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Regular Article

Strongly convergent unitary representations of limit groups

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We prove that all finitely generated fully residually free groups (limit groups) have a sequence of finite dimensional unitary representations that 'strongly converge' to the regular representation of the group. The corresponding statement for finitely generated free groups was proved by Haagerup and Thorbjørnsen in 2005. In fact, we can take the unitary representations to arise from representations of the group by permutation matrices, as was proved for free groups by Bordenave and Collins.

As for Haagerup and Thorbjørnsen, the existence of such representations implies that for any non-abelian limit group, the Ext-invariant of the reduced C^*-algebra is not a group (has non-invertible elements).

An important special case of our main theorem is in application to the fundamental groups of closed orientable surfaces of genus at least two. In this case, our results can be used as an input to the methods previously developed by the authors of the appendix. The output is a variation of our previous proof of Buser's 1984 conjecture that there exist a sequence of closed hyperbolic surfaces with genera tending to infinity and first eigenvalue of the Laplacian tending to $\frac{1}{4}$. In this variation of

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the proof, the systoles of the surfaces are bounded away from zero and the surfaces can be taken to be arithmetic.

Contents

1. Introduction

A discrete group Γ is *fully residually free* (FRF) if for any finite set $S \subset \Gamma$, there exists a homomorphism $\Gamma \to \mathbf{F}$ that is injective on S where $\mathbf{F}$ is a free group. Finitely generated FRF groups are known to coincide with Sela's *limit groups* [25], so we use these two notions interchangeably in the sequel.

For $N \in \mathbf{N}$ let $\mathbf{U}(N)$ denote the group of $N \times N$ complex unitary matrices. For a discrete group Γ, $\lambda_\Gamma : \Gamma \to \mathrm{End}(\ell^2(\Gamma))$ is the left regular representation. It was an open problem for some years, popularized by Voiculescu in [26, Qu. 5.12], whether for a finitely generated free group $\mathbf{F}$, there exists a sequence of unitary representations $\{\rho_i : \mathbf{F} \to \mathbf{U}(N_i)\}_{i=1}^{\infty}$ such that for any element $z \in \mathbf{C}[\mathbf{F}]$,

$$\limsup_{i\to\infty} \|\rho_i(z)\| \le \|\lambda_{\mathbf{F}}(z)\|.$$

The norm on the left is the operator norm on $\mathbf{C}^{N_i}$ with respect to the standard Hermitian metric, and the norm on the right is the operator norm on $\ell^2(\mathbf{F})$. This problem was solved in the affirmative in a huge breakthrough by Haagerup and Thorbjørnsen [16].

In fact, following [26], given that the reduced C^*-algebra of $\mathbf{F}$ is simple by a result of Powers [23], the inequality above can be improved automatically[1] to

$$\lim_{i\to\infty} \|\rho_i(z)\| = \|\lambda_{\mathbf{F}}(z)\| \quad \forall z \in \mathbf{C}[\mathbf{F}]. \tag{1.1}$$

This notion of convergence of a sequence of finite dimensional unitary representations given by (1.1) applies equally as well to any discrete group Γ and we refer to this as *strong convergence.*

Theorem 1.1. *Any limit group Γ has a sequence of finite dimensional unitary representations that strongly converge to the regular representation of Γ. In fact, these unitary representations can be taken to factor through*

$$\Gamma \to S_N \xrightarrow{\text{std}} \mathbf{U}(N-1) \tag{1.2}$$

for some varying N, where S_N is the group of permutations of N letters, and std *is the $N-1$ dimensional irreducible component of the representation of S_N by 0-1 matrices.*

Remark 1.2. Some authors require *weak convergence* as part of the definition of strong convergence, meaning that there is pointwise convergence of the normalized traces. By a folklore result, for large classes of groups, including non-abelian limit groups, weak convergence follows from our definition of strong convergence here. We provide a proof of this in §1.10. We also note that the existence of permutation representations of limit groups weakly converging to the regular representation (implying in particular the soficity of limit groups) is an easy consequence of their residual finiteness, which in turn, is an easy consequence of the residual finiteness of free groups.

It was proved by G. Baumslag in [1] that the fundamental groups Λ_g of closed orientable surfaces are FRF, and it is also known [2, pp. 414-415] that the fundamental groups of non-orientable surfaces S with $\chi(S) \leq -2$ are FRF. This gives the following corollary of Theorem 1.1.

Corollary 1.3. *Let Γ denote the fundamental group of a connected closed surface S that is either orientable with no constraint on $\chi(S)$, or non-orientable with $\chi(S) \leq -2$. Then Γ has a sequence of finite dimensional unitary representations that strongly converge to the regular representation. Moreover, they can be taken to be of the form (1.2) for some varying N.*

Corollary 1.3 leaves open the cases of connected non-orientable surfaces with $\chi = 1$ ($\mathbf{R}P^2$), $\chi = 0$ (the Klein bottle $\mathbf{R}P^2 \# \mathbf{R}P^2$), and $\chi = -1$ ($(\mathbf{R}P^2)^{\#3}$). In all these cases the corresponding fundamental groups are not FRF.[2] The fundamental group of $\mathbf{R}P^2$ is

[1] See proof of Theorem 1.1 below for details.
[2] The first two cases are easy to check, and the case of $\chi = -1$ is due to Lyndon [21].

$\mathbf{Z}/2\mathbf{Z}$, and its regular representation is finite dimensional. We prove the following as an addendum to our main result.

Proposition 1.4. *The fundamental groups* $\pi_1((\mathbf{R}P^2)^{\#2}) = \langle a, b \,|\, b^{-1} = aba^{-1} \rangle$ *and* $\pi_1((\mathbf{R}P^2)^{\#3}) = \langle a, b, c \,|\, a^2b^2c^2 \rangle$ *have sequences of finite dimensional unitary representations that strongly converge to their respective regular representations.*

The proof of Theorem 1.1 revolves around the following potential property of discrete groups that we introduce here.

Definition 1.5. A discrete group Γ is C^**-residually free* if for any finite set $S \subset \Gamma$ and $\epsilon > 0$, there is a homomorphism $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(z))\| \leq \|\lambda_\Gamma(z)\| + \epsilon,$$

for all $z \in \mathbf{C}[\Gamma]$ supported on S with unit ℓ^1 norm.

Example 1.6. Any extension $N \to G \xrightarrow{\phi} \mathbf{F}$ of a free group by an amenable group N is C^*-residually free. Indeed, since N is amenable 1 is weakly contained in the regular representation of N. Then by Fell's continuity of induction ([13], [7, Thm. F.3.5]) we have that the quasi-regular representation of G on $\ell^2(G/N)$ is weakly contained in the regular representation G, hence by [7, Thm. F.4.4] for any $z \in \mathbf{C}[G]$

$$\|\lambda_{G/N}(zN)\| = \|\lambda_{\mathbf{F}}(\phi(z))\| \leq \|\lambda_G(z)\|.$$

Here we prove the following.

Theorem 1.7. *Limit groups are* C^**-residually free.*

The converse to Theorem 1.7 does not hold: Example 1.6 shows that $\mathbf{Z} \times \mathbf{F}$ is C^*-residually free, but it is easy to see that it is not FRF. It is, however, also easy to see that it is residually free. Furthermore, the group

$$\langle a, b, c \,|\, b^{-1} = aba^{-1},\ [c, b]\rangle$$

is C^*-residually free by Example 1.6 since it is an extension of the free group $\langle b, c\rangle$ by $\mathbf{Z} \cong \langle a\rangle$. On the other hand, it is not even residually free since it contains an embedded Klein bottle subgroup $\langle a, b\rangle$. It is an interesting question, not pursued here, to give some alternative characterization of a group being C^*-residually free.

Given a free group $\mathbf{F}$, and a basis X of $\mathbf{F}$, we write $|f|_X$ for the word length of f in the basis X. In any discrete group Γ with generating set Y we write $B_Y(r)$ for the elements of Γ that can be written as a product of at most r elements of $Y \cup Y^{-1}$. The proof of Theorem 1.7 relies on the following key proposition.

Proposition 1.8. *Let Γ be a limit group with a fixed finite generating set Y. There is a free group $\mathbf{F}$ with a basis X and $D = D(\Gamma, Y) > 0$ and $C = C(\Gamma, Y) > 0$ such that for any $r > 0$ there is an epimorphism $f : \Gamma \to \mathbf{F}$ which is injective on $B_Y(r)$ and*

$$\max_{g \in B_Y(r)} |f(g)|_X \leq Cr^D.$$

1.1. Further consequences I: spectral gaps

A hyperbolic surface is a complete Riemannian surface (without boundary) of constant curvature -1. Given a hyperbolic surface X, we write Δ_X for the Laplace-Beltrami operator on $L^2(X)$. If X is closed this operator's spectrum $\operatorname{spec}(\Delta_X)$ consists of eigenvalues $0 = \lambda_0(X) \leq \lambda_1(X) \leq \cdots \leq \lambda_k(X) \leq \cdots$ with $\lambda_k(X) \to \infty$ as $k \to \infty$. It was a conjecture of Buser [11] whether there exist a sequence of closed hyperbolic surfaces X_i with genera tending to infinity and with

$$\lambda_1(X_i) \to \frac{1}{4}$$

where λ_1 denotes the first non-zero eigenvalue of the Laplacian. The value $\frac{1}{4}$ is the asymptotically optimal one by a result of Huber [17]. See [15, Introduction] for an overview of the rich history of this problem. Buser's conjecture was settled in [15]. The proof therein does not allow us to take the surfaces to be arithmetic, and requires the surfaces to have very short curves. The results of this work in conjunction with the ideas in [15] allow us, along with Hide, to prove:

Theorem 1.9. *There exists a sequence of closed arithmetic hyperbolic surfaces $\{X_i\}_{i \in \mathbb{N}}$ with $g(X_i) \to \infty$, systoles uniformly bounded away from zero, and with*

$$\lambda_1(X_i) \to \frac{1}{4}.$$

In fact the X_i can be taken to be covering spaces of a fixed arithmetic hyperbolic surface X.

Theorem 1.9 is proved in the Appendix[3] by the second named author (MM) and Hide, as a consequence of the following corollary of Theorem 1.1.

Corollary 1.10 *(Matrix coefficients version of Theorem 1.1). Let Γ be a limit group. There exist a sequence of finite dimensional representations ρ_i such that for any $r \in \mathbf{N}$ and finitely supported map $a : \Gamma \to \operatorname{Mat}_{r \times r}(\mathbf{C})$, we have*

[3] In fact, the Appendix proves a more general statement about coverings of *any* hyperbolic surface; see Theorem A.1.

$$\lim_{i\to\infty} \| \sum_{\gamma\in\Gamma} a(\gamma)\otimes \rho_i(\gamma)\| = \| \sum_{\gamma\in\Gamma} a(\gamma)\otimes\lambda(\gamma)\|.$$

The norm on the left hand side is the operator norm for the tensor product of (r and N_i-dimensional) ℓ^2 norms. The norm on the right is the operator norm for the tensor product of ℓ^2 and the inner product on $\ell^2(\Gamma)$.

The proof of Corollary 1.10 from Theorem 1.1 is fairly standard and based on the fact that there is a unique C^*-algebra norm on $\mathrm{Mat}_{n\times n}(A)$ where A is a C^*-algebra. See [16, §9] for details.

1.2. Further consequences II: $\mathrm{Ext}(C^*_r(\Gamma))$ is not a group

In [5,6], Brown, Douglas, and Fillmore introduced and studied a homological/K-theoretic invariant $\mathrm{Ext}(\mathcal{A})$ of a unital separable C^*-algebra $\mathcal{A}$. By definition, $\mathrm{Ext}(\mathcal{A})$ is the collection of $*$-homomorphisms

$$\pi : \mathcal{A} \to B(\ell^2(\mathbf{N}))/\mathcal{K}$$

modulo conjugation of unitary operators on $\ell^2(\mathbf{N})$, where $B(\ell^2(\mathbf{N}))$ is the bounded operators on $\ell^2(\mathbf{N})$ and $\mathcal{K}$ is the ideal of compact operators therein. This is naturally a semigroup with multiplication arising from $(\pi_1,\pi_2)\mapsto \pi_1\oplus\pi_2$ composed with an isomorphism $\ell^2(\mathbf{N})\oplus\ell^2(\mathbf{N})\cong\ell^2(\mathbf{N})$.

One of the motivations of the work of Haagerup and Thorbjørnsen [16] was to prove that there are non-invertible elements of $\mathrm{Ext}(C^*_r(\mathbf{F}))$ when $\mathbf{F}$ is a finitely generated non-abelian free group, i.e., $\mathrm{Ext}(C^*_r(\mathbf{F}))$ is not a group.

The passage from the existence of strongly convergent unitary representations of $\mathbf{F}$ to this statement uses the following result proved by Voiculescu in [26, §§5.14] (see [16, Rmk. 8.6] for another exposition).

Proposition 1.11. *If Γ is a discrete, countable, non-amenable group with a sequence of finite dimensional unitary representations that strongly converge to the regular representation of Γ, then $\mathrm{Ext}(C^*_r(\Gamma))$ is not a group.*

Since non-abelian limit groups Γ are C^*-simple (Lemma 5.2), they are non-amenable. Indeed, an amenable group Γ has a C^*-algebra morphism $C^*_r(\Gamma)\to\mathbf{C}$ by [7, Thm. F.4.4] whose kernel contradicts simplicity. Hence combining Theorem 1.1 with Proposition 1.11 we obtain the following extension of ‘$\mathrm{Ext}(C^*_r(\mathbf{F}))$ is not a group’:

Corollary 1.12. *If Γ is a non-abelian limit group, then $\mathrm{Ext}(C^*_r(\Gamma))$ is not a group.*

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2. Background

Groups

We write e for the identity in any group. For any group Γ, $\mathbf{C}[\Gamma]$ denotes the group algebra of Γ with complex coefficients. For a free group $\mathbf{F}$ with a fixed set of generators X, for each $h \in \mathbf{F}$, we write $|h|_X$ for the reduced word length of h with respect to $X \cup X^{-1}$. If the generators X are clear we write $|h|$ for this reduced word length.

If Y is a generating set of any group Γ, we write $B_Y(r) \subset \Gamma$ for the elements of Γ that can be written as the product of at most r elements of $Y \cup Y^{-1}$.

Analysis

Given a discrete group Γ, $\lambda_\Gamma : \Gamma \to \mathrm{End}(\ell^2(\Gamma))$ is the *left regular representation*

$$\lambda_\Gamma(g)[f](h) \stackrel{\text{def}}{=} f(g^{-1}h).$$

This representation extends by linearity to one of the convolution algebra $\ell^1(\Gamma)$. For $\psi \in \ell^1(\Gamma)$, since λ_Γ is unitary we have the basic inequality

$$\|\lambda_\Gamma(\psi)\| \le \|\psi\|_{\ell^1} \tag{2.1}$$

where the norm on the left is the operator norm. The *reduced C^*-algebra* of Γ, denoted $C_r^*(\Gamma)$, is the closure of $\lambda_\Gamma(\ell^1(\Gamma))$ with respect to the operator norm topology. A *tracial state* on a unital C^* algebra $\mathcal{A}$ is a linear functional τ such that $\tau(1) = 1$, $\tau(a^*a) \ge 0$ (in particular, is real) for all $a \in \mathcal{A}$, and $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$.

An important inequality due to Haagerup [14] links the operator norm in $\mathrm{End}(\ell^2(\mathbf{F}))$ and the ℓ^2 norm in $\mathbf{C}[\mathbf{F}]$.

Lemma 2.1 *(Haagerup). Let X denote a finite generating set for a free group $\mathbf{F}$. Suppose that $a \in \mathbf{C}[\mathbf{F}]$ is supported on $B_X(r)$. Then*

$$\|\lambda_{\mathbf{F}}(a)\| \le (r+1)^{\frac{3}{2}} \|a\|_{\ell^2}.$$

Proof. Haagerup in [14, Lemma 1.4] proved that

$$\|\lambda_{\mathbf{F}}(a)\| \le \sum_{i=0}^{\infty} (i+1)\|a_i\|_{\ell^2}$$

where a_i is the function a multiplied pointwise by the indicator function of $B_X(i)\backslash B_X(i-1)$, i.e. the sphere of radius i. If a is supported on $B_X(r)$ then using Cauchy-Schwarz above gives the result, since $\sum_{i=0}^{r} \|a_i\|_{\ell^2}^2 = \|a\|_{\ell^2}^2$. □

There is also a more basic inequality in the reverse direction that holds for arbitrary discrete groups. Suppose that Γ is a discrete group. Then let $\delta_e \in \ell^2(\Gamma)$ denote the indicator function of the identity. We have for $a \in \mathbf{C}[\Gamma]$

$$\|a\|_{\ell^2}^2 = \langle \lambda_\Gamma(a)\delta_e, \lambda_\Gamma(a)\delta_e \rangle \leq \|\lambda_\Gamma(a)\|^2. \tag{2.2}$$

3. Proof of Proposition 1.8

The proof of Proposition 1.8 relies on the deep fact that any limit group embeds in an iterated extension of centralizers of a free group, and quantified versions of theorems of Gilbert and Benjamin Baumslag.

Definition 3.1. Let Γ be a limit group, $A < \Gamma$ a maximal abelian subgroup. A group $\Gamma' = \Gamma *_A B$, $B = A \times \langle t \rangle$ is an *extension of centralizers* of Γ. A group Γ is an *iterated extension of centralizers* if there is a chain of subgroups

$$\mathbf{F} = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_n = \Gamma$$

such that Γ_{i+1} is an extension of centralizers of Γ_i. The *height* of the extension is n.

Any iterated extension of centralizers is fully residually free, and so are their finitely generated subgroups, hence such subgroups are limit groups. Amazingly, the converse holds: any limit group actually embeds in a (finitely) iterated extension of centralizers. This was first claimed by Kharlampovich and Myasnikov in their papers on the Tarski problem [20, Theorem 4]. For a proof following Sela see [12, Theorem 4.2]. The forward implication seems to be contained in Lyndon's original paper on his free exponential group [22, last two paragraphs, page 533], which is the direct limit over the family of all iterated extensions of centralizers of $\mathbf{F}$, ordered by inclusion. See also [9, Theorem C1].

Let Γ be a limit group with some fixed generating set Y. The *distortion function of* Γ *with respect to* Y is the function

$$d_Y(r) = \min_{\substack{f\colon \Gamma \to \mathbf{F} \\ X \subset \mathbf{F}}} \max_{g \in B_Y(r)} |f(g)|_X ,$$

where the minimum is over all free groups $\mathbf{F}$, X which are bases of $\mathbf{F}$, and homomorphisms $f\colon \Gamma \to \mathbf{F}$ that are injective on $B_Y(r)$. The proof of Proposition 1.8 is a recapitulation of the proof that an iterated extension of centralizers is fully residually free in a way that lets us bound the distortion function by a polynomial whose degree depends on the height. We start with an improvement of Baumslag's power lemma.

Lemma 3.2 *(cf. [1, Proposition 1]). Fix a free group* $\mathbf{F}$ *with basis* X*. Let* $u, b_0, b_1, \dots, b_n \in \mathbf{F}$*, with* u *also cyclically reduced, nontrivial, and not a proper power of another element. If*

$$w = \prod_{i=0}^{n} u^{k_i} b_i = e \tag{3.1}$$

for

$$\min_{i>0}\{|k_i|\} > (8n+2) \cdot \max_{i \geq 0}\{1, |b_i|/|u|\},$$

then $[u, b_i] = e$ *for some* i*.*

G. Baumslag proved the same thing if $w = e$ for infinitely many integral values of each of the k_i. See also the proof of [27, Lemma 4.13], which has, implicitly, an effective version of Lemma 3.2 in it.

Proof. The proof is by induction on n. Clearly for $n = 0$, if $u^{k_0} b_0 = e$ then b_0 is a power of u and hence commutes with u.

We begin by manipulating our hypothesis to a more convenient form for the induction. If

$$\min_{i>0}\{|k_i|\} > (8n+2) \cdot \max_{i \geq 0}\{1, |b_i|/|u|\}$$

then

$$|w| > |u| \cdot \sum_{i>0} |k_i| > (8n+2) \cdot \max_{i \geq 0}\{|u|, |b_i|\}. \tag{3.2}$$

(Here $|w|$ is the non-reduced length of w.)

Let W denote the (non-reduced) combinatorial word formed by concatenating the reduced expressions for the u^{k_i} and the b_i defining w. Represent W as a directed cycle with a subinterval for each letter of W, where each subinterval is labelled by the corresponding $x^{\pm 1}$, $x \in X$, that appears at this point of the word. Because W reduces to the identity in $\mathbf{F}$, if we draw this circle on the boundary of a disc, it is possible to create a perfect matching of the subintervals by disjoint arcs[4] in the disc such that every arc has endpoints in interiors of two intervals labelled respectively x, x^{-1} for some $x \in X$. (Indeed, some consecutive pair of letters can be matched at first and then one iterates.) Fix such a matching. The arcs cut the disc into further topological discs.

The tree of cancellations T for w is the dual graph to this decomposition of the disc. A vertex of T is *special* if its corresponding topological disc meets an endpoint of one of

[4] An arc is an embedded interval with boundary in the boundary of the disc.

the subwords u^{k_i}, one of the b_i, or has valence at least three. An embedded segment in T with special endpoints and no special vertices in its interior is a *long edge*.

Every valence one vertex in T is special, so there are at most $2n+2$ of them. We now work out the maximal number of long edges in a tree with at most $2n+2$ valence one vertices, which will happen when the number $q_{\geq 3}$ of vertices of valence at least 3 is maximized. Let q_m be the number of valence m vertices in T. Then

$$1 = \chi(T) = \sum q_m(1 - m/2) \leq \frac{2n+2}{2} - \frac{1}{2} q_{\geq 3}$$

implies $q_{\geq 3} \leq 2n$, there are at most $4n+2$ special vertices, and there are at most $4n+1$ long edges.

The sum of the lengths of the long edges is $|w|/2$, so there is a long edge of length at least $|w|/(8n+2)$, which from (3.2) is at least

$$\max_{i\geq 0}\{|u|, |b_i| + 1\}.$$

If this is the case, since the endpoints of the b_i are special, the long edge is covered only by subsegments of powers of u. Because u is not a proper power, the segment (with a fixed direction) corresponds to a unique reduced expression of the form $u_0 u^a u_1$ where u_0 and u_1 are proper subwords of u and $a > 0$. (Otherwise, one is led to the conclusion that u can be written as a reduced product of reduced words $u = pq = qp$, and by [24, Lemma 2.2], this contradicts u not being a proper power.) Let us now fix the direction of the long edge so $a > 0$.

The upshot of this unique expression is that the term u_0 corresponds to a terminal subsegment of a u as written in (3.1) (part of a u^{k_i} with $k_i > 0$), for each time the long edge is traversed in its given direction. If the long edge is traversed in the other way by the path of w, then the u_0 segment corresponds to an initial subsegment of a u^{-1} in a u^{k_i} with $k_i < 0$.

Fix an endpoint v of the u_0 segment in the long edge. Consider the subpaths of the path of w punctuated by returns to v. After cutting the tree at v, there must be at least one b_i subpath on either half of the resulting forest. So there must be some closed subpath of w beginning and ending at v and corresponding, possibly after cyclic rotation of w, to a subsequence as indicated below by the underbrace

$$u^{k_0} b_0 u^{k_1} b_1 \cdots u^{k_j - a} \underbrace{u^a b_j u^{k_{j+1}} \cdots u^{k_l} b_l u^c}_{\downarrow} u^{k_{l+1} - c} b_{l+1} \cdots u^{k_n} b_n$$

with $l - j < n$, and

$$u^a b_j u^{k_{j+1}} \cdots u^{k_l} b_l u^c = e,$$

which implies

$$u^{a+c} b_j u^{k_{j+1}} \cdots u^{k_l} b_l = e.$$

Reducing $a+c$, we can use the inductive hypothesis to conclude that for some j, $[u, b_j] = e$. (Note that this is where the minimum of k_i only over $i > 0$ is useful in the induction; $a + c$ could in principle be very small.) □

A similar result holds when u is not necessarily cyclically reduced and is a power, e.g., $u = ps^l p^{-1}$, with s cyclically reduced and $|u| = 2|p| + l|s|$. Rewrite the expression for w as

$$e = w = \prod ps^{lk_i} p^{-1} b_i$$

conjugate by p^{-1}, and absorb the p's into the b's to get

$$e = w' = \prod s^{lk_i} b'_i .$$

Then the same conclusion clearly holds when

$$l \min_{i \geq 0} \{|k_i|\} > (8n + 2) \cdot \max_{i \geq 0} \{1, (|b_i| + 2|p|)/|s|\} .$$

For the applications, since $|u| = l|s| + 2|p| > 2|p|$ so we can use instead the easier to use yet still sufficient inequality

$$\min_{i \geq 0} \{|k_i|\} \geq (8n + 2) \cdot \max_{i \geq 0} \{|b_i| + |u|\} , \tag{3.3}$$

which gives the same conclusion. Note that this minimum of k_i is now over all i, and not just $i > 0$. (The latter was just more convenient for the previous induction.)

In what follows, Γ is a limit group with a fixed finite generating set Y, A is a maximal abelian subgroup in Γ, and Γ' is the extension of centralizers $\Gamma' = \Gamma *_A B$, where $B = A \times \langle t \rangle$. Let $Y' = Y \cup \{t\}$.

Any element $\gamma' \in \Gamma'$ can be written in shortest form w.r.t. Y' as

$$\gamma' = \prod_{i=0}^{m} t^{n_i} \gamma_i$$

where γ_i are shortest form words in Y and

$$\sum_i |n_i| + |\gamma_i|_Y = |\gamma'|_{Y'}. \tag{3.4}$$

If any γ_{i_0} corresponds to an element of A and there is more than one γ_i then we can combine two γ_i and two t^{n_i} by commuting γ_{i_0} with one of its neighbouring t^{n_j}. This decreases m by one and does not increase

$$\sum_i |n_i| + |\gamma_i|_Y .$$

Therefore it must remain the same ($= |\gamma'|_{Y'}$). At the end of this process either all $\gamma_i \in \Gamma \backslash A$ and (3.4) still holds or

$$\gamma' = t^n \alpha$$

with $\alpha \in A$ and

$$|n| + |\alpha|_Y = |\gamma'|_{Y'} .$$

In either case we call this expression normal form. The previous discussion shows:

Fact 3.3. *Any element of γ can be written as a normal form word in Y' whose length is the shortest amongst all expressions for γ in Y'.*

Lemma 3.4 *(cf. [2, Lemma 7, Theorem 8]). Fix $a \in A \backslash \{e\}$. Then*

$$d_{Y'}(r) \leq (8r^2 + 4r) d_Y (2(r + |a|))^2 .$$

If $d_Y(r)$ is a polynomial of degree D then $d_{Y'}(r)$ is bounded above by a polynomial of degree $2D + 2$.

This is essentially a version of B. Baumslag's generalization of G. Baumslag's version of Lemma 3.2 from free groups to limit groups, where we keep track of the constants and avoid the phrases "sufficiently large" and "as large as we like."

Proof. Let $\pi \colon \Gamma' \to \Gamma$ be the retraction to Γ defined by $\pi(t) = e$, let τ be the automorphism of Γ' fixing Γ with

$$\tau(t) \stackrel{\text{def}}{=} ta.$$

We will find an $h \colon \Gamma' \to \mathbf{F}$ which is injective on normal forms in Y' of length at most r and doesn't stretch too much.

Assume first that $|g|_{Y'} \leq r$ and by Fact 3.3 we can assume g has the normal form

$$\prod t^{n_i} v_i ,$$

which is a product of at most $\lfloor r/2 \rfloor$ terms $t^{n_i} v_i$ with

$$\sum_i |n_i| + |v_i|_Y \leq r.$$

Then

$$f \circ \pi \circ \tau^m(g) = \prod f(a)^{mn_i} f(v_i)\,.$$

In order to use Lemma 3.2 we need to choose f so that $f([a, v_i]) \neq e$. In the worst case the commutator $[a, v_i]$ has length at most

$$L \stackrel{\text{def}}{=} 2(r + |a|).$$

Choose $f\colon \Gamma \to \mathbf{F}$ and a basis X of $\mathbf{F}$ such that

$$d_Y(L) = \max_{g \in B_Y(L)} |f(g)|_X$$

and f embeds $B_Y(L) \subset \Gamma$. By (3.3), with $k_i = mn_i$, $n \leq \lfloor r/2 \rfloor$, $u = f(a)$, and $b_i = f(v_i)$, as long as

$$\min\{|mn_i|\} \geq (4r + 2) \cdot \max\{|f(v_i)|_X + |f(a)|_X\}$$

$f \circ \pi \circ \tau^m(g)$ is nontrivial. In the worst case $n_i = 1$ for all i and $f(a)$ and $f(v_i)$ have length $d_Y(L)$, so choose

$$m = m(r, |a|) = (4r + 2) \cdot 2d_Y(L)$$

and let

$$h = f \circ \pi \circ \tau^m.$$

We continue to use the same basis X for $\mathbf{F}$. Now overestimate the length of $h(g)$: the normal form which can be expanded the most is t^r, so we have $r \cdot m$ terms whose images have length at most $d_Y(L)$, and therefore

$$|h(g)|_X \leq r \cdot \underbrace{(8r + 4) \cdot d_Y(L)}_{m} \cdot d_Y(L) = (8r^2 + 4r) d_Y(L)^2\,.$$

If g is of the form $t^n\alpha$ then the worst that can happen is $n = 1$ and α has length at most $r - 1 < L$, $h(g) = f(a)^m f(\alpha)$, but in this case

$$m \cdot d_Y(L) \geq m \cdot |f(a)|_X \geq |h(t)|_X \geq m > d_Y(L) \geq |f(\alpha)|_X\,,$$

so $h(g)$ is nontrivial — $h(t)$ and $f(\alpha)$ cannot fully cancel since $m > d_Y(L)$ — and $h(g)$ is not longer than $m \cdot d_Y(L) + d_Y(L)$, which is at most $r \cdot m \cdot d_Y(L)$ as required.

The final statement about degrees follows since L is linear in r. □

Corollary 3.5. *Let Γ be a limit group, and suppose Γ embeds in an extension of centralizers of height n. Then $d_Y(r)$ is bounded above by a polynomial in r of degree*

$$D(n) = 2^{n+2} - 2^n - 2\,.$$

Proof. For height 0, the distortion function is just r. Clearly by Lemma 3.4 and induction a polynomial of degree $D(n)$ suffices. Now embed Γ in an iterated extension of centralizers of height n:

$$\Gamma \hookrightarrow \Gamma_n > \Gamma_{n-1} > \cdots > \Gamma_1 > \mathbf{F}\,.$$

Since the embedding $\Gamma \hookrightarrow \Gamma_n$ expands lengths at most linearly, Γ has distortion function bounded above by a polynomial of degree $D(n)$ as well. □

Proposition 1.8 follows immediately.

4. Proof of Theorem 1.7

Proof of Theorem 1.7. Fix a set Y of generators of Γ. It suffices to prove the theorem for the finite set $B_Y(R)$ for arbitrary $R > 0$. We are given $\epsilon > 0$. Let $S_Y(R) \subset \mathbf{C}[\Gamma]$ denote the ℓ^1-unit sphere of the elements supported on $B_Y(R)$. Our task is to prove that there is a homomorphism $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(a))\| \leq \|\lambda_\Gamma(a)\| + \epsilon, \tag{4.1}$$

for all $a \in S_Y(R)$. The set $S_Y(R)$ is compact with respect to the ℓ^1 norm. Take a finite $\frac{\epsilon}{3}$-net $\{a_i\}_{i\in\mathcal{I}}$ for $S_Y(R)$ w.r.t. the ℓ^1 norm.

Due to the inequality (2.1) and the triangle inequality, the functions $a \mapsto \|\lambda_{\mathbf{F}}(a)\|$ and $a \mapsto \|\lambda_\Gamma(a)\|$ are 1-Lipschitz on $S_Y(R)$ with respect to the ℓ^1 norm and hence if we can prove the existence of $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(a_i))\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3} \tag{4.2}$$

for all $i \in \mathcal{I}$ then (4.1) will follow for all $a \in S_Y(R)$ as required. So we now set out to prove (4.2).

Let C and D be the constants from Proposition 1.8. Choose $m = m(\epsilon) \in \mathbf{N}$ large enough so that

$$\left[C(2mR)^D + 1\right]^{\frac{3}{4m}} \leq 1 + \frac{\epsilon}{3}. \tag{4.3}$$

We apply Proposition 1.8 with $r = 2mR$ to get an epimorphism $\phi : \Gamma \to \mathbf{F}$ injective on $B_Y(2mR)$, and a generating set X of $\mathbf{F}$ such that

$$\phi(B_Y(2mR)) \subset B_X\left(C(2mR)^D\right). \tag{4.4}$$

Let $b_i \stackrel{\text{def}}{=} \phi(a_i)$ for each $i \in \mathcal{I}$.

Note that

$$\|\lambda_\Gamma(a_i)\|^{2m} = \|\lambda_\Gamma\left(a_i^* a_i\right)\|^m = \|\lambda_\Gamma\left(a_i^* a_i\right)^m\|$$

and similarly, $\|\lambda_{\mathbf{F}}(b_i)\|^{2m} = \|\lambda_{\mathbf{F}}\left(b_i^* b_i\right)^m\|$. Each $\left(b_i^* b_i\right)^m$ is supported on $B_X(C(2mR)^D)$ by (4.4), hence by Haagerup's inequality (Lemma 2.1) we have

$$\begin{aligned}
\|\lambda_{\mathbf{F}}(b_i)\|^{2m} &= \|\lambda_{\mathbf{F}}\left(b_i^* b_i\right)^m\| \\
&\leq [C(2mR)^D + 1]^{\frac{3}{2}} \|\left(b_i^* b_i\right)^m\|_{\ell^2} \\
&= [C(2mR)^D + 1]^{\frac{3}{2}} \|\left(a_i^* a_i\right)^m\|_{\ell^2} \\
&\leq [C(2mR)^D + 1]^{\frac{3}{2}} \|\lambda_\Gamma(a_i)\|^{2m}.
\end{aligned}$$

The equality on the third line used that ϕ is injective on $B_Y(2mR)$, and the final inequality used (2.2). Hence

$$\begin{aligned}
\|\lambda_{\mathbf{F}}(b_i)\| &\leq [C(2mR)^D + 1]^{\frac{3}{4m}} \|\lambda_\Gamma(a_i)\| \\
&\leq \left(1 + \frac{\epsilon}{3}\right) \|\lambda_\Gamma(a_i)\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3}
\end{aligned}$$

by our choice of m in (4.3); the last inequality used that $\|a_i\|_{\ell^1} = 1$ and (2.1). □

5. Proof of Theorem 1.1

Here we split into cases when Γ is abelian or not. Limit groups cannot have torsion, so abelian limit groups are of the form $\mathbf{Z}^r$ for some $r \in \mathbf{N}$.

5.1. *Proof when* $\Gamma = \mathbf{Z}^r$

The case when $\Gamma = \mathbf{Z}^r$ must be dealt with by hand here.

Lemma 5.1. *Theorem 1.1 holds when* $\Gamma = \mathbf{Z}^r$. *Moreover, for this sequence of representations we can ensure that for all* $z \in \mathbf{C}[\Gamma]$

$$\lim_{i \to \infty} \frac{\operatorname{Tr}(\rho_i(z))}{N_i} = \tau(z) \tag{5.1}$$

where $\tau(g) \stackrel{\text{def}}{=} \delta_{eg}$ *is the delta function at the identity, extended linearly to a tracial state on* $C_r^*(\mathbf{Z}^r)$.

Proof. Let $T^r \stackrel{\text{def}}{=} (S^1)^r$ be the standard r-dimensional flat torus. The Fourier transform gives an isomorphism of C^*-algebras

$$\mathcal{F}\colon C_r^*(\mathbf{Z}^r) \to C(T^r).$$

For $q \in \mathbf{N}$ let T_q^r denote the subtorus $(\mathbf{Z}/q\mathbf{Z})^r \subset T^r$. We obtain, via restriction and Fourier transform, a finite dimensional representation

$$C_r^*(\mathbf{Z}^r) \xrightarrow{\rho_q} C(T_q^r)$$

that restricts to finite dimensional unitary representation of $\mathbf{Z}^r$. For any $z \in \mathbf{C}[\mathbf{Z}^r]$ we have

$$\|\rho_q(z)\| = \max_{x \in T_q^r} |\mathcal{F}[z](x)| \to \max_{x \in T^r} |\mathcal{F}[z](x)| = \|\lambda_{\mathbf{Z}^r}(z)\|$$

as $q \to \infty$. We have only used here the fact that T_q^r Hausdorff converges to T^r as $q \to \infty$.

We also have

$$\frac{\mathrm{Tr}(\rho_q(z))}{\dim \rho_q(z)} = \frac{1}{|T_q^r|} \sum_{x \in T_q^r} \mathcal{F}[z](x) \xrightarrow[q\to\infty]{} \int_{T^r} \mathcal{F}[z] d\mu = \tau(z)$$

where $d\mu$ is Lebesgue probability measure on T^r, and the convergence is by the definition of the Riemann integral; the last equality is by Fourier inversion.

Finally we remark that these representations factor through the permutation representation of $\mathbf{Z}^r$ acting on $\mathbf{Z}^r/q\mathbf{Z}^r$ by (left) multiplication. After we remove the trivial representation of $\mathbf{Z}^r$ from this, the previous arguments still work, and thus we can obtain the conclusion with representations of the form (1.2). □

5.2. Proof for non-abelian limit groups

In the following, $\mathbf{F}$ will always denote some (not always the same) free group, and Γ will be a fixed limit group.

Lemma 5.2. *If Γ is a non-abelian limit group, then the reduced C^*-algebra of Γ is simple (has no non-trivial closed ideals) and has a unique tracial state.*

Proof. We claim that any non-abelian FRF group Γ has the P_{nai} property of Bekka, Cowling, and de la Harpe [4, Def. 4]. This states that for any finite set $S \subset \Gamma \backslash \{e\}$, there is $y \in \Gamma$ of infinite order such that for every $x \in S$, x and y are free generators of a free rank 2 subgroup of Γ.

Proof of Claim. It is easy to check that since Γ is FRF, two elements x and y are free generators of a free rank 2 subgroup of Γ if and only if they do not commute. So to check property P_{nai} above, it remains to check that given any finite subset $S \subset \Gamma \backslash \{e\}$, there is an infinite order y not commuting with any element of S.

Because Γ is non-abelian, there are two elements $a, b \in \Gamma$ with $[a, b] \neq e$. By the FRF condition, there is a epimorphism[5] $\phi : \Gamma \to \mathbf{F}$ that is an injection on $S \cup \{e\} \cup \{[a, b]\}$. In particular, the rank of $\mathbf{F}$ must be at least 2. Since $\phi(S)$ is a finite subset of $\mathbf{F}$ not containing the identity, there is an (necessarily infinite order) element f not commuting with any element of $\phi(S)$. Then any preimage of f, say y, is infinite order and does not commute with any element of S. *This ends the proof of the claim.*

The proof of Lemma 5.2 now concludes by using [4, Lemmas 2.1 and 2.2]. □

Proof of Theorem 1.1. The upshot of Lemma 5.2 is that proving the existence of a sequence of unitary representations $\{\rho_i : \Gamma \to \mathbf{U}(N_i)\}_{i=1}^{\infty}$ strongly converging to the regular representation reduces to proving the existence of a sequence with

$$\limsup_{i\to\infty} \|\rho_i(z)\| \leq \|\lambda_\Gamma(z)\| \tag{5.2}$$

for all $z \in \mathbf{C}[\Gamma]$ of unit ℓ^1 norm. We give a proof of this passage that was also mentioned in the Introduction.

Suppose (5.2) holds. Then for any non-principal ultrafilter $\mathcal{F}$, we form the ultraproduct[6] C^*-algebra $\mathcal{U} \stackrel{\text{def}}{=} \prod_{\mathcal{F}} \rho_i(\mathbf{C}[\Gamma])$. There is a natural $*$-algebra map $\iota : \mathbf{C}[\Gamma] \to \mathcal{U}$. The inequality (5.2) implies

$$\|\iota(z)\|_{\mathcal{U}} \leq \|\lambda_\Gamma(z)\| \tag{5.3}$$

for all $z \in \mathbf{C}[\Gamma]$. If $\mathcal{U}_1$ denotes the closure of $\iota(\mathbf{C}[\Gamma])$ in $\mathcal{U}$, then inequality (5.3) implies that the map ι extends continuously to a C^*-algebra map from $C_r^*(\Gamma)$ to $\mathcal{U}_1$. But since we know $C_r^*(\Gamma)$ is simple by Lemma 5.2, this map must be injective. But injective C^*-algebra maps are isometries (to their images), so we have for all $z \in \mathbf{C}[\Gamma]$

$$\|\lambda_\Gamma(z)\| = \|\iota(z)\|_{\mathcal{U}} = \lim_{i\to\mathcal{U}} \|\rho_i(z)\|.$$

Since this holds for arbitrary non-principal ultrafilters, it holds also that $\|\lambda_\Gamma(z)\| = \lim_{i\to\infty} \|\rho_i(z)\|$.

This reduces our task to proving (5.2), which we begin now. Fix a finite set of generators of Γ with respect to which we will define balls. Given $\epsilon > 0$ we will prove that there is a unitary representation $\rho = \rho(\epsilon) : \Gamma \to \mathbf{U}(N)$ with $N = N(\epsilon)$ such that

$$\|\rho(z)\| \leq \|\lambda_\Gamma(z)\| + \epsilon$$

for $z \in \mathbf{C}[\Gamma]$ with support in $B\left(\frac{1}{\epsilon}\right)$ and $\|z\|_{\ell^1} = 1$. By taking $\epsilon \to 0$, this will imply the existence of a sequence ρ_i satisfying (5.2) for any z.

---

[5] Because subgroups of free groups are free, the homomorphism provided by the FRF condition has free image.

[6] For background on ultrafilters and ultraproducts, see [10, Appendix A].

As in the proof of Theorem 1.7 (§4), by taking an $\frac{\epsilon}{3}$-net of the unit ℓ^1 sphere of the elements in $\mathbf{C}[\Gamma]$ supported on $B\left(\frac{1}{\epsilon}\right)$, it suffices to prove

$$\|\rho(a_i)\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3}$$

for a finite collection $\{a_i\}_{i\in\mathcal{I}}$ of elements of $\mathbf{C}[\Gamma]$ with $\|a_i\|_{\ell^1} = 1$.

We apply Theorem 1.7 with $S = B\left(\frac{1}{\epsilon}\right)$ to obtain a homomorphism $\phi : \Gamma \to \mathbf{F}$ with $\mathbf{F}$ free such that

$$\|\lambda_{\mathbf{F}}(\phi(a_i))\| \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{6}, \tag{5.4}$$

for all $i \in \mathcal{I}$. Let $b_i \stackrel{\text{def}}{=} \phi(a_i) \in \mathbf{C}[\mathbf{F}]$.

The remainder of the proof splits into three cases.

A. If $\mathbf{F}$ is rank 1, i.e. $\mathbf{F} = \mathbf{Z}$ then Lemma 5.1 tells that there is a finite dimensional unitary representation π of $\mathbf{F}$ such that

$$\|\pi(b_i)\| \leq \|\lambda_{\mathbf{F}}(b_i)\| + \frac{\epsilon}{6} \tag{5.5}$$

for all $i \in \mathcal{I}$.

B. Otherwise, if one only wants unitary representations in Theorem 1.1, then by Haagerup and Thorbjørnsen [16, Thm. B] there is a finite dimensional unitary representation π of $\mathbf{F}$ such that (5.5) holds for all $i \in \mathcal{I}$.

C. If one wants the full strength of Theorem 1.1 and $\mathbf{F}$ has rank at least 2, then unitary representations factoring through S_N as in (1.2) and satisfying (5.5) for all $i \in \mathcal{I}$ exist by the work of Bordenave and Collins [3].

Then let $\rho \stackrel{\text{def}}{=} \pi \circ \phi$, a finite dimensional unitary representation of Γ. Since $\rho(a_i) = \pi(b_i)$, using (5.4) we obtain

$$\|\rho(a_i)\| = \|\pi(b_i)\| \leq \|\lambda_{\mathbf{F}}(b_i)\| + \frac{\epsilon}{6} \leq \|\lambda_\Gamma(a_i)\| + \frac{\epsilon}{3}$$

for all $i \in \mathcal{I}$ as required. □

6. Strong implies weak

Lemma 6.1 *(Strong convergence implies weak convergence). Let Γ be a finitely generated discrete group such that $C_r^*(\Gamma)$ has a unique tracial state. If $\{\rho_i : \Gamma \to \mathbf{U}(N_i)\}_{i=1}^\infty$ is a sequence of finite dimensional unitary representations that strongly converge to the regular representation of Γ, then for any $z \in \mathbf{C}[\Gamma]$*

$$\lim_{i\to\infty} \frac{\mathrm{Tr}(\rho_i(z))}{N_i} = \tau(z),$$

where τ is the unique tracial state on $C_r^*(\Gamma)$. Tr *denotes the usual matrix trace on* $\mathbf{U}(N_i)$ *extended linearly to* $\mathbf{C}[\mathbf{U}(N_i)]$.

We heard this lemma stated by Benoît Collins in a talk in Northwestern University in June 2022. The proof is to our knowledge not in the literature so we give it here.

Proof of Lemma 6.1. Consider any non-principal ultrafilter $\mathcal{F}$ on $\mathbf{N}$, and form the ultraproduct C^*-algebra $\mathcal{U} \stackrel{\text{def}}{=} \prod_{\mathcal{F}} \rho_i(\mathbf{C}[\Gamma])$. Let $\mathcal{U}_1$ denote the C^*-subalgebra in $\mathcal{U}$ generated by the images $\hat{\gamma}_i$ in $\mathcal{U}$ of the generators γ_i of Γ. Strong convergence implies that the natural map from $\mathbf{C}[\Gamma]$ to $\mathcal{U}_1$ is an isometric embedding with respect to the norm on $\mathbf{C}[\Gamma]$ coming from $C_r^*(\Gamma)$, and hence extends to an isomorphism between $C_r^*(\Gamma)$ and $\mathcal{U}_1$. On the other hand,

$$\lim_{i \to \mathcal{F}} \frac{\operatorname{Tr} \circ \rho_i}{N_i}$$

defines a tracial state on $\mathcal{U}_1$, and when transferred to $C_r^*(\Gamma)$ must coincide with the unique tracial state there. Since the convergence holds for all non-principal ultrafilters, the convergence must hold in general. □

7. Proof of Proposition 1.4

If $\Gamma \leq \Lambda$ are countable groups and $\rho : \Gamma \to \mathbf{U}(H)$ is a unitary representation of Γ on a separable Hilbert space H, we view ρ as making H a left Γ-module. Via the right regular representation, $\ell^2(\Lambda)$ is a right Γ-module, and via the left regular representation, it is a left Λ-module. The induced representation is defined to be the left Λ-module

$$\operatorname{Ind}_\Gamma^\Lambda \rho \stackrel{\text{def}}{=} \ell^2(\Lambda) \otimes_\Gamma H$$

where the tensor product is the completed (Hilbert space) one. This module has an invariant Hermitian inner product for which $g_i \otimes_\Gamma e_j$ is an orthonormal basis, where $g_1, \ldots, g_K, \ldots$ denote left coset representatives for Γ in Λ and $\{e_j\}_{j=1}^{\dim(H)}$ are an orthonormal basis for H.

The proof of both cases of Proposition 1.4 rely on the following lemma, which may be of independent interest.

Lemma 7.1. *Let Λ be any discrete group, Γ a finite index subgroup, and $\rho_i : \Gamma \to \mathbf{U}(N_i)$ finite dimensional unitary representations for which the conclusion of Corollary 1.10 holds. Then the induced unitary representations $\operatorname{Ind}_\Gamma^\Lambda \rho_i$ strongly converge to the regular representation of Λ.*

Proof. Let $\rho : \Gamma \to \mathbf{U}(H)$ be any unitary representation of Γ on a Hilbert space H. Let $g_1, \ldots, g_K$ denote left coset representatives for Γ in Λ. For each $1 \leq k \leq K$ and $h \in \Lambda$ we have

$$h[g_k \otimes_\Gamma v] = g_{\kappa(k,h)}\gamma(k,h) \otimes_\Gamma v = g_{\kappa(k,h)} \otimes_\Gamma \rho(\gamma(k,h))v$$

where $\kappa(k,h) \in \{1,\ldots,K\}$ and $\gamma(k,h) \in \Gamma$ are uniquely defined by

$$hg_k = g_{\kappa(k,h)}\gamma(k,h).$$

The map above is isometrically conjugate to the map

$$\sum_k E_{k,\kappa(k,h)} \otimes_{\mathbf{C}} \rho(\gamma(k,h)) \in \mathrm{End}(\ell^2(\Lambda/\Gamma)) \otimes_{\mathbf{C}} \mathrm{End}(H),$$

where $E_{p,q}(g_i) \stackrel{\text{def}}{=} \delta_{ip} g_q$ are the elementary matrices. The isometric conjugacy does not depend on ρ or h, only Λ and Γ and the choice of coset representatives. This means for any $z \in \mathbf{C}[\Gamma]$ the map $[\mathrm{Ind}_\Gamma^\Lambda \rho](z)$ is conjugate to some

$$\sum_{g\in\Gamma} a(g) \otimes \rho(g) \in \mathrm{End}(\ell^2(\Lambda/\Gamma)) \otimes_{\mathbf{C}} \mathrm{End}(H) \tag{7.1}$$

where $g \mapsto a(g)$ is finitely supported and the coefficients $a(g)$ do not depend on ρ; i.e. as ρ varies they may be taken the same for a fixed z.

Applying the conclusion of Corollary 1.10 to (7.1) for a sequence of $\rho_i : \Gamma \to \mathbf{U}(N_i)$ we learn that

$$\lim_{i\to\infty} \|[\mathrm{Ind}_\Gamma^\Lambda \rho_i](z)\| = \|\sum_{g\in\Gamma} a(g) \otimes \lambda_\Gamma(g)\|.$$

But now applying our previous argument in reverse, $\sum_{g\in\Gamma} a(g) \otimes \lambda_\Gamma(g)$ is isometrically conjugate to $[\mathrm{Ind}_\Gamma^\Lambda \lambda_\Gamma](z) = \lambda_\Lambda(z)$, since $\ell^2(\Lambda) \otimes_\Gamma \ell^2(\Gamma) \cong \ell^2(\Lambda)$ as a left Λ-module. Hence the conclusion

$$\lim_{i\to\infty} \|[\mathrm{Ind}_\Gamma^\Lambda \rho_i](z)\| = \|\lambda_\Lambda(z)\|. \quad \square$$

Proof of Proposition 1.4. The Klein bottle has an orientable double cover with Euler characteristic $0 = 2 \times 0$, hence a torus. The non-orientable surface $(\mathbf{R}P^2)^{\#3}$ with Euler characteristic -1 has an orientable double cover with Euler characteristic -2, hence it is a genus 2 orientable surface. This means that their fundamental groups have index 2 subgroups respectively isomorphic to $\mathbf{Z}^2$ and Γ_2, the fundamental group of an orientable genus 2 surface. Since $\mathbf{Z}^2$ and Γ_2 are both limit groups, Corollary 1.10 applies to both of them. Therefore Lemma 7.1 implies that the fundamental groups of both the Klein bottle and $(\mathbf{R}P^2)^{\#3}$ have finite dimensional unitary representations that strongly converge to their regular representations. $\square$

Appendix A. Spectral gaps of hyperbolic surfaces

The purpose of this appendix is to explain how the following theorem can be deduced from Corollary 1.10.

Theorem A.1. *Let X be a compact hyperbolic surface. There exists a sequence of Riemannian covers $\{X_i\}_{i\in\mathbb{N}}$ of X with genera $g(i)\to\infty$ as $i\to\infty$ such that for any $\epsilon>0$, for i large enough depending on ϵ,*

$$\operatorname{spec}\left(\Delta_{X_i}\right)\cap\left[0,\frac{1}{4}-\epsilon\right)=\operatorname{spec}\left(\Delta_{X}\right)\cap\left[0,\frac{1}{4}-\epsilon\right),$$

where the multiplicities are the same on either side.

To see that we can take all surfaces to be arithmetic we use the following argument. Let

$$\Gamma_0(15)\stackrel{\text{def}}{=}\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbf{Z})\,:\,c\equiv 0 \bmod 15\right\}.$$

The cusped hyperbolic surface $Y_0(15)\stackrel{\text{def}}{=}\Gamma_0(15)\backslash\mathbb{H}$ has no spectrum in $(0,\frac{1}{4})$ by a result of Huxley [18, Thm., pg. 250]. Let $D_{3,5}$ denote the quaternion algebra over $\mathbf{Q}$ generated by i,j,k such that

$$i^2=3,\ j^2=5,\ ij=-ji=k.$$

Then $D_{3,5}$ is a division algebra with discriminant 15 [8, Ex. 8.27]. Let $\mathcal{O}$ denote a maximal order[7] in $D_{3,5}$ and $\mathcal{O}^1$ the elements of norm 1 in $\mathcal{O}$. Then $\mathcal{O}^1$ embeds as a cocompact subgroup of $\mathrm{PSL}_2(\mathbf{R})$; let $X=\mathcal{O}^1\backslash\mathbb{H}$. By the work of Jacquet and Langlands [19] (see [8, Thm. 8.18] for a convenient concise reference) every eigenvalue of X is an eigenvalue of $Y_0(15)$ and hence X has no eigenvalues in $(0,\frac{1}{4})$.

Taking this X in Theorem A.1, one obtains a different proof of [15, Corollary 1.3] with a slightly stronger conclusion, i.e. there exists a sequence of compact *arithmetic* hyperbolic surfaces $\{X_i\}_{i\in\mathbb{N}}$ with genera $g\left(X_i\right)\to\infty$ and $\lambda_1\left(X_i\right)\to\frac{1}{4}$. Such a sequence of covering surfaces also have systoles uniformly bounded away from 0, also in contrast to the proof of [15, Corollary 1.3] (this conclusion on the systole is independent of arithmeticity).

[7] See [8, Ex. 8.27] for an explicit maximal order.

A.1. Set up

For any $n \in \mathbb{N}$, let $[n] \overset{\text{def}}{=} \{1, \ldots, n\}$ and S_n denote the group of permutations of $[n]$. Let X be a fixed compact hyperbolic surface with genus $g \geqslant 2$. We view X as

$$X = \Gamma \backslash \mathbb{H},$$

where Γ is a discrete, torsion free subgroup of $\mathrm{PSL}_2(\mathbb{R})$, isomorphic to the surface group Λ_g. Given any $\phi \in \mathrm{Hom}(\Gamma, S_n)$ we define an action of Γ on $\mathbb{H} \times [n]$ by

$$\gamma(z, x) \overset{\text{def}}{=} (\gamma z, \phi(\gamma)[x]).$$

Then we obtain a degree n covering space X_ϕ of X by

$$X_\phi \overset{\text{def}}{=} \Gamma \backslash_\phi (\mathbb{H} \times [n]). \tag{A.1}$$

Let $V_n \overset{\text{def}}{=} \ell^2([n])$ and $V_n^0 \subset V_n$ the subspace of functions with zero mean. Then S_n acts on V_n via std, the standard representation by 0-1 matrices, and V_n^0 is the $n-1$ dimensional irreducible component. Throughout this appendix, we let $\{\rho_i\}_{i\in\mathbb{N}}$ be a sequence of N_i-dimensional unitary representations of Γ that factor through S_{N_i} by

$$\Gamma \xrightarrow{\phi_i} S_{N_i} \xrightarrow{\text{std}} \mathrm{End}\left(V_{N_i}^0\right), \tag{A.2}$$

such that for any $r \in \mathbf{N}$ and finitely supported map $a : \Gamma \to \mathrm{Mat}_{r\times r}(\mathbf{C})$, we have

$$\limsup_{i\to\infty} \| \sum_{\gamma\in\Gamma} a(\gamma) \otimes \rho_i(\gamma) \| \leq \| \sum_{\gamma\in\Gamma} a(\gamma) \otimes \lambda(\gamma) \|, \tag{A.3}$$

as provided by Corollary 1.10. Note that by approximation by finite-rank operators on either side (as in [15, Proof of Prop. 6.3]) the property in (A.3) extends easily to the case of

$$a : \Gamma \to \mathcal{K}$$

where $\mathcal{K}$ are the compact operators on a separable Hilbert space. We use this extension in the sequel.

Then through $\{\rho_i\}_{i\in\mathbb{N}}$, we obtain a sequence of degree-N_i covering surfaces $\{X_i\}_{i\in\mathbb{N}}$ from (A.1).

A.2. Function spaces

For the convenience of the reader we recall the following function spaces from [15, Section 2.2]. We define $L^2_{\text{new}}(X_i)$ to be the space of L^2 functions on X_i orthogonal to all lifts of L^2 functions from X. Then

$$L^2(X_i) \cong L^2_{\text{new}}(X_i) \oplus L^2(X).$$

We fix F to be a Dirichlet fundamental domain for X. Let $C^\infty\left(\mathbb{H}; V^0_{N_i}\right)$ denote the smooth $V^0_{N_i}$-valued functions on $\mathbb{H}$. There is an isometric linear isomorphism between

$$C^\infty(X_i) \cap L^2_{\text{new}}(X_i),$$

and the space of smooth $V^0_{N_i}$-valued functions on $\mathbb{H}$ satisfying

$$f(\gamma z) = \rho_i(\gamma) f(z),$$

for all $\gamma \in \Gamma$, with finite norm

$$\|f\|^2_{L^2(F)} \stackrel{\text{def}}{=} \int_F \|f(z)\|^2_{V^0_{N_i}} d\mu_{\mathbb{H}}(z) < \infty.$$

Here $d\mu_{\mathbb{H}}$ denotes the hyperbolic volume form. We denote the space of such functions by $C^\infty_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$. The completion of $C^\infty_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ with respect to $\|\bullet\|_{L^2(F)}$ is denoted by $L^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$; the isomorphism above extends to one between $L^2_{\text{new}}(X_i)$ and $L^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$.

We introduce the following Sobolev spaces. Let $H^2(\mathbb{H})$ denote the completion of $C^\infty_c(\mathbb{H})$ with respect to the norm

$$\|f\|^2_{H^2(\mathbb{H})} \stackrel{\text{def}}{=} \|f\|^2_{L^2(\mathbb{H})} + \|\Delta f\|^2_{L^2(\mathbb{H})}.$$

Let $C^\infty_{c,\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ denote the subset of $C^\infty_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ consisting of functions which are compactly supported modulo Γ. We let $H^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ denote the completion of $C^\infty_{c,\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ with respect to the norm

$$\|f\|^2_{H^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)} \stackrel{\text{def}}{=} \|f\|^2_{L^2(F)} + \|\Delta f\|^2_{L^2(F)}.$$

We let $H^2(X_i)$ denote the completion of $C^\infty_c(X_i)$ with respect to the norm

$$\|f\|^2_{H^2(X_i)} \stackrel{\text{def}}{=} \|f\|^2_{L^2(X_i)} + \|\Delta f\|^2_{L^2(X_i)}.$$

Viewing $H^2(X_i)$ as a subspace of $L^2(X_{\phi_i})$, we let

$$H^2_{\text{new}}(X_i) \stackrel{\text{def}}{=} H^2(X_i) \cap L^2_{\text{new}}(X_i).$$

There is an isometric isomorphism between $H^2_{\text{new}}(X_i)$ and $H^2_{\phi_i}\left(\mathbb{H}; V^0_{N_i}\right)$ that intertwines the two relevant Laplacian operators.

A.3. Operators on $\mathbb{H}$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, let

$$R_{\mathbb{H}}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H}),$$
$$R_{\mathbb{H}}(s) \stackrel{\text{def}}{=} (\Delta_{\mathbb{H}} - s(1-s))^{-1},$$

be the resolvent on the upper half plane. Then $R_{\mathbb{H}}(s)$ is an integral operator with radial kernel $R_{\mathbb{H}}(s;r)$. Let $\chi_0 : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\chi_0(t) = \begin{cases} 1 & \text{if } t \leqslant 0, \\ 0 & \text{if } t \geqslant 1. \end{cases}.$$

For $T > 0$, we define a smooth cutoff function χ_T by $\chi_T(t) \stackrel{\text{def}}{=} \chi_0(t-T)$. We then define the operator $R_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ to be the integral operator with radial kernel

$$R_{\mathbb{H}}^{(T)}(s;r) \stackrel{\text{def}}{=} \chi_T(r) R_{\mathbb{H}}(s;r).$$

Following [15, Section 5.2] we define $L_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ to be the integral operator with radial kernel

$$\mathbb{L}^{(T)}(s;r) \stackrel{\text{def}}{=} \left(-\frac{\partial^2}{\partial r^2}[\chi_T] - \frac{1}{\tanh r}\frac{\partial}{\partial r}[\chi_T]\right) R_{\mathbb{H}}(s;r) - 2\frac{\partial}{\partial r}[\chi_T]\frac{\partial R_{\mathbb{H}}}{\partial r}(s;r).$$

It is proved in [15, Lemma 5.3] that for any $f \in C_c^\infty(\mathbb{H})$ and $s \in \left[\frac{1}{2}, 1\right]$, we have

1. $R_{\mathbb{H}}^{(T)}(s) f \in H^2(\mathbb{H})$.
2. $(\Delta - s(1-s)) R_{\mathbb{H}}^{(T)}(s) f = f + \mathbb{L}_{\mathbb{H}}^{(T)}(s) f$ as equivalence classes of L^2 functions.

It is also proved, as a consequence of [15, Lemma 5.2], that for any $s_0 > \frac{1}{2}$ we can choose a $T = T(s_0)$ such that for all $s \in [s_0, 1]$ we have

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})} \leq \frac{1}{8}. \tag{A.4}$$

A.4. Proof of Theorem A.1

Recall that $\{\rho_i\}_{i\in\mathbb{N}}$ is a sequence of strongly convergent representations of the form (A.2) that satisfy (A.3) as guaranteed by Corollary 1.10. As in [15, Section 5.3], we define

$$R_{\mathbb{H},i}^{(T)}(s;x,y) \stackrel{\text{def}}{=} R_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_{N_i}^0},$$
$$\mathbb{L}_{\mathbb{H},i}^{(T)}(s;x,y) \stackrel{\text{def}}{=} \mathbb{L}_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_{N_i}^0}.$$

We define $R_{\mathbb{H},i}^{(T)}(s)$, $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ to be the corresponding integral operators. We have the following analogue of [15, Lemma 5.5].

Lemma A.2. *For all* $s \in \left[\frac{1}{2}, 1\right]$*,*

1. *The integral operator* $R_{\mathbb{H},i}^{(T)}(s)$ *is well-defined on* $C_{c,\phi_i}^{\infty}\left(\mathbb{H}; V_{N_i}^0\right)$ *and extends to a bounded operator*
$$R_{\mathbb{H},i}^{(T)}(s) : L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right) \to H_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right).$$
2. *The integral operator* $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ *is well-defined on* $C_{c,\phi_i}^{\infty}\left(\mathbb{H}; V_{N_i}^0\right)$ *and extends to a bounded operator on* $L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right)$*.*
3. *We have*
$$[\Delta - s(1-s)]\, R_{\mathbb{H},i}^{(T)}(s) = 1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s)$$
as an identity of operators on $L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right)$*.*

The proof of Lemma A.2 easily follows from the proof of [15, Lemma 5.5], simplified in places by the compactness of the fundamental domain F in the current setting.

We have an isomorphism of Hilbert spaces

$$L_{\phi_i}^2\left(\mathbb{H}; V_{N_i}^0\right) \cong L^2(F) \otimes V_{N_i}^0,$$

given by

$$f \mapsto \sum_{e_i} \langle f|_F, e_i\rangle \otimes e_i,$$

where $\{e_j\}_{j=1}^{N_i-1}$ is some choice of basis for $V_{N_i}^0$. After conjugation by this isomorphism, the operator $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ becomes

$$\mathbb{L}_{\mathbb{H},i}^{(T)}(s) \cong \sum_{\gamma \in S} a_\gamma^{(T)}(s) \otimes \rho_i\left(\gamma^{-1}\right), \tag{A.5}$$

where

$$a_\gamma^{(T)}(s) : L^2(F) \to L^2(F),$$
$$a_\gamma^{(T)}(s)[f](x) \overset{\text{def}}{=} \int\limits_{y \in F} \mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)\, f(y)\, d\mu_{\mathbb{H}}(y).$$

Since $\mathbb{L}_{\mathbb{H}}^{(T)}(s, \gamma x, y)$ is only non-zero when $d(\gamma x, y) \leqslant T + 1$, in (A.5) one can take $S = S(T) \subset \Gamma$ to be finite. Since $\mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)$ is smooth and bounded it follows that the operators $a_{\gamma}^{(T)}(s)$ are Hilbert-Schmidt and therefore compact. We define

$$\mathcal{L}_{s,\infty}^{(T)} \stackrel{\text{def}}{=} \sum_{\gamma \in S} a_{\gamma}^{(T)}(s) \otimes \lambda\left(\gamma^{-1}\right).$$

Under the isomorphism

$$L^2(F) \otimes \ell^2(\Gamma) \cong L^2(\mathbb{H}),$$
$$f \otimes \delta_{\gamma} \mapsto f \circ \gamma^{-1},$$

(with $f \circ \gamma^{-1}$ extended by zero from a function on γF) the operator $\mathcal{L}_{s,\infty}^{(T)}$ is conjugated to

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H}).$$

To prove Theorem A.1, we need to replace the probabilistic bound [15, Lemma 6.3] by a deterministic one.

Proposition A.3. *For any $s_0 > \frac{1}{2}$ there is a $T = T(s_0) > 0$ such that for any fixed $s \in [s_0, 1]$ there is an $I(s_0, s)$ with*

$$\|\mathcal{L}_{s,\phi_i}^{(T)}\|_{L^2(F)\otimes V_{N_i}^0} \leq \frac{1}{4},$$

for all $i \geqslant I$.

Proof. Let $s_0 > \frac{1}{2}$ and a fixed $s \in [s_0, 1]$ be given. By (A.4) we can find a $T(s_0)$ such that

$$\|\mathcal{L}_{s,\infty}^{(T)}\|_{L^2(F)\otimes\ell^2(\Gamma)} \leq \frac{1}{8}. \tag{A.6}$$

Recall that the coefficients $a_{\gamma}(s)$ are supported on a finite set $S = S(T) \subset \Gamma$. Because the $a_{\gamma}(s)$ are compact, we apply (A.3) (and the following remark) to the operators $\mathbb{L}_{\mathbb{H},i}^{(T)}(s)$ to find that there is $I \in \mathbf{N}$ such that for all $i \geq I(s_0, s)$

$$\|\mathcal{L}_{s,\phi_i}^{(T)}\|_{L^2(F)\otimes V_{N_i}^0} \leq \|\mathcal{L}_{s,\infty}^{(T)}\|_{L^2(F)\otimes\ell^2(\Gamma)} + \frac{1}{8} \leq \frac{1}{4}. \quad \square$$

We can now prove Theorem A.1.

Proof of Theorem A.1. Given $\epsilon > 0$ let $s_0 = \frac{1}{2} + \sqrt{\epsilon}$ so that $s_0(1 - s_0) = \frac{1}{4} - \epsilon$. Let $T = T(s_0)$ be the value provided by Proposition A.3 for this s_0.

We use a finite net to control all values of $s \in [s_0, 1]$. Using [15, Lemma 6.1] as in [15, Proof of Thm. 1.1] tells us that there is a finite set $Y = Y(s_0)$ of points in $[s_0, 1]$ such that for any $s \in [s_0, 1]$, there is $s' \in Y$ with

$$\|\mathcal{L}_{s,\phi_i}^{(T)} - \mathcal{L}_{s',\phi_i}^{(T)}\| \leq \frac{1}{4} \tag{A.7}$$

for all i.

Combining (A.7) with Proposition A.3 applied to $\mathcal{L}_{s,\phi_i}^{(T)}$ for every $s \in Y$ we find that there is an $I(s_0)$ such that for all $s \in [s_0, 1]$ and $i \geq I(s_0)$

$$\|\mathbb{L}_{\mathbb{H},i}^{(T)}(s)\|_{L^2_{\text{new}}(X_i)} = \|\mathcal{L}_{s,\phi_i}^{(T)}\|_{L^2(F)\otimes V_{N_i}^0} \leq \frac{1}{2}. \tag{A.8}$$

By Lemma A.2, for $s > \frac{1}{2}$ $R_{\mathbb{H},i}^{(T)}(s)$ is a bounded operator from $L^2_{\text{new}}(X_i)$ to $H^2_{\text{new}}(X_i)$. By Lemma A.2 we have that

$$\left(\Delta_{X_\phi} - s(1-s)\right) R_{\mathbb{H},i}^{(T)}(s) = 1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s),$$

on $L^2_{\text{new}}(X_i)$. From (A.8) for all $i \geqslant I(s_0)$ and $s \in [s_0, 1]$, $\left(1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s)\right)^{-1}$ exists as a bounded operator on $L^2_{\text{new}}(X_i)$. We now get that for all $i \geqslant I(s_0)$ and all $s \in [s_0, 1]$,

$$\left(\Delta_{X_i} - s(1-s)\right) R_{\mathbb{H},i}^{(T)}(s) \left(1 + \mathbb{L}_{\mathbb{H},i}^{(T)}(s)\right)^{-1} = 1,$$

and we conclude that $(\Delta_{X_i} - s(1-s))$ has a bounded right inverse from $L^2_{\text{new}}(X_i)$ to $H^2_{\text{new}}(X_i)$, implying that for $i \geqslant I(s_0)$, Δ_{X_i} has no new eigenvalues λ with $\lambda \leq s_0(1-s_0) = \frac{1}{4} - \epsilon$. $\square$

Data availability

No data was used for the research described in the article.

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