# Optimal Planning in Habit Formation Models with Multiple Goods

Mauro Bambi[*], Daria Ghilli [†], Fausto Gozzi [‡], Marta Leocata [§]

**Abstract**

In this paper we investigate a model with habit formation and two types of substitute goods. We are inspired by the classical models in e.g. [COW00] where on the contrary only one good is considered. Such a family of models, even in the case of 1 good, are difficult to study since their utility function is not concave in the interesting cases (see e.g. [BG20]), hence the first order conditions are not sufficient. We are inspired by the situation in which there is a lockdown in the economy and one sector closes whereas habits develop on the second good. This more elaborate model will be the subject of future work. In the present paper, we carry out a first analysis in the case of no lockdown. We introduce and explain the model which considers two goods where one of the two is related to habit stock and where the utility function is expressed as the sum of two utility functions. For this model, we provide some first results using the dynamic programming approach. We prove that the value function is a viscosity solution of the Hamilton-Jacobi-Bellman equation, and also some results on the qualitative behaviour of the value function are furnished. Such results will form a solid ground over which a deep study of the features of the solutions can be performed.

*JEL Code: E21, E30, E60, I18.*

*Keywords: Habits, 2-Sector Economy, Dynamic Programming, Viscosity solution.*

[*]Economics and Finance Department, Durham University, Durham, UK. email: mauro.bambi@durham.ac.uk

[†]Università degli Studi di Pavia, Dipartimento di Science Economiche e Aziendali, Pavia, IT. email: daria.ghilli@unipv.it

[‡]Economics and Finance Department, LUISS Guido Carli University, Rome, IT. email: fgozzi@luiss.it

[§]Scuola Normale Superiore, Pisa, IT. email: marta.leocata@sns.it

# Contents



# 1 Introduction

The main aim of this paper is to formulate and study a growth model with habits formation which takes into account the presence of two types of different goods. The model we propose is inspired by similar (classical) models with habits formation studied in the literature so far, see [COW00, COW97, BG20]. Concerning the above mentioned literature, our aim is to focus on a main difference, that is, the presence of two different goods. This case has not been studied so far in the literature on growth models with habits formation.

We choose to focus on the case of two different goods inspired by the situation in which there is a lockdown on the economy and the sector producing one good closes whereas habits develop on the other good. A real-world example of the situation we have in mind might be the case of one good being cinema and the other good being Netflix. In the case of a lockdown, the sector producing the first good (cinema) closes and habits develop on the second good (Netflix). This case has been fully investigated in the paper [BGGL23] by the authors, with a different utility function. However, in the present paper, we do not investigate the impact of a lockdown on the economy. This will be the subject of future works. This paper is a first step towards this more elaborate model, to understand first what happens in the economy in our different setting in the case of no lockdown.

A major difficulty of the present paper that we aimed at focusing on consists in the fact that the utility we choose (following [COW00, COW97, BG20]) is not concave. Note that on the contrary in [BGGL23] the parameters in the utility are chosen for concavity to be respected. More in detail, in the previous above mentioned papers, it has been observed the crucial relation between internal habit formation and increase in savings in a particular parameters' set, where the multiplicative utility function is never jointly concave in consumption and habits. In particular, joint concavity is never possible when the coefficient of relative risk aversion is bigger than one, i.e. $\sigma > 1$, and if the agent's weighting habit is less than one, i.e. $\gamma < 1$. The lack of concavity makes this class of model difficult to study, even in the case of one good (see e.g. [BG20, YZ+14, ACCR05, DN12, KR07]).

This implies that sufficient optimality conditions of maximum principle type are missing, see page 103 in [SS86].

The aim of the present paper is to investigate the case of non concavity in our setting.

Therefore, in this paper, we address the problem using the dynamic programming approach. Our methods are inspired by [BG20] where similar methods are used to study the case of only one good. We emphasize that the advantage of this approach is that it provides optimality conditions independent of the concavity assumption. Although this method is a rather powerful tool for providing the uniqueness of the optimal control strategy, a uniqueness result is not demonstrated in this paper. We plan to present some results in this direction in some future work.

This approach was developed in the seminal work of Crandall and Lions, see [Lio81, CL83] and it has found a recent application to macroeconomic problems (see [AHL+22, FGP+08b, CGL+23]). In this paper, we apply this theory using the homogeneity properties of the problem (for a similar approach see [FGP+08b, BG20, CS89, BD+97]).

The model under analysis is a variant of the model proposed in [COW00] and later also studied in [BG20], in particular, we take into account two goods to be consumed, and only one of them is related to a habit stock. The instantaneous utility function that we examine for habit-related consumption was introduced in [Abe90], and it is

$$\frac{\left(\frac{c_1}{h^\gamma}\right)^{1-\sigma}}{1-\sigma},$$

while the utility function for the second good is the classical CRRA utility function

$$\frac{(c_2)^{1-\sigma}}{1-\sigma}.$$

We stress the fact that the utility function composed by the sum of the two utility functions above is not strictly concave when $\sigma > 1$ and $\gamma < 1$.

We decided to focus on this specific utility function since it is a typical utility function considered classically in economic growth models, in particular in the above mentioned papers (see in particular [BG20]) which have been our starting point. Indeed, we chose a model that has been fully motivated in the literature to focus on the two difficulties we decided to tackle, the presence of two goods and the nonconcavity of the utility. We think that it would be interesting to try to extend our approach to a more general utility function not satisfying the concavity property, but this is out of the scope of the present paper.

Our main contributions are on the line of [BG20] (see Proposition 1 of [BG20]) except for the fact that, due to our different setting (that is, the presence of two goods), we are not able to prove the regularity of the value function and then to find the optimal feedback control. Indeed the formula for the optimal feedback control depends on the gradient of the value function and hence it is not valuable if the value function is not differentiable. As a consequence, we cannot find an explicit formula for the optimal feedback control as, on the contrary, in Theorem 1 of [BG20]. We refer to Section 4, Proposition 6 and related remarks for further detail.

Before analyzing the complete model, we propose some results on a further simplified model where the utility of consumption in the second goods is neglected. Then, by Dynamic Programming, we furnish some results on the complete model: we prove that the value function is a viscosity solution of the HJB equation, and we derive some properties on the qualitative behaviour of the value function. We think that such results constitute a reliable basis for studying satisfactorily the properties of the optimal paths.

Let us just mention that the literature on habits formation models is very large and varied. One of the very first papers that is worth mentioning introducing the qualitative peculiarities of the models has been [Bec92]. Later on, the literature spread out and considered different approaches. Since the papers [COW00, COW97], several authors have investigated how the usage of "multiplicative" habits can be efficient in several fields. In [Fuh00], it has been observed that habit formation improves significantly the effects of spending and inflation on monetary-policy actions; in [DPMRR03] the role of habit formation in shaping precautionary savings and the wealth distribution in economies with heterogeneous agents have been studied; finally, (among others) multiplicative habits appear also in portfolio choice models, for example [GM03] among others. Finally, since our paper is strongly inspired by [BG20] and by [BGGL23], it is appropriate to refer to the introduction of [BG20] for other references related to the particular (nonconcave) problem we consider and to [BGGL23] for a complete survey of the literature related to growth models with habits formation and multiple sectors.

The paper is organized as follows. In Section 2, the optimal control problem is introduced. In Section 3, we propose a simplification of the optimal control problem and we prove that if the set of controls is unbounded, i.e. $(\underline{c}, +\infty)$, then the value function is trivially null and we furnish some results suggesting that if the set of controls is bounded, i.e. $(\underline{c}, \bar{c})$, then corner solutions are obtained. In Section 4, first, the Maximum Principle is stated and the existence and uniqueness of the steady state are proven. Then by dynamic programming approach, we prove that the value function $V$ is a viscosity solution of the HJB equation and we also derive some qualitative properties of $V$, namely we prove that $V$ is negative, decreasing, locally Lipschitz and $V(+\infty) = -\infty$.

# 2 Model Setup

A consumer faces the following problem:

$$\max_{c_1, c_2} \int_0^{\infty} e^{-\rho t} U(c_1(t), c_2(t), h^{c_1, h_0}(t)) dt$$

subject to the following constraints

$$\dot{h} = \phi(c_1 - h) \quad with\ h(0) = h_0 \tag{1}$$

$$c_1 + Bc_2 = 1 \tag{2}$$

$$c_1 \in (\underline{c}, \bar{c}) \tag{3}$$

where $\phi$ is a parameter related to the velocity of the development of habits. The problem can be rewritten as

$$\max_{c_1(\cdot) \in \mathcal{A}} \int_0^{\infty} e^{-\rho t} U(c_1(t), B^{-1}(1 - c_1(t)), h^{c_1, h_0}(t)) dt \tag{4}$$

where $\mathcal{A}$ is the set of admissible controls (which will be specified in the following sections), subject to the following constraints

$$\dot{h} = \phi(c_1 - h) \quad with\ h(0) = h_0 \tag{5}$$

$$c_1 \in (\underline{c}, \bar{c}). \tag{6}$$

where $\phi, \underline{c}, \bar{c}$ are positive constants.

We specify the following utility function :

$$U(c_1, c_2, h) = u(c_1, h) + v(c_2) = \frac{\left(\frac{c_1}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma}$$

where $\gamma \in (0,1)$ and $\sigma > 1$. Using the constraint (2) we get

$$U(c_1, B^{-1}(1-c_1), h) \equiv u(c_1, h) + v(c_1) = \frac{\left(\frac{c_1}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} + \tilde{B}\frac{(1-c_1)^{1-\sigma}}{1-\sigma}$$

where $\tilde{B} \equiv B^{\sigma-1}$. The parameter $\gamma$ signifies the importance of habits in the utility function. When $\gamma = 0$, habits have no influence, and utility is solely determined by the level of consumption. Conversely, when $\gamma = 1$, habits matter as much as consumption in determining utility. For intermediate values of $\gamma$, where $\gamma \in (0,1)$, habits affect utility, but less than consumption. Our analysis focuses on this last and more realistic case.

This functional form of the utility function was introduced by Carroll et al [COW97] to solve several issues arising with another popular utility function where habits enter in a subtractive form. We choose the same type of utility function also for $c_2$ since our aim is to focus on the models studied in [BG20] so far but in our different settings. We remark that this utility function is popular in economic growth models.

Notice that $u(c_1, h)$ is monotonically increasing in $c_1$ and concave in $c_1$ for any choice of the parameters, i.e. $u_{c_1} > 0$ and $u_{c_1c_1} < 0$. On the other hand, $u(c_1, h)$ is monotonically decreasing in the habits, i.e. $u_h < 0$, meaning that habits are harmful.[1] Concavity of $u(c_1, h)$ with respect to $h$, i.e. $u_{hh} < 0$, implies that

$$\sigma > 1 + \frac{1}{\gamma}. \tag{7}$$

This condition prevents $u(c_1, h)$ to be jointly concave in $(c_1, h)$ as $det(D^2u(c_1, h)) \geq 0$ if and only if $\sigma \leq \frac{\gamma}{1-\gamma}$ (see also [BG20] page 10). Nevertheless, this does not necessarily imply that the solution to the problem is not a maximum. Finally, $v(c_2)$ is monotonically increasing and concave in $c_2$. To sum up, in all the work we will assume the following hypothesis holding on the coefficients.

**Assumption 1.** *We assume that* $\sigma > 1$, $\gamma < 1$ *and* $\gamma(\sigma - 1) > 1$.

[1] More generally we could assume that $\gamma \in (-1, 1)$ so that in the interval $\gamma \in (-1, 0)$ the habits becomes beneficial since $u_h > 0$.

In the core of the text, we will also use the following notation,

$$u(c_1, h) = u_1(h)u_2(c_1)$$

where $u_1(h) = \frac{h^{\gamma(\sigma-1)}}{1-\sigma}$, $u_2(c_1) = c_1^{1-\sigma}$.

To stress the relevance of such a utility function, first, we consider a simpler version where the second term is not considered. In particular, we take

$$\widetilde{U}(c_1, h) = u(c_1, h) = \frac{\left(\frac{c_1}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} \tag{8}$$

# 3 Some results on a simpler model

The following section deals with a simpler version of our optimization problem, namely when the second term in the utility function is not considered. This section aims to help to understand where the problems arise in the use of dynamic programming in the most complicated model. Moreover, it helps to understand how a dynamic programming problem works in the case of a "corner" solution (see Proposition 2).

We consider the following optimal control problem for a typical consumer,

$$W(h_0) = \sup_{c_1(\cdot)\in\mathcal{A}} \int_0^\infty e^{-\rho t}\widetilde{U}\left(c_1(t), h^{h_0,c_1}(t)\right) dt \tag{9}$$

subject to the following state equation

$$\dot{h} = \phi(c_1 - h) \ \ with \ h(0) = h_0 \tag{10}$$

and where the set of admissible controls is,

$$\mathcal{A} = \{c_1 \in L^1_{loc}(\mathbb{R}_+) \text{ s.t. } c_1(t) \in [\underline{c}, \bar{c}], \text{ and } h^{h_0,c_1}(t) \geq 0, \ \forall t \geq 0\}.$$

Notice that the state constraint prescribed in the definition of the set $\mathcal{A}$ is somehow fictitious. Indeed, the state $h$ remains positive for each $c_1 \in L^1_{loc}(\mathbb{R}_+)$. The set of admissible controls can be rewritten simply as

$$\mathcal{A} = \{c_1 \in L^1_{loc}(\mathbb{R}_+) \text{ s.t. } c_1(t) \in [\underline{c}, \bar{c}]\}. \tag{11}$$

We first study the problem by considering $\bar{c} = +\infty$.

**Lemma 1.** *Let $W$ be the value function defined in* (9) *with $\tilde{U}$ defined in* (8)*. Then*

*(i) $W$ is negative,*

*(ii) $W$ is decreasing.*

*Proof.* (i) Since the utility function is negative, $W \leq 0$.
(ii) By the linearity of the state equation we get that if $h_1 < h_2$, then $h^{h_1,c_1}(t) < h^{h_2,c_1}(t)$ for all $t \in \mathbb{R}_+$. Since the utility function is decreasing with respect to the variable $h$, we conclude that

$$W(h_1) - W(h_2) > 0.$$

□

**Proposition 1.** *If $\phi(1-\sigma) < \rho$ and $\bar{c} = +\infty$, then the value function defined in* (9) *is null, i.e. $W \equiv 0$.*

*Proof.* To prove the result, we just find a sequence of admissible controls, $c_n$, such that $J(c_n) \to 0$ and $c_n \to \infty$ for $n \to \infty$. By choosing $c_n = n$, one can check that

$$J(c_n) = \int_0^\infty e^{(-\rho+\phi(1-\sigma))t} \frac{n^{1-\sigma}}{1-\sigma} \left( h_0 + n \frac{e^{\phi t} - 1}{\phi} \right)^{\gamma(1-\sigma)} dt \to 0, \quad n \to \infty$$

□

**Remark 1.** *Notice that if we assume also that $\underline{c} = 0$, then the value function $W$ is $(1-\gamma)(1-\sigma)$-homogenous. Indeed, by the linearity of the state equation, we get that for each $\alpha > 0$ and $h_0 \in \mathbb{R}_+$,*

$$\begin{aligned}
W(\alpha h_0) &= \sup_{c_1(\cdot)\in\mathcal{A}} \int_0^\infty e^{-\rho t} \frac{(c_1(t))^{1-\sigma}}{1-\sigma} \left(h^{\alpha h_0, c_1}(t)\right)^{-\gamma(1-\sigma)} dt \\
&= \sup_{c_1(\cdot)\in\mathcal{A}} \int_0^\infty e^{-\rho t} \frac{\left(\alpha \frac{c_1(t)}{\alpha}\right)^{1-\sigma}}{1-\sigma} \left(h^{\alpha h_0, \alpha \frac{c_1}{\alpha}}(t)\right)^{-\gamma(1-\sigma)} dt \\
&= \sup_{\frac{c_1(\cdot)}{\alpha}\in\mathcal{A}} \int_0^\infty e^{-\rho t} \frac{\left(\alpha \frac{c_1(t)}{\alpha}\right)^{1-\sigma}}{1-\sigma} \left(h^{\alpha h_0, \alpha \frac{c_1}{\alpha}}(t)\right)^{-\gamma(1-\sigma)} dt \\
&= \sup_{\tilde{c_1}(\cdot)\in\mathcal{A}} \int_0^\infty e^{-\rho t} \frac{(\alpha \tilde{c_1}(t))^{1-\sigma}}{1-\sigma} \left(h^{\alpha h_0, \alpha \tilde{c_1}}(t)\right)^{-\gamma(1-\sigma)} dt \\
&= \alpha^{(1-\gamma)(1-\sigma)} \sup_{\tilde{c_1}(\cdot)\in\mathcal{A}} \int_0^\infty e^{-\rho t} \frac{(\tilde{c_1}(t))^{1-\sigma}}{1-\sigma} \left(h^{h_0, \tilde{c_1}}\right)^{-\gamma(1-\sigma)} dt = \alpha^{(1-\gamma)(1-\sigma)} W(h_0).
\end{aligned}$$

*Coherently with the result of Proposition 1, we observe that the only function negative, decreasing, and $(1-\gamma)(1-\sigma)$-homogenous is the null function.*

Note that in the paper by Carroll et al. [COW00], a zero-value function cannot be a solution because the agents in their model have the opportunity to postpone current consumption by investing in the capital stock, thereby increasing future utility. This particular channel is absent in our paper, setting our approach apart from theirs.

If $\bar{c} \neq +\infty$, the problem is more difficult to study. Although we do not present a complete study for this problem, we will show some partial results that seem to confirm that in this framework the optimum is reached at $c_1(t) \equiv \bar{c}$. In the following proposition, we prove that, if only constant controls are considered, our conjecture is true.

**Proposition 2.** *Consider $\bar{c} \neq +\infty$, and as set of admissible control, $\tilde{\mathcal{A}} \subset \mathcal{A}$*

$$\tilde{\mathcal{A}} = \{c_1(t) \equiv c_1,\ c_1 \in [\underline{c}, \bar{c}]\}.$$

*Then the problem*

$$\tilde{W}(h_0) = \sup_{c_1(\cdot)\in\tilde{\mathcal{A}}} \int_0^\infty e^{-\rho t} \frac{\left(\frac{c_1(t)}{h(t)^\gamma}\right)^{1-\sigma}}{1-\sigma} dt \tag{12}$$

*subject to the state equation* (10), *attains its maximum for $c_1 = \bar{c}$, i.e. $\tilde{W}(h_0) = J(\bar{c})$.*

*Proof.* The result is a consequence of the monotonicity of the function,

$$F(c_1) = \frac{c_1^{1-\sigma}}{1-\sigma}\left[e^{-\phi t} h_0 + \phi c_1 \int_0^t e^{-\phi(t-s)} ds\right]^{\gamma(\sigma-1)}.$$

We notice that

$$F'(c_1) = c_1^{-\sigma}(a(t)+b(t)c_1)^{\gamma(\sigma-1)}\left[\frac{a(t)+b(t)c_1(1-\gamma)}{a(t)+b(t)}\right],$$

with $a(t) = e^{-\phi t} h_0$, $b(t) = 1 - e^{-\phi t}$. Since $a(t), b(t) > 0$ and $\gamma < 1$, we have that $F$ is an increasing function, and so it is the objective function. □

**Remark 2.** *Now we observe that the result stated above can actually be generalized to a different class of control, in particular a class of unbounded control. Consider as set of admissible control*

$$\tilde{\mathcal{A}} = \{c(\cdot) : [0, +\infty) \to \mathbb{R},\ c(t) = ke^t,\ k \in (0, +\infty)\}.$$

*For this type of control the integrand of the objective function can be rewritten as*

$$\hat{F}(k) = \frac{k^{1-\sigma}e^{t(1-\sigma)}}{1-\sigma}\left[e^{-\phi t}h_0 + \phi k e^{-\phi t}\int_0^t e^{s(\phi+1)}ds\right]^{\gamma(\sigma-1)}.$$

*We have*

$$\hat{F}'(k) = k^{-\sigma}(a(t)+kb(t)^{\gamma(\sigma-1)}e^{t(1-\sigma)}\left[\frac{a(t)+kb(t)(1-\gamma)}{a(t)+kb(t)}\right].$$

*where* $a(t) = e^{-\phi t}h_0, b(t) = \frac{\phi}{\phi+1}(e - e^{-\phi t})$*. Since* $a(t), b(t) > 0$ *and* $\gamma < 1$ *we have that* $\hat{F}$ *is increasing. As a consequence we deduce the same result as in the simplified version analysed in Proposition 2. In particular we deduce that the maximum is attained for* $k \to \infty$*. Notice that a similar result is not obtained in [BG20] because in [BG20] the capital is also a state variable, neglected here.*

**Proposition 3.** *Let* $W$ *be the value function defined in* (9) *with* $\bar{c} \neq +\infty$*. Then for* $h_0 \to \infty$,

$$a_2\frac{(h_0 \wedge \bar{c})^{\gamma(\sigma-1)}}{1-\sigma} \leq W(h_0) \leq a_1\frac{h_0^{\gamma(\sigma-1)}}{1-\sigma}$$

*Proof.* First, we observe that

$$\begin{aligned} V(h_0) &\geq \int_0^\infty e^{-\rho t}\frac{(\bar{c})^{1-\sigma}}{1-\sigma}(h^{h_0,\bar{c}}(t))^{-\gamma(1-\sigma)}dt \\ &= \int_0^\infty e^{-\rho t}\frac{(\bar{c})^{1-\sigma}}{1-\sigma}(e^{-\phi t}(h_0-\bar{c})+\bar{c})^{-\gamma(1-\sigma)}dt \end{aligned}$$

Since if $h_0 \geq \bar{c}$, then $\bar{c} \leq e^{-\phi t}(h_0-\bar{c})+\bar{c} \leq h_0$ and if $h_0 \leq \bar{c}$, then $h_0 \leq e^{-\phi t}(h_0-\bar{c})+\bar{c} \leq \bar{c}$, we get that

$$V(h_0) \geq \frac{\bar{c}^{1-\sigma}}{1-\sigma}\frac{(h_0 \wedge \bar{c})^{\gamma(\sigma-1)}}{\rho}.$$

Since $h^{h_0,c_1} \geq h_0$ and moreover the function $\frac{c_1^{1-\sigma}}{1-\sigma}$ is increasing and negative, we have

$$\begin{aligned} V(h_0) &= \sup_{c_1(\cdot)\in\tilde{\mathcal{A}}}\int_0^\infty e^{-\rho t}\frac{c_1(t)^{1-\sigma}}{1-\sigma}(h^{h_0,c_1}(t))^{-\gamma(1-\sigma)}dt \\ &\leq \int_0^\infty e^{-\rho t}\frac{(\bar{c})^{1-\sigma}}{1-\sigma}(h_0)^{-\gamma(1-\sigma)}dt \\ &= \frac{1}{\rho}\frac{(\bar{c})^{1-\sigma}}{1-\sigma}(h_0)^{-\gamma(1-\sigma)} \end{aligned}$$

□

# 4 Some results on the optimization problem (4)

We now present some results on the richer model with two consumption goods. We start with some preliminary analysis based on the Maximum Principle. It is essential to note that while the steady-state results we will present adhere to the necessary conditions for optimality, we cannot ascertain whether they also meet the sufficiency conditions. This issue arises due to the non-concavity of the utility function. Our future purpose is to investigate whether the dynamic programming approach helps to understand whether the optimal strategies, trajectories, and costate, coincide with the unique solution of the Maximum Principle.

## 4.1 Existence and uniqueness of steady state

To study the existence and uniqueness of the steady state, we derive the current value Hamiltonian:

$$\begin{aligned} H_{CV}(c_1, h, \mu) &\equiv u(c_1, h) + v(c_1) + \mu\phi(c_1 - h) && (13) \\ &= \frac{\left(\frac{c_1}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} + \tilde{B}\frac{(1-c_1)^{1-\sigma}}{1-\sigma} + \mu\phi(c_1 - h) && (14) \end{aligned}$$

which is not concave as the utility function $u(.)$ is not jointly concave in $(c_1, h)$. We apply the maximum principle, which we recall in the following.

**Proposition 4.** *Let $c_1$ be a solution of the optimal control problem (4). Then $c_1$ solves*

$$\dot{\mu} = -u_h(c_1, h) + \mu(\phi + \rho) \tag{15}$$

$$u_{c_1}(c_1, h) + v_{c_1}(c_1) + \mu\phi = 0 \tag{16}$$

$$\dot{h} = \phi(c_1 - h) \tag{17}$$

$$\lim_{t\to+\infty} h\mu e^{-\rho t} = 0 \tag{18}$$

Consider now a stationary steady state of the economy where $\dot{h} = \dot{\mu} = 0$. From the state equation, we have immediately that $h = c_1 = h^*$ with $h^*$ indicating the steady-state value.

**Proposition 5.** *A unique steady state $h = h^*$ always exists.*

*Proof.* At the steady state, the two first-order conditions of (15) become:

$$u_{c_1}(h^*) + v_{c_1}(h^*) + \mu^*\phi = 0 \tag{19}$$

$$\mu^* = \frac{u_h(h^*)}{\phi+\rho} < 0 \tag{20}$$

Substituting the latter into the first condition leads to

$$u_{c_1}(h^*) + v_{c_1}(h^*) + \frac{\phi}{\phi+\rho}u_h(h^*) = 0$$

which rewrites as it follows when we consider the CES utility function:

$$(h^*)^{-\sigma(1-\gamma)-\gamma} - \tilde{B}(1-h^*)^{-\sigma} - \frac{\phi\gamma}{\rho+\phi}(h^*)^{-\sigma(1-\gamma)-\gamma} = 0$$

or equivalently (rearranging the terms):

$$\left(\frac{\tilde{B}(\rho+\phi)}{\rho+\phi(1-\gamma)}\right)^{\frac{1}{\sigma}}(h^*)^{\frac{\sigma(1-\gamma)+\gamma}{\sigma}} = 1-h^*$$

Notice that the LHS of the equation is a function of $h^*$ which is zero at $h^* = 0$ and is monotonically increasing as long as $\frac{\sigma(1-\gamma)+\gamma}{\sigma} > 0$ or equivalently $\sigma > -\frac{\gamma}{1-\gamma}$ which is always respected since $\gamma \in (0,1)$ and $\sigma$ is positive. On the other hand, the RHS of the equation is a linear decreasing function having value 1 when $h^*$ is zero. Therefore there always exists a unique $h^*$ when the two functions intersect each other. □

## 4.2 Dynamic Programming

The value function of the problem is

$$V(h_0) = \sup_{c_1(\cdot)\in\mathcal{A}(h_0)} \int_0^{+\infty} e^{-\rho t} U(c_1(t), B^{-1}(1-c_1(t)), h(t))dt \tag{21}$$

subject to the following state equation

$$\dot{h} = \phi(c_1 - h) \quad h(0) = h_0, \tag{22}$$

where $\mathcal{A}(h_0)$ is defined in (11). The HJB equation associated to the optimal control problem (4) is

$$\rho V = \max_{c_1\in[\underline{c},\bar{c}]} H_{CV}(c_1, h, DV) \tag{23}$$

where $H_{CV}(c_1, h, p)$ is defined in (13).

We are going to prove that the value function is a viscosity solution to (23). First, we recall the definition of viscosity solution.

**Definition 2.** *A function $u \in C([0, +\infty))$ is a viscosity subsolution of* (23) *if, for any $\phi \in C^1([0, +\infty))$*

$$\rho V(h_0) \leq \max_{c_1 \in [\underline{c}, \bar{c}]} H_{CV}(c_1, h_0, D\phi(h_0)) \tag{24}$$

*at any local maximum point $h_0 \in [0, +\infty)$ of $u - \phi$. Similarly, $u \in C^([0, +\infty))$ is a viscosity supersolution of* (23) *if, for any $\phi \in C^1([0, +\infty))$*

$$\rho V(h_0) \geq \max_{c_1 l \in [\underline{c}, \bar{c}]} H_{CV}(c_1, h_0, D\phi(h_0)) \tag{25}$$

*at any local minimum point $h_0 \in [0, +\infty)$ of $V - \phi$. Finally, $u$ is a viscosity solution of* (23) *if it is simultaneously a viscosity sub- and supersolution.*

In the following proposition, we will prove some fundamental properties of the value function, among which that the value function is a viscosity solution of (23). As underlined in the introduction, in order to find the optimal control in feedback form, we should first prove the differentiability of the value function. However, we are not able to derive this result in our setting. Indeed the classical argument to prove regularity used in this class of problems (as e.g. in [BG20, FGP08a, CS89]) relies on the strict convexity in the gradient variable $\mu$ of the Hamiltonian in the Hamilton-Jacobi-Bellman equation (see equation (23)). Computing the maximum point in the Hamiltonian gives the following equation in the consumption $c_1$:

$$(c_1 h^{-\gamma})^{-\sigma} h^{-\gamma} - \tilde{B}(1 - c_1)^{-\sigma} + \mu\phi = 0$$

which is not explicitly solvable in $c_1$. Hence we cannot find an explicit expression of the Hamiltonian in $\mu$ and verify the strict convexity. As it is evident from the previous computation, the argument would work in the case of just one good (as in [BG20]).

**Proposition 6.** *Let $V$ be the value function defined in* (21)*. Then*

*(i) $V$ is negative, $V(0^+) > -\infty$, and $V$ is decreasing and continuous in* $[0, +\infty)$.

*(ii) $V$ is a viscosity solution in* $(0, +\infty)$ *of the HJB equation* (23).

*(iii) $V$ is locally Lipschitz in* $(0, +\infty)$.

*(iv) $V(+\infty) = -\infty$.*

**Remark 3.** *We remark that we did not prove that $V$ is the unique solution to (23) for technical issues which are out of the scope of the present paper. The reason is that our Hamiltonian has a $(1-\sigma)$-growth, $\sigma > 0$, in the control. The literature on uniqueness of Hamilton-Jacobi-Bellman equations has focused up to now just on the case p-growth with $p > 1$ (see [DLL06, DLL11]), and the complementary case appears to be though to tackle.*

*Proof.* (i). Since $\sigma > 1$, it is straightforward to see that $V < 0$. Moreover

$$V(0^+) \geq \int_0^{+\infty} e^{-\rho t} \left( \frac{\underline{c}^{1-\sigma}}{1-\sigma} e^{-\phi\gamma(\sigma-1)t}(e^{\phi t}-1)^{\gamma(\sigma-1)} + \tilde{B}\frac{(1-\bar{c})^{1-\sigma}}{1-\sigma} \right) dt > -\infty$$

from which we conclude the second inequality.

Note first that the set of control $\mathcal{A}(h_0)$ is independent of $h_0$, since

$$h(t) = e^{-\phi t} \left[ h_0 + \phi \int_0^t c_1(s) e^{\phi s} ds \right] \geq 0$$

since $h_0 \geq 0, \phi > 0$ and $c_1 \in [\underline{c}, \bar{c}]$.

Then, it is easy to see that $V(\cdot)$ is decreasing since the set of controls $\mathcal{A}(h_0)$ is independent of $h_0$ while the utility function decreases (since $\gamma > 0$ and $\sigma > 1$) in $h(t)$, hence in $h_0$ as the equation for $h(t)$ is linear. The continuity follows by sequences by using that $U(c_1(s), B^{-1}(1-c_1(s)), \cdot)$ for $s > 0$ is locally Lipschitz in $(0, +\infty)$ since $\gamma(\sigma-1) \geq 1$ and the continuity of $h(t)$ for all $t > 0$ with respect to the initial datum.

(ii). Note first that the state constraint can be easily proved to hold for every control $c_1$ since $\phi > 0$ and $\underline{c} > 0$. Then the fact that $V$ is a viscosity solution of the HJB equation (23) follows by standard arguments in viscosity theory (see for example [BD+97] Chapter III Proposition 2.8). However, differently from the standard case (see [BD+97]) here $U(c_1(s), B^{-1}(1-c_1(s)), \cdot)$ for $s > 0$ is not uniformly continuous but just locally Lipschitz in $(0, +\infty)$ if $\gamma(\sigma-1) \geq 1$. We give the proof for completeness. For convenience of notation, in the following proof we will use the notation $h^{h_0,c_1}$ to denote the trajectory solution to (22). First note that the Dynamic Programming Principle (DPP) can be proved as in Proposition 2.5 Chapter 3 of [BD+97], that is

$$V(h_0) = \sup_{c_1(\cdot)\in\mathcal{A}(h_0)} \left\{ \int_0^t U(c_1(s), B^{-1}(1-c_1(s)), h^{h_0,c_1}(s)) e^{-\rho s} ds + V(h^{h_0,c_1}(t)) e^{-\rho t} \right\}.$$

Now we prove that the value function is a viscosity solution of (23). First, we prove that

$V$ is a subsolution. Let $\phi \in C^1([0,+\infty))$ and $h_0$ be a local maximum point of $V-\phi$, that is, for some $r>0$

$$V(h_0) - V(z) \geq \phi(h_0) - \phi(z), \qquad \text{for all } z \in B(h_0, r) \cap [0,+\infty). \tag{26}$$

Then for each $\epsilon > 0$ and $t > 0$ by the inequality $\leq$ in the DPP, there exists $\tilde{c}_1 \in \mathcal{A}(h_0)$ (depending on $\epsilon$ and $t$) such that

$$V(h_0) \leq \int_0^t U(\tilde{c}_1(s), B(1-\tilde{c}_1(s)), h^{h_0,c_1}(s))e^{-\rho s} ds + V(h^{h_0,c_1}(t))e^{-\rho t} + t\epsilon. \tag{27}$$

Since $\gamma(\sigma-1) > 1$ we have $U(c_1(s), B^{-1}(1-c_1(s)), \cdot)$ is locally Lipschitz in $(0,+\infty)$ for any $s \geq 0$. For s enough small we can suppose that $h^{h_0,c_1}(s) \in B(h_0, r)$.Then

$$\begin{aligned} |U(\tilde{c}_1(s), B^{-1}(1-\tilde{c}_1(s)), h^{h_0,c_1}(s)) - U(\tilde{c}_1(s), B^{-1}(1-\tilde{c}_1(s)), h_0)| &\leq C_r |h^{h_0,c_1}(s) - h_0| \\ &\leq C_r r. \end{aligned}$$

Then the integral in the righthand side of (27) can be rewritten as

$$\int_0^t U(\tilde{c}_1(s), B^{-1}(1-\tilde{c}_1(s)), h_0)e^{-\rho s} ds + o(t), \qquad \text{as } t \to 0$$

where $o(t)$ indicates a function $g(t)$ such that $\lim_{t\to 0^+} \frac{|g(t)|}{t} = 0$. Then (26) with $z = h^{h_0,c_1}(t)$ and (27) give

$$\phi(h_0) - \phi(h^{h_0,c_1}(t)) - \int_0^t U(\tilde{c}_1(s), B(1-\tilde{c}_1(s)), h_0)e^{-\rho s} ds + V(h^{h_0,c_1}(t))(1-e^{-\rho t}) \leq t\epsilon + o(t) \tag{28}$$

Moreover

$$\begin{aligned} \phi(h_0) - \phi(h^{h_0,\tilde{c}_1}(t)) &= -\int_0^t \frac{d}{ds}\phi(h^{h_0,\tilde{c}_1}(s)) ds \\ &= -\int_0^t D\phi(h^{h_0,c_1}(s)) \cdot \phi(\tilde{c}_1(s) - h^{h_0,\tilde{c}_1}(s)) ds \\ &= -\int_0^t D\phi(h_0) \cdot \phi(\tilde{c}_1(s) - h^{h_0,c_1}(s)) ds + o(t) \end{aligned} \tag{29}$$

Plugging (29) into (28) and adding $\pm \int_0^t U(\tilde{c}_1(s), B^{-1}(1-\tilde{c}_1(s)), h_0) ds$ we get

$$\int_0^t \left\{ -D\phi(h_0) \cdot \phi(\tilde{c}_1(s) - h^{h_0,c_1}(s)) - U(\tilde{c}_1(s), B^{-1}(1-\tilde{c}_1(s)), h_0) \right\} ds$$
$$+ \int_0^t U(\tilde{c}_1(s), B^{-1}(1-\tilde{c}_1(s)), h_0)(1-e^{-\rho s}) ds + V(h^{h_0,c_1}(t))(1-e^{-\rho t}) \le t\epsilon + o(t).$$

The term in brackets in the first integral is estimated below by

$$\min_{c_1 \in [\underline{c}, \bar{c}]} \left\{ -D\phi(h_0) \cdot \phi(c_1 - h_0) - U(c_1, B^{-1}(1-c_1), h_0) \right\}$$

and the second integral is $o(t)$, so we can divide by $t$ and pass to the limit to get

$$\min_{c_1 \in [\underline{c}, \bar{c}]} \left\{ -D\phi(h_0) \cdot \phi(c_1 - h_0) - U(c_1, B^{-1}(1-c_1), h_0) \right\} + \rho V(h_0) \le \epsilon$$

where we have used the continuity of $V$ at $h_0$ and of $h^{h_0,c_1}$ at 0. Note that the previous inequality is equivalent to (24). Since $\epsilon$ is arbitrary, the proof that $V$ is a subsolution is complete.

Now we prove that $V$ is a supersolution to (23). Let $\phi \in C^1([0,+\infty))$ and $h_0$ be a local minimum point of $V - \phi$, that is, for some $r > 0$ , $V(h_0) - V(z) \le \phi(h_0) - \phi(z)$ for all $z \in B(h_0, r)$. Fix an arbitrary $c_1 \in [\underline{c}, \bar{c}]$ and let $h^{h_0,c_1}(t)$ be the solution corresponding to the constant control $c_1(t) = c$ for all $t$. For $t$ small enough $h^{h_0,c_1}(t) \in B(h_0, r)$ and then

$$\phi(h_0) - \phi(h^{h_0,c_1}(t)) \ge V(h_0) - V(h^{h_0,c_1}(t)) \quad \text{for all } 0 \le t \le t_0.$$

By using the inequality "$\ge$" in the DPP, we get

$$\phi(h_0) - \phi(h^{h_0,c_1}(t)) \ge \int_0^t U(c_1, B^{-1}(1-c_1), h^{h_0,c_1}(s)) e^{-\rho s} ds + V(h^{h_0,c_1}(s))(e^{-\rho t} - 1).$$

Therefore dividing by $t > 0$ and letting $t \to 0$, we obtain, by the differentiability of $\phi$ and the continuity of $V, h^{h_0,c_1}$ and $U$

$$-D\phi(h_0) \cdot (h^{h_0,c_1})'(0) = -D\phi(h_0) \cdot \phi(c_1 - h_0) \ge U(c_1, B^{-1}(1-c_1), h_0) - \rho V(h_0).$$

Since $c_1 \in [\underline{c}, \bar{c}]$ is arbitrary, we have proved that

$$\rho V(h_0) + \min_{c_1 \in [\underline{c}, \bar{c}]} \left\{ -\phi(c_1 - h_0) \cdot D\phi(h_0) - U(c_1, B^{-1}(1-c_1), h_0) \right\} \ge 0,$$

and being the previous inequality equivalent to (25), we get that $V$ is a viscosity supersolution to (23).

(iii) Let $h_1 > h_2$. For every $\delta > 0$, there exists $\hat{c}(\cdot) \in \mathcal{A}$ such that

$$\begin{aligned} 0 \leq V(h_2) - V(h_1) &\leq \int_0^{+\infty} e^{-\rho t} \left[u(h_2(t), \hat{c}(t)) - u(h_1(t), \hat{c}(t))\right] dt + \delta \\ &= \int_0^{+\infty} e^{-\rho t} \frac{\hat{c}(t)^{1-\sigma}}{1-\sigma} \left[h_2(t)^{\gamma(\sigma-1)} - h_1(t)^{\gamma(\sigma-1)}\right] dt + \delta, \end{aligned}$$

where for convenience of notation we denote by $h_1(\cdot), h_2(\cdot)$ the trajectories $h^{h_1,\hat{c}}(\cdot), h^{h_2,\hat{c}}(\cdot)$, respectively. Let

$$a(t) = \int_0^t e^{\phi s} \hat{c}(s) ds.$$

Then by the Lagrange theorem there exists $\xi \in (h_2, h_1)$ such that

$$\begin{aligned} h_1(t)^{\gamma(\sigma-1)} - h_2(t)^{\gamma(\sigma-1)} &= \left(e^{-\phi t}\left[h_1 + a(t)\right]\right)^{\gamma(\sigma-1)} - \left(e^{-\phi t}\left[h_2 + a(t)\right]\right)^{\gamma(\sigma-1)} \\ &= e^{\phi\gamma(1-\sigma)t}(\xi + a(t))^{\gamma(\sigma-1)-1}(-\gamma(1-\sigma))(h_1 - h_2). \end{aligned}$$

Then we have

$$\begin{aligned} h_2(t)^{\gamma(\sigma-1)} - h_1(t)^{\gamma(\sigma-1)} &\geq e^{\phi\gamma(1-\sigma)t}(\xi + a(t))^{\gamma(\sigma-1)-1}\gamma(1-\sigma)(h_1 - h_2) \\ &\geq e^{\phi\gamma(1-\sigma)t}(h_1 + a(t))^{\gamma(\sigma-1)-1}\gamma(1-\sigma)(h_1 - h_2) \end{aligned}$$

and then

$$\frac{h_2(t)^{\gamma(\sigma-1)} - h_1(t)^{\gamma(\sigma-1)}}{1-\sigma} \leq e^{\phi\gamma(1-\sigma)t}(h_1 + a(t))^{\gamma(\sigma-1)-1}\gamma(h_1 - h_2)$$

Note

$$h_1 + a(t) \leq h_1 - \frac{\bar{c}}{\phi} + \frac{\bar{c}}{\phi} e^{\phi t} \leq 2\max\left\{h_1 - \frac{\bar{c}}{\phi}, \frac{\bar{c}}{\phi}\right\} e^{\phi t} := a e^{\phi t}.$$

Note that $a$ depends on $h_1$. Then

$$0 \leq V(h_2) - V(h_1) \leq \left(\underline{c}^{1-\sigma}\gamma c^{\gamma(\sigma-1)-1} \int_0^{+\infty} e^{-(\rho+\phi)t} dt\right)(h_1 - h_2) + \delta \tag{30}$$

Since (30) holds for every $\delta > 0$, we have

$$0 \leq \frac{V(h_2) - V(h_1)}{h_1 - h_2} \leq \underline{c}^{1-\sigma}\gamma(\rho+\phi)^{-1}c^{\gamma(\sigma-1)-1},$$

completing the proof.

(iv) We observe that $u_1, u_2$ are increasing functions and $v$ is a decreasing function. So

$$u_1(h_t^{h_0,\underline{c}}) \leq u_1(h_t^{h_0,c_1}) \leq u_1(h_t^{h_0,\bar{c}}), \quad u_2(\underline{c}) \leq u_2(c_1) \leq u_2(\bar{c})$$

$$v(\bar{c}) \leq v(c_1) \leq v(\underline{c_1})$$

and then,

$$u_2(c_1)u_1(h_t^{h_0,\underline{c}}) \leq u_2(\bar{c})u_1(h_t^{h_0,\underline{c}})$$

We can estimate the value function,

$$V(h_0) \leq \int_0^\infty e^{-\rho t}u_2(\bar{c})u_1(h_t^{h_0,\underline{c}})dt + v(\underline{c}) = \int_0^\infty e^{-\rho t}u_2(\bar{c})(e^{-\phi t}(h_0 - \underline{c}) + \underline{c}))^{-\gamma(1-\sigma)}dt + v(\underline{c}).$$

The right-hand side converges to $+\infty$ and we conclude that $V(+\infty) = -\infty$. □

# 5 Conclusion

In this paper, we explore an intertemporal optimization problem where a consumer must decide the consumption levels of two goods, with habit formation occurring with one of these goods. A key feature of our study is that the utility function is not jointly concave with respect to the state and control variables. Initially, we simplify the model by disregarding the utility derived from the consumption of the good without habit formation. This simplification allows us to comprehensively analyze the dynamics of consumption and habits.

Subsequently, we tackle the more complex scenario involving both goods. We employ a dynamic programming approach and establish that the value function is a viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation. Additionally, we derive several qualitative properties of the value function. These findings are crucial preliminary steps for future research, where we plan to incorporate a lockdown scenario modelled as a temporary, random-duration minimum provision of one of the two goods. The insights gained in this paper are vital not only for the analytical determination of consumption and habit dynamics but also for any numerical analysis, as the existence of a unique viscosity solution ensures the accuracy and uniqueness of the numerical solution to the HJB equation. Moreover, the results achieved in the present paper constitute a preliminary and necessary step to study the more elaborate model where a *lockdown* of random arrival and length, that is the shutdown of one sector of the economy, is investigated.

## Declarations

**Conflict of interest.** The authors declare that they have no conflict of interest.

# References


[Abe90] Andrew B Abel. Asset prices under habit formation and catching up with the joneses, 1990.

[ACCR05] Jaime Alonso-Carrera, Jordi Caballé, and Xavier Raurich. Growth, habit formation, and catching-up with the joneses. *European Economic Review*, 49(6):1665–1691, 2005.

[AHL+22] Yves Achdou, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. Income and wealth distribution in macroeconomics: A continuous-time approach. *The review of economic studies*, 89(1):45–86, 2022.

[BD+97] Martino Bardi, Italo Capuzzo Dolcetta, et al. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, volume 12. Springer, 1997.

[Bec92] Gary Stanley Becker. Habits, addictions, and traditions. *Kyklos*, 45:327–345, 1992.

[BG20] Mauro Bambi and Fausto Gozzi. Internal habits formation and optimality. *Journal of Mathematical Economics*, 91:165–172, 2020.

[BGGL23] Mauro Bambi, Daria Ghilli, Fausto Gozzi, and Marta Leocata. Habits and demand changes after covid19. *Journal of Mathematical Economics*, 2023.

[CGL+23] Alessandro Calvia, Fausto Gozzi, Marta Leocata, Georgios I Papayiannis, Anastasios Xepapadeas, and Athanasios N Yannacopoulos. An optimal control problem with state constraints in a spatio-temporal economic growth model on networks. *arXiv preprint arXiv:2304.11568*, 2023.

[CL83] Michael G Crandall and Pierre-Louis Lions. Viscosity solutions of hamilton-jacobi equations. *Transactions of the American mathematical society*, 277(1):1–42, 1983.

[COW97] Christopher D Carroll, Jody Overland, and David N Weil. Comparison utility in a growth model. *Journal of economic growth*, 2:339–367, 1997.

[COW00] Christopher D Carroll, Jody Overland, and David N Weil. Saving and growth with habit formation. *American Economic Review*, 90(3):341–355, 2000.

[CS89] Piermarco Cannarsa and Halil Mete Soner. Generalized one-sided estimates for solutions of hamilton-jacobi equations and applications. *Nonlinear Analysis: Theory, Methods & Applications*, 13(3):305–323, 1989.

[DLL06] F. Da Lio and O. Ley. Uniqueness results for second-order bellman–isaacs equations under quadratic growth assumptions and applications. *SIAM Journal on Control and Optimization*, 45(1), 2006.

[DLL11] F. Da Lio and O. Ley. Uniqueness results for convex hamilton-jacobi equations under $p > 1$ growth conditions on data. *Appl. Math. Optim.*, 63(3):309–339, 2011.

[DN12] W Davis Dechert and Kazuo Nishimura. A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Nonlinear Dynamics in Equilibrium Models: Chaos, Cycles and Indeterminacy*, pages 237–257, 2012.

[DPMRR03] Antonia Diaz, Josep Pijoan-Mas, and José-Victor Rios-Rull. Precautionary savings and wealth distribution under habit formation preferences. *Journal of Monetary Economics*, 50:1257–1291, 2003.

[FGP08a] Giuseppe Freni, Fausto Gozzi, and Cristina Pignotti. Optimal strategies in linear multisector models: Value function and optimality conditions. *Journal of Mathematical Economics*, 44(1):55–86, 2008.

[FGP+08b] Giuseppe Freni, Fausto Gozzi, Cristina Pignotti, et al. A multisector ak model with endogenous growth: value function and optimality conditions. *Journal of Mathematical Economics*, 44:55–86, 2008.

[Fuh00] Jeffrey C. Fuhrer. Habit formation in consumption and its implications for monetary-policy models. *American Economic Review*, 90:367–390, 2000.

[GM03] Francisco Gomes and Alexander Michaelides. Portfolio choice with internal habit formation: a life-cycle model with uninsurable labor income risk. *Journal of Monetary Economics*, 6:729–766, 2003.

[KR07] Takashi Kamihigashi and Santanu Roy. A nonsmooth, nonconvex model of optimal growth. *Journal of Economic Theory*, 132(1):435–460, 2007.

[Lio81] PL Lions. Solutions generalisees des equations hamilton-jacobi du premier ordre. 1981.

[SS86] Atle Seierstad and Knut Sydsaeter. *Optimal control theory with economic applications*. Elsevier North-Holland, Inc., 1986.

[YZ+14] Zaifu Yang, Rong Zhang, et al. *Rational addictive behavior under uncertainty*. Department of Economics and Related Studies, University of York, 2014.