisations* p''_i/q''_i *and* r''_j/s''_j *of* p'_i/q'_i *and* r'_j/s'_j *respectively such that*

$$\begin{pmatrix} p''_i \\ q''_i \end{pmatrix} = A \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad \begin{pmatrix} r''_j \\ s''_j \end{pmatrix} = (A^T)^{-1} \begin{pmatrix} r_j \\ s_j \end{pmatrix}.$$

Proof First, as $M = (p_i/q_i) \cdot (r_j/s_j)$ is equivalent to $M' = (p'_i/q'_i) \cdot (r'_j/s'_j)$, there exists normalisations p''_i/q''_i and r''_j/s''_j of p'_i/q'_i and r'_j/s'_j such that $M = (p''_i/q''_i) \cdot (r''_j/s''_j)$.

Next, let $X, Y \in SL_2(\mathbb{Z}[\sigma])$ be matrices such that

$$X \begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Y \begin{pmatrix} p''_0 & p''_1 \\ q''_0 & q''_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the following two pairs of sequences $\big((\overline{p}_i/\overline{q}_i), (\overline{r}_j/\overline{s}_j)\big)$ and $\big((\overline{p}_i{}''/\overline{q}_i{}''), (\overline{r}_j{}''/\overline{s}_j{}'')\big)$ given by:

$$\begin{pmatrix} \overline{p}_i \\ \overline{q}_i \end{pmatrix} = X \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad \begin{pmatrix} \overline{r}_j \\ \overline{s}_j \end{pmatrix} = (X^T)^{-1} \begin{pmatrix} r_j \\ s_j \end{pmatrix}$$

and

$$\begin{pmatrix} \overline{p}_i{}'' \\ \overline{q}_i{}'' \end{pmatrix} = Y \begin{pmatrix} p''_i \\ q''_i \end{pmatrix}, \qquad \begin{pmatrix} \overline{r}_j{}'' \\ \overline{s}_j{}'' \end{pmatrix} = (Y^T)^{-1} \begin{pmatrix} r''_j \\ s''_j \end{pmatrix}.$$

By Remark 5.12 these two pairs of sequences coincide, so we get the condition of the lemma satisfied for $A = Y^{-1}X$. □

Theorem 5.18 *The scalar product provides a bijection between equivalence classes of tame $SL_2(\mathbb{Z}[\sigma])$-tilings and pairs of paths in $\mathcal{T}$ considered up to the simultaneous action of $SL_2(\mathbb{Z}[\sigma])$ by $A \in SL_2(\mathbb{Z}[\sigma])$ on one of the paths and $(A^T)^{-1}$ on the other.*

Proof By Proposition 5.8 the scalar product maps any pair of normalised paths in $\mathcal{T}$ to a tame $SL_2(\mathbb{Z}[\sigma])$-tiling. We will prove that this map provides the required bijection.

First, notice that by Definition 5.16 different normalisations of the paths lead to equivalent $SL_2(\mathbb{Z}[\sigma])$-tilings. Next, by Lemma 5.14 pairs of paths equivalent under the action of $SL_2(\mathbb{Z}[\sigma])$ result in the same $SL_2(\mathbb{Z}[\sigma])$-tiling. This implies that the scalar product is a well-defined map from pairs of paths in $\mathcal{T}$ (up to the simultaneous action of $SL_2(\mathbb{Z}[\sigma])$) to equivalence classes of $SL_2(\mathbb{Z}[\sigma])$-tilings. This map is surjective by Proposition 5.11 and injective by Lemma 5.17. □

Remark 5.19 In [30], a similar bijection was constructed for $SL_2(\mathbb{Z})$-tilings using determinant instead of the scalar product, i.e. using the map $((p_i/q_i), (r_j/s_j)) \mapsto (m_{i,j})$ given by

$$m_{i,j} = p_i s_j - q_i r_j.$$

Our construction can also be formulated in these terms. Namely, replace the path (r_j/s_j) by the path $(s_j/-r_j)$. Then the scalar product is exactly replaced by computing the determinant. Notice that the transformation $(r_j/s_j) \to (s_j/-r_j)$ is given by the map $z \to -1/z$, or, in other words, by the action of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ applied to the path (r_j/s_j).

Note that the construction with determinant is also invariant under isometries, i.e. given a pair of normalised paths $((p_i/q_i), (r_j/s_j))$ and a matrix $A \in SL_2(\mathbb{Z}[\sigma])$, the paths $(A(p_i/q_i), A(r_j/s_j))$ obtained by simultaneous action of A on both paths define the same $SL_2(\mathbb{Z}[\sigma])$-tiling as $((p_i/q_i), (r_j, s_j))$ (cf. Lemma 5.14).

Furthermore, given standard horospheres at all points of $\widehat{\mathbb{Q}}(\sigma)$, the absolute value of the element $m_{i,j} = p_i s_j - q_i r_j$ of the constructed $SL_2(\mathbb{Z}[\sigma])$-tiling is the λ-length of the arc connecting p_i/q_i with r_j/s_j (see Theorem 4.12).

In view of Remark 5.19, the result of Theorem 5.18 can be reformulated as follows.

Theorem 5.20 *The map $((p_i/q_i), (r_j/s_j)) \mapsto (m_{i,j} = p_i s_j - q_i r_j)$ provides a bijection between equivalence classes of tame $SL_2(\mathbb{Z}[\sigma])$-tilings and pairs of paths in $\mathcal{T}$ considered up to the simultaneous action of $SL_2(\mathbb{Z}[\sigma])$ on both paths.*

Example 5.21 The $SL_2(\mathbb{Z}[\sigma])$-tiling constructed in Example 5.10 can be obtained using the procedure above from periodic paths $(u_i) = (\dots, \infty, 0, 1, \sigma, \infty, \dots)$, and $(v_j) = (\dots, \infty, \sigma, 1, 0, \infty, \dots)$.

Definition 5.22 We say that two paths (p_i/q_i) and (r_j/s_j) are *coplanar in $\mathcal{T}$* if there exists a hyperbolic plane Π containing a face of some fundamental tetrahedron in $\mathcal{T}$ and such that $p_i/q_i, r_j/s_j \in \Pi$ for all i, j.

The following is an immediate corollary of Theorem 5.20.

Corollary 5.23 *Let (p_i/q_i) and (r_j/s_j) be two normalised paths on $\mathcal{T}$, and let $(m_{i,j}) = (p_i s_j - q_i r_j)$ be the $SL_2(\mathbb{Z}[\sigma])$-tiling defined by these paths. Then the following are equivalent:*

(a) paths (p_i/q_i) and (r_j/s_j) are coplanar;

(b) $m_{i,j} \in \mathbb{Z}$ *for all* $i, j \in \mathbb{Z}$;
(c) *there are two consecutive rows* $k, k+1$ *and two consecutive columns* $l, l+1$, *such that* $m_{i,l}, m_{i,l+1} \in \mathbb{Z}$, *and* $m_{k,j}, m_{k+1,j} \in \mathbb{Z}$ *for all* $i, j \in \mathbb{Z}$.

Proof • "(a)⇒(b)": Given two paths lying in a plane Π, there exists $A \in SL_2(\mathbb{Z}[\sigma])$ taking Π to the vertical plane containing the real line. Then all the points $A(p_i/q_i)$ and $A(r_j/s_j)$ are real, so the resulting paths lie in a copy of the Farey graph contained in $A(\Pi)$, which implies (b).

- "(b)⇒(c)": This is a trivial implication.
- "(c)⇒(a)": We show that condition (c) implies that both paths (p_i/q_i) and (r_j/s_j) are real rational numbers (and hence (a) follows). Showing this is equivalent to proving that both (p_i/q_i) and $(-s_j/r_j)$ are real. In terms of the latter pair of paths, the $SL_2(\mathbb{Z}[\sigma])$-tiling $(m_{i,j})$ is the scalar product of paths. Now, we apply the construction from Proposition 5.11, and get that p_i/q_i and $-s_j/r_j$ are real rationals. □

6 Paths in $\mathcal{T}$ and sequences in $\mathbb{Z}[\sigma]$

In this section we show that paths in $\mathcal{T}$ can be parameterised by (bi-infinite) sequences of elements of $\mathbb{Z}[\sigma]$. This relates the results of the previous section to classification of SL_2-tilings obtained in [3].

Definition 6.1 A path $(v_i)_{i=m}^n$ in $\mathcal{T}$ represented by irreducible fractions $v_i = p_i/q_i$ for all i is called *skew-normalised* if

$$\det \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = (-1)^i$$

for all admissible i.

Similarly to the case of normalised paths, the property of being skew-normalised is preserved under the action of $SL_2(\mathbb{Z}[\sigma])$.

Definition 6.2 Consider a skew-normalised path (v_{i-1}, v_i, v_{i+1}) in $\mathcal{T}$ with $v_i = p_i/q_i$ irreducible fractions in $\mathbb{Z}[\sigma]$. Assume that

$$\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a \end{pmatrix}.$$

We say that a is the *oriented* $\mathcal{T}$*-angle of* $v_{i-1}v_iv_{i+1}$ and denote it by $\angle_{\mathcal{T}}(v_{i-1}v_iv_{i+1})$.

Proposition 6.3 *(a) Given a skew-normalised path* (v_{i-1}, v_i, v_{i+1}) *in* $\mathcal{T}$, *there exists a unique* $a \in \mathbb{Z}[\sigma]$ *such that* $\angle_{\mathcal{T}}(v_{i-1}v_iv_{i+1}) = a$.
(b) *For every skew-normalised path* (v_{i-1}, v_i) *in* $\mathcal{T}$ *and every* $a \in \mathbb{Z}[\sigma]$ *there exists a unique* $v_{i+1} \in \mathcal{T}_0$ *such that* (v_{i-1}, v_i, v_{i+1}) *is a skew-normalised path and* $\angle_{\mathcal{T}}(v_{i-1}v_iv_{i+1}) = a$.
(c) *For every skew-normalised path* (v_i, v_{i+1}) *in* $\mathcal{T}$ *and every* $a \in \mathbb{Z}[\sigma]$ *there exists a unique* $v_{i-1} \in \mathcal{T}_0$ *such that* (v_{i-1}, v_i, v_{i+1}) *is a skew-normalised path and* $\angle_{\mathcal{T}}(v_{i-1}v_iv_{i+1}) = a$.

Proof To prove (a), note that

$$\begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = \begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} \cdot \begin{pmatrix} 0 & \xi \\ 1 & \alpha \end{pmatrix} \tag{6.1}$$

for some $\alpha, \xi \in \mathbb{C}$ which are defined uniquely.

Since the path (v_{i-1}, v_i, v_{i+1}) is skew-normalised,

$$\det\begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} = (-1)^{i-1}, \qquad \det\begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = (-1)^i,$$

which implies that $\xi = 1$. Equation (6.1) can now be rewritten as

$$\begin{pmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{pmatrix} = \begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix}.$$

Therefore, we conclude

$$\angle_{\mathcal{T}}(v_{i-1}v_iv_{i+1}) = \alpha = \frac{\det\begin{pmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{pmatrix}}{\det\begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix}} = (-1)^{i-1}\det\begin{pmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{pmatrix},$$

which proves (a).

Parts (b) and (c) now follow from (6.1) with $\xi = 1$ and $\alpha = a$. □

Definition 6.4 Given a skew-normalised path (v_i), define $a_i = \angle_{\mathcal{T}}(v_{i-1}v_iv_{i+1})$. We call the sequence (a_i) the *$\mathcal{T}$-angle sequence* of the path (v_i).

Remark 6.5 $\mathcal{T}$-angle sequence is a counterpart of *itinerary* appearing in [30] and *quiddity sequence* appearing in the context of friezes [6]. This sequence (up to change of certain signs) also appears in [3, Section 3].

Proposition 6.6 *(a) Every bi-infinite sequence (a_i), $a_i \in \mathbb{Z}[\sigma]$, is a $\mathcal{T}$-angle sequence of some skew-normalised path.*

(b) Two skew-normalised paths (p_i/q_i) and (r_j/s_j) have the same $\mathcal{T}$-angle sequence if and only if the paths are equivalent under $SL_2(\mathbb{Z}[\sigma])$-action.

Proof (a) follows from parts (b) and (c) of Proposition 6.3.

To prove (b), note that invariance of $\mathcal{T}$-angle sequences under the action of $SL_2(\mathbb{Z}[\sigma])$ follows immediately from Definition 6.2: left multiplication by the same matrix does not affect the validity of the equation.

Conversely, consider a skew-normalised path (v_i). Parts (b) and (c) of Proposition 6.3 imply that the fractions $v_0 = p_0/q_0$ and $v_1 = p_1/q_1$ together with the sequence (a_i) uniquely define the whole path (v_i). Note that there exists $A \in SL_2(\mathbb{Z}[\sigma])$ taking (v_0, v_1) to $(\frac{1}{0}, \frac{0}{1})$. Therefore, given two skew-normalised paths (v_i) and (v_i'), there exists an element of $SL_2(\mathbb{Z}[\sigma])$ taking (v_i) to (v_i'), which completes the proof. □

Remark 6.7 Given a path (v_i) in $\mathcal{T}$, there exist 6 of its skew-normalisations obtained from one of them by multiplying both numerator and denominator of v_i by $\sigma^{(-1)^i\cdot k}$ (cf. Remark 5.3). The $\mathcal{T}$-angle sequence of the new path is then obtained by multiplying a_i by $\sigma^{(-1)^{i+1}\cdot 2k}$.

Definition 6.8 Consider two sequences (a_i) and (b_i) where $a_i, b_i \in \mathbb{Z}[\sigma]$. We say that (a_i) and (b_i) are *equivalent* if there exists $k \in \{-1, 1\}$ such that $b_i = a_i\sigma^{(-1)^{i+1}\cdot 2k}$.

An equivalence class of sequences corresponds to a path in $\mathcal{T}$ considered up to the action of $PSL_2(\mathbb{Z}[\sigma])$.

We introduce one more notion with the aim to reformulate Theorem 5.20 in terms of $\mathcal{T}$-angle sequences. Namely, the paths from Theorem 5.20 are replaced by their $\mathcal{T}$-angle sequences, and the relative position of the paths with respect to each other is indicated by a matrix in $SL_2(\mathbb{Z}[\sigma])$.

Definition 6.9 Consider a triple $\big((a_i), (b_j), X\big)$, where (a_i) and (b_j) are equivalence classes of sequences, $a_i, b_j \in \mathbb{Z}[\sigma]$, and $X \in SL_2(\mathbb{Z}[\sigma])$. We say that two triples $\big((a_i), (b_j), X\big)$ and $\big((\tilde{a}_i), (\tilde{b}_j), \widetilde{X}\big)$ are *equivalent* if the following two conditions hold:

(1) (a_i) is equivalent to $(\tilde{a}_i)$, and (b_j) is equivalent to $(\tilde{b}_j)$;
(2) if $\tilde{a}_0 = a_0\sigma^{-2k}$ and $\tilde{b}_0 = b_0\sigma^{-2l}$, then $\widetilde{X} = \pm \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^{-l} \end{pmatrix} X \begin{pmatrix} \sigma^{-k} & 0 \\ 0 & \sigma^k \end{pmatrix}$.

Remark 6.10 Substituting X with $-X$ corresponds to changing the sign of all numerators and denominators in one of the normalised paths. In particular, a generic equivalence class of triples consists of 18 elements.

Theorem 6.11 *There exists a bijection between equivalence classes of tame $SL_2(\mathbb{Z}[\sigma])$-tilings and equivalence classes of triples $\big((a_i), (b_j), X\big)$, where (a_i) and (b_j) are sequences, $a_i, b_j \in \mathbb{Z}[\sigma]$, and $X \in SL_2(\mathbb{Z}[\sigma])$.*

Proof Given a triple $\big((a_i), (b_j), X\big)$ we construct an $SL_2(\mathbb{Z}[\sigma])$-tiling as follows.

According to Proposition 6.6, there exists a unique skew-normalised path (u_i) with $\mathcal{T}$-angle sequence (a_i) and $u_0 = p_0/q_0$, $u_1 = p_1/q_1$, where

$$\frac{p_0}{q_0} = \begin{cases} 1/0 & \text{if } 0 \le \arg(a_0) < 2\pi/3, \\ \sigma^{-1}/0 & \text{if } 2\pi/3 \le \arg(a_0) < 4\pi/3, \\ \sigma^{-2}/0 & \text{if } 4\pi/3 \le \arg(a_0) < 2\pi; \end{cases} \qquad \frac{p_1}{q_1} = \frac{0}{p_0^{-1}}.$$

Similarly, there exists a unique skew-normalised path (v_j) with $\mathcal{T}$-angle sequence (b_j) and $v_0 = r_0/s_0$, $v_1 = r_1/s_1$, where

$$\begin{pmatrix} r_0 & r_1 \\ s_0 & s_1 \end{pmatrix} = \begin{cases} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} & \text{if } 0 \le \arg(b_0) < 2\pi/3, \\ \begin{pmatrix} \sigma^{-1}x_{1,1} & \sigma x_{1,2} \\ \sigma^{-1}x_{2,1} & \sigma x_{2,2} \end{pmatrix} & \text{if } 2\pi/3 \le \arg(b_0) < 4\pi/3, \\ \begin{pmatrix} \sigma^{-2}x_{1,1} & \sigma^2 x_{1,2} \\ \sigma^{-2}x_{2,1} & \sigma^2 x_{2,2} \end{pmatrix} & \text{if } 4\pi/3 \le \arg(b_0) < 2\pi. \end{cases}$$

We can now construct a normalised path (u_i') according to the following rule: if $u_i = p_i/q_i$ then

$$u_i' = \begin{cases} \frac{p_i}{q_i} & \text{if } i \equiv 0, 1 \pmod 4, \\ \frac{-p_i}{-q_i} & \text{if } i \equiv 2, 3 \pmod 4. \end{cases}$$

Similarly, we use the skew-normalised path (v_j) to construct a normalised path (v_j'). Now, a tame $SL_2(\mathbb{Z}[\sigma])$-tiling is defined by $m_{i,j} = p_i's_j' - q_i'r_j'$, where $u_i' = p_i'/q_i'$, $v_j' = r_j'/s_j'$ (see Remark 5.19).

If sequences $(\tilde{a}_i)$ and (a_i) are equivalent, then, by Remark 6.7, the corresponding skew-normalised paths $(\tilde{u}_i)$ and (u_i) are distinct skew-normalisations of the same path in $\mathcal{T}$. Hence, $(\tilde{u}_i')$ and (u_i') are distinct normalisations of the same path in $\mathcal{T}$. Since X is substituted with $\widetilde{X}$, the sequence (b_j) still leads to the same path (v_j) in $\mathcal{T}$. Therefore, by Theorem 5.20, the triple $\big((\tilde{a}_i), (\tilde{b}_j), \widetilde{X}\big)$ leads to an equivalent $SL_2(\mathbb{Z}[\sigma])$-tiling. Similarly, changing (b_j) (and X) also leads to an equivalent $SL_2(\mathbb{Z}[\sigma])$-tiling.

Therefore, we have constructed a well-defined map from equivalence classes of triples $\big((a_i),(b_j),X\big)$ to equivalence classes of $SL_2(\mathbb{Z}[\sigma])$-tilings.

To prove surjectivity of this map, consider any two paths in $\mathcal{T}$ and the corresponding class of $SL_2(\mathbb{Z}[\sigma])$-tilings. Choose any skew-normalisations (u_i) and (v_j) of these paths and set (a_i) and (b_j) to be the corresponding $\mathcal{T}$-angle sequences. Set X to be the element of $SL_2(\mathbb{Z}[\sigma])$ taking u_0 to v_0 and u_1 to v_1. Then the triple $\big((a_i),(b_j),X\big)$ produces an $SL_2(\mathbb{Z}[\sigma])$-tilings from the required equivalence class. This proves surjectivity.

Notice that taking different skew-normalisations of the paths in the construction above (and, hence, the corresponding matrix X) leads to an equivalent triple. This shows injectivity. □

The next theorem is a very particular case of Proposition 3 of [3] where it is proved in purely linear-algebraic terms (cf. also [21, Theorem 2]).

Theorem 6.12 *There exists a bijection between tame $SL_2(\mathbb{Z}[\sigma])$-tilings and triples $\big((a_i),(b_j),X\big)$, where (a_i) and (b_j) are sequences, $a_i, b_j \in \mathbb{Z}[\sigma]$, and $X \in SL_2(\mathbb{Z}[\sigma])$.*

Proof The map constructed in the proof of Theorem 6.11 takes equivalence classes of triples $\big((a_i),(b_j),X\big)$ to equivalence classes of tame $SL_2(\mathbb{Z}[\sigma])$-tilings. Due to Proposition 5.11, the same map is a surjective map from triples $\big((a_i),(b_j),X\big)$ to tame $SL_2(\mathbb{Z}[\sigma])$-tilings. We now observe that generically equivalence classes of triples and equivalence classes of $SL_2(\mathbb{Z}[\sigma])$-tilings consist of 18 elements each (see Remarks 5.15 and 6.10), which shows that the map is one-to-one in the generic case.

We are left to deal with singular equivalence classes. The small classes of tilings appear when $m_{i,j}=0$ for all $i+j$ even (or all $i+j$ odd). In every such class there are precisely six tilings, and it is easy to see that corresponding $\mathcal{T}$-angle sequences (a_i) and (b_j) are both identically zero. The matrix X in that case can take precisely six values, so that the equivalence class of triples is also of size 6. This completes the proof of the theorem. □

7 Further comments

7.1 Skew-normalised paths and continued fractions

Consider a finite skew-normalised path $(v_i)_{i=0}^{n+2}$ and let $v_0 = 1/0$ and $v_1 = 0/1$. Let $(a_i)_{i=1}^{n+1}$ be its $\mathcal{T}$-angle sequence. Then $(a_i)_{i=1}^{n+1}$ provides a continued fraction expansion for v_{n+2}:

$$v_{n+2} = [a_1; a_2, \ldots, a_{n+1}] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{n+1}}}}}.$$

Indeed, as it was shown in the proof of Proposition 6.3,

$$\begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = \begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix},$$

and the statement becomes a classical property of continuants for $[a_1; a_2, \ldots, a_{n+1}]$.

Corollary 7.1 *A skew-normalised path (u_i) visits the same vertex of $\mathcal{T}$ twice if and only if its $\mathcal{T}$-angle sequence (a_i) contains a finite subsequence $(a_i)_{i=k}^{k+n}$, $k, n \in \mathbb{Z}$ such that $0 = [a_k; a_{k+1}, a_{k+2}, \ldots, a_{k+n}]$.*

7.2 Some further properties of $\mathcal{T}$-angles

Definition 6.2 of $\mathcal{T}$-angle $\angle_{\mathcal{T}}(v_0v_1v_2)$ can be generalised to the situation when v_0, v_1, v_2 are any vertices of $\mathcal{T}$: here, we drop the requirement on v_0v_1 and v_1v_2 to be edges of $\mathcal{T}$. Namely, following the proof of Proposition 6.3, one can define $a \in \mathbb{Q}(\sigma)$ such that

$$\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} \cdot \begin{pmatrix} \xi \\ a \end{pmatrix}$$

for some $\xi \in \mathbb{Q}(\sigma)$ (notice that a may not belong to $\mathbb{Z}[\sigma]$ anymore).
Next, we list some properties of $\mathcal{T}$-angles.

Proposition 7.2 *Let $v_i = p_i/q_i$, where $p_i, q_i \in \mathbb{Z}[\sigma]$, $i = 1, 2, 3$. Let $\det_{i,j} = p_iq_j - p_jq_i$. Then*

(1) $\angle_{\mathcal{T}}(v_0v_1v_2) = \frac{\det_{0,2}}{\det_{0,1}}$;
(2) $\angle_{\mathcal{T}}(v_0v_1v_2) \cdot \angle_{\mathcal{T}}(v_1v_2v_0) \cdot \angle_{\mathcal{T}}(v_2v_0v_1) = -1$.

Proof The first property is a result of a direct computation, the second follows from the first. □

There is also the following version of the Ptolemy relation for $\mathcal{T}$-angles.

Proposition 7.3 *Let $v_i = p_i/q_i$, where $p_i, q_i \in \mathbb{Z}[\sigma]$, $i \in \{1, 2, 3, 4\}$ be four distinct points lying on one circle or line in $\mathbb{C}$ in the cyclic order $v_1v_2v_3v_4$. Let $v_0 = p_0/q_0$ be any other point in $\widehat{\mathbb{Q}}(\sigma)$. Define*

$$x_{i,j} = \sqrt{|\angle_{\mathcal{T}}(v_iv_0v_j) \cdot \angle_{\mathcal{T}}(v_jv_0v_i)|}.$$

Then $x_{1,3}x_{2,4} = x_{1,2}x_{3,4} + x_{2,3}x_{1,4}$.

Proof We prove the identity with the assumption that all p_i/q_i are irreducible. This does not affect the validity of the statement as the identity is homogeneous.

Denote $\det_{i,j} = p_iq_j - p_jq_i$. In view of Proposition 7.2(a),

$$x_{i,j} = x_{j,i} = \sqrt{\left|\frac{\det_{i,j}\det_{j,i}}{\det_{i,0}\det_{j,0}}\right|} = \frac{|\det_{i,j}|}{\sqrt{|\det_{i,0}\det_{j,0}|}}.$$

Theorem 4.12 implies that the determinants satisfy the Ptolemy relation, i.e.

$$|\det_{1,3}||\det_{2,4}| = |\det_{1,2}||\det_{3,4}| + |\det_{2,3}||\det_{1,4}|.$$

By dividing every term of this equation by $\sqrt{|\det_{1,0}\det_{2,0}\det_{3,0}\det_{4,0}|}$, we obtain the required relation for $x_{i,j}$. □

7.3 Other imaginary quadratic fields

Let K be an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, where $d \in \mathbb{Z}_+$ is square-free, and let $\widehat{K} = K \cup \{\infty\}$. Let $\mathcal{O}_K$ be the ring of integers of K, consider K as the field of fractions of $\mathcal{O}_K$. So, every point of $\widehat{K}$ can be written as an irreducible fraction p/q, where $p, q \in \mathcal{O}_K$. We consider $\widehat{K}$ as points of the boundary of $\mathbb{H}^3$, with Bianchi group $Bi(d)$ acting on $\widehat{K}$.

Following [27], one can consider a graph G with vertices at $\widehat{K}$ and edges between p/q and r/s whenever $|ps - rq| = 1$. As for $d = 3$, G can be constructed as the orbit of the classical Farey graph $\mathcal{F}$ under the action of $Bi(d)$. Note that the graph is not always connected, see [33].

Triangular and quadrilateral faces of G are described in [27] (up to the action of $Bi(d)$).

7.3.1 Lambda lengths

As for $d = 3$, we can assign a standard horosphere at every point of $\mathcal{O}_K$ (i.e., a sphere of Euclidean radius 1/2), and then apply $SL_2(\mathcal{O}_K)$ to obtain a standard horosphere at every point of $\widehat{K}$. Two horospheres are tangent if and only if their centres are adjacent in G, and disjoint otherwise.

Proceeding as in Remark 4.14 and using the results of [23], we can deduce that the det-lengths still coincide with λ-lengths with respect to the standard horospheres.

In particular, this leads to the following corollary: given a normalised path (p_i/q_i), the value $|p_iq_{i+2} - p_{i+2}q_i|$ coincides with the λ-length between p_i/q_i and p_{i+2}/q_{i+2}.

7.3.2 SL_2-tilings

The results of Sect. 5 are also valid for any imaginary quadratic field K. The only difference is that one needs to consider all units of K rather than powers of σ (for example, we need to do so in defining equivalent $SL_2(\mathcal{O}_K)$ tilings as in Definition 5.16).

In particular, given two normalised paths in G, we can construct an $SL_2(\mathcal{O}_K)$-tiling by taking determinants of the pairs of entries. The construction provides a bijection between equivalence classes of tame $SL_2(\mathcal{O}_K)$-tilings and pairs of paths in G considered up to the simultaneous action of $SL_2(\mathcal{O}_K)$ on both paths.

The counterpart of Theorem 6.12 for any K is proved in [3], while the geometric approach via pairs of paths and their corresponding sequences of elements of $\mathcal{O}_K$ also works. In particular, modulus of i-th entry of the corresponding quiddity sequence will be equal to the λ-length between $(i-1)$-th and $(i+1)$-th vertices of the path.

7.3.3 Friezes

Consider an $SL_2(\mathbb{Z})$-tiling constructed via taking determinants (see Theorem 5.20), where two paths coincide. If the path is periodic with period $m > 3$, we obtain a tame *frieze pattern* of height $m - 3$ (see [19]). It follows from [30] that all tame friezes can be obtained in this way. This can be generalized to $SL_2(\mathcal{O}_K)$-tilings for all imaginary quadratic fields using Schmidt arrangements described above: tame friezes of height $m - 3$ without zeroes correspond precisely to non-self-intersecting closed (normalised) paths of length m.

Recently, Cuntz and Holm [9, 10] proved that the number of tame friezes of given height without zeroes over $\mathcal{O}_K$ is finite for any $K = \mathbb{Q}(\sqrt{-d})$, and for $d \notin \{1, 2, 3, 7, 11\}$ all entries of friezes are actually rational integers. We would like to note that an independent proof of this result can be obtained by looking at the geometry of the corresponding graphs.

The finiteness of the number of friezes of given height without zeroes is equivalent to the finiteness of the number of closed non-self-intersecting paths up to the action of the Bianchi group. The latter is implied by the following observation: in a closed non-self-intersecting path of length m, the modulus of any entry of the quiddity sequence does not exceed $m - 2$ (due to their relation to λ-lengths).

Further, it follows from [33, Theorem 5.1] that for $d \notin \{1, 2, 3, 7, 11\}$ all vertices of any closed non-self-intersecting path in the corresponding graph belong to an image of $\widehat{\mathbb{Q}}$ under an element of $Bi(d)$, which immediately implies that all entries of the frieze are integers.

Also, all tame non-zero friezes over Eisenstein integers can be enumerated by closed paths in the tetrahedral graph (up to the symmetry group of the graph). It would be interesting to have a combinatorial way of a complete enumeration of all closed paths of a given length.

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Declarations

Conflict of interest. No conflict of interest to declare.

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