Kelvin - Voigt fluid models in double diffusive porous convection

Brian Straughan[1*]

[1*]Department of Mathematics, University of Durham, Stockton Road, Durham, DH1 3LE, U.K..

Corresponding author(s). E-mail(s): brian.straughan@durham.ac.uk;

Abstract

We investigate problems of convection with double diffusion in a saturated porous medium, where the saturating fluid is one of viscoelastic type, being specifically a Navier - Stokes - Voigt fluid, or a Kelvin - Voigt fluid. The double diffusion problem is analysed for a porous medium with Darcy and Brinkman terms, for a Navier - Stokes - Voigt fluid, and then for a general Kelvin - Voigt fluid of order N. The case where N has the value one is analysed in detail. We also propose a theory where the fluid and solid temperatures may be different, i.e. a local thermal non-equilibrium (LTNE) theory for a porous medium saturated by a Kelvin - Voigt fluid. A further generalization to include heat transfer by a model due to C. I. Christov is analysed in the context of Kelvin - Voigt fluids in porous media. Finally we examine the question of whether a Navier - Stokes - Voigt theory should be used for nonlinear flows, or whether a suitable objective derivative is required.

Keywords: Navier - Stokes - Voigt, Kelvin - Voigt, thermal convection, objective derivative, linear instability, global nonlinear stability, double diffusion, porous convection

MSC Classification: 76E06 , 74F10 , 35B35 , 35Q30

1 Introduction

Donald Nield has been responsible for many of the major developments in the theory of convection in porous media. Rarely can someone have entered an area and produced a fundamental contribution which has then had such a striking influence on the work which has followed. We mention five areas where he initiated developments which have motivated me personally, but also countless others. Firstly he developed the theory of double diffusive convection in a porous medium where convective motion is driven by a temperature gradient but is simultaneously affected by a gradient of solute, Nield [1, 2]. The second area where he was instrumental in producing the original article is in thermal convection in a fluid overlying a saturated porous material, Nield [3]. Donald Nield was also heavily involved with work in dual porosity, or bidisperse, porous media, see Nield [4], and much of this was developed with Andrey Kuznetsov, see e.g. Nield and Kuznetsov [5]. The fourth field in which Donald was at the initial forefront is to shear flow in a saturated porous medium, Nield [6]. The final area we mention is local thermal nonequilibrium (LTNE) convection, where Donald was heavily involved, see Nield [7], Nield and Bejan [8], and much of this was developed alongside Andrew Rees, Banu and Rees [9], see also the book by Straughan [10]. A very lucid historical account of LTNE convection is contained in the very readable article by Bidin and Rees [11].

We could list many, many subsequent works which have employed developments from the original articles of Nield in each of these five areas. For example, in double diffusive convection, Barletta and Nield [12], Harfash and Hill [13], Kuznetsov et al. [14], Deepika and Narayana [15], Nield and Kuznetsov [16], Matta et al. [17], Deepika [18], Kumar et al. [19], Straughan [20], Capone et al. [21], Capone et al. [22], Wang and Chen [23], Deepika et al. [24]. In two layer convection with a fluid overlying a porous layer we quote Chen and Chen [25], Chang et al. [26], Hill and Straughan [27, 28], Payne and Straughan [29], Samanta [30], McCurdy et al. [31], and Mirbod et al. [32]. This is an area with much application to real life, for example, in renewable energy and the creation of electricity and desalinized water via a solar pond, see e.g. Kumaravel et al. [33], Mbelu et al. [34], Hill and Carr [35]. Bidispersive convection is now a huge area, see e.g. Nield and Kuznetsov [36, 37], Kuznetsov and Nield [38], Gentile and Straughan [39, 40], Capone et al. [41], Capone and Massa [42], Siddabasappa et al. [43], Bhavyashree et al. [44], Straughan [45], and the many references therein, and in tridisperse convection, see Kuznetsov and Nield [46], Gentile and Straughan [47]. Shear flow in a porous channel saturated with a fluid has been analysed by Avramenko et al. [48], Hill and Straughan [49], Straughan and Harfash [50] and Samanta [51], and many other references to this still highly active area are contained in these papers. The field of LTNE is another area which has grown enormously, see e.g. Rees et al. [52], Barletta and Rees [53], Nield et al. [54], Kuznetsov et al. [14], Eltayeb [55], Capone et al. [56], Capone and Gianfrani [57], Freitas et al. [58], Shivakumara et al. [59], Shivakumara and Raghunatha [60], Badday and Harfash [61], Noon and Haddad [62].

Within the realm of a clear fluid, i.e. one which is not occupying a porous medium, there has been extensive work by the Russian school of mathematics on what might be termed generalized Navier - Stokes equations. This is aimed at describing certain classes of fluids with memory, or mixtures of polymers. In particular we are focussing on the class of fluids known as Kelvin - Voigt fluids of order N, where a particular case is a Kelvin - Voigt fluid of order zero which is also known as a Navier - Stokes - Voigt fluid. The aim of this work is to develop and describe double diffusive convection with a Navier - Stokes - Voigt fluid, or more generally, with a Kelvin - Voigt fluid, where this saturates a porous material. The porous material will be of Brinkman - Darcy type, i.e. it will contain both Brinkman and Darcy terms.

From a mathematical viewpoint Kelvin - Voigt fluids have been extensively studied with respect to existence and regularity of a solution, and asymptotic properties, notably by Oskolkov [63, 64, 65, 66, 67, 68], Oskolkov and Shadiev [69], Baranovskii [70], Berselli and Bisconti [71], Celebi et al. [72], Damázio et al. [73], Di Plinio et al. [74], Kalantarov and Titi [75, 76], Kalantarov et al. [77], Krasnoschok et al. [78], Layton and Rebholz [79], Niche [80], Sukacheva [81, 82], Sviridyuk and Sukacheva [83], Sukacheva and Kondyukov [84], Sukacheva and Matveeva [85], Sukacheva and Sviridyuk [86], Zvyagin [87, 88].

A model for a type of Navier - Stokes - Voigt fluid which incorporates temperature effects is developed by Oskolkov [65, 66], and Sukacheva [81] develops this for what is now called a Navier - Stokes - Voigt fluid, see also the non isothermal development for a general Kelvin - Voigt fluid by Sukacheva and Matveeva [85, 89]. Explicit quantitative stability analyses of thermal and double diffusive convection with a Navier - Stokes - Voigt fluid, not in a porous medium, are given by Straughan [90, 91], and for a Kelvin - Voigt fluid by Straughan [92, 93].

In the context of a pure fluid the Navier - Stokes - Voigt system of equations modifies the momentum equation in the Navier - Stokes equations by addition of a term $-\hat{\lambda}\Delta\partial\mathbf{v}/\partial t$ to the acceleration $\dot{\mathbf{v}}$, where v_i is the velocity field and a dot denotes the material derivative. The constant $\hat{\lambda} > 0$ is the Kelvin - Voigt coefficient and Δ is the Laplacian. In this work we modify these equations to be applicable to flow through a saturated porous medium. In this way we develop a system of equations which contains the Darcy friction term and the Brinkman term, both familiar in porous media analysis, but the saturating fluid is a viscoelastic one of Navier - Stokes - Voigt type.

2 Navier - Stokes - Voigt model in porous media

Throughout, we employ standard indicial notation together with the Einstein summation convention. For example,

$$v_{i,i} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\,,$$

where $\mathbf{v} = (u, v, w) \equiv (v_1, v_2, v_3)$. Also,

$$v_j v_{i,j} = \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = u\frac{\partial v_i}{\partial x} + v\frac{\partial v_i}{\partial y} + w\frac{\partial v_i}{\partial z}\,, \qquad i = 1, 2, 3,$$

and

$$v_j T_{,j} = \sum_{j=1}^{3} v_j \frac{\partial T}{\partial x_j} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\,,$$

and

$$v_{i,jj} = \Delta v_i = \frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial y^2} + \frac{\partial^2 v_i}{\partial z^2}\,, \qquad i = 1,2,3.$$

We now present the equations for a Navier - Stokes - Voigt theory of flow in a porous medium with temperature and salt effects present. Let $v_i(\mathbf{x},t)$, $T(\mathbf{x},t)$, $C(\mathbf{x},t)$, $p(\mathbf{x},t)$ denote the pore averaged fluid velocity, temperature, concentration of a dissolved salt, and the pressure, at a point $\mathbf{x}$ at time t. Let $\mu, K, \tilde{\mu}, \alpha, g, \kappa, \kappa_s, \hat{\lambda}, \alpha_s$ be positive constants which correspond physically to the dynamic viscosity of the fluid, the permeability of the porous medium, the Brinkman coefficient, the thermal expansion coefficient of the fluid, gravity, thermal diffusivity, salt diffusivity, the Kelvin - Voigt coefficient, and the coefficient of salt in the density. We employ a Boussinesq approximation, see Barletta [94, 95], Breugem and Rees [96], Nield and Barletta [97]. The relevant equations then take the form, cf. Straughan [10, equations (1.101)],

$$\begin{aligned}
&A\frac{\partial v_i}{\partial t} - \hat{\lambda}\Delta\frac{\partial v_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\mu}{K\rho_0}v_i + \tilde{\mu}\Delta v_i + \alpha g T k_i - \alpha_s g C k_i\,,\\
&\frac{\partial v_i}{\partial x_i} = 0,\\
&\frac{1}{M}\frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i} = \kappa\Delta T,\\
&\epsilon\frac{\partial C}{\partial t} + v_i\frac{\partial C}{\partial x_i} = \epsilon\kappa_s\Delta C.
\end{aligned} \tag{1}$$

In these equations ρ_0 is the constant reference density, ϵ is the porosity, M is defined in Straughan [10, page 35], and A is the inertia coefficient. The A term is the local acceleration, the $\hat{\lambda}$ term is the viscoelastic contribution due to the Navier - Stokes - Voigt theory, the μ term is the Darcy contribution, and the $\tilde{\mu}$ term arises from the Brinkman contribution.

Equation $(1)_1$ may be thought of as the momentum equation in the form

$$\rho_0 A\frac{\partial v_i}{\partial t} = \sigma_{ji,j} + \rho f_i - \frac{\mu}{K}v_i\,, \tag{2}$$

where $\rho(T,C)$ is the density and σ_{ij} is the Cauchy stress. The Cauchy stress would have form

$$\sigma_{ij} = -p\delta_{ij} + 2\tilde{\mu}d_{ij} + 2\hat{\lambda}d_{ij,t}\,, \tag{3}$$

where d_{ij} is the symmetric part of the velocity gradient, i.e.

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \tag{4}$$

Damázio et al. [73] criticize the representation (3) due to the fact that $d_{ij,t}$ is not an objective time derivative. We return to this important point in section 7 but for now we observe that provided the flow speed in the porous medium is not too large then (3) should be a reasonably accurate approximation.

2.1 Double diffusive convection

Here we analyse the problem of double diffusive convection in a horizontal layer of fluid saturated porous media contained between the planes $z = 0$ and $z = d$. The boundaries are maintained at constant temperatures and concentrations T_L, C_L when $z = 0$ and T_U, C_U when $z = d$. We consider the case where $T_L > T_U$ and $C_L > C_U$. Thus, we are considering the heated and salted below case where heat and salt are giving opposing contributions to the possibility of convective motion. The salt is stabilizing while the temperature will destabilize. The steady state is

$$\bar{\mathbf{v}} \equiv 0, \qquad \bar{T} = -\beta z + T_L\,, \qquad \bar{C} = -\beta_s z + C_L\,,$$

where $\beta = (T_L - T_U)/d$ and $\beta_s = (C_L - T_U)/d$ and they are both positive.

Perturbations are introduced in equations (1) and the perturbations to $(\bar{v}_i, \bar{T}, \bar{C}, \bar{p})$ are denoted by (u_i, θ, ϕ, π). The resulting equations are non - dimensionalized with the scales

$$\begin{aligned}
&x_i = x_i^* d, \qquad t = t^* \mathcal{T}, \qquad u_i = u_i^* U, \qquad \theta = \theta^* T^\sharp, \qquad \phi = \phi^* C^\sharp,\\
&\pi = \pi^* P, \qquad U = \frac{\kappa}{d}\,, \qquad \mathcal{T} = \frac{d^2}{\kappa M}\,, \qquad T^\sharp = Ud\sqrt{\frac{\beta\mu}{\kappa\alpha g K \rho_0}}\,,\\
&C^\sharp = Ud\sqrt{\frac{\mu}{\epsilon\kappa_s\alpha_s g K\rho_0}}\,, \qquad \lambda = \frac{\hat{\lambda} K\rho_0}{\mu d^2 \mathcal{T}}\,,\\
&B = \frac{K\rho_0\tilde{\mu}}{d^2\mu}\,, \qquad Va = \frac{\mu\mathcal{T}}{AK\rho_0}\,, \qquad Le = \frac{\kappa}{\kappa_s}\,,
\end{aligned}$$

where Le is the Lewis number, Va is the Vadasz number, B is the non-dimensional Brinkman coefficient, and λ is the non-dimensional Kelvin - Voigt parameter. The Rayleigh number $Ra = R^2$, and the salt Rayleigh number $\mathcal{C}^2$ are given by

$$R = \sqrt{\frac{\beta\alpha g K d^2\rho_0}{\mu\kappa}}\,, \qquad \mathcal{C} = \sqrt{\frac{d^2\alpha_s g K\rho_0}{\mu\kappa_s}}\,.$$

Then, omitting stars, the non-dimensional perturbation equations have the form

$$\begin{aligned}
&\frac{1}{Va}u_{i,t} - \lambda\Delta u_{i,t} = -\pi_{,i} - u_i + B\Delta u_i + R\theta k_i - \mathcal{C}\phi k_i\,,\\
&u_{i,i} = 0,\\
&\theta_{,t} + u_i\theta_{,i} = Rw + \Delta\theta,\\
&MLe\phi_{,t} + \frac{Le}{\epsilon}u_i\phi_{,i} = \mathcal{C}w + \Delta\phi.
\end{aligned} \tag{5}$$

These equations hold on $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\}$ for $t > 0$. The boundary conditions are

$$u_i = 0, \quad \theta = 0, \quad \phi = 0, \qquad z = 0, 1, \tag{6}$$

together with the fact that u_i, θ, ϕ, π satisfy a plane tiling periodicity in the x, y - plane.

To analyse linear instability we discard the nonlinear terms in (5) and we introduce a time dependence like $e^{\sigma t}$, i.e. we put $u_i = u_i(\mathbf{x})e^{\sigma t}$, with similar forms for θ, ϕ, π. The pressure is then removed from the resulting equations and one may arrive at the following system of equations in (w, θ, ϕ),

$$\begin{aligned}
&-\frac{\sigma}{Va}\Delta w + \lambda\sigma\Delta^2 w = \Delta w - B\Delta^2 w - R\Delta^*\theta + \mathcal{C}\Delta^*\phi,\\
&\sigma\theta = Rw + \Delta\theta,\\
&MLe\sigma\phi = \mathcal{C}w + \Delta\phi,
\end{aligned} \tag{7}$$

where Δ^* is the horizontal Laplacian.

For two stress free surfaces (7) may be solved with a sin series solution and one finds the stationary convection boundary is given by

$$R^2_{stat} = \mathcal{C}^2 + \frac{\Lambda^2}{a^2} + B\frac{\Lambda^3}{a^2}\,, \tag{8}$$

where $\Lambda = \pi^2 + a^2$, cf. Rees [98]. The stationary convection instability boundary is found by minimizing R^2 in a^2.

To obtain the oscillatory convection boundary one solves the determinant equation from (7) to show that R^2 satisfies the equation

$$R^2 = \sigma^2\frac{\tilde{\Lambda}}{a^2} + \sigma\frac{(\Psi + \Lambda\tilde{\Lambda})}{a^2} + \frac{\Psi\Lambda}{a^2} + \mathcal{C}^2\Big(\frac{\sigma + \Lambda}{M_1\sigma + \Lambda}\Big)\,, \tag{9}$$

where

$$M_1 = MLe, \qquad \Psi = \Lambda + B\Lambda^2, \qquad \tilde{\Lambda} = \frac{\Lambda}{Va} + \lambda\Lambda^2.$$

Since R^2 must be real we put $\sigma = i\omega$ and take the real and imaginary parts of (9). The imaginary part yields the following equation for ω^2,

$$\omega^2 = -\mathcal{C}^2\frac{\Lambda a^2}{M_1^2(\Psi + \Lambda\tilde{\Lambda})} + \mathcal{C}^2\frac{\Lambda a^2}{M_1(\Psi + \Lambda\tilde{\Lambda})} - \frac{\Lambda^2}{M_1^2}\,. \tag{10}$$

The real part then yields

$$R^2_{osc} = \mathcal{C}^2\left(\frac{\Lambda^2 + M_1\omega^2}{\Lambda^2 + M_1^2\omega^2}\right) + \frac{\Psi\Lambda}{a^2} - \omega^2\frac{\tilde{\Lambda}}{a^2}\,. \tag{11}$$

To find the critical oscillatory convection thresholds one minimizes R^2_{osc} in a^2, checking from (10) that $\omega^2 > 0$ at criticality.

Numerical values for this minimization are given in section 8.

The Soret effect and slip type boundary conditions are considered for thermosolutal convection in a Navier - Stokes - Voigt fluid by Badday and Harfash [61].

3 Kelvin - Voigt models in porous media

In this section we describe the problem of double diffusive convection in a porous medium when the saturating fluid is one of Kelvin - Voigt order L, for $L \geq 0$, L an integer.

As Straughan [93] points out there is a very interesting class of complex (viscoelastic) materials associated with the names of Kelvin and of Voigt, cf. Chirita and Zampoli [99], Layton and Rebholz [79]. This class of fluid was proposed by Oskolkov [100], and thorough analyses of solvability issues such as existence and regularity are given in Oskolkov [67, 68], Oskolkov and Shadiev [69], and in Sukacheva [101, 102]. The work of Sukacheva [102] describes a more complicated model of Kelvin - Voigt fluid than we consider here, but it is very much of interest in its own right. Kelvin - Voigt models which incorporate thermal effects are given by Sukacheva [81, 82], Sukacheva and Matveeva [85], Matveeva [103], Sukacheva and Kondyukov [84].

We now describe the equations for double diffusive convection in a Kelvin - Voigt fluid of order $0, 1, 2, \ldots, L$, which saturates a porous medium. The Kelvin - Voigt fluid of order 0 is exactly the same as the Navier - Stokes - Voigt fluid examined in section 2. This class of fluid is also sometimes known as an Oskolkov fluid, cf. Sviridyuk [104, 105, 106], Sviridyuk and Sukacheva [107]. The general equations have form

$$\begin{aligned}
&A\frac{\partial v_i}{\partial t} - \hat{\lambda}\Delta\frac{\partial v_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\mu}{K\rho_0}v_i + \tilde{\mu}\Delta v_i + \alpha g T k_i - \alpha_s g C k_i + \sum_{\alpha=1}^{L}\beta_\alpha \Delta W_i^\alpha\,,\\
&\frac{\partial v_i}{\partial x_i} = 0,\\
&\frac{1}{M}\frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i} = \kappa\Delta T,\\
&\epsilon\frac{\partial C}{\partial t} + v_i\frac{\partial C}{\partial x_i} = \epsilon\kappa_s\Delta C,\\
&\frac{\partial W_i^\alpha}{\partial t} + \hat{\gamma}^\alpha W_i^\alpha = v_i \qquad \alpha = 1, \ldots, L.
\end{aligned} \tag{12}$$

The difference with the Navier - Stokes - Voigt equations given in (1) are the addition of the terms in W_i^α in $(12)_1$ and the last line of (12) which essentially defines W_i^α. To understand this set of equations we observe that the Darcy piece involving v_i arises from a mixture theory of a fluid and a solid as is lucidly explained in Morro [108]. Actually the velocity field here is really the fluid velocity minus the solid velocity so that the difference velocity is an objective quantity. However, the solid skeleton is assumed fixed so the fluid flows through the porous body. Oskolkov [67], Oskolkov [68] shows that effectively the W_i^α terms arise from a stress relation of form

$$S_{ij} = \kappa_3\frac{\partial d_{ij}}{\partial t} + \nu d_{ij} + \sum_{\beta=1}^{L}\xi_\beta\int_{-\infty}^{t}\exp\{-\hat{\gamma}_\beta(t-s)\}\,d_{ij}\,ds,$$

where $\kappa_3 > 0$ and $\xi_\beta > 0$ are constants, d_{ij} is the symmetric part of the velocity gradient, and S_{ij} is the Cauchy extra stress tensor.

We set up the double diffusion convection problem in a horizontal layer of saturated porous medium exactly as in section 2.1. We denote by q_i^α the perturbations to W_i^α and non - dimensionalize with the scalings of section 2.1 with, in addition, the extra scalings

$$q_i^\alpha = q_i^{\alpha *} Q, \quad Q = U\mathcal{T}, \quad \gamma^\alpha = \hat{\gamma}^\alpha \mathcal{T}, \quad \epsilon^\alpha = \frac{\beta_\alpha K \rho_0 \mathcal{T}}{d^2 \mu}.$$

The steady solution is as in section 2.1 with the additional request that $\bar{W}_i^\alpha = 0$. When this is done we derive the non - dimensional perturbation equations for double diffusive convection in a porous medium with a Kelvin - Voigt fluid of order L to be

$$\begin{aligned}
&\frac{1}{Va} u_{i,t} - \lambda \Delta u_{i,t} = -\pi_{,i} - u_i + B\Delta u_i + R\theta k_i - \mathcal{C}\phi k_i + \sum_{\alpha=1}^{L} \epsilon_\alpha \Delta q_i^\alpha \,, \\
&u_{i,i} = 0, \\
&\theta_{,t} + u_i \theta_{,i} = Rw + \Delta\theta, \\
&MLe\phi_{,t} + \frac{Le}{\epsilon} u_i \phi_{,i} = \mathcal{C}w + \Delta\phi, \\
&q_{i,t}^\alpha + \gamma^\alpha q_i^\alpha = u_i \,, \qquad \alpha = 1, \dots, L.
\end{aligned} \tag{13}$$

These equations hold on $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\}$ for $t > 0$. The boundary conditions are

$$u_i = 0, \quad \theta = 0, \quad \phi = 0, \quad q_i^\alpha = 0, \ \alpha = 1, \dots, L, \qquad z = 0, 1, \tag{14}$$

together with the fact that $u_i, \theta, \phi, \pi, q_i^\alpha$ satisfy a plane tiling periodicity in the x, y - plane.

We do not proceed further with the general case where L is arbitrary. Instead we now analyse double diffusive convection in a porous medium saturated with a Kelvin - Voigt fluid of order one. As we shall see the case $L = 1$ is already somewhat involved.

4 Double diffusive porous convection with a Kelvin - Voigt model of order one

The basic governing system of equations when $L = 1$ follows from (12). We commence with the non - dimensional perturbation equations which one finds from (13) when $L = 1$. We simplify notation and put $\delta = \epsilon_1$, $\gamma = \gamma^1$ and $q_i = q_i^1$. The fully nonlinear perturbation equations are then

$$
\begin{aligned}
&\frac{1}{Va}u_{i,t}-\lambda\Delta u_{i,t}=-\pi_{,i}-u_i+B\Delta u_i+R\theta k_i-\mathcal{C}\phi k_i+\delta\Delta q_i\,,\\
&u_{i,i}=0,\\
&\theta_{,t}+u_i\theta_{,i}=Rw+\Delta\theta,\\
&M_1\phi_{,t}+\frac{Le}{\epsilon}u_i\phi_{,i}=\mathcal{C}w+\Delta\phi,\\
&q_{i,t}+\gamma q_i=u_i\,.
\end{aligned}
\tag{15}
$$

To proceed with a linear instability analysis we discard the nonlinear terms in (15) and seek a time dependence like $e^{\sigma t}$. This results in having to solve the equations

$$
\begin{aligned}
&\frac{\sigma}{Va}u_i-\lambda\sigma\Delta u_i=-\pi_{,i}-u_i+B\Delta u_i+R\theta k_i-\mathcal{C}\phi k_i+\delta\Delta q_i\,,\\
&u_{i,i}=0,\\
&\sigma\theta=Rw+\Delta\theta,\\
&M_1\sigma\phi=\mathcal{C}w+\Delta\phi,\\
&(\sigma+\gamma)q_i=u_i\,.
\end{aligned}
\tag{16}
$$

Next remove the pressure from (16) and substitute for q_i to derive the system of equations

$$
\begin{aligned}
&-\frac{\sigma}{Va}\Delta w+\lambda\sigma\Delta^2 w=\Delta w-B\Delta^2 w-R\Delta^*\theta+\mathcal{C}\Delta^*\phi-\frac{\delta}{(\sigma+\gamma)}\,\Delta^2 w\,,\\
&\sigma\theta=Rw+\Delta\theta,\\
&M_1\sigma\phi=\mathcal{C}w+\Delta\phi.
\end{aligned}
\tag{17}
$$

For two stress free surfaces the stationary convection boundary follows from (17) and one finds

$$
R^2_{stat}=\mathcal{C}^2+\frac{\Lambda^2}{a^2}+\Big(B+\frac{\delta}{\gamma}\Big)\,\frac{\Lambda^3}{a^2}
\tag{18}
$$

and the critical Rayleigh numbers of stationary convection are derived by minimizing in a^2.

The oscillatory convection boundary may be found for two stress free surfaces from (17). This leads to the fourth order equation

$$
\begin{aligned}
M_1\tilde{\Lambda}\sigma^4&+\sigma^3\big\{M_1(\gamma\tilde{\Lambda}+\Psi)+\Lambda\tilde{\Lambda}(M_1+1)\big\}\\
&+\sigma^2\big\{\tilde{\Lambda}\Lambda^2+M_1\delta\Lambda^2+\Lambda(M_1+1)(\gamma\tilde{\Lambda}+\Psi)\big\}\\
&\sigma\big\{\Lambda^2(\gamma\tilde{\Lambda}+\Psi)+\Lambda(M_1+1)(\gamma\Psi+\delta\Lambda^2)\big\}\\
&+(\gamma\Psi+\delta\Lambda^2)\Lambda^2+\mathcal{C}^2a^2\big(\sigma^2+\sigma[\gamma+\Lambda]+\gamma\Lambda\big)\\
&=R^2a^2\big(M_1\sigma^2+\sigma[\gamma M_1+\Lambda]+\gamma\Lambda\big)\,.
\end{aligned}
\tag{19}
$$

Now put $\sigma = i\omega$ in (19) and take the real and imaginary parts. One may arrive at the equations

$$\begin{aligned}
M_1\tilde{\Lambda}\omega^4 - \omega^2\big\{\tilde{\Lambda}\Lambda^2 + M_1\delta\Lambda^2 + \Lambda(M_1+1)(\gamma\tilde{\Lambda}+\Psi)\big\}& \\
+ (\gamma\Psi + \delta\Lambda^2)\Lambda^2 - \mathcal{C}^2a^2\omega^2 + \mathcal{C}^2a^2\gamma\Lambda& \\
= -R^2a^2\omega^2M_1 + R^2a^2\gamma\Lambda\,,&
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
-\,\omega^2\big\{M_1(\gamma\tilde{\Lambda}+\Psi) + \Lambda\tilde{\Lambda}(M_1+1)\big\}& \\
\Lambda^2(\gamma\tilde{\Lambda}+\Psi) + \Lambda(M_1+1)(\gamma\Psi+\delta\Lambda^2)& \\
+\,\mathcal{C}^2a^2(\gamma+\Lambda) = R^2a^2(\gamma M_1+\Lambda)\,.&
\end{aligned} \tag{21}$$

By eliminating ω^2 in (20) one arrives at a quadratic equation for R^2a^2. The critical oscillatory convection values are then found from the resulting expression by solving the quadratic for R^2 and by carefully minimizing R^2 taking care to ensure $\omega^2 > 0$, otherwise the values are meaningless.

Critical values for thermal convection in a Kelvin - Voigt fluid of order one and of order two are computed in Straughan [93], and for double diffusion in a Kelvin - Voigt fluid of order one by Straughan [92], and for double diffusion in a Kelvin - Voigt fluid of order two by Dhumd and Haddad [109], and thermosolutal convection in a bidisperse porous material is investigated by Badday and Harfash [110].

5 Navier - Stokes - Voigt model with LTNE

Local thermal non-equilibrium (LTNE) models in thermal convection and in double diffusive convection are discussed in Banu and Rees [9], Bidin and Rees [11], and in the book by Straughan [10]. We here develop such a theory for a viscoelastic fluid of Navier - Stokes - Voigt type saturating a porous medium where the temperature in the fluid, T^f, may be different to that in the solid porous skeleton, T^s. While we could develop a model for thermal convection in the presence of a salt field we restrict attention to thermal convection alone to emphasize the novelty of Navier - Stokes - Voigt theory.

Motivated by equations (1) and the fundamental theory of Banu and Rees [9], we write the governing equations as

$$\begin{aligned}
&A\frac{\partial v_i}{\partial t} - \hat{\lambda}\Delta\frac{\partial v_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\mu}{K\rho_0}v_i + \tilde{\mu}\Delta v_i + \alpha g T^f k_i\,,\\
&\frac{\partial v_i}{\partial x_i} = 0,\\
&\epsilon(\rho c)_f\frac{\partial T^f}{\partial t} + (\rho c)_f v_i\frac{\partial T^f}{\partial x_i} = \epsilon k_f\Delta T^f + h(T^s - T^f),\\
&(1-\epsilon)(\rho c)_s\frac{\partial T^s}{\partial t} = (1-\epsilon)k_s\Delta T^s - h(T^s - T^f),
\end{aligned} \tag{22}$$

where the notation is as in section 2, with additionally $(\rho c)_f$ being the product of the fluid density and the specific heat of the fluid at constant pressure, and $(\rho c)_s$ being analogous quantities for the solid porous matrix. The coefficients k_f and k_s are the thermal conductivities of the fluid and solid, respectively, and h is a heat transfer coefficient between the fluid and solid.

System (22) holds in the horizontal layer $\{(x,y)\in\mathbb{R}^2\}\times\{z\in(0,d)\}$ for $t>0$. The boundary conditions are

$$v_i=0,\quad z=0,d;\qquad T^f=T^s=T_L\,,\quad z=0;\qquad T^f=T^s=T_U\,,\quad z=d;$$

for T_L,T_U constants with $T_L>T_U$. The steady conduction solution is

$$\bar{T}^f=\bar{T}^s=-\beta z+T_L\,,\qquad \bar{v}_i\equiv 0.$$

To analyse stability and instability of this solution we introduce perturbations u_i,θ,φ,π to v_i,T^f,T^s,p. These perturbations are non-dimensionalized with the scales

$$\begin{aligned}
&\theta=T^\sharp\theta^*\,,\quad \varphi=T^\sharp\varphi^*\,,\quad T^\sharp=Ud\sqrt{\frac{\beta\mu c_f}{\epsilon k_f\alpha gK}}\,,\quad u_i=Uu_i^*\,,\quad x_i=x_i^*d\,,\\
&t=t^*\mathcal{T},\quad \mathcal{T}=\frac{(\rho c)_f d^2}{k_f}\,,\quad U=\frac{\epsilon k_f}{(\rho c)_f d}\,,\quad \pi=\pi^*P,\quad P=\frac{d\mu U}{K}\,,\\
&Va=\frac{\mu\mathcal{T}}{K\rho_0 A}\,,\quad \lambda=\frac{K_0\rho_0\hat{\lambda}}{d^2\mathcal{T}\mu}\,,\quad B=\frac{\tilde{\mu}K\rho_0}{\mu d^2}\,,\\
&K_1=\frac{(\rho c)_s}{(\rho c)_f}\,\frac{k_f}{k_s}\,,\quad K_2=\Big(\frac{\epsilon}{1-\epsilon}\Big)\frac{k_f}{k_s}\,.
\end{aligned}$$

We also define the Rayleigh number $Ra=R^2$ as

$$Ra=\frac{d^2\beta c_f\rho_0\alpha gK}{\epsilon k_f\mu}\,,$$

and the Rees number H by

$$H=\frac{d^2h}{\epsilon k_f}\,.$$

Dropping stars the non-dimensional perturbation equations are

$$\begin{aligned}
&\frac{1}{Va}u_{i,t}-\lambda\Delta u_{i,t}=-\pi_{,i}-u_i+B\Delta u_i+R\theta k_i\,,\\
&u_{i,i}=0,\\
&\theta_{,t}+u_i\theta_{,i}=Rw+\Delta\theta+H(\varphi-\theta),\\
&K_1\varphi_{,t}=\Delta\varphi-HK_2(\varphi-\theta).
\end{aligned}\tag{23}$$

These equations hold on $\mathbb{R}^2\times\{z\in(0,1)\}$ for $t>0$. The boundary conditions are

$$u_i=0,\quad \theta=\varphi=0,\qquad z=0,1,$$

and u_i,θ,φ,π are periodic in x,y.

One may write equations (23) in the abstract form

$$AU_t = LU + N(U),$$

where $U = (u, v, w, \theta, \varphi)^T$ and where A and L are the linear operators and N represents the nonlinear term $u_i\theta_{,i}$. For equations (23) both operators A and L are symmetric, for example L is equivalent to

$$L = \begin{pmatrix} -I + B\Delta & 0 & 0 & 0 & 0 \\ 0 & -I + B\Delta & 0 & 0 & 0 \\ 0 & 0 & -I + B\Delta & R & 0 \\ 0 & 0 & R & \Delta - H & \\ 0 & 0 & 0 & H & \dfrac{1}{K_2}\Delta - H \end{pmatrix}$$

Then one may show as in Straughan [111] that exchange of stabilities holds and the linear instability boundary is the same as the global nonlinear stability one. This is important because now one may infer the stability boundary from the numerical results in Banu and Rees [9], even when the saturating fluid is one of Navier - Stokes - Voigt type.

If we write down the analogous model for double - diffusive convection with LTNE and a Navier - Stokes - Voigt fluid saturating the porous medium, in the heated and salted below configuration, then the linear operator L is not symmetric and oscillatory instabilities will occur. In this case the linear instability boundaries will be different from those when no salt field is present.

6 Navier - Stokes - Voigt - Christov model

Much modern research centres on flow and heat transfer in domains with very small dimensions, the area of thermofluidic flow. When this flow is through a porous solid then it may be argued that classical Fourier theory is not sufficient. Christov [112] presents an argument where one should also consider higher gradients in the heat flux law. For an isotropic body he begins with a generalization of Fourier's law of form

$$-\chi_2\Delta q_i + q_i = -k\frac{\partial T}{\partial x_i} + k_2\Delta\frac{\partial T}{\partial x_i}\,. \tag{24}$$

The general theory of Christov [112] then gives rise to a heat equation which contains four spatial derivatives. However, he also allows the possibility of having $k_2 = 0$. Since the balance of energy in the body has equation

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial q_i}{\partial x_i}\,,$$

this particular case gives rise to an equation for the temperature field of form

$$\frac{\partial T}{\partial t} - \chi_2 \Delta \frac{\partial T}{\partial t} = k\Delta T\,, \tag{25}$$

for coefficients $k, \chi_2 > 0$. Clearly, (25) is in some sense analogous to the addition of the Kelvin - Voigt term in Navier - Stokes - Voigt theory, in that this equation incorporates a memory effect in the temperature field. In view of the above, we now develop a model for a porous layer saturated with a Navier - Stokes - Voigt fluid, but when the temperature equation also has a form analogous to (25). Thus, we commence with a model deriving from (1) in section 2 and analyse a double diffusion convection problem analogous to that of equations (5) in section 2.1.

We suppose the fluid saturated porous medium satisfies the equations

$$\begin{aligned}
&A\frac{\partial v_i}{\partial t} - \hat{\lambda}\Delta\frac{\partial v_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\mu}{K\rho_0}v_i + \tilde{\mu}\Delta v_i + \alpha g T k_i - \alpha_s g C k_i\,,\\
&\frac{\partial v_i}{\partial x_i} = 0,\\
&\frac{1}{M}\frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i} - \frac{\chi_2}{M}\Delta\frac{\partial T}{\partial t} = \kappa\Delta T,\\
&\epsilon\frac{\partial C}{\partial t} + v_i\frac{\partial C}{\partial x_i} = \epsilon\kappa_s\Delta C.
\end{aligned} \tag{26}$$

These equations hold in the layer $\mathbb{R}^2 \times \{z \in (0, d)\}$ for $t > 0$. The steady state and the boundary conditions are the same as in section 2.1. The non-dimensional perturbation equations are derived as in section 2.1, but now have form

$$\begin{aligned}
&\frac{1}{Va}u_{i,t} - \lambda\Delta u_{i,t} = -\pi_{,i} - u_i + B\Delta u_i + R\theta k_i - \mathcal{C}\phi k_i\,,\\
&u_{i,i} = 0,\\
&\theta_{,t} + u_i\theta_{,i} - \chi\Delta\theta_{,t} = Rw + \Delta\theta,\\
&MLe\phi_{,t} + \frac{Le}{\epsilon}u_i\phi_{,i} = \mathcal{C}w + \Delta\phi,
\end{aligned} \tag{27}$$

where χ is a non-dimensional form of χ_2/M. The boundary conditions on the perturbations are as (6).

To develop a linear instability analysis one proceeds as in section 2.1 but now the relevant equations instead of (7) are

$$\begin{aligned}
&-\frac{\sigma}{Va}\Delta w + \lambda\sigma\Delta^2 w = \Delta w - B\Delta^2 w - R\Delta^*\theta + \mathcal{C}\Delta^*\phi,\\
&\sigma\theta - \chi\sigma\Delta\theta = Rw + \Delta\theta,\\
&MLe\sigma\phi = \mathcal{C}w + \Delta\phi.
\end{aligned} \tag{28}$$

For two stress free surfaces this leads to the following equation for R^2,

$$R^2 = \frac{1}{a^2}(\sigma\tilde{\Lambda} + \Psi)(\sigma\Lambda^* + \Lambda) + \mathcal{C}^2\Big(\frac{\sigma\Lambda^* + \Lambda}{\sigma_1 M_1 + \Lambda}\Big) \tag{29}$$

where

$$\Lambda^* = 1 + \chi\Lambda.$$

The last term in (29) is rearranged and then one recognises R^2 has to be real. One then takes the real and imaginary parts of equation (29). The imaginary part leads to the following equation for ω^2,

$$\omega^2 = \mathcal{C}^2 a^2 \,\frac{\Lambda(M_1 - \Lambda^*)}{M_1^2(\Lambda^*\Psi + \tilde{\Lambda}\Lambda)} - \frac{\Lambda^2}{M_1^2}\,. \tag{30}$$

The real part of (29) gives the following relation for R^2,

$$R^2 = -\omega^2\frac{\Lambda\Lambda^*}{a^2} + \frac{\Psi\Lambda}{a^2} + \mathcal{C}^2\Big(\frac{\Lambda^2 + M_1\Lambda^*}{\Lambda^2 + M_1^2}\Big)\,. \tag{31}$$

The oscillatory convection Rayleigh numbers are found by minimizing (31) in a^2, controlling the fact that ω^2 should be positive at criticality by using (30).

7 Fully nonlinear models

We pointed out in section 2 that Damázio et al. [73] argue that (3) cannot, in general, be correct since $d_{ij,t}$ is not an objective derivative. Objective derivatives are discussed in detail in Morro [113], and in particular in the context of viscoelastic fluids by Giorgi and Morro [114]. A very detailed account of objective time derivatives with explicit reference to porous media is given by Morro [108], who develops porous media theory from a mixture of a fluid and a solid, taking care to ensure the use of objective derivatives is compatible with the principles of continuum thermodynamics and with the Clausius - Duhem entropy inequality.

One way around the critiscism is to replace $d_{ij,t}$ by a suitable objective derivative. However, there are many such derivatives as Morro [113] indicates. Morro [108] precisely points out that (3) could employ a corotational derivative, see his equation (35). To understand this let a superposed dot denote the material derivative, and let w_{ij} denote the skew-symmetric part of $v_{i,j}$, i.e.

$$w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}),$$

the symmetric part being defined in (4). If we identify d_{ij} with the tensor D and w_{ij} with the tensor W, then the corotational derivative of D is

$$\overset{\circ}{D} = \dot{D} - WD - DW^T. \tag{32}$$

Thus, equation (2) would be replaced by

$$\begin{aligned}\rho_0 A\Big(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\Big) &= \rho f_i - \frac{\mu}{K}v_i - \frac{\partial p}{\partial x_i} + \tilde{\mu}\Delta v_i + \hat{\lambda}\Delta\frac{\partial v_i}{\partial t} \\ &+2\hat{\lambda}\Big\{\frac{\partial}{\partial x_j}\Big(v_k\frac{\partial d_{ij}}{\partial x_k}\Big) - \frac{\partial}{\partial x_j}(w_{ik}d_{kj}) - \frac{\partial}{\partial x_j}(d_{ik}w_{jk})\Big\}\,.\end{aligned} \tag{33}$$

Clearly, equation (33) is very different from equation (2). In (33) there are four new nonlinear terms.

It may well be that for many situations, especially in a dense porous medium where the porosity is not too high, the nonlinear terms will be small and can be neglected. Certainly in a linear instability analysis of thermal convection problems where the steady state is one of zero velocity then these extra terms will not be present. However, one should be very careful with any nonlinear analysis, be it an energy stability analysis, weakly nonlinear analysis, or by some other means, it will be necessary to check the behaviour of the nonlinear terms.

8 Numerical results

We now report on numerical results for the model of section 2.1. Thus, we minimize in a^2 numerically (8) and (11), taking care to employ (10) to ensure $\omega^2 > 0$ at the oscillatory convection threshold. The liquid is water, the solute is salt (NaCl) and the porous material is sand. The physical values for these materials are taken from Straughan [115]. The coefficients M and κ are

$$M = \frac{(\rho_0 c_p)_f}{(\rho_0 c)_m}\,, \qquad \kappa = \frac{k_m}{(\rho_0 c_p)_f}\,,$$

where

$$(\rho_0 c)_m = \epsilon(\rho_0 c_p)_f + (1-\epsilon)(\rho_0 c)_s\,, \qquad k_m = k_s(1-\epsilon) + k_f\epsilon\,,$$

ϵ being the porosity which we take as $\epsilon = 0.3$. The coefficients k_s and k_f are the thermal conductivity of sand and water, respectively. The values we adopt are

$$\begin{aligned} &k_f = k_{water} = 0.606\ W/mK, &&k_s = k_{sand} = 0.25\ W/mK,\\ &c_{p\,water} = 4187\ J/kgC, &&c_{p\,sand} = 830\ J/kgC,\\ &\rho_{0\,water} = 998\ kg\,m^{-3}\,, &&\rho_{0\,sand} = 1922\ kg\,m^{-3}\,,\end{aligned}$$

and the diffusion coefficient of NaCl in water is

$$k_c = 2.66\times 10^{-9}\ m^2\,s^{-1}.$$

These values yield

$$Le = 32.1003, \qquad M = 2.6194, \qquad M_1 = 84.0835.$$

The numerical results for the transition from stationary to oscillatory convection are presented in table 1. Figure 1 shows the linear instability curves with Brinkman number $B = 0$, Vadasz number $Va = 1$ and the Kelvin - Voigt parameter λ taking the values 0, 0.5 and 1. Each curve should be interpreted as linear instability is by stationary convection until a critical value of $\mathcal{C}^2 = C$ is reached and the the linear instability is the oscillatory convection curve which branches thereafter. For example, when $\lambda = 1$ the transition is when $C = 128.49$ and the critical Rayeligh number is $Ra = 167.968$. The stabilizing effect of the Kelvin - Voigt term is clearly evident.

In table 1 the first three lines show the variation in the transition value $Ra_{stat} = Ra_{osc}$ as the Brinkman number B varies. The increase of Brinkman number increases the transition value as one would expect. The values in lines 3,4,5 and 6 show the variation with the Vadasz number, and this is seen to not be a dominant factor. The last three lines display the variation of the transition threshold as the Kelvin - Voigt parameter λ varies. Again, the variation of λ has a relatively strong effect.

9 Conclusions

We have analysed the behaviour of the instability transition to convective motion for a variety of models of Navier - Stokes - Voigt type or Kelvin - Voigt type when such a fluid saturates a porous medium. One should note that a Navier - Stokes - Voigt fluid is also a Kelvin - Voigt fluid of order zero, and is alternatively known as an Oskolkov fluid. We specifically look at the Navier - Stokes - Voigt model (Kelvin - Voigt of order 0), the Kelvin - Voigt model of order N, the Navier - Stokes - Voigt model when local thermal non-equilibrium effects are present, the Navier - Stokes - Voigt model when the heat flux satisfies a Christov relation, and we briefly discuss the implications of employing an objective time derivative in the Navier - Stokes - Voigt cases.

We have not seen such models presented in the context of porous media flow and we believe they will be of use in applications of viscoelastic fluids saturating a porous medium under non - isothermal conditions.

Acknowledgments. I would like to thank very much four anonymous referees for helpful suggestions which have markedly improved the manuscript.
Author contributions. This work was performed 100 per cent by B. Straughan, who also completely wrote the article.
Conflict of interest. There are no conflicts of interest.
Funding. This work was supported by the Leverhulme Trust, grant number EM/2019-022/9.
Data Availability. Any sources of data are indicated in the text.

References

[1] Nield, D.A.: The thermohaline Rayleigh - Jeffreys problem. J. Fluid Mech **29**, 545–558 (1967)

[2] Nield, D.A.: Onset of thermohaline convection in a porous medium. Water Resources Res. **4**, 553–560 (1968)

[3] Nield, D.A.: Onset of convection in a fluid layer overlying a layer of a porous medium. J. Fluid Mech **81**, 513–522 (1977)

[4] Nield, D.A.: Effects of local thermal non-equilibrium in steady convection processes in saturated porous media: forced convection in a chanel. J. Porous Media **1**, 181–186 (1998)

[5] Nield, D.A., Kuznetsov, A.V.: The interaction of thermal nonequilibrium and heterogeneous conductivity effects in forced convection in layered porous media. Int. J. Heat Mass Transfer **112**, 4369–4373 (2016)

[6] Nield, D.A.: The stability of flow in a chanel or duct occupied by a porous medium. Int. J. Heat Mass Transfer **46**, 4351–4354 (2003)

[7] Nield, D.A.: A note on modelling of local thermal nonequilibrium in a structured porous medium. Int. J. Heat Mass Transfer **45**, 4367–4368 (2002)

[8] Nield, D.A., Bejan, A.: Convection in Porous Media, 4th edn. Springer, New York (2013)

[9] Banu, N., Rees, D.A.S.: Onset of Darcy-Bénard convection using a thermal non-equilibrium model. Int. J. Heat Mass Transfer **45**, 2221–2228 (2002)

[10] Straughan, B.: Convection with Local Thermal Non-equilibrium and Microfluidic Effects. Advances in Mechanics and Mathematics Series, vol. 32. Springer, Cham, Switzerland (2015)

[11] Bidin, B., Rees, D.A.S.: Pattern selection for Darcy - Bénard convection with local thermal nonequilibrium. Int. J. Heat Mass Transfer **153**, 119539 (2020)

[12] Barletta, A., Nield, D.A.: Thermosolutal convective instability and viscous dissipation effect in a fluid - saturated porous medium. Int. J. Heat Mass Transfer **54**, 1641–1648 (2011)

[13] Harfash, A.J., Hill, A.A.: Simulation of three dimensional double diffusive throughflow in internally heated anisotropic porous media. Int. J. Heat Mass Transfer **72**, 609–615 (2014)

[14] Kuznetsov, A.V., Nield, D.A., Barletta, A., Celli, M.: Local thermal non-equilibrium and heterogeneity effects on the onset of double diffusive convection in an internally heated and soluted porous medium. Trans. Porous Media **109**, 393–409 (2015)

[15] Deepika, N., Narayana, P.A.L.: Nonlinear stability of double-diffusive convection in a porous layer with throughflow and concentration based internal heat source. Transport in Porous Media **111**, 751–762 (2016)

[16] Nield, D.A., Kuznetsov, A.V.: Do isoflux boundary conditions inhibit oscillatory

double - diffusive convection. Transport in Porous Media **112**, 609–618 (2016)

[17] Matta, A., Narayana, P., Hill, A.A.: Double diffusive Hadley - Prats flow in a horizontal layer with a concentration based internal heat source. J. Math. Anal. Appl. **452**, 1005–1018 (2017)

[18] Deepika, N.: Linear and nonlinear stability of double-diffusive convection with the soret effect. Transport in Porous Media **121**, 93–108 (2018)

[19] Kumar, G., Narayana, P.A.L., Sahn, K.C.: Linear and nonlinear thermosolutal instabilities in an inclined porous layer. Proc. Roy. Soc. London A **476**, 20190705 (2020)

[20] Straughan, B.: Heated and salted below porous convection with generalized temperature and solute boundary conditions. Trans. Porous Media **131**, 617–631 (2020)

[21] Capone, F., De Luca, R., Vadasz, P.: Onset of thermosolutal convection in rotating horizontal nanofluid layers. Acta Mech. **233**, 2237–2247 (2022)

[22] Capone, F., De Luca, R., Massa, G.: The onset of double diffusive convection in a rotating bidisperse porous medium. Eur. Phys. J. Plus **137**, 1034 (2022)

[23] Wang, C.C., Chen, F.: On the double - diffusive layer formation in the vertical annulus driven by radial thermal and salinity gradients. Mech. Res. Comm. **100**, 103991 (2022). https://doi.org/10.1016/j.mechrescom.2022.103991

[24] Deepika, N., Narayana, P.A.L., Hill, A.A.: The nonlinear stability analysis of double - diffusive convection with viscous dissipation effect. Transport in Porous Media **150**, 215–227 (2023)

[25] Chen, F., Chen, C.F.: Onset of finger convection in a horizontal porous layer underlying a fluid layer. J. Heat Transfer **110**, 403–409 (1988)

[26] Chang, M.H., Chen, F., Straughan, B.: Instability of Poiseuille flow in a fluid overlying a porous layer. J. Fluid Mech. **564**, 287–303 (2006)

[27] Hill, A.A., Straughan, B.: Poiseuille flow in a fluid overlying a porous medium. J. Fluid Mech. **603**, 137–149 (2008)

[28] Hill, A.A., Straughan, B.: Global stability for thermal convection in a fluid overlying a highly porous material. Proc. Roy. Soc. London A **465**, 207–217 (2009)

[29] Payne, L.E., Straughan, B.: Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modelling questions. J. Math. Pures Appl. **77**, 317–354 (1998)

[30] Samanta, A.: Linear stability of a plane Couette - Poiseuille flow overlying a porous layer. Int. J. Multiphase Flow **123**, 103160 (2020)

[31] McCurdy, M., Moore, N.J., Wang, X.: Predicting convection configurations in coupled fluid - porous systems. J. Fluid Mech. **953**, 23 (2022)

[32] Mirbod, P., Hooshyar, S., Taheri, E., Yoshikawa, H.N.: On the instability of particle - laden flows in channels with porous walls. Phys. Fluids **36**, 044105 (2024)

[33] Kumaravel, S., Nagaraj, M., Barmavatu, P.: Experimental and theoretical investigation to optimize the performance of solar still. Desalination and Water Treatment **318**, 100343 (2024)

[34] Mbelu, O.V., Adeyinka, A.M., Yahya, D.I., Adediji, Y.B., Njoku, H.: Advances in solar pond technology and prospects of efficiency improvement methods. Sustainable Energy Research **11**, 18 (2024)

[35] Hill, A.A., Carr, M.: Stabilizing solar ponds by using porous materials. Advances in Water Resources **60**, 1–6 (2013)

[36] Nield, D.A., Kuznetsov, A.V.: Forced convection in a bidisperse porous medium channel: a conjugate problem. Int. J. Heat Mass Transfer **47**, 5375–5380 (2004)

[37] Nield, D.A., Kuznetsov, A.V.: The onset of convection in a bidisperse porous. Int. J. Heat Mass Transfer **49**, 3068–3074 (2006)

[38] Kuznetsov, A.V., Nield, D.A.: The onset of double - diffusive nanofluid convection in a layer of a saturated porous medium. Trans. Porous Media **85**, 941–951 (2010)

[39] Gentile, M., Straughan, B.: Bidispersive vertical convection. Proc. Roy. Soc. A **473**, 20170481 (2017)

[40] Gentile, M., Straughan, B.: Bidispersive thermal convection with relatively large macropores. J. Fluid Mech. **898**, 14 (2020)

[41] Capone, F., De Luca, R., Gentile, M.: Coriolis effect on thermal convection in a rotating bidispersive porous layer. Proc. Roy. Soc. London A **476**, 20190875 (2020)

[42] Capone, F., Massa, G.: The effects of Vadasz term, anisotropy and rotation on bidisperse convection. Int. J. Nonlinear Mech. **135**, 103749 (2021)

[43] Siddabasappa, C., Siddheshwar, P.G., Mallikarjunaiah, S.M.: Analytical study of the Brinkman - Bénard convection in a bidisperse porous medium: linear and weakly nonlinear study. Thermal Sci. Engng. Prog. **39**, 101696 (2023)

[44] Bhavyashree, S.M., Ragoju, R., Reddy, G.S.K.: Effect of viscous dissipation on thermal convection in bidispersive porous media with vertical throughflow: gloabl stability analysis. Phys. Fluids **36**, 084108 (2024)

[45] Straughan, B.: Anisotropic bidispersive convection. Proc. Roy. Soc. London A **475**, 20190206 (2019)

[46] Kuznetsov, A.V., Nield, D.A.: The onset of convection in a tridisperse porous medium. Int. J. Heat Mass Transfer **54**, 3120–3127 (2011)

[47] Gentile, M., Straughan, B.: Tridisperse thermal convection. Nonlinear Anal. Real World Appl. **42**, 378–386 (2018)

[48] Avramenko, A.A., Kuznetsov, A.V., Basok, B.I., Blinov, D.G.: Investigation of stability of laminar flow in a parallel - plate channel filled with a fluid saturated porous medium. Phys. Fluids **17**, 094102 (2005)

[49] Hill, A.A., Straughan, B.: Stability of Poiseuille flow in a porous medium. In: Rannacher, R., Sequeira, A. (eds.) Advances in Mathematical Fluid Mechanics vol. 465, pp. 287–293. Springer, Heidelberg (2010)

[50] Straughan, B., Harfash, A.J.: Instability in Poiseuille flow in a porous medium with slip boundary conditions. Microfluidics and Nanofluidics **15**, 109–115 (2013)

[51] Samanta, A.: Nonmodal stability analysis of Poiseuille flow through a porous medium. Advances in Water Resources **100**, 1–20 (2024)

[52] Rees, D.A.S., Bassom, A.P., Siddheshwar, P.G.: Local thermal non-equilibrium effects arising from the injection of a hot fluid into a porous medium. J. Fluid Mech. **594**, 379–398 (2008)

[53] Barletta, A., Rees, D.A.S.: Local thermal non-equilibrium effects in the Darcy - Bénard instability with isoflux boundary conditions. Int. J. Heat Mass Transfer **55**, 384–394 (2012)

[54] Nield, D.A., Kuznetsov, A.V., Barletta, A., Celli, M.: The effects of double diffusion and local thermal non-equilibrium on the onset of convection in a layered porous medium: non - oscillatory instability. Trans. Porous Media **107**, 261–279 (2015)

[55] Eltayeb, I.A.: Stability of porous Bénard - Brinkman layer in local thermal non - equilibrium with Cattaneo effects in the solid. Int. J. Thermal Sci. **98**, 208–218 (2015)

[56] Capone, F., Gentile, M., Gianfrani, J.A.: Optimal stability thresholds in rotating fully anisotropic porous medium with LTNE. Transport in Porous Media **139**,

185–201 (2021)

[57] Capone, F., Gianfrani, J.A.: Natural convection in a fluid saturating an anisotropic porous medium in LTNE: effect of depth - dependent viscosity. Acta Mech. **233**, 4535–4548 (2022)

[58] Freitas, R.B., Brandao, P.V., Alves, C.S.D.B., Celli, M., Barletta, A.: The effect of local thermal non - equilibrium on the onset of thermal instability for a metallic foam. Phys. Fluids **34**, 034105 (2022)

[59] Shivakumara, I.S., Raghunatha, K.R., Dhananjaya, M., Vinod, Y.: Lack of local thermal nonequilibrium effects on convection in a porous medium saturated with an Ellis fluid. Transport Por. Media **143**, 300–400 (2022)

[60] Shivakumara, I.S., Raghunatha, A.: Changes in the onset of double - diffusive local thermal nonequilibrium convection due to the introduction of a of third component. Transport in Porous Media **143**, 225–242 (2022)

[61] Badday, A.J., Harfash, A.J.: The effects of the Soret and slip boundary conditions on thermosolutal convection with a Navier - Stokes - Voigt fluid. Phys. Fluids **35**, 014101 (2023)

[62] Noon, N.J., Haddad, S.A.: Stability analysis of double - diffusive convection in local thermal non - equilibrium porous medium with internal heat source and reaction effects. J. Non-Equilibrium Thermodyn. **48**, 25–39 (2023)

[63] Oskolkov, A.P.: The uniqueness and solvability of boundary value problems for the equations of motion for aqueous solutions of polymers. Zap. Nauc. Sem. Leningrad. Otdel. Mat. Inst. Steklov **38**, 98–136 (1973)

[64] Oskolkov, A.P.: A nonstationary quasilinear system with a small parameter, regularizing a system of Navier - Stokes equations. J. Soviet Math. **6**, 51–57 (1976)

[65] Oskolkov, A.P.: Some quasilinear systems occurring in the study of the motion of viscous fluids. J. Soviet Math. **9**, 765–790 (1978)

[66] Oskolkov, A.P.: Some nonstationary linear and quasilinear systems occurring in the investigation of the motion of viscous fluids. J. Soviet Math. **10**, 299–355 (1978)

[67] Oskolkov, A.P.: Initial - boundary value problems for the equations of Kelvin - Voigt fluids and Oldroyd fluids. Proc. Steklov Inst. Math. **179**, 126–164 (1988)

[68] Oskolkov, A.P.: Nonlocal problems for the equations of motion of Kelvin - Voigt fluids. J. Math. Sciences. **75**, 2058–2078 (1995)

[69] Oskolkov, A.P., Shadiev, R.: Towards a theory of global solvability on $[0, \infty)$ of

initial - boundary value problems for the equations of motion of Oldroyd and Kelvin - Voigt fluids. J. Math. Sciences **68**, 240–253 (1994)

[70] Baranovskii, E.S.: The Navier - Stokes - Voigt equations with position dependent slip boundary conditions. ZAMP **74**, 6 (2023)

[71] Berselli, L.C., Bisconti, L.: On the structural stability of the Euler - Voigt and Navier - Stokes - Voigt models. Nonlinear Analysis **75**, 117–130 (2012)

[72] Celebi, A.O., Kalantarov, V.K., Polat, M.: Global attractors for 2D Navier - Stokes - Voigt equations in an unbounded domain. Applicable Analysis **88**, 381–392 (2009)

[73] Damázio, P.D., Manholi, P., Silvestre, A.L.: L^q theory of the Kelvin - Voigt equations in bounded domains. J. Differential Equations **260**, 8242–8260 (2016)

[74] Di Plinio, F., Giorgini, A., Pata, V., Temam, R.: Navier - Stokes - Voigt equations with memory in 3D lacking instantaneous kinematic viscosity. J. Nonlin. Sci. **28**, 656–686 (2018)

[75] Kalantarov, V.K., Titi, E.S.: Global attractors and determining modes for the 3D Navier - Stokes - Voigt equations. Chinese Annals of Math. **30**, 697–714 (2009)

[76] Kalantarov, V.K., Titi, E.S.: Global stabilization of the Navier - Stokes - Voigt and the damped nonlinear wave equations by a finite number of feedback controllers. Discrete and Continuous Dynamical Systems B **23**, 1325–1345 (2018)

[77] Kalantarov, V.K., Levant, B., Titi, E.S.: Gevrey regularity of the global attractor of the 3D Navier - Stokes - Voigt equations. J. Nonlin. Sci. **19**, 133–152 (2009)

[78] Krasnoschok, M., Pata, V., Siryk, S.V., Vasylyeva, N.: A sub-diffusive Navier - Stokes - Voigt system. Physica D **409**, 132503 (2020)

[79] Layton, W.J., Rebholz, L.G.: On relaxation times in the Navier - Stokes - Voigt model. Int. J. Comput. Fluid Dyn. **27**, 184–187 (2013)

[80] Niche, C.J.: Decay characterization of solutions to Navier - Stokes - Voigt equations in terms of the initial datum. J. Differential Equations **260**, 4440–4453 (2016)

[81] Sukacheva, T.G.: Solvability of a nonstationary thermal convection problem for a viscoelastic incompressible fluid. Differ. Uravn. **36**, 1106–1112 (2000)

[82] Sukacheva, T.G.: Oskolkov models and Sobolev type equations. Bull. South Ural State Tech. Univ., Ser. Math. Modelling, Programming and Computer Software **15**, 5–22 (2022)

[83] Sviridyuk, G.A., Sukacheva, T.G.: On the solvability of a nonstationary problem describing the dynamics of an incompressible viscoelastic fluid. Mathematical Notes **63**, 388–395 (1998)

[84] Sukacheva, T.G., Kondyukov, A.O.: On a class of Sobolev type equations. Bull. South Ural State Tech. Univ., Ser. Math. Modelling and Programming **7**, 5–21 (2014)

[85] Sukacheva, T.G., Matveeva, O.P.: On a homogeneous model of the non - compressible viscoelastic Kelvin - Voigt fluid of the non - zero order. J. Samara State Tech. Univ., Ser. Phys. Math. Sci. **5**, 33–41 (2010)

[86] Sukacheva, T.G., Sviridyuk, G.A.: The Avalos - Triggiani problem for the linear Oskolkov system and a system of wave equations ii. J. Comp. Eng. Math. **9**, 67–72 (2022)

[87] Zvyagin, A.V.: Study of solvability of a thermoviscoelastic model describing the motion of weakly concentrated water solutions of polymers. Siberian Math. J. **59**, 843–859 (2018)

[88] Zvyagin, A.V.: Weak solvability of non-linearly viscous Pavlovskii model. Izv. Vyssh. Uchebn. Zaved. Mat. **6**, 87–93 (2022)

[89] Sukacheva, T.G., Matveeva, O.P.: Problem of heat convection of the incompressible viscoelastic Kelvin - Voigt fluid of nonzero order. Izvestiya VUZ, Matematika **45**, 44–51 (2001)

[90] Straughan, B.: Thermosolutal convection with a Navier - Stokes - Voigt fluid. Appl. Math. Optimization **83**, 2587–2599 (2021)

[91] Straughan, B.: Nonlinear stability for convection with temperature dependent viscosity in a Navier - Stokes - Voigt fluid. Eur. Phys. J. Plus **138**, 4380 (2023)

[92] Straughan, B.: Competitive double diffusive convection in a Kelvin - Voigt fluid of order one. Appl. Math. Optimization **84**, 631–650 (2021)

[93] Straughan, B.: Instability thresholds for thermal convection in a Kelvin - Voigt fluid of variable order. Rend. Circ. Matem. Palermo Ser. 2 **71**, 187–206 (2022)

[94] Barletta, A.: Local energy balance, specific heats and the Oberbeck - Boussinesq approximation. Int. J. Heat Mass Transfer **270**, 5266–5270 (2015)

[95] Barletta, A.: The Boussinesq approximation for buoyant flows. Mech. Research Comm. **124**, 103939 (2022)

[96] Breugem, W.P., Rees, D.A.S.: A derivation of the volume-averaged Boussinesq equations for flow in porous media with viscous dissipation. Trans. Porous Media **63**, 1–12 (2006)

[97] Nield, D.A., Barletta, A.: Extended Oberbeck - Boussinesq approximation study of convective instabilities in a porous layer with horizontal flow and bottom heating. Int. J. Heat Mass Transfer **53**, 577–585 (2010)

[98] Rees, D.A.S.: The onset of Darcy - Brinkman convection in a porous layer: an asymptotic analysis. Int. J. Heat Mass Transfer **45**, 2213–2220 (2002)

[99] Chirita, S., Zampoli, V.: On the forward and backward in time problems in the Kelvin - Voigt thermoelastic materials. Mech. Res. Comm. **68**, 25–30 (2015)

[100] Oskolkov, A.P.: Theory of nonstationary flows of Kelvin - Voigt fluids. Zap. Nauchn. Sem. LOMI **115**, 191–202 (1982)

[101] Sukacheva, T.G.: On a model of the motion of an incompressible viscoelastic Kelvin - Voigt fluid of nonzero order. Differ. Uravn. **33**, 552–557 (1997)

[102] Sukacheva, T.G.: On the solvability of a nonstationary viscoelastic Kelvin - Voigt fluid of nonzero grade. Izv. Vyssh. Uchebn. Zaved. Mat. **3**, 47–54 (1998)

[103] Matveeva, O.P.: Model of thermoconvection of incompressible viscoelastic fluid of non - zero order - computational experiment. Bull. South Ural State Tech. Univ., Ser. Math. Modelling and Programming **6**, 134–138 (2013)

[104] Sviridyuk, G.A.: On the manifold of solutions of a problem on the dynamics of an incompressible viscoelastic fluid. Differ. Uravn. **24**, 1832–1834 (1988)

[105] Sviridyuk, G.A.: Manifolds of solutions of a class of evolution and dynamical equations. Dokl. Akad. Nauk. SSSR **304**, 301–304 (1989)

[106] Sviridyuk, G.A.: A problem of the dynamics of a viscoelastic incompressible fluid. Differ. Uravn. **26**, 1992–1998 (1990)

[107] Sviridyuk, G.A., Sukacheva, T.G.: Cauchy problem for a class of semilinear equations of Sobolev type. Siberian Math. J. **31**, 794–802 (1990)

[108] Morro, A.: On the modelling of thermal convection in porous media through rate - type equations. Annali Univ. Ferrara **70**, 547–563 (2024)

[109] Dhumd, D.Z., Haddad, S.A.: Onset of double - diffusive convection with a Kelvin - Voigt fluid of variable order. Special Topics and Reviews in Porous Media **15**, 1–11 (2024)

[110] Badday, A.J., Harfash, A.J.: Thermosolutal convection in a Brinkman - Darcy - Kelvin - Voigt fluid with a bidisperse porous medium. Phys. Fluids **36**, 014119 (2024)

[111] Straughan, B.: Global nonlinear stability in porous convection with a thermal non-equilibrium model. Proc. Roy. Soc. London A **462**, 409–418 (2006)

[112] Christov, C.I.: On a higher gradient generalization of Fourier's law of heat conduction. In: Amer. Inst. Phys. Conf. Proc., vol. 946, pp. 11–22 (2007)

[113] Morro, A.: Modelling elastic heat conductors via objective rate equations. Cont. Mech. Thermodyn. **30**, 1231–1243 (2018)

[114] Giorgi, C., Morro, A.: On the modelling of compressible viscous fluids via Burgers and Oldroyd derivatives. Stud. Appl. Math. **176**, 127701–253 (2024)

[115] Straughan, B.: Bidispersive double diffusive convection. Int. J. Heat Mass Transfer **126**, 504–508 (2018)

Table 1 Transition values between stationary and oscillatory convection

Ra	a^2_{stat}	a^2_{osc}	ω^2	C	λ	Va	B
6757.86	4.951	4.930	1.022×10^{-5}	138.36	0.5	100	10
769.014	5.094	4.893	1.750×10^{-4}	67.325	0.5	100	1
103.488	9.870	4.669	2.978×10^{-3}	64.01	0.5	100	0
104.198	9.870	4.671	2.938×10^{-3}	64.72	0.5	10	0
103.406	9.870	4.669	2.982×10^{-3}	63.93	0.5	10^{10}	0
103.410	9.870	4.669	2.977×10^{-3}	63.93	0.5	10^3	0
111.327	9.870	4.685	2.602×10^{-3}	71.85	0.5	1	0
48.955	9.870	8.377	1.396×10^{-3}	9.48	0	1	0
167.968	9.870	4.145	1.822×10^{-3}	128.49	1	1	0

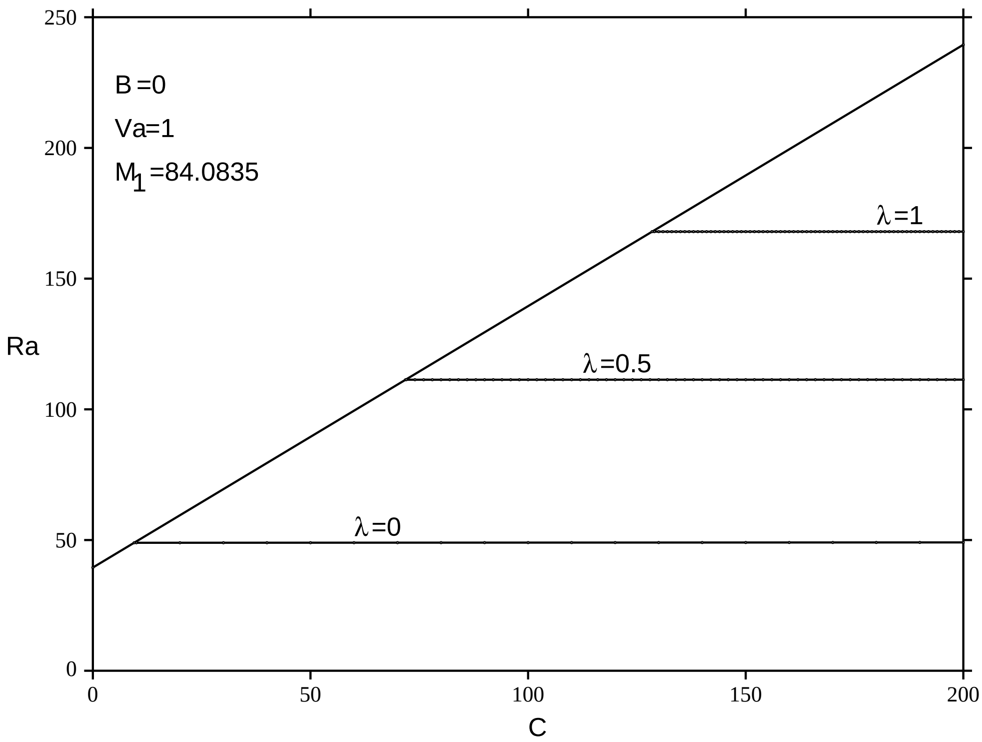

Fig. 1 Graph of Ra against C with $Le = 32.1003$, $M = 2.6194$. The diagonal line running from the lower left to the upper right represents the stationary convection curve. The transition points to oscillatory convection are $Ra = 48.955, C = 9.48$ when $\lambda = 0$, $Ra = 111.327, C = 71.85$ when $\lambda = 0.5$, and $Ra = 167.968, C = 128.49$ when $\lambda = 1$.