Published for SISSA by Springer

Received: August 30, 2024
Accepted: November 21, 2024
Published: December 27, 2024

Gauging the diamond: integrable coset models from twistor space

Lewis T. Cole,[a] **Ryan A. Cullinan**,[b] **Ben Hoare**,[b] **Joaquin Liniado**[c] **and Daniel C. Thompson**[a,d]

[a] *Department of Physics, Swansea University, Swansea SA2 8PP, U.K.*

[b] *Department of Mathematical Sciences, Durham University, Durham DH1 3LE, U.K.*

[c] *Instituto de Física La Plata (CONICET and Universidad Nacional de La Plata), CC 67 (1900) La Plata, Argentina*

[d] *Theoretische Natuurkunde, Vrije Universiteit Brussel, and the International Solvay Institutes, Pleinlaan 2, B-1050 Brussels, Belgium*

E-mail: l.t.cole@pm.me, ryan.a.cullinan@durham.ac.uk, ben.hoare@durham.ac.uk, jliniado@iflp.unlp.edu.ar, d.c.thompson@swansea.ac.uk

Abstract: Recent work has shown that certain integrable and conformal field theories in two dimensions can be given a higher-dimensional origin from holomorphic Chern-Simons in six dimensions. Along with anti-self-dual Yang-Mills and four-dimensional Chern-Simons, this gives rise to a diamond correspondence of theories. In this work we extend this framework to incorporate models realised through gaugings. As well as describing a higher-dimensional origin of coset CFTs, by choosing the details of the reduction from higher dimensions, we obtain rich classes of two-dimensional integrable models including homogeneous sine-Gordon models and generalisations that are new to the literature.

Keywords: Chern-Simons Theories, Integrable Field Theories

ArXiv ePrint: 2407.09479

https://doi.org/10.1007/JHEP12(2024)202

JHEP12(2024)202

Contents

1 Introduction

Quantum field theories (QFTs) in two dimensions have both direct applications in condensed matter systems and as the worldsheet theories of strings, and can provide a tractable sandpit for the study of quantum field theory more generally. Special examples are provided by conformal field theories (CFTs) and integrable field theories (IFTs), for which powerful infinite-dimensional symmetries enable us to exactly determine certain key properties and observables.

One longstanding goal has been to provide a constructive origin of these integrable systems from some putative parent theory, perhaps in higher dimensions. For instance, Ward suggested [1] that all integrable equations may arise as reductions of the 4d anti-self-dual Yang-Mills (ASDYM) equation. Given a choice of complex structure on $\mathbb{R}^4$, the ASDYM equation are

$$F^{2,0} = 0 = F_{zw}\,, \tag{1.1}$$

$$F^{0,2} = 0 = F_{\bar{z}\bar{w}}\,, \tag{1.2}$$

$$\varpi \wedge F^{1,1} = 0 = F_{z\bar{z}} + F_{w\bar{w}}\,, \tag{1.3}$$

where ϖ is the Kähler form. There are (at the very least) two senses in which ASDYM can be viewed as an integrable theory in its own right. The first is that the ASDYM equations can be exactly solved by the ADHM construction [2]. The second is that these equations admit a zero curvature formulation in terms of a Lax pair of differential operators [3]:

$$L = \nabla_z - \zeta \nabla_{\bar{w}}\,, \quad M = \nabla_w + \zeta \nabla_{\bar{z}}\,, \qquad [L, M] = 0 \quad \forall \zeta \iff F = - \star F\,. \tag{1.4}$$

Accordingly, in this work, we will denote four-dimensional QFTs whose equations of motion can be recast as the anti-self duality of some connection as IFT_4.

A prominent example in this class of theories is the 4d Wess-Zumino-Witten model (WZW_4) [4–7], which arises as a partial gauge fixing of the ASDYM equation. Up to a gauge transformation, we may parametrise a generic connection that solves equations (1.1) and (1.2) as $A = -\bar{\partial} g g^{-1}$, where the group-valued field g becomes the fundamental field of WZW_4. The remaining ASDYM equation (1.3) becomes $\varpi \wedge \partial(\bar{\partial} g g^{-1}) = 0$, which are the equations of motion of WZW_4, also known as Yang's equation. The more well-known WZW_2 also arises as a reduction of WZW_4, and Yang's equation reduces to the familiar holomorphic conservation law characterising this CFT_2. Another example is found by solving equations (1.1) and (1.3), leaving equation (1.2) as the dynamical equation of motion. In this case, the IFT_4 is known as the LMP model [8, 9], which gives the pseudo-dual of the principal chiral model (PCM) after reduction.

Alternatively, motivated by the similarity between Reidemeister moves in knot theory and the Yang-Baxter equation that underpins integrability, Witten suggested [10] that integrable models might have a description in terms of Chern-Simons theory. The realisation of this idea came some years later, with Costello's understanding [11, 12] (see also [13]) that the gauge theory description should combine the topological nature of Chern-Simons theory with the holomorphic nature of the spectral parameter characterising IFTs. The theory proposed in [11, 12] was extended and developed in a sequence of papers [14–16] describing a Chern-Simons theory, which we denote by CS_4, defined over a four-manifold $\Sigma \times C$ with the action

$$S_{\mathrm{CS}_4}[A] = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right). \tag{1.5}$$

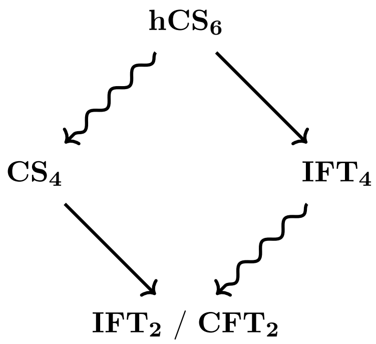

Figure 1. The diamond correspondence of integrable avatars, in which wavy arrows indicate a descent by reduction and straight arrows involve localisation i.e. integration over $\mathbb{CP}^1$. In this paper the IFT$_4$ will either be WZW$_4$, the LMP model or their gaugings. The landscape of IFT$_2$ produced in this fashion will be rich and varied.

Here, ω is a meromorphic differential on the complex curve C, which we will take to be $C = \mathbb{CP}^1$. Specifying boundary conditions at the poles of ω, the dynamics can be 'localised' to take place on Σ, which is identified with the spacetime of the IFT$_2$, and the curve C is associated to spectral parameter of the Lax connection (see [17] for a pedagogical introduction).

An elegant origin of both the CS$_4$ and the ASDYM descriptions was provided in the work of Bittleston and Skinner [18] in terms of a six-dimensional holomorphic Chern-Simons theory (hCS$_6$), first proposed in [19, 20]. The theory is defined over (the Euclidean slice of) Penrose's twistor space [21] with the action functional

$$S_{\text{hCS}_6}[\mathscr{A}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\left(\mathscr{A} \wedge \bar{\partial}\mathscr{A} + \frac{2}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right), \tag{1.6}$$

in which Ω is a meromorphic $(3,0)$ form. This action is supplemented by a choice of boundary conditions at the poles of Ω. The various lower-dimensional descriptions follow from exploiting the fibration structure $\mathbb{CP}^1 \hookrightarrow \mathbb{PT} \twoheadrightarrow \mathbb{R}^4$. Reducing along two directions within $\mathbb{R}^4$, hCS$_6$ descends to CS$_4$. Alternatively, we can instead choose to first localise over $\mathbb{CP}^1$, which leads to IFT$_4$ in the ASDYM sense. Indeed, the integrability properties of ASDYM are fundamentally tied to this twistorial origin and evidence suggests that at a quantum level it is natural to consider the formulation on twistor space [20, 22]. Applying the reduction along $\mathbb{R}^4$ to this IFT$_4$ produces an IFT$_2$, which can also be recovered by localising the CS$_4$ description. In this way, we have a diamond correspondence of theories illustrated in figure 1. Other recent work on hCS$_6$ includes [20, 23, 24].

Given an IFT$_2$ or CFT$_2$ it is sometimes possible to obtain another I/CFT$_2$ via gauging. Perhaps the most famous examples are the GKO G/H coset CFTs [25], which can be given a Lagrangian description by taking a WZW$_2$ CFT on G and gauging a (vectorially acting) H subgroup [26–29]. This motivates the core question of this work:

How can the diamond correspondence be gauged?

Resolving this question dramatically expands the scope of theories that can be given a higher-dimensional avatar. A significant clue is given by the rather remarkable Polyakov-Wiegmann

JHEP12(2024)202

(PW) identity, which shows that the G/H gauged WZW_2 model is actually equivalent to the difference of a G WZW_2 model and an H WZW_2 model. This points towards a general resolution that certain integrable gauged models might be obtained as differences of ungauged models. This is less obvious than it might first seem; it was noted in [7] that for a PW identity to apply for WZW_4 it is necessary for the gauging to be performed by connections with field strength restricted to type $(1,1)$. The six-dimensional origin of such a constraint is rather intriguing and will be elucidated in this paper. In the context of CS_4, Stedman recently proposed [30] considering the difference of CS_4 to give rise to gaugings of IFT_2. We will recover this construction as a reduction of hCS_6 theory in the present work, as well as uncovering some additional novelties in the CS_4 description.

At the top of the diamond, we will consider a theory of two connections, $\mathscr{A} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathscr{B} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{h}$ for a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The action of this theory is

$$S_{\mathrm{ghCS}_6} = S_{\mathrm{hCS}_6}[\mathscr{A}] - S_{\mathrm{hCS}_6}[\mathscr{B}] + S_{\mathrm{int}}[\mathscr{A}, \mathscr{B}]\,, \tag{1.7}$$

in which the term S_{int} couples the two gauge fields. We will develop this story by means of two explicit examples: choosing Ω to have two double poles, we will study the diamond relevant to the gauged WZW theory, and with Ω containing a single fourth-order pole we will study the gauged LMP model. This seemingly simple setup gives rise to a rich story whose results we now summarise:

1. Starting from the holomorphic theory on twistor space (1.7), we localise to arrive at an action for a gauged version of WZW_4 (denoted gWZW_4). After localising, the gauge field B is constrained to satisfy two of the three anti-self-dual Yang-Mills equations, namely $F^{2,0}[B] = 0$ and $F^{0,2}[B] = 0$, and the resulting gWZW_4 is an IFT_4.[1]

2. The two gauge fields $\mathscr{A}$ and $\mathscr{B}$ of the gauged hCS_6 theory (denoted ghCS_6) source various degrees of freedom in gWZW_4. In particular, as well as the fundamental field g and the 4d gauge field B, auxiliary degrees of freedom enter as Lagrange multipliers for the constraints $F^{2,0}[B] = 0$ and $F^{0,2}[B] = 0$.

3. Reducing by two dimensions, we recover a variety of IFT_2 including the special case of the gauged WZW_2 model (denoted gWZW_2). In general, we find a model coupling a gauged IFT_2 and a Hitchin system [31] involving the gauge field B and a pair of adjoint scalar fields. These scalars may source a potential for the gWZW_2 in which case we recover the complex sine-Gordon model and more broadly the homogeneous sine-Gordon models [32]. At the special point associated to the 2d PCM, Lagrange multipliers ensure that the gauge field is flat and hence trivial — this is essential as the gauged PCM is not generically integrable.

4. We also use this formalism to perform an integrable gauging of the LMP model. Just as in the gauging of WZW_4, the field strength of the gauge field must be constrained to obey two of the anti-self-dual Yang-Mills equations, this time $F^{2,0}[B] = 0$ and $\varpi \wedge F^{1,1}[B] = 0$. It is noteworthy that the two equations that are enforced by Lagrange multipliers agree

[1] This indicates that general unconstrained gaugings of WZW_4 will break integrability in the sense outlined above.

with the two equations that are identically solved in the ungauged case. This is true for both WZW_4 and the LMP model. In addition, we show that the gauged LMP model obeys a PW-like identity such that it may be expressed as the difference of two LMP models on $\mathfrak{g}$ and $\mathfrak{h}$.

Let us outline the structure of this paper. We begin in section 2 with a review of the diamond correspondence of theories for the ungauged WZW model. In section 3, we introduce the gauging of this diamond concentrating in particular on the right hand side. We recover the gauged IFT_4 and demonstrate that its equations of motion may be rewritten as the ASDYM equations. The wide array of IFT_2 are explored in section 4 where we also show that they are integrable and provide the associated Lax connection. Following the gauging of WZW_4, section 5 elaborates on the left hand side of the diamond, connecting to CS_4 by first reducing, and then to the IFT_2 by localisation. Section 6 describes the diamond for the gauged LMP theory. We conclude with a brief outlook in section 7. Although the subject matter necessarily entails a degree of technical complexity, we have endeavoured to keep the main presentation streamlined and complement this with a number of technical appendices.

2 The ungauged WZW diamond

In this section, we briefly describe the diamond correspondence of theories in which the two-dimensional theory is the WZW_2 CFT. This is a summary of part of the analysis first presented in [18], which will serve to fix conventions and review key steps relevant to later sections.

2.1 hCS_6 with double poles

We begin at the top of the diamond with 6d holomorphic Chern-Simons theory (hCS_6) whose fundamental field is an algebra-valued connection $\mathscr{A} \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$. The six-dimensional action is given by

$$S_{hCS_6}[\mathscr{A}] = \frac{1}{2\pi i} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\left(\mathscr{A} \wedge \bar{\partial}\mathscr{A} + \frac{2}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A} \right) , \tag{2.1}$$

in which we have introduced a meromorphic $(3,0)$-form Ω. As a real manifold, there is an isomorphism $\mathbb{PT} \cong \mathbb{R}^4 \times \mathbb{CP}^1$ and we introduce coordinates $x^{a\dot{a}} \in \mathbb{R}^4$ and $\pi_a \in \mathbb{CP}^1$. In these coordinates, the meromorphic $(3,0)$-form, which we take to have two double poles at $\alpha_a, \beta_a \in \mathbb{CP}^1$,[2] is given by[3]

$$\Omega = \frac{1}{2}\Phi(\pi)\, \epsilon_{\dot{a}\dot{b}}\, \pi_a \mathrm{d}x^{a\dot{a}} \wedge \pi_b \mathrm{d}x^{b\dot{b}} \wedge \langle \pi \mathrm{d}\pi \rangle , \qquad \Phi = \frac{\langle \alpha\beta \rangle^2}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} . \tag{2.2}$$

The poles of Ω in $\mathbb{CP}^1$ play the role of boundaries in hCS_6 because total derivatives pick up a contribution from $\bar{\partial}\Omega$ which is a distribution with support at these poles. To ensure

[2] To have the correct weight Φ should have four poles on $\mathbb{CP}^1$. The relevant configuration for WZW_4 and its gauging is two double poles [18]. The LMP model that we consider in section 6 requires a fourth-order pole. One could also include simple poles, which would lead to deformed models as discussed in [24].

[3] Spinor contractions are defined by $\langle \alpha\beta \rangle = \epsilon^{ab}\alpha_a\beta_b$, see appendix A for further details of spinor conventions.

JHEP12(2024)202

a well-defined variational principle, we impose boundary conditions on the gauge field at these poles given by

$$\mathscr{A}|_{\pi=\alpha} = 0\,, \qquad \mathscr{A}|_{\pi=\beta} = 0\,. \tag{2.3}$$

Turning to the symmetries of this model, the theory is invariant under gauge transformations acting as

$$\hat{\gamma}: \quad A \mapsto (A)^{\hat{\gamma}} = \hat{\gamma}^{-1}\mathscr{A}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\,, \tag{2.4}$$

so long as they preserve the boundary conditions. This implies restrictions on the allowed transformations at the poles of Ω, which are given by

$$\pi^a\partial_{a\dot{a}}\hat{\gamma}|_{\pi=\alpha} = 0\,, \quad \pi^a\partial_{a\dot{a}}\hat{\gamma}|_{\pi=\beta} = 0\,. \tag{2.5}$$

2.2 Localisation of $\mathbf{hCS_6}$ with double poles to $\mathbf{WZW_4}$

Surprisingly, all of the physical degrees of freedom in hCS_6 can be captured by a four-dimensional integrable field theory (IFT_4). This field theory is derived by localising the hCS_6 action, integrating out the $\mathbb{CP}^1$ and landing on a theory on $\mathbb{R}^4$. For the choice of meromorphic $(3,0)$-form Ω and boundary conditions given above, this 4d theory is WZW_4. This localisation is possible because of the substantial gauge symmetry in Chern-Simons theories. Indeed, the dynamical fields arise precisely where this gauge symmetry is broken, namely at the poles of Ω. Fields capturing these degrees of freedom are known as 'edge modes' and enter via the field redefinition

$$\mathscr{A} = (\mathscr{A}')^{\hat{g}} = \hat{g}^{-1}\mathscr{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}\,. \tag{2.6}$$

Expressing the action $S_{\mathrm{hCS}_6}[\mathscr{A}]$ in terms of the fields $\mathscr{A}'$ and $\hat{g}$ one obtains

$$\begin{aligned} S_{\mathrm{hCS}_6}[\mathscr{A}] = S_{\mathrm{hCS}_6}[\mathscr{A}'] &+ \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathscr{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) \\ &- \frac{1}{6\pi\mathrm{i}}\int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g}\big)\,, \end{aligned} \tag{2.7}$$

where, with a slight abuse of notation, we are also denoting by $\hat{g}$ a smooth homotopy to a constant map in the last term (this will be perpetuated later without further comment). Notably, the edge mode $\hat{g}$ only appears in this action against the 4-form $\bar{\partial}\Omega$, which is a distribution with support at the poles of Ω. This means that the action only depends on $\hat{g}$ through its value (and $\mathbb{CP}^1$-derivative) at the poles of Ω, which we denote by

$$\hat{g}|_{\pi=\alpha} = g\,, \qquad \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\alpha} = u\,, \qquad \hat{g}|_{\pi=\beta} = \tilde{g}\,, \qquad \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\beta} = \tilde{u}\,. \tag{2.8}$$

Let us consider the symmetries of the theory in this new parametrisation. The gauge transformation (2.4) acts trivially on $\mathscr{A}'$ while $\hat{g}$ transforms with a right action as

$$\hat{\gamma}: \quad \mathscr{A}' \mapsto \mathscr{A}'\,, \qquad \hat{g} \mapsto \hat{g}\hat{\gamma}\,. \tag{2.9}$$

JHEP12(2024)202

In addition, the new parametrisation has introduced a redundancy, which we dub an internal gauge symmetry, acting as

$$\check{\gamma}: \quad \mathscr{A}' \mapsto \check{\gamma}^{-1}\mathscr{A}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma}\,, \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g}\,. \tag{2.10}$$

We can exploit these symmetries to impose gauge fixing conditions on the fields $\mathscr{A}'$ and $\hat{g}$. Let us fix $\mathscr{A}'$ such that it has no $\mathbb{CP}^1$ leg, and fix the value of $\hat{g}$ at $\pi = \beta$ to the identity.[4] The surviving edge mode at the other pole $g = \hat{g}|_{\pi=\alpha}$ will become the fundamental field of WZW_4.

Returning to the action (2.7), the first term is a genuine six-dimensional bulk term that we eliminate by going on-shell. We find the bulk equation of motion $\bar{\partial}_0\mathscr{A}'_{\dot{a}} = 0$, which implies that these components are holomorphic. This may be solved in terms of $\mathbb{CP}^1$-independent components $A'_{a\dot{a}}$ as

$$\mathscr{A}' = \pi^a A'_{a\dot{a}}\bar{e}^{\dot{a}}\,, \qquad \bar{e}^{\dot{a}} = \frac{\hat{\pi}_a \mathrm{d}x^{a\dot{a}}}{\langle \pi\hat{\pi}\rangle}\,. \tag{2.11}$$

In this expression, $\bar{e}^{\dot{a}}$ is a basis $(0,1)$-form on twistor space defined in appendix B. This completely specifies the $\mathbb{CP}^1$-dependence of $\mathscr{A}'$, and the boundary conditions (2.3) may be solved to determine $A'_{a\dot{a}}$ in terms of g,

$$A'_{a\dot{a}} = -\frac{\beta_a \alpha^b}{\langle \alpha\beta\rangle}\,\partial_{b\dot{a}} g g^{-1}\,. \tag{2.12}$$

From these components, we can construct a 4d connection $A' = A'_{a\dot{a}}\mathrm{d}x^{a\dot{a}}$. This parametrisation of A' in terms of g is known as Yang's parametrisation with g being called Yang's matrix. This solution for $\mathscr{A}'$ may now be substituted into the action and the integral over $\mathbb{CP}^1$ can be computed explicitly. The second and third term of (2.7) localise to a four-dimensional action (the detailed derivation is presented in appendix D) and we land on the WZW_4 theory defined by

$$S_{\mathrm{WZW}_4} = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g) + \int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathscr{L}_{\mathrm{WZ}}[g]\,. \tag{2.13}$$

In the second term, we have introduced a 2-form defined by

$$\omega_{\alpha,\beta} = \frac{1}{\langle\alpha\beta\rangle}\,\alpha_a\beta_b\,\epsilon_{\dot{a}\dot{b}}\,\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}}\,, \tag{2.14}$$

and the WZ 3-form

$$\mathscr{L}_{\mathrm{WZ}}[g] = \frac{1}{3}\,\mathrm{Tr}(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g})\,, \tag{2.15}$$

defined, as usual, using a suitable extension $\tilde{g}$ of g.

The equations of motion of this theory are given by

$$\mathrm{d}\left(\star - \omega_{\alpha,\beta}\wedge\right)\mathrm{d}g g^{-1} = 0 \quad \Leftrightarrow \quad \epsilon^{\dot{a}\dot{b}}\beta^a\partial_{a\dot{a}}\left(\alpha^b\partial_{b\dot{b}} g g^{-1}\right) = 0\,. \tag{2.16}$$

[4] At this point, we may further fix the $\mathbb{CP}^1$-derivative of $\hat{g}$ at both $\pi = \alpha$ and $\pi = \beta$ to zero. However, such terms drop out of the action anyway in the ungauged case without specifying this.

JHEP12(2024)202

The six-dimensional gauge transformations (constrained by boundary conditions) descend to semi-local symmetries of the action (2.13), which act as

$$g \to \gamma_L^{-1} \cdot g \cdot \gamma_R \,, \qquad \alpha^a \partial_{a\dot{a}} \gamma_R = 0 \,, \qquad \beta^a \partial_{a\dot{a}} \gamma_L = 0 \,, \tag{2.17}$$

where $\gamma_L = \hat{\gamma}|_\beta$ and $\gamma_R = \hat{\gamma}|_\alpha$. Of particular interest is the case where $\beta = \hat{\alpha}$, i.e. the poles of Ω are antipodal on $\mathbb{CP}^1$, in which case $\omega_{\alpha,\hat{\alpha}} = \varpi$ is proportional to the Kähler form on $\mathbb{R}^4$. Here, we are referring to the Kähler form with respect to the complex structure $\mathscr{J}_\alpha$ that is defined[5] by the point $\alpha \in \mathbb{CP}^1$. In this case, the semi-local symmetries can be interpreted as a holomorphic left action and anti-holomorphic right action (akin to the 2d WZW current algebra).

2.3 Interpretation as ASDYM

A 4d Yang-Mills connection A' with curvature $F[A'] = \mathrm{d}A' + A' \wedge A'$ is said to be anti-self dual if it obeys $F = -\star F$. After converting to bi-spinor notation, the anti-self-dual Yang-Mills (ASDYM) equations can be expressed as

$$\pi^a \pi^b F_{a\dot{a}b\dot{b}} = 0 \qquad \forall\, \pi_a \in \mathbb{CP}^1 \,. \tag{2.18}$$

This contains three independent equations that can be extracted by introducing basis spinors α_a and β_a satisfying $\langle \alpha\beta \rangle \neq 0$. The three independent equations are then expressed in terms of contractions with these basis spinors as

$$\alpha^a \alpha^b F_{a\dot{a}b\dot{b}} = 0 \,, \tag{2.19}$$

$$\beta^a \beta^b F_{a\dot{a}b\dot{b}} = 0 \,, \tag{2.20}$$

$$(\alpha^a \beta^b + \beta^a \alpha^b) F_{a\dot{a}b\dot{b}} = 0 \,. \tag{2.21}$$

The six-dimensional origin of WZW_4 (and indeed all IFT_4 constructed in this way) ensures that the connection A' introduced in the previous section satisfies the ASDYM equation when evaluated on solutions to the WZW_4 equations of motion. This follows from the six-dimensional equation $\Omega \wedge \mathscr{F}[\mathscr{A}'] = 0$, which encodes both the holomorphicity of $\mathscr{A}'$ and eq. (2.18). To see this explicitly for WZW_4 where the connection A' in given by eq. (2.12), we note that the β-contracted eq. (2.20) holds because $\langle \beta\beta \rangle = 0$, and the α-contracted eq. (2.19) holds due to the Maurer-Cartan identity. The remaining equation (2.21) yields the equations of motion of WZW_4 (2.16).

2.4 Reduction of WZW_4 to WZW_2

Next, we will apply a two-dimensional reduction to WZW_4 specified by two vector fields V_i on $\mathbb{R}^4$ with $i = 1, 2$. The idea of reduction is to restrict to field configurations that are invariant under the flow of these vector fields. The two-dimensional dynamics of the reduced theory will be specified by the Lagrangian $\mathscr{L}_{\mathrm{IFT}_2} = (V_1 \wedge V_2) \vee \mathscr{L}_{\mathrm{IFT}_4}$ where $\mathscr{L}_{\mathrm{IFT}_4}$ is the Lagrangian of the parent theory and we denote the contraction of a vector field V with a differential form X by $V \vee X$.

[5] Recall that $\mathbb{R}^4$ is a hyper-Kähler manifold, which has a $\mathbb{CP}^1$s worth of complex structures, see appendix A, eq. (A.8).

Let us introduce a pair of unit norm spinors γ_a and $\kappa_{\dot a}$ and define the basis of 1-forms on $\mathbb{R}^4$

$$\mathrm{d}z = \gamma_a \kappa_{\dot a} \mathrm{d}x^{a\dot a}\,, \qquad \mathrm{d}\bar z = \hat\gamma_a \hat\kappa_{\dot a} \mathrm{d}x^{a\dot a}\,, \qquad \mathrm{d}w = \gamma_a \hat\kappa_{\dot a} \mathrm{d}x^{a\dot a}\,, \qquad \mathrm{d}\bar w = -\hat\gamma_a \kappa_{\dot a} \mathrm{d}x^{a\dot a}\,. \tag{2.22}$$

These are adapted to the complex structure $\mathcal{J}_\gamma$ defined by $\gamma_a \in \mathbb{CP}^1$. We choose to reduce along the vector fields dual to $\mathrm{d}z$ and $\mathrm{d}\bar z$ by demanding that $\partial_z g = \partial_{\bar z} g = 0$.[6] Contracting the WZW_4 Lagrangian with these vector fields results in the two-dimensional action of a principal chiral model (PCM) plus Wess-Zumino (WZ) term:

$$S_{\mathrm{PCM}+\mathcal{k}\mathrm{WZ}_2}[g] = \frac{1}{2}\int_\Sigma \mathrm{Tr}(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g) + \frac{\mathrm{i}\mathcal{k}}{3}\int_{\Sigma\times[0,1]} \mathrm{Tr}(\hat g^{-1}\mathrm{d}\hat g \wedge \hat g^{-1}\mathrm{d}\hat g \wedge \hat g^{-1}\mathrm{d}\hat g)\,. \tag{2.23}$$

In this action, the relative coefficient between the WZ term and the PCM term is given by

$$\mathcal{k} = \frac{\alpha+\beta}{\alpha-\beta}\,, \qquad \alpha = \frac{\langle\gamma\alpha\rangle}{\langle\alpha\hat\gamma\rangle}\,, \qquad \beta = \frac{\langle\gamma\beta\rangle}{\langle\beta\hat\gamma\rangle}\,. \tag{2.24}$$

Varying the basis spinor γ_a in these expressions changes the choice of reduction vector fields and parametrises a family of two-dimensional theories interpolating between WZW_2 and the PCM. The WZW_2 CFT limit is obtained when $\mathcal{k} \to 1$ with $\alpha\beta$ held fixed. This can be achieved by starting at the Kähler point in 4d, with $\beta = \hat\alpha$, and choosing the reduction to be aligned with the complex structure, i.e. setting $\gamma = \alpha$. An alternative reduction that turns off the WZ term and recovers the PCM is achieved by setting $\beta = -\alpha$.

For general choices of reduction, the four-dimensional semi-local symmetries descend to a global $G_L \times G_R$ symmetry. This is because, for example, the conditions $\alpha^a \partial_{a\dot a}\gamma_R = 0$ and $\partial_z \gamma_R = \partial_{\bar z}\gamma_R = 0$ generically contain four independent constraints leaving only constant solutions. However, when the reduction is taken to the CFT point, this system of four constraints is not linearly independent, and chiral symmetries emerge satisfying $\partial_w \gamma_R = 0$ (and vice versa for γ_L).

Lax connection. A virtue of this approach is that a $\mathfrak{g}^{\mathbb{C}}$-valued Lax connection for the dynamics of the resultant IFT_2 may be derived from the 4d connection A':

$$\begin{aligned} \mathscr{L}_{\bar w} &= \frac{1}{\langle\pi\hat\gamma\rangle}\hat\kappa^{\dot a}\pi^a(\partial_{a\dot a} + A'_{a\dot a}) = \partial_{\bar w} + \frac{(\beta-\zeta)}{(\alpha-\beta)}\partial_{\bar w} g g^{-1}\,, \\ \mathscr{L}_{w} &= \frac{1}{\langle\pi\gamma\rangle}\kappa^{\dot a}\pi^a(\partial_{a\dot a} + A'_{a\dot a}) = \partial_{w} + \frac{\alpha(\beta-\zeta)}{\zeta(\alpha-\beta)}\partial_{w} g g^{-1}\,, \end{aligned} \tag{2.25}$$

where the spectral parameter is given by $\zeta = \frac{\langle\gamma\pi\rangle}{\langle\pi\hat\gamma\rangle}$. Flatness of this connection for all values of ζ is equivalent to the equations of motion of the PCM plus WZ term

$$\alpha\partial_{\bar w}(\partial_w g g^{-1}) - \beta\partial_w(\partial_{\bar w} g g^{-1}) = 0 \quad \Leftrightarrow \quad \mathrm{d}(\star - \mathrm{i}\mathcal{k})\mathrm{d}g g^{-1} = 0\,. \tag{2.26}$$

Notice that in the CFT limit $\mathcal{k} \to 1$ with $\beta \to \infty$, $\alpha \to 0$ the Lax connection becomes chiral and spectral parameter independent.

[6]In this case for reality we have $\mathscr{L}_{\mathrm{IFT}_2} = \mathrm{i}(\partial_z \wedge \partial_{\bar z}) \vee \mathscr{L}_{\mathrm{IFT}_4}$.

2.5 Reduction of hCS_6 to CS_4

Instead of first integrating over $\mathbb{CP}^1$ and then reducing to two dimensions, we could instead directly apply the reduction to hCS_6. This gives the action of CS_4,

$$S_{CS_4}[A] = \frac{1}{2\pi i}\int_{\Sigma\times\mathbb{CP}^1} \omega \wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) . \tag{2.27}$$

Here Σ is the $\mathbb{R}^2 \subset \mathbb{R}^4$ with coordinates $w, \bar{w}$, and the meromorphic 1-form ω is given by

$$\omega = i(\partial_z \wedge \partial_{\bar{z}}) \vee \Omega . \tag{2.28}$$

A crucial feature here is that this contraction introduces zeroes in ω to complement its poles, as required by the Riemann-Roch theorem. For the case at hand, ω is given explicitly by

$$\omega = i\,\frac{\langle\alpha\beta\rangle^2\langle\pi\gamma\rangle\langle\pi\hat{\gamma}\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}\,\langle\pi d\pi\rangle , \tag{2.29}$$

and the zeroes are introduced at the points $\pi_a = \gamma_a, \hat{\gamma}_a$. The details of the reduction show that, while our six-dimensional gauge field was regular, the connection A entering in CS_4 develops poles at the zeroes of ω. In particular, the component A_w will have a simple pole at $\pi_a = \gamma_a$ and $A_{\bar{w}}$ will have a simple pole at $\pi_a = \hat{\gamma}_a$. The four-dimensional Chern-Simons connection is subject to the same boundary conditions as its parent, namely it vanishes at the points $\alpha, \beta \in \mathbb{CP}^1$. The subsequent localisation of CS_4 then gives the same PCM plus WZ term derived by reducing WZW_4 (see for instance [16] section 10).

3 The gauged WZW diamond

We now come to the main results of this paper. In this section, we will construct a diamond correspondence of theories which realises the gauged WZW_2 model, i.e. the G/H coset CFT.

3.1 Gauged WZW models

First, let us review the gauging of the WZW model and the crucial Polyakov-Wiegmann identity. Letting G be a Lie group and $g \in C^\infty(\Sigma, G)$ a smooth G-valued field, the WZW_2 action is[7]

$$S_{WZW_2}[g] = \frac{1}{2}\int_\Sigma \mathrm{Tr}_{\mathfrak{g}}(g^{-1}dg\wedge\star g^{-1}dg) + \frac{1}{3}\int_{\Sigma\times[0,1]} \mathrm{Tr}_{\mathfrak{g}}(\hat{g}^{-1}d\hat{g}\wedge\hat{g}^{-1}d\hat{g}\wedge\hat{g}^{-1}d\hat{g}) . \tag{3.1}$$

Gauging a vectorial H-action of the PCM term is straightforward. We introduce an $\mathfrak{h}$-valued connection $B \in \Omega^1(\Sigma) \otimes \mathfrak{h}$ transforming as

$$\ell \in C^\infty(\Sigma, H): \quad B \mapsto \ell^{-1}B\ell + \ell^{-1}d\ell , \quad g \mapsto \ell^{-1}g\ell , \tag{3.2}$$

with field strength $F[B] = dB + B\wedge B$. The PCM term is then gauged by replacing the exterior derivatives with covariant derivatives $dg \to Dg = dg + [B, g]$. Less trivially, the gauge completion of the WZ 3-form is [29, 33–35]

$$\mathscr{L}_{gWZ}[g, B] = \mathscr{L}_{WZ}[g] + d\,\mathrm{Tr}_{\mathfrak{g}}(g^{-1}dg\wedge B + dgg^{-1}\wedge B + g^{-1}Bg\wedge B) . \tag{3.3}$$

[7]To minimise factors of imaginary units we momentarily adopt Lorentzian signature. Schematically, we have $S_{\mathrm{Lorentz}} = -iS_{\mathrm{Euclid}}|_{\star\to i\star}$.

Adding these two pieces together gives the gauged WZW_2 action,

$$S_{\mathrm{gWZW}_2}[g,B]=S_{\mathrm{WZW}_2}[g]+\int_\Sigma \mathrm{Tr}_{\mathfrak{g}}(g^{-1}\mathrm{d}g\wedge(1-\star)B+\mathrm{d}gg^{-1}\wedge(1+\star)B+B\wedge\star B+g^{-1}Bg\wedge(1-\star)B). \tag{3.4}$$

Notice that chiral couplings between currents and gauge fields emerge from combinations of the PCM and WZ contributions. The identity

$$\mathscr{L}_{\mathrm{WZ}}[g_1g_2]=\mathscr{L}_{\mathrm{WZ}}[g_1]+\mathscr{L}_{\mathrm{WZ}}[g_2]+\mathrm{d}\,\mathrm{Tr}_{\mathfrak{g}}\left(\mathrm{d}g_2g_2^{-1}\wedge g_1^{-1}\mathrm{d}g_1\right), \tag{3.5}$$

ensures that (3.4) can be recast as the difference of two WZW_2 models. To see this we choose a parametrisation of the gauge field B in terms of two smooth H-valued fields

$$B=\frac{1+\star}{2}\,a^{-1}\mathrm{d}a+\frac{1-\star}{2}\,b^{-1}\mathrm{d}b\,,\quad a,b\in C^\infty(\Sigma,H)\,. \tag{3.6}$$

In two dimensions, this is not a restriction on the field content of the gauge field, but simply a way of parametrising the two independent components of B. With such a parametrisation, if we then further define $\tilde{g}=agb^{-1}\in C^\infty(\Sigma,G)$ and $\tilde{h}=ab^{-1}\in C^\infty(\Sigma,H)$ the gauged model (3.4) can be written as the difference of two WZW_2 models:

$$S_{\mathrm{gWZW}_2}[g,B]=S_{\mathrm{WZW}_2}[\tilde{g}]-S_{\mathrm{WZW}_2}[\tilde{h}]\,. \tag{3.7}$$

This is known as the Polyakov-Wiegmann (PW) identity [36].

3.2 Gauging of the WZW_4 model

Let us now consider the four-dimensional WZW model, given by eq. (2.13). The gauging procedure follows in the exact same manner, producing an analogous gauged WZW_4 action,

$$S^{(\alpha,\beta)}_{\mathrm{gWZW}_4}[g,B]=\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{Tr}(g^{-1}\nabla g\wedge\star g^{-1}\nabla g)+\int_{\mathbb{R}^4\times[0,1]}\omega_{\alpha,\beta}\wedge\mathscr{L}_{\mathrm{gWZ}}[g,B]\,. \tag{3.8}$$

Here, we denote the covariant derivative by $\nabla g=\mathrm{d}g+[B,g]$. A critical difference between two and four dimensions is the applicability of the PW identity as was pointed out in [7]. In two dimensions, this mapping relies on the relation (3.6). To extend it to four dimensions, we consider the operator on 1-forms

$$J_{\alpha,\beta}(\sigma)=-\mathrm{i}\star(\omega_{\alpha,\beta}\wedge\sigma)\,. \tag{3.9}$$

Checking that $J^2_{\alpha,\beta}=-\mathrm{id}$, we can introduce the useful projectors

$$P=\frac{1}{2}\left(\mathrm{id}-\mathrm{i}J\right),\qquad \bar{P}=\frac{1}{2}\left(\mathrm{id}+\mathrm{i}J\right), \tag{3.10}$$

which furnish a range of identities detailed in appendix C. With these in mind, we can write a four-dimensional analogue to (3.6),

$$B=P\left(a^{-1}\mathrm{d}a\right)+\bar{P}\left(b^{-1}\mathrm{d}b\right),\quad a,b\in C^\infty(\mathbb{R}^4,H)\,. \tag{3.11}$$

With this parametrisation of the gauge field, it is indeed possible to use the composite fields $\tilde{g} = agb^{-1} \in C^\infty(\mathbb{R}^4, G)$ and $\tilde{h} = ab^{-1} \in C^\infty(\mathbb{R}^4, H)$ to express the gauged WZW_4 action in a fashion akin to eq. (3.7) as

$$S^{(\alpha,\beta)}_{\mathrm{gWZW}_4}[g, B] = S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{g}] - S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{h}] \, . \tag{3.12}$$

However, unlike in two dimensions, the parametrisation of the gauge field in eq. (3.11) is not generic. It implies a restriction on the connection, namely that its curvature satisfies

$$\alpha^a \alpha^b F_{a\dot{a}b\dot{b}}[B] = 0 \, , \quad \beta^a \beta^b F_{a\dot{a}b\dot{b}}[B] = 0 \, . \tag{3.13}$$

This can be thought of as an analogue of imposing that F is strictly a $(1,1)$-form, which indeed is the case when $\beta = \hat{\alpha}$ and the WZW_4 is taken at the Kähler point. It is noteworthy that these constraints on the background gauge field agree with two of the three ASDYM equations; the same two equations that were identically satisfied by the Yang parametrisation of the connection A'. In the forthcoming analysis, we will see how this arises from the hCS_6 construction.

3.3 A six-dimensional origin

We now turn to the six-dimensional holomorphic Chern-Simons theory on twistor space that will descend to the above gauged WZW models in two and four dimensions. Given the factorisation of gWZW_2 to the difference of WZW_2 models, a natural candidate is to simply consider the difference of hCS_6 theories to generalise the six-dimensional action introduced in [18–20, 23]. Indeed, a similar idea was proposed in [30] to construct 2d coset models from the difference of CS_4 theories. However, how this should work in six dimensions is less clear as the factorisation of gWZW_4 requires the curvature of the gauge field to be constrained.

The fundamental fields of our theory are two connections $\mathscr{A} \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathscr{B} \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{h}$, which appear in the six-dimensional action

$$S_{\mathrm{ghCS}_6}[\mathscr{A}, \mathscr{B}] = S_{\mathrm{hCS}_6}[\mathscr{A}] - S_{\mathrm{hCS}_6}[\mathscr{B}] - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\mathscr{A} \wedge \mathscr{B}) \, , \tag{3.14}$$

where the functional S_{hCS_6} is defined in eq. (2.1). As well as the bulk hCS_6 functionals, we have also included a coupling term between the two connections that contributes on the support of $\bar{\partial}\Omega$, i.e. at the poles of Ω. We will shortly provide a motivation for this boundary term related to the boundary conditions we impose on the theory.

This definition is slightly imprecise; strictly speaking, the inner product denoted by 'Tr' should be defined separately for each algebra, i.e. $\mathrm{Tr}_{\mathfrak{g}}$ and $\mathrm{Tr}_{\mathfrak{h}}$. In the coupling term, where $\mathscr{B}$ enters inside $\mathrm{Tr}_{\mathfrak{g}}$, we should first act on $\mathscr{B}$ with a Lie algebra homomorphism from $\mathfrak{h}$ to $\mathfrak{g}$, and, in principle, this homomorphism could be chosen differently at each pole of Ω. We discuss more general gaugings, beyond the vectorial gauging considered here, in appendix E. It will also be useful for us to assume that we have an orthogonal decomposition of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k} \, , \qquad \mathrm{Tr}(X \cdot Y) = \mathrm{Tr}(X^{\mathfrak{h}} \cdot Y^{\mathfrak{h}}) + \mathrm{Tr}(X^{\mathfrak{k}} \cdot Y^{\mathfrak{k}}) \, , \tag{3.15}$$

and that the homogeneous space G/H is reductive,

$$[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k} \, . \tag{3.16}$$

To complete the specification of the theory, we must supply boundary conditions that ensure the vanishing of the boundary term in the variation of (3.14),

$$\delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big((\delta\mathscr{A} + \delta\mathscr{B}) \wedge (\mathscr{A} - \mathscr{B})\big)\,. \tag{3.17}$$

Since $\bar{\partial}\Omega$ only has support at the poles of Ω, the integral over $\mathbb{CP}^1$ may be computed explicitly in this term. As well as contributions proportional to delta-functions on $\mathbb{CP}^1$, this will also include $\mathbb{CP}^1$-derivatives of delta-functions since the poles in Ω are second order. Using the localisation formula in the appendix D, we find

$$\begin{aligned} \delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = -\int_{\mathbb{R}^4} \bigg[& \frac{\alpha_a \beta_b \Sigma^{ab}}{\langle \alpha\beta\rangle} \wedge \mathrm{Tr}\big((\delta\mathscr{A} + \delta\mathscr{B}) \wedge (\mathscr{A} - \mathscr{B})\big) \\ & + \frac{1}{2}\,\alpha_a\alpha_b\Sigma^{ab} \wedge \partial_0 \mathrm{Tr}\big((\delta\mathscr{A} + \delta\mathscr{B}) \wedge (\mathscr{A} - \mathscr{B})\big)\bigg] + \alpha \leftrightarrow \beta\,. \end{aligned} \tag{3.18}$$

In this expression, we have introduced a basis for the self-dual 2-forms defined by $\Sigma^{ab} = \varepsilon_{\dot{a}\dot{b}} \mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}}$. To attain the vanishing of the boundary variation, we consider the boundary conditions

$$\mathscr{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0\,, \qquad \mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \mathscr{B}\big|_{\alpha,\beta}\,, \qquad \partial_0\mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \partial_0\mathscr{B}\big|_{\alpha,\beta}\,, \tag{3.19}$$

where the superscripts $\mathfrak{k}$ and $\mathfrak{h}$ denote projections corresponding to the decomposition (3.15). This completes our definition of the gauged hCS_6 theory.

We might choose to think of the boundary term in the variation as being a potential for a 'symplectic' form[8,9]

$$\Theta = \delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}}\,, \qquad \Omega = \delta\Theta = -\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \Big(\mathrm{Tr}_{\mathfrak{g}}(\delta\mathscr{A}\wedge\delta\mathscr{A}) - \mathrm{Tr}_{\mathfrak{h}}(\delta\mathscr{B}\wedge\delta\mathscr{B})\Big)\,, \tag{3.20}$$

such that our boundary conditions define a Lagrangian (i.e. maximal isotropic) subspace. We would like to interpret this as a symplectic form on an appropriate space of fields defined over $\mathbb{R}^4$. Evaluating the integral over $\mathbb{CP}^1$ and writing $\Omega = \Omega_{\mathscr{A}} - \Omega_{\mathscr{B}}$, this symplectic form is given by

$$\Omega_{\mathscr{A}} = \int_{\mathbb{R}^4} \bigg[\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle} \wedge \mathrm{Tr}_{\mathfrak{g}}(\delta\mathscr{A}\wedge\delta\mathscr{A})\big|_{\alpha} + \frac{1}{2}\alpha_a\alpha_b\Sigma^{ab} \wedge \partial_0 \mathrm{Tr}_{\mathfrak{g}}(\delta\mathscr{A}\wedge\delta\mathscr{A})\big|_{\alpha}\bigg] + \alpha\leftrightarrow\beta\,, \tag{3.21}$$

with an analogous expression for $\Omega_{\mathscr{B}}$. Since our boundary conditions are identical at each pole, we concentrate only on the contributions associated to the pole at α. The symplectic form is not sensitive to the entire field configuration $\mathscr{A} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{g}$, but rather to the evaluation of $\mathscr{A}$ at the poles and its first $\mathbb{CP}^1$-derivative,

$$\vec{\mathscr{A}} = (\mathscr{A}|_{\alpha}, \partial_0\mathscr{A}|_{\alpha})\,. \tag{3.22}$$

This data may be interpreted as defining a 1-form (or more precisely a $(0,1)$-form with respect to the complex structure defined by α) on $\mathbb{R}^4$ valued in the Lie algebra[10] $\vec{\mathfrak{g}} = \mathfrak{g} \ltimes \mathbb{R}^{\dim(G)}$.

[8] Precedent in the literature dictates that we denote the symplectic form by Ω; we trust that context serves to disambiguate from the meromorphic differential Ω.

[9] This is slightly loose as the 2-form is degenerate; strictly speaking we should restrict to symplectic leaves.

[10] The dimension of $\vec{\mathfrak{g}}$ is $2\dim(G)$, so it must be isomorphic to $\mathbb{R}^{\dim(G)} \oplus \mathbb{R}^{\dim(G)}$ as a vector space. The Lie algebra structure may be derived by considering consecutive infinitesimal gauge transformations. In the CS_4 literature, these structures have been studied under the name 'defect Lie algebras' [37, 38].

With this in mind, it is more accurate to say that the contribution from the pole at α in Ω is a symplectic form on the space of configurations

$$(\vec{\mathscr{A}},\vec{\mathscr{B}}) \in \Omega^{0,1}(\mathbb{R}^4)\otimes\big(\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}\big)\,. \tag{3.23}$$

This symplectic form may be succinctly written by introducing an inner product on the Lie algebra $\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}$, and our boundary conditions describe an isotropic subspace with respect to this inner product.[11]

To be explicit, we associate $\mathbb{R}^{\dim G}$ with the dual $\mathfrak{g}^*$ and denote the natural pairing of the algebra and its dual by $\tilde{x}(x)\in\mathbb{R}$ for $x\in\mathfrak{g}$ and $\tilde{x}\in\mathfrak{g}^*$. We let $\vec{X}=(x,\tilde{x})$ and $\vec{Y}=(y,\tilde{y})$ be elements of $\vec{\mathfrak{g}}$ such that the bracket on $\vec{\mathfrak{g}}$ is defined by

$$[\vec{X},\vec{Y}]_{\vec{\mathfrak{g}}} = ([x,y], \mathrm{ad}^*_x\tilde{y}-\mathrm{ad}^*_y\tilde{x})\,, \tag{3.24}$$

where the co-adjoint action is $\mathrm{ad}^*_y\tilde{x}(x)=\tilde{x}([x,y])$. We equip $\vec{\mathfrak{g}}$ with the inner product

$$\langle\vec{X},\vec{Y}\rangle_{\vec{\mathfrak{g}}} = \frac{\langle\beta\hat{\alpha}\rangle}{\langle\alpha\beta\rangle\langle\alpha\hat{\alpha}\rangle}\,\mathrm{Tr}_{\mathfrak{g}}(x\cdot y)+\frac{1}{2}\Big(\tilde{y}(x)+\tilde{x}(y)\Big)\,, \tag{3.25}$$

such that the contribution from the pole at α to $\Omega_{\mathscr{A}}$ is given by

$$\Omega_{\mathscr{A}} = \int_{\mathbb{R}^4}\mu_\alpha\wedge\langle\delta\vec{\mathscr{A}},\delta\vec{\mathscr{A}}\rangle_{\vec{\mathfrak{g}}}\,, \tag{3.26}$$

where $\mu^\alpha=\alpha_a\alpha_b\Sigma^{ab}$ is the $(2,0)$-form defined by the complex structure associated to $\alpha\in\mathbb{CP}^1$.

In a similar fashion we let $\vec{U}=(u,\tilde{u})$ and $\vec{V}=(v,\tilde{v})$ be elements of $\vec{\mathfrak{h}}$, which is equipped with a bracket and pairing via the same recipe. We consider the commuting direct sum $\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}$ equipped with pairing and bracket

$$\langle\!\langle(\vec{X},\vec{U}),(\vec{Y},\vec{V})\rangle\!\rangle = \langle\vec{X},\vec{Y}\rangle_{\vec{\mathfrak{g}}}-\langle\vec{U},\vec{V}\rangle_{\vec{\mathfrak{h}}}\,,\qquad [\![(\vec{X},\vec{U}),(\vec{Y},\vec{V})]\!] = ([\vec{X},\vec{Y}]_{\vec{\mathfrak{g}}},[\vec{U},\vec{V}]_{\vec{\mathfrak{h}}})\,, \tag{3.27}$$

such that the total symplectic form coming from the pole at α is just

$$\Omega = \int_{\mathbb{R}^4}\mu_\alpha\wedge\langle\!\langle(\delta\vec{\mathscr{A}},\delta\vec{\mathscr{B}}),(\delta\vec{\mathscr{A}},\delta\vec{\mathscr{B}})\rangle\!\rangle\,. \tag{3.28}$$

Then, our boundary conditions can be expressed as $(\vec{\mathscr{A}},\vec{\mathscr{B}})\in\Omega^{0,1}(\mathbb{R}^4)\otimes L$ where we introduce a subspace

$$L = \big\{(\vec{X},\vec{U})\in\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}\mid x=u\,,\ P^*_{\mathfrak{h}}\tilde{x}=\tilde{u}\big\}\,, \tag{3.29}$$

in which $P^*_{\mathfrak{h}}$ is dual to the projector $P_{\mathfrak{h}}$ onto the subalgebra, i.e. $P^*_{\mathfrak{h}}\tilde{x}(x)=\tilde{x}(P_{\mathfrak{h}}x)$. As L is defined by $\dim\mathfrak{g}+\dim\mathfrak{h}$ constraints, it is half-dimensional and it is also isotropic with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, hence defines a Lagrangian subspace. Moreover, assuming that G/H is reductive, L

[11] This need not be the case since our boundary conditions could generically intertwine constraints on the algebra and spacetime components, meaning they would not be captured by a subspace of the algebra alone. They would always, however, define an isotropic subspace of $\Omega^{0,1}(\mathbb{R}^4)\otimes\big(\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}\big)$ by definition. Examples of this more general type of boundary condition can be found in [24].

is a subalgebra.[12] Pre-empting the following section, this analysis indicates that there will be a residual $\vec{\mathfrak{h}}$ gauge symmetry associated to the pole at α, and similarly at β.

We can make one further observation[13] on the role of the boundary term from a symplectic perspective, which is best illustrated by a finite-dimensional analogy. Recall that the cotangent bundle $M = T^*X$ is a symplectic manifold; if we let $\{x^i\}$ be local coordinates on X and $\{\xi_i\}$ the components of a 1-form $\xi = \xi_i dx^i \in T^*_x X$, then $p = (x^i, \xi_i)$ provide local coordinates for M in terms of which the canonical symplectic form is $\Omega = \mathrm{d}\xi_i \wedge \mathrm{d}x^i$. The tautological potential, which admits a coordinate free definition in terms of the projection $\pi : T^*X \to X$, for this is given by $\Theta = \xi_i \mathrm{d}x^i$. The zero section, i.e. points $p = (x^i, \xi_i = 0)$ of T^*X, is a Lagrangian submanifold and we note that Θ vanishes trivially here. Now Weinstein's tubular neighbourhood theorem ensures that in the vicinity of a Lagrangian submanifold L, any symplectic manifold M locally looks like T^*L with L given by the zero section. In the case at hand, our boundary conditions are of the schematic form $\xi = \mathscr{A} - \mathscr{B} = 0$, and the effect of including the additional boundary contribution in the action (3.14) ensures that the resultant symplectic potential is the tautological one.

To close this section, let us comment that at the special point $\alpha = \hat{\beta}$, one of the terms in the inner product (3.25) vanishes. This allows for a larger class of admissible boundary conditions, even in the ungauged model, including the examples

$$\mathscr{A}\big|_{\hat{\alpha}} = 0\,, \qquad \partial_0\mathscr{A}\big|_{\alpha} = 0 \qquad \text{or} \qquad \partial_0\mathscr{A}\big|_{\hat{\alpha}} = 0\,, \qquad \partial_0\mathscr{A}\big|_{\alpha} = 0\,. \tag{3.30}$$

We leave these for future development.

3.4 Localisation to gWZW$_4$

The localisation procedure follows in a similar fashion to the ungauged model. However, given that there are now two gauge fields $\mathscr{A}$ and $\mathscr{B}$, some care is required to account for degrees of freedom and residual symmetries.

We introduce a new pair of connections $\mathscr{A}' \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathscr{B}' \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{h}$, along with group-valued fields $\hat{g} \in C^\infty(\mathbb{PT}, G)$ and $\hat{h} \in C^\infty(\mathbb{PT}, H)$ related to the original gauge fields by

$$\begin{aligned} \mathscr{A} &= \hat{g}^{-1}\mathscr{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \equiv \mathscr{A}'^{\hat{g}}\,, \\ \mathscr{B} &= \hat{h}^{-1}\mathscr{B}'\hat{h} + \hat{h}^{-1}\bar{\partial}\hat{h} \equiv \mathscr{B}'^{\hat{h}}\,. \end{aligned} \tag{3.31}$$

The redundancy in this parametrisation is given by the action of $\check{\gamma} \in C^\infty(\mathbb{PT}, G)$ and $\check{\eta} \in C^\infty(\mathbb{PT}, H)$:

$$\mathscr{A}' \mapsto \check{\gamma}^{-1}\mathscr{A}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma}\,, \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g}\,, \tag{3.32}$$

$$\mathscr{B}' \mapsto \check{\eta}^{-1}\mathscr{B}'\check{\eta} + \check{\eta}^{-1}\bar{\partial}\check{\eta}\,, \qquad \hat{h} \mapsto \check{\eta}^{-1}\hat{h}\,, \tag{3.33}$$

which leave $\mathscr{A}$ and $\mathscr{B}$ invariant. As before, this is partially used to fix away the $\mathbb{CP}^1$ legs

$$\mathscr{A}'_0 = \mathscr{B}'_0 = 0\,. \tag{3.34}$$

[12] If $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ is not assumed to be reductive then the stabiliser of L consists of elements of the form

$$\mathrm{stab}_L = \left\{ (\vec{X}, \vec{U}) \in \vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}} \mid x = u\,,\ P^*_{\mathfrak{h}}\tilde{x} = \tilde{u}\,,\ [u, \mathfrak{k}] = 0\,,\ ([\mathfrak{h}, \mathfrak{k}], \tilde{u}) = 0 \right\}.$$

[13] We thank A. Arvanitakis for this suggestion.

.

The localisation procedure will produce a four-dimensional boundary theory with fields given by the evaluations of $\hat{g}, \hat{h}$ and their $\mathbb{CP}^1$-derivatives at the poles α and β of Ω. Since the $\mathbb{CP}^1$-derivatives will play an important role, we denote them

$$\hat{u} = \hat{g}^{-1}\partial_0\hat{g}\,, \quad \hat{v} = \hat{h}^{-1}\partial_0\hat{h}\,. \tag{3.35}$$

After fixing (3.34), we note that there is still some remaining symmetry given by internal gauge transformations (3.32) and (3.33) that are $\mathbb{CP}^1$-independent. We use this residual symmetry to fix

$$\hat{g}|_\beta = \mathrm{id}\,, \quad \hat{h}|_\beta = \mathrm{id}\,. \tag{3.36}$$

On the other hand, the action (3.14) is invariant under gauge transformations acting on $\mathscr{A}$ and $\mathscr{B}$ that preserve the boundary conditions (3.19). These are given by smooth maps $\hat{\gamma} \in C^\infty(\mathbb{PT}, G)$ and $\hat{\eta} \in C^\infty(\mathbb{PT}, H)$ satisfying[14]

$$\hat{\gamma}|_{\alpha,\beta} = \hat{\eta}|_{\alpha,\beta}\,, \quad \partial_0\hat{\gamma}|_{\alpha,\beta} = \partial_0\hat{\eta}|_{\alpha,\beta}\,. \tag{3.37}$$

The induced action of these gauge transformations on the new field content is

$$\mathscr{A}' \mapsto \mathscr{A}'\,, \qquad \hat{g} \mapsto \hat{g}\hat{\gamma}\,, \qquad \hat{u} \mapsto \hat{\gamma}^{-1}\hat{u}\hat{\gamma} + \hat{\gamma}^{-1}\partial_0\hat{\gamma}\,, \tag{3.38}$$

$$\mathscr{B}' \mapsto \mathscr{B}'\,, \qquad \hat{h} \mapsto \hat{h}\hat{\eta}\,, \qquad \hat{v} \mapsto \hat{\eta}^{-1}\hat{v}\hat{\eta} + \hat{\eta}^{-1}\partial_0\hat{\eta}\,. \tag{3.39}$$

We would like to use this symmetry to further fix degrees of freedom. Note that, while the right action on the fields $\hat{g}$ and $\hat{h}$ at α is entirely unconstrained, the action at β should preserve the gauge fixing condition (3.36). This is achieved by performing both an internal and external gauge transformation simultaneously, and requiring $\hat{\gamma}|_\beta = \check{\gamma}$ and $\hat{\eta}|_\beta = \check{\eta}$. This results in an induced left action on the fields $\hat{g}$ and $\hat{h}$ at α. In summary, introducing some notation for simplicity, we have our boundary degrees of freedom

$$\hat{g}|_\alpha \coloneqq g\,, \qquad \hat{g}|_\beta = \mathrm{id}\,, \qquad \hat{u}|_\alpha \coloneqq u\,, \qquad \hat{u}|_\beta \coloneqq \tilde{u}\,, \tag{3.40}$$

$$\hat{h}|_\alpha \coloneqq h\,, \qquad \hat{h}|_\beta = \mathrm{id}\,, \qquad \hat{v}|_\alpha \coloneqq v\,, \qquad \hat{v}|_\beta \coloneqq \tilde{v}\,, \tag{3.41}$$

and boundary gauge transformations

$$\hat{\gamma}|_\alpha = \hat{\eta}|_\alpha \coloneqq r\,, \qquad \hat{\gamma}^{-1}\partial_0\hat{\gamma}|_\alpha = \hat{\eta}^{-1}\partial_0\hat{\eta}|_\alpha \coloneqq \epsilon\,, \tag{3.42}$$

$$\hat{\gamma}|_\beta = \hat{\eta}|_\beta \coloneqq \ell^{-1}\,, \qquad \hat{\gamma}^{-1}\partial_0\hat{\gamma}|_\beta = \hat{\eta}^{-1}\partial_0\hat{\eta}|_\beta \coloneqq \tilde{\epsilon}\,, \tag{3.43}$$

which act on the boundary fields as

$$g \mapsto \ell g r\,, \quad u \mapsto r^{-1}ur + \epsilon\,, \quad \tilde{u} \mapsto \ell\tilde{u}\ell^{-1} + \tilde{\epsilon}\,, \tag{3.44}$$

$$h \mapsto \ell h r\,, \quad v \mapsto r^{-1}vr + \epsilon\,, \quad \tilde{v} \mapsto \ell\tilde{v}\ell^{-1} + \tilde{\epsilon}\,, \tag{3.45}$$

with $\ell, r \in C^\infty(\mathbb{R}^4, H)$ and $\epsilon, \tilde{\epsilon} \in C^\infty(\mathbb{R}^4, \mathfrak{h})$. Based on our expectation of a gauge theory containing a G-valued field and a vectorial H-gauge symmetry, we use the above symmetries to fix

$$h = \mathrm{id}\,, \qquad v = \tilde{v} = 0\,. \tag{3.46}$$

[14] Here we use that the homogeneous space G/H is reductive (3.16) to ensure that the boundary conditions are preserved.

We are thus left with a residual symmetry $r = \ell^{-1}$ acting as

$$g \mapsto \ell g \ell^{-1}\,, \quad u \mapsto \ell u \ell^{-1}\,, \quad \tilde{u} \mapsto \ell \tilde{u} \ell^{-1}\,, \quad B \mapsto \ell B \ell^{-1} - \mathrm{d}\ell \ell^{-1}\,, \tag{3.47}$$

which will become the H-gauge symmetry of our 4d theory.

We now proceed with the localisation of the six-dimensional action. As with the ungauged model, the first step is to write the action in terms of $\mathscr{A}', \mathscr{B}'$ and $\hat{g}, \hat{h}$. Given that the localisation formula (D.9) introduces at most one ∂_0 derivative, all dependence on $\hat{h}$ will drop out due to our gauge fixing choices (3.41) and (3.46). Hence there will be no contribution from $S_{\mathrm{hCS}_6}[\mathscr{B}]$ to the four-dimensional action. As per eq. (2.7), we find that the bulk equations of motion (i.e. contributions to the variation of the action that are not localised at the poles of Ω) enforce $\bar{\partial}_0 \mathscr{A}'_{\dot{a}} = \bar{\partial}_0 \mathscr{B}'_{\dot{a}} = 0$. This implies that the components $\mathscr{A}'_{\dot{a}}, \mathscr{B}'_{\dot{a}}$ are holomorphic, which (combined with the fact that they have homogeneous weight 1) allows us to deduce that

$$\mathscr{A}'_{\dot{a}} = \pi^a A'_{a\dot{a}}\,, \quad \mathscr{B}'_{\dot{a}} = \pi^a B'_{a\dot{a}}\,, \tag{3.48}$$

where $A'_{a\dot{a}}, B'_{a\dot{a}}$ are $\mathbb{CP}^1$-independent. Imposing the bulk equations of motion and the gauge fixings described above, the remaining contributions in the action (3.14) are given by

$$\begin{aligned} S_{\mathrm{ghCS}_6}[\mathscr{A},\mathscr{B}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} & \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathscr{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - (\hat{g}^{-1}\mathscr{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}) \wedge \mathscr{B}'\big) \\ & - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g})\,. \end{aligned} \tag{3.49}$$

In the ungauged model, the next step was to solve the boundary conditions for $\mathscr{A}'$ in terms of $\hat{g}$. Here, the boundary conditions on $\mathscr{A}$ and $\mathscr{B}$, i.e. excluding those relating the $\mathbb{CP}^1$-derivatives of the gauge fields, do not fully determine $A'_{a\dot{a}}, B'_{a\dot{a}}$ and instead relate them as[15]

$$A'_{a\dot{a}} = B'_{a\dot{a}} + \Theta_{a\dot{a}} \coloneqq B'_{a\dot{a}} - \frac{1}{\langle \alpha\beta\rangle}\, \beta_a \alpha^b \nabla_{b\dot{a}} g g^{-1}\,, \tag{3.50}$$

where the covariant derivative is given by $\nabla_{a\dot{a}} g g^{-1} = \partial_{a\dot{a}} g g^{-1} + B'_{a\dot{a}} - \mathrm{Ad}_g B'_{a\dot{a}}$. The relation (3.50) allows us to express (3.49) entirely in terms of $\mathscr{B}'$, $\Theta = \pi^a \Theta_{a\dot{a}} \bar{e}^{\dot{a}}$ and $\hat{g}$. Many of the terms combine to produce a gauged Wess-Zumino contribution (3.3) with the result

$$S_{\mathrm{ghCS}_6}[\mathscr{A},\mathscr{B}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\Big(\Theta \wedge \Big(\nabla \hat{g}\hat{g}^{-1} - \mathscr{B}'\Big)\Big) - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathscr{L}_{\mathrm{gWZ}}[\hat{g},\mathscr{B}']\,. \tag{3.51}$$

Given that both $B'_{a\dot{a}}$ and $\Theta_{a\dot{a}}$ are $\mathbb{CP}^1$-independent, we have that

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\Theta \wedge \mathscr{B}') = 0\,, \tag{3.52}$$

[15] The boundary conditions relating the $\mathbb{CP}^1$-derivatives of the gauge fields impose

$$\frac{\alpha^a}{\langle \alpha\beta\rangle} \left(\nabla_{a\dot{a}} g g^{-1}\right)^{\mathfrak{h}} = -\beta^a \nabla_{a\dot{a}} \tilde{u}^{\mathfrak{h}}\,, \qquad \frac{\beta^a}{\langle \alpha\beta\rangle} \left(g^{-1} \nabla_{a\dot{a}} g\right)^{\mathfrak{h}} = -\alpha^a \nabla_{a\dot{a}} u^{\mathfrak{h}}\,,$$

which, in principle, can be solved for $B'_{a\dot{a}}$. However, we will not invoke these since they will follow as equations of motion of the 4d theory due to the addition of the boundary term in the gauged hCS_6 action (3.14). See appendix E for more details.

with cancelling contributions from the two end points of the integral. Hence we are left with a manifestly covariant result

$$S_{\text{ghCS}_6}[\mathscr{A},\mathscr{B}] = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\left(\Theta \wedge \left(\nabla \hat{g}\hat{g}^{-1}\right) - L_{\text{gWZ}}[\hat{g},\mathscr{B}']\right). \tag{3.53}$$

Applying the localisation formula (D.13) in appendix D yields the four-dimensional action

$$\begin{aligned} S_{\text{IFT}_4} = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}(\nabla g g^{-1} \wedge \star \nabla g g^{-1}) + \int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta}\wedge \mathscr{L}_{\text{gWZ}}[g,B'] \\ - \int_{\mathbb{R}^4} \mu_\alpha \wedge \mathrm{Tr}(u\cdot F[B']) + \mu_\beta\wedge \mathrm{Tr}(\tilde{u}\cdot F[B']). \end{aligned} \tag{3.54}$$

At this point only the $\mathfrak{h}$-components of u and $\tilde{u}$ contribute to the action, and so henceforth, to ease notation and without loss of generality, we set their projection onto $\mathfrak{k}$ to zero.

Something rather elegant has happened; we have found that the localisation of the six-dimensional theory returns not only the gauging of the WZW_4 model, but also residual edge modes serving as Lagrange multipliers constraining the field strength to obey exactly those conditions (3.13) that ensure the theory can be written as the difference of WZW_4 models. The constraints $F^{2,0}=0$ and $F^{0,2}=0$ have also been imposed by Lagrange multipliers in the context of 5d Kähler Chern-Simons theory [5, 6]. This theory bears a similar relationship to WZW_4 as 3d Chern-Simons theory bears to WZW_2. This poses a natural question: what is the direct relationship between this 5d Kähler Chern-Simons theory and 6d holomorphic Chern-Simons theory? We suspect the mechanism here is rather similar to that which relates CS_4 and CS_3 [39] and comment on this further in the outlook.

3.5 Equations of motion and ASDYM

Making use of the projectors previously introduced in eq. (3.10), the equations of motion following from the action (3.54) read

$$\begin{aligned} \delta B':&\quad 0 = \bar{P}\nabla g g^{-1}|_{\mathfrak{h}} - P g^{-1}\nabla g|_{\mathfrak{h}} + \star\left(\mu_\alpha\wedge\nabla u + \mu_\beta \wedge \nabla\tilde{u}\right),\\ \delta g:&\quad 0 = \nabla \star \nabla g g^{-1} - \omega_{\alpha,\beta}\wedge \nabla(\nabla g g^{-1}) + 2\omega_{\alpha,\beta}\wedge F[B'],\\ \delta u:&\quad 0 = \mu_\alpha \wedge F[B'],\\ \delta\tilde{u}:&\quad 0 = \mu_\beta\wedge F[B']. \end{aligned} \tag{3.55}$$

We can exploit the projectors to extract two independent contributions from the B' equation of motion:

$$\delta B': \quad \begin{aligned} 0 &= \bar{P}\left(\nabla g g^{-1}|_{\mathfrak{h}} + \star(\mu_\beta\wedge\nabla\tilde{u})\right),\\ 0 &= P\left(g^{-1}\nabla g|_{\mathfrak{h}} - \star(\mu_\alpha\wedge \nabla u)\right). \end{aligned} \tag{3.56}$$

As expected from the discussion in appendix E, these are exactly the conditions that arise from the boundary conditions relating the $\mathbb{CP}^1$-derivatives of the gauge fields, $\partial_0\mathscr{A}^{\mathfrak{h}}|_{\alpha,\beta} = \partial_0\mathscr{B}|_{\alpha,\beta}$, justifying a posteriori why we did not impose them in the localisation procedure.

Making use of the identity

$$\nabla(\omega_{\alpha,\beta}\wedge\star(\mu_\beta\wedge\nabla\tilde{u})) = \nabla(\mu_\beta\wedge\nabla\tilde{u}) = \mu_\beta\wedge F[B']\cdot\tilde{u}, \tag{3.57}$$

JHEP12(2024)202

we obtain an on-shell integrability condition for the first equation in (3.56), namely that

$$\nabla(\omega_{\alpha,\beta} \wedge \bar{P}(\nabla g g^{-1}|_{\mathfrak{h}})) = 0\,. \tag{3.58}$$

Hence, using the projection of the δg equation of motion onto $\mathfrak{h}$, we have that $\omega_{\alpha,\beta} \wedge F[B'] = 0$ follows on-shell.

Let us return to the ASDYM equations, which we can recast as

$$\mu_\alpha \wedge F = 0\,, \quad \mu_\beta \wedge F = 0\,, \quad \omega_{\alpha,\beta} \wedge F = 0\,. \tag{3.59}$$

In differential form notation, the relation (3.50) can be written as

$$A' = B' - \bar{P}(\nabla g g^{-1})\,. \tag{3.60}$$

By virtue of the identities obeyed by the projectors in eqs. (C.3) to (C.5) and the covariant Maurer-Cartan identity obeyed by $R^\nabla = \nabla g g^{-1}$,

$$\nabla R^\nabla - R^\nabla \wedge R^\nabla = (1 - \mathrm{Ad}_g) F[B']\,, \tag{3.61}$$

we can readily establish

$$\mu_\beta \wedge F[A'] = \mu_\beta \wedge F[B']\,, \tag{3.62}$$

$$\mu_\alpha \wedge F[A'] = \mu_\alpha \wedge \mathrm{Ad}_g F[B']\,, \tag{3.63}$$

$$\begin{aligned} 2\omega_{\alpha,\beta} \wedge F[A'] &= 2\omega_{\alpha,\beta} \wedge F[B'] + 2\omega_{\alpha,\beta} \wedge \nabla \bar{P}(R^\nabla) \\ &= 2\omega_{\alpha,\beta} \wedge F[B'] - \nabla(\star \nabla g g^{-1}) + \omega_{\alpha,\beta} \wedge \nabla(\nabla g g^{-1})\,. \end{aligned} \tag{3.64}$$

Hence we conclude that the $\delta g, \delta u, \delta \tilde{u}$ equations of motion are equivalent to the ASDYM equations for the connection A'. Demanding that the B' connection is also ASD requires in addition that $\omega_{\alpha,\beta} \wedge F[B'] = 0$, which is indeed a consequence of the B' equations of motion as shown above.

As we have seen, the equations of motion of gWZW_4 (3.54) are equivalent to the ASDYM equations for the two connections, A' and B'. This ensures that the gauging of the WZW_4 is compatible with integrability. Indeed, from these connections we can construct Lax pairs of differential operators as in eq. (1.4), where the spectral parameter can be interpreted as the coordinate on $\mathbb{CP}^1 \subset \mathbb{PT}$.

3.6 Constraining then reducing

We now proceed to the bottom of the diamond by reduction of the IFT_4. In this section, we shall first implement the constraints imposed by the Lagrange multipliers $u, \tilde{u}$ in the 4d theory and then reduce. While not the most general reduction, this will allow us to directly recover the gauged WZW coset CFT. In section 4, we will investigate more general reductions, in particular, what happens if we reduce without first imposing constraints.

Imposing the reduction ansatz that $\partial_z = \partial_{\bar{z}} = 0$ in the complex coordinates of eq. (2.22), we have that the solution to the constraints $B = P(a^{-1}\mathrm{d}a) + \bar{P}(b^{-1}\mathrm{d}b)$ becomes

$$\begin{aligned} B' = B'_{a\dot{a}} \mathrm{d}x^{a\dot{a}} = \frac{1}{\alpha - \beta}\left(\alpha b^{-1}\partial_w b - \beta a^{-1}\partial_w a\right) \mathrm{d}w - \frac{1}{\alpha - \beta}\left(\beta b^{-1}\partial_{\bar{w}} b - \alpha a^{-1}\partial_{\bar{w}} a\right) \mathrm{d}\bar{w} \\ + \frac{1}{\alpha - \beta}\left(b^{-1}\partial_{\bar{w}} b - a^{-1}\partial_{\bar{w}} a\right) \mathrm{d}z + \frac{\alpha\beta}{\alpha - \beta}\left(b^{-1}\partial_w b - a^{-1}\partial_w a\right) \mathrm{d}\bar{z}\,. \end{aligned} \tag{3.65}$$

For simplicity, let us first consider the Kähler point and align the reduction to the complex structure (implemented by taking $\alpha \to 0$ and $\beta \to \infty$). In this scenario, the reduction ansatz enforces that $B'_z = B'_{\bar{z}} = 0$ with the remaining components of B' parametrising a generic two-dimensional gauge field. Effectively, we can simply ignore the constraints altogether but impose $B'_z = B'_{\bar{z}} = 0$ as part of the reduction ansatz. This could be interpreted as demanding $D_z = D_{\bar{z}} = 0$ acting on fields. In this case, it is immediate that the 4d gauged WZW reduces to a 2d gauged WZW.

Away from the Kähler point and aligned reduction, i.e. not fixing α and β, we need to keep track of contributions coming from B'_z and $B'_{\bar{z}}$. We can still view the B'_w and $B'_{\bar{w}}$ components of eq. (3.65) as a parametrisation of a generic 2d gauge field, but there is no way in which we can view the B'_z and $B'_{\bar{z}}$ as a local combination of B'_w and $B'_{\bar{w}}$. We are forced to work with the variables a and b rather than a 2d gauge field. Fortunately, however, the reduction can still be performed immediately if we use the composite fields $\tilde{g} = agb^{-1}$ and $\tilde{h} = ab^{-1}$. These composite fields are invariant under the H-gauge symmetry, but a new semi-local symmetry emerges given by $a \to \ell a$, $b \to br^{-1}$ with $\alpha^b \partial_{b\dot{b}} r = \beta^b \partial_{b\dot{b}} \ell = 0$. These leave B', g, h invariant but act as $\tilde{g} \to \ell \tilde{g} r$ and $\tilde{h} \to \ell \tilde{h} r$. At the Kähler point and aligned reduction, these symmetries descend to affine symmetries, but in general descend only to global transformations. Recall that in terms of the composite fields gWZW_4 becomes

$$S^{(\alpha,\beta)}_{\mathrm{gWZW}_4}[g, B'] = S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{g}] - S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{h}] \,. \tag{3.66}$$

It is then immediate that this reduces to the difference of PCM plus WZ term theories with action (2.23) and WZ coefficient $\rlap{\,/}k$:

$$S_{\mathrm{IFT}_2}[\tilde{g}, \tilde{h}] = S_{\mathrm{PCM}+\rlap{\,/}k\mathrm{WZ}_2}[\tilde{g}] - S_{\mathrm{PCM}+\rlap{\,/}k\mathrm{WZ}_2}[\tilde{h}] \,. \tag{3.67}$$

Away from the CFT point, $\rlap{\,/}k = 1$, this cannot be recast in terms of a deformation of gWZW_2 expressed as a local functional of B', g.

Lax formulation. To obtain the Lax connection of the resulting IFT_2 we first note that the four-dimensional gauge fields, upon solving the constraints on B', are gauge equivalent to

$$A'_{a\dot{a}} = -\frac{1}{\langle \alpha\beta \rangle} \beta_a \alpha^b \partial_{b\dot{a}} \tilde{g} \tilde{g}^{-1} \,, \quad B'_{a\dot{a}} = -\frac{1}{\langle \alpha\beta \rangle} \beta_a \alpha^b \partial_{b\dot{a}} \tilde{h} \tilde{h}^{-1} \,.$$

Therefore, we may simply follow the construction of the Lax connection from the ungauged model in eq. (2.25), with the connection A' producing a Lax for the $S_{\mathrm{PCM}+\rlap{\,/}k\mathrm{WZ}_2}[\tilde{g}]$ and B' producing one for $S_{\mathrm{PCM}+\rlap{\,/}k\mathrm{WZ}_2}[\tilde{h}]$.

4 More general IFT_2 from IFT_4: reducing then constraining

In the previous section, we reduced from the gauged WZW_4 model to an IFT_2, but prior to reduction we enforced the constraints imposed by the Lagrange multiplier fields. These constraints determine implicit relations between the components of the gauge field as per eq. (3.65). In the simplest case, where we work at the Kähler point and align the reduction directions with the complex structure, the constraints enforce $B'_z = B'_{\bar{z}} = 0$. However, if we do not impose the constraints in 4d, the standard reduction ansatz would only require that B'_z and $B'_{\bar z}$ are functionally independent of z and $\bar z$, a weaker condition.

In this section, we explore the consequences of reducing without first constraining. Denoting the reduction by $\rightsquigarrow$ we anticipate that the lower-dimensional description will include additional fields as[16]

$$\begin{aligned} B'_w(w,\bar w,z,\bar z) &\rightsquigarrow B_w(w,\bar w)\,, \qquad & B'_{\bar w}(w,\bar w,z,\bar z) &\rightsquigarrow B_{\bar w}(w,\bar w)\,,\\ B'_z(w,\bar w,z,\bar z) &\rightsquigarrow \bar\Phi(w,\bar w)\,, & B'_{\bar z}(w,\bar w,z,\bar z) &\rightsquigarrow \Phi(w,\bar w)\,, \end{aligned} \tag{4.1}$$

where Φ and $\bar\Phi$ will be adjoint scalars in the lower-dimensional theory (sometimes called Higgs fields in the literature). These will enter explicitly in the lower-dimensional theory through the reduction of covariant derivatives

$$\nabla_z g g^{-1} \rightsquigarrow \bar\Phi - g\bar\Phi g^{-1}\,, \qquad \nabla_{\bar z} g g^{-1} \rightsquigarrow \Phi - g\Phi g^{-1}\,. \tag{4.2}$$

On-shell, the 4d gauge field B' is ASD and couples to matter in the gWZW$_4$ model. It is well-known that the reduction of an ASDYM connection leads to the Hitchin system, and we will see this feature in the lower-dimensional dynamics below.

The two-dimensional Lagrangian that arises from reducing eq. (3.54) without first constraining is[17]

$$\begin{aligned} L_{\mathrm{IFT}_2} = &\frac{1}{2}\mathrm{Tr}(g^{-1}D_w g g^{-1} D_{\bar w} g) + \frac{1}{2}\frac{\alpha+\beta}{\alpha-\beta} L_{\mathrm{gWZ}} + \mathrm{Tr}\Big(\Phi\bar\Phi + \frac{\alpha}{\alpha-\beta}\Phi\mathrm{Ad}_g\bar\Phi - \frac{\beta}{\alpha-\beta}\Phi\mathrm{Ad}_g^{-1}\bar\Phi\Big)\\ &+\frac{1}{\alpha-\beta}\mathrm{Tr}(\Phi(g^{-1}D_{\bar w}g + D_{\bar w}g g^{-1}) + \alpha\beta\bar\Phi(g^{-1}D_w g + D_w g g^{-1}))\\ &+\mathrm{Tr}(\tilde u(F_{\bar w w} - \beta^{-1}D_{\bar w}\Phi - \beta D_w\bar\Phi - [\bar\Phi,\Phi])) + \mathrm{Tr}(u(F_{\bar w w} - \alpha^{-1}D_{\bar w}\Phi - \alpha D_w\bar\Phi - [\bar\Phi,\Phi]))\,, \end{aligned} \tag{4.3}$$

where we denote the 2d covariant derivative as $D = \mathrm{d} + \mathrm{ad}_B$ and note that we have rescaled $\tilde u \to \frac{\tilde u}{\langle\beta\gamma\rangle\langle\beta\hat\gamma\rangle}$ and $u \to \frac{u}{\langle\alpha\gamma\rangle\langle\alpha\hat\gamma\rangle}$. The fields of the IFT$_2$ are $g \in G$ and $B_{w,\bar w}, \Phi, \bar\Phi, u, \tilde u \in \mathfrak{h}$. In addition to the overall coupling, the IFT$_2$ (4.3) only depends on a single parameter. This can be seen by introducing[18]

$$\hbar = \frac{\alpha+\beta}{\alpha-\beta}\,, \qquad \hbar' = -\frac{2\sqrt{\alpha\beta}}{\alpha-\beta}\,, \qquad \hbar^2 - \hbar'^2 = 1\,, \tag{4.4}$$

rescaling $\Phi \to \sqrt{\alpha\beta}\,\Phi$ and $\bar\Phi \to \frac{1}{\sqrt{\alpha\beta}}\,\bar\Phi$, and defining $X^- = \hbar'^{-1}(u+\tilde u)$ and $\tilde X^+ = \hbar'^{-1}(u-\tilde u)$. The Lagrangian (4.3) can then be rewritten as

$$\begin{aligned} L_{\mathrm{IFT}_2} = &\frac{1}{2}\mathrm{Tr}(g^{-1}D_w g g^{-1}D_{\bar w}g) + \frac{\hbar}{2}L_{\mathrm{gWZ}} + \mathrm{Tr}(\Phi\mathcal{O}\bar\Phi + \Phi V_{\bar w} + \bar\Phi V_w)\\ &+\mathrm{Tr}(X^-(\hbar'(F_{\bar w w} - [\bar\Phi,\Phi]) + \hbar(D_w\bar\Phi + D_{\bar w}\Phi))) + \mathrm{Tr}(\tilde X^+(D_w\bar\Phi - D_{\bar w}\Phi))\,, \end{aligned} \tag{4.5}$$

[16] Note, we are dropping the prime on the 2d gauge field B.

[17] The 2d Lagrangians are defined as $S_{\mathrm{IFT}_2} = 2\mathrm{i}\int_{\mathbb{R}^2}\mathrm{d}w\wedge\mathrm{d}\bar w\, L_{\mathrm{IFT}_2}$. We have also introduced the scalar densities L_{WZ} and L_{gWZ} where $\int\mathrm{d}w\wedge\mathrm{d}\bar w\, L_{(\mathrm{g})\mathrm{WZ}}(g) = \int_{\mathbb{R}^2\times[0,1]}\mathscr{L}_{(\mathrm{g})\mathrm{WZ}}(\hat g)$ and the 3-forms $\mathscr{L}_{\mathrm{WZ}}$ and $\mathscr{L}_{\mathrm{gWZ}}$ are defined in eqs. (2.15) and (3.3) respectively. Explicitly, we have

$$L_{\mathrm{gWZ}} = L_{\mathrm{WZ}}(g) + \mathrm{Tr}\big((g^{-1}\partial_w g + \partial_w g g^{-1})B_{\bar w} - (g^{-1}\partial_{\bar w}g + \partial_{\bar w}g g^{-1})B_w + B_w\mathrm{Ad}_g B_{\bar w} - B_w\mathrm{Ad}_g^{-1}B_{\bar w}\big)\,,$$

which we include for convenience.

[18] Here, we have implicitly assumed that $\alpha\beta \geq 0$, which implies that $|\hbar| \geq 1$. The other regime of interest, $\alpha\beta \leq 0$ and $|\hbar| \leq 1$ is related by the analytic continuation $\hbar' \to -\mathrm{i}\hbar'$.

where

$$\mathcal{O} = 1 - \frac{\cancel{k}+1}{2}\,\mathrm{Ad}_g + \frac{\cancel{k}-1}{2}\,\mathrm{Ad}_g^{-1}\,, \qquad V_{w,\bar{w}} = -\frac{\cancel{k}'}{2}\,(g^{-1}D_{w,\bar{w}}g + D_{w,\bar{w}}gg^{-1})\,. \tag{4.6}$$

Note that the CFT points $\cancel{k}=1$ or $\cancel{k}=-1$ correspond to taking $\gamma \to \beta$ or $\gamma \to \alpha$, i.e. when the zeroes of the twist function coincide with the poles.

By construction, as the reduction of gWZW_4, the equations of motion of this theory are expected to be equivalent to the zero curvature of Lax connections, whose components are given by the $\mathrm{d}w$ and $\mathrm{d}\bar{w}$ legs of the 4d gauge fields. Explicitly, these Lax connections are given by

$$\begin{aligned} \mathscr{L}_w^{(A)} &= \partial_w + B_w - \frac{\cancel{k}+1}{2}\,K_w - \frac{1}{\zeta}\left(\Phi + \frac{\cancel{k}'}{2}\,K_w\right)\,, \\ \mathscr{L}_{\bar{w}}^{(A)} &= \partial_{\bar{w}} + B_{\bar{w}} + \frac{\cancel{k}-1}{2}\,K_{\bar{w}} + \zeta\left(\bar{\Phi} + \frac{\cancel{k}'}{2}\,K_{\bar{w}}\right)\,, \end{aligned} \tag{4.7}$$

$$\mathscr{L}_w^{(B)} = \partial_w + B_w - \frac{1}{\zeta}\,\Phi\,, \qquad \mathscr{L}_{\bar{w}}^{(B)} = \partial_{\bar{w}} + B_{\bar{w}} + \zeta\bar{\Phi}\,, \tag{4.8}$$

where we have also redefined the spectral parameter $\zeta \to \sqrt{\alpha\beta}\,\zeta$ compared to section 2.4 and introduced the currents

$$K_w = D_w g g^{-1} + \frac{\cancel{k}-1}{\cancel{k}'}\,(1-\mathrm{Ad}_g)\Phi\,, \qquad K_{\bar{w}} = D_{\bar{w}} g g^{-1} - \frac{\cancel{k}+1}{\cancel{k}'}\,(1-\mathrm{Ad}_g)\bar{\Phi}\,. \tag{4.9}$$

It is natural to ask whether first reducing and then constraining leads to a consistent truncation since we are dropping certain parts of the 4d constraints. In the language of Kaluza-Klein compactifications these would be the higher-mode constraints. Since every term in a higher-mode constraint will depend on the higher modes of some field, it follows that setting all the higher modes to zero is expected to be a consistent truncation. Another viewpoint is that of symmetry reduction; since the action is invariant under shifts of z and $\bar{z}$ we can consistently set all fields to be independent of them. If the truncation is consistent then we expect the resulting 2d theory to be integrable, which we now check explicitly.

4.1 Lax formulation

Before analysing the Lagrangian (4.5) in more detail, let us show explicitly that its equations of motion are indeed equivalent to the zero-curvature condition for the Lax connections (4.7) and (4.8). The equations of motion that follow from the Lagrangian (4.5) by varying $\tilde{X}^+$, X^- and g are

$$\begin{aligned} \delta\tilde{X}^+ :\quad & \mathscr{E}_+ \equiv D_w\bar{\Phi} - D_{\bar{w}}\Phi = 0\,, \\ \delta X^- :\quad & \mathscr{E}_- \equiv \cancel{k}'(F_{\bar{w}w} - [\bar{\Phi},\Phi]) + \cancel{k}(D_w\bar{\Phi} + D_{\bar{w}}\Phi) = 0\,, \\ \delta g g^{-1} :\quad & \mathscr{E}_g \equiv \frac{\cancel{k}-1}{2}\left(D_w K_{\bar{w}} + \frac{\cancel{k}+1}{\cancel{k}'}\,[\bar{\Phi},K_w]\right) - \frac{\cancel{k}+1}{2}\left(D_{\bar{w}}K_w - \frac{\cancel{k}-1}{\cancel{k}'}\,[\Phi,K_{\bar{w}}]\right) \\ & \qquad + \frac{\cancel{k}}{\cancel{k}'}\,\mathscr{E}_- - \frac{1}{\cancel{k}'}\,(D_w\bar{\Phi} + D_{\bar{w}}\Phi) = 0\,. \end{aligned} \tag{4.10}$$

We also have the Bianchi identity following from the zero-curvature of the Maurer-Cartan form $\mathrm{d}gg^{-1}$

$$\mathscr{Z} \equiv D_w K_{\bar{w}} + \frac{\cancel{k}+1}{\cancel{k}'}[\bar{\Phi}, K_w] - D_{\bar{w}} K_w + \frac{\cancel{k}-1}{\cancel{k}'}[\Phi, K_{\bar{w}}] + [K_{\bar{w}}, K_w] + \frac{1}{\cancel{k}'}(1-\mathrm{Ad}_g)(\mathscr{E}_- + \mathscr{E}_+) = 0\,. \tag{4.11}$$

The zero curvature of the A-Lax eq. (4.7) gives rise to three equations that are linear combinations of the equations of motion eq. (4.10) and the Bianchi identity eq. (4.11):

$$\begin{aligned} 0 &= \frac{\cancel{k}-1}{2}\mathscr{Z}' - \mathscr{E}_g + \frac{\cancel{k}}{\cancel{k}'}\mathscr{E}_- - \frac{1}{\cancel{k}'}\mathscr{E}_+\,,\\ 0 &= \cancel{k}'^2\mathscr{Z}' - 2\cancel{k}\mathscr{E}_g + 2\cancel{k}'\mathscr{E}_-\,,\\ 0 &= \frac{\cancel{k}+1}{2}\mathscr{Z}' - \mathscr{E}_g + \frac{\cancel{k}}{\cancel{k}'}\mathscr{E}_- + \frac{1}{\cancel{k}'}\mathscr{E}_+\,, \end{aligned} \tag{4.12}$$

where we have defined $\mathscr{Z}' \equiv \mathscr{Z} - \frac{1}{\cancel{k}'}(1-\mathrm{Ad}_g)(\mathscr{E}_- + \mathscr{E}_+)$. On the other hand, the zero curvature of the B-Lax (4.8) defines the Hitchin system:

$$0 = D_{\bar{w}}\Phi\,, \qquad 0 = F_{\bar{w}w} - [\bar{\Phi}, \Phi]\,, \qquad 0 = D_w\bar{\Phi}\,, \tag{4.13}$$

which can be rewritten as the three equations $\mathscr{E}_\pm = 0$ and $\mathscr{E}_0 \equiv D_w\bar{\Phi} + D_{\bar{w}}\Phi = 0$. Therefore, the two Lax connections give rise to five independent equations, which are linear combinations of the equations of motion (4.10), the Bianchi identity (4.11), and the additional equation $\mathscr{E}_0 = 0$.

To recover this final equation from the equations of motion, let us consider the variational equations for $B_w, B_{\bar{w}}, \bar{\Phi}, \Phi$:

$$\begin{aligned} \delta B_w:&\quad \mathscr{E}_B \equiv \cancel{k}' D_{\bar{w}} X^- - [\bar{\Phi}, \tilde{X}^+ + \cancel{k}X^-] + \frac{\cancel{k}-1}{2}P_{\mathfrak{h}}K_{\bar{w}} + \frac{\cancel{k}+1}{2}P_{\mathfrak{h}}\mathrm{Ad}_g^{-1}K_{\bar{w}} - \frac{\cancel{k}+1}{\cancel{k}'}P_{\mathfrak{h}}(1-\mathrm{Ad}_g^{-1})\bar{\Phi} = 0,\\ \delta B_{\bar{w}}:&\quad \mathscr{E}_{\bar{B}} \equiv \cancel{k}' D_w X^- - [\Phi, \tilde{X}^+ - \cancel{k}X^-] + \frac{\cancel{k}+1}{2}P_{\mathfrak{h}}K_w + \frac{\cancel{k}-1}{2}P_{\mathfrak{h}}\mathrm{Ad}_g^{-1}K_w - \frac{\cancel{k}-1}{\cancel{k}'}P_{\mathfrak{h}}(1-\mathrm{Ad}_g^{-1})\Phi = 0,\\ \delta\Phi:&\quad \mathscr{E}_\Phi \equiv D_{\bar{w}}(\tilde{X}^+ - \cancel{k}X^-) + \cancel{k}'[\bar{\Phi}, X^-] - \frac{\cancel{k}'}{2}P_{\mathfrak{h}}(1+\mathrm{Ad}_g^{-1})K_{\bar{w}} + P_{\mathfrak{h}}(1-\mathrm{Ad}_g^{-1})\bar{\Phi} = 0,\\ \delta\bar{\Phi}:&\quad \mathscr{E}_{\bar{\Phi}} \equiv D_w(\tilde{X}^+ + \cancel{k}X^-) + \cancel{k}'[\Phi, X^-] + \frac{\cancel{k}'}{2}P_{\mathfrak{h}}(1+\mathrm{Ad}_g^{-1})K_w - P_{\mathfrak{h}}(1-\mathrm{Ad}_g^{-1})\Phi = 0. \end{aligned} \tag{4.14}$$

These can be understood as a first-order system of equations for $\tilde{X}^+$ and X^-. Consistency of the system implies that they should satisfy the integrability conditions $[D_{\bar{w}}, D_w]\tilde{X}^+ = [F_{\bar{w}w}, \tilde{X}^+]$ and $[D_{\bar{w}}, D_w]X^- = [F_{\bar{w}w}, X^-]$. We find that

$$\cancel{k}'[D_{\bar{w}}, D_w]X^- - \cancel{k}'[F_{\bar{w}w}, X^-] = [X^+, \mathscr{E}_+] + [X^-, \mathscr{E}_-] + P_{\mathfrak{h}}(1-\mathrm{Ad}_g^{-1})\mathscr{E}_g + \cancel{k}P_{\mathfrak{h}}\mathrm{Ad}_g^{-1}\mathscr{Z}\,, \tag{4.15}$$

hence, using the Bianchi identity (4.11), this vanishes on the equations of motion for $\tilde{X}^+$, X^- and g (4.10). On the other hand, we have

$$\cancel{k}'[D_{\bar{w}}, D_w]\tilde{X}^+ - \cancel{k}'[F_{\bar{w}w}, \tilde{X}^+] = [X^+, \mathscr{E}_-] + [X^-, \mathscr{E}_+] + \frac{2\cancel{k}}{\cancel{k}'}\mathscr{E}_- - \frac{2}{\cancel{k}'}\mathscr{E}_0 - P_{\mathfrak{h}}(1+\mathrm{Ad}_g^{-1})\mathscr{E}_g + \cancel{k}P_{\mathfrak{h}}\mathrm{Ad}_g^{-1}\mathscr{Z}\,. \tag{4.16}$$

Here, we see that in addition to the Bianchi identity (4.11) and equations of motion (4.10), we also require $\mathscr{E}_0 = 0$, recovering the final equation of the Lax system.

4.2 Relation to known models

As we will shortly see, if we take H to be abelian, the Lagrangian (4.5) can be related to known models, including the homogeneous sine-Gordon models and the PCM plus WZ term. However, for non-abelian H this model has not been considered before, and defines a new integrable field theory in two dimensions. Moreover, by integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$, it leads to an integrable sigma model for the fields g, $\tilde{X}^+$ and X^-. We leave the study of these models for future work.

To recover a sigma model from the Lagrangian (4.5) for abelian H, after integrating out $B_w, B_{\bar{w}}$, we have two options. The first is to integrate out $\Phi, \bar{\Phi}$. The second is to solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$. For abelian H the Lagrangian (4.5) simplifies to

$$\begin{aligned} L^{\text{ab}}_{\text{IFT}_2} = \frac{1}{2}\operatorname{Tr}(g^{-1}D_w g g^{-1}D_{\bar{w}}g) + \frac{\hbar}{2}L_{\text{gWZ}} + \operatorname{Tr}(\Phi\mathcal{O}\bar{\Phi} + \Phi V_{\bar{w}} + \bar{\Phi}V_w) \\ + \operatorname{Tr}\big((X^-(\hbar' F_{\bar{w}w} + \hbar(\partial_w\bar{\Phi} + \partial_{\bar{w}}\Phi))\big) + \operatorname{Tr}\big(\tilde{X}^+(\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi)\big)\,. \end{aligned} \tag{4.17}$$

This takes the form of a first-order action in the Büscher procedure, and it follows that the two sigma models will be T-dual to each other with dual fields X^+ and $\tilde{X}^+$. Explicitly the Lagrangians, before integrating out $B_w, B_{\bar{w}}$, are

$$\begin{aligned} L^{\tilde{X}}_{\text{IFT}_2} = \frac{1}{2}\operatorname{Tr}(g^{-1}D_w g g^{-1}D_{\bar{w}}g) + \frac{\hbar}{2}L_{\text{gWZ}} + \hbar'\operatorname{Tr}(X^- F_{\bar{w}w}) \\ + \operatorname{Tr}\big((\partial_w\tilde{X}^+ - V_w + \hbar\partial_w X^-)\mathcal{O}^{-1}(\partial_{\bar{w}}\tilde{X}^+ + V_{\bar{w}} - \hbar\partial_{\bar{w}}X^-)\big)\,, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} L^{X}_{\text{IFT}_2} = \frac{1}{2}\operatorname{Tr}(g^{-1}D_w g g^{-1}D_{\bar{w}}g) + \frac{\hbar}{2}L_{\text{gWZ}} + \hbar'\operatorname{Tr}(X^- F_{\bar{w}w}) \\ + \frac{1}{4}\operatorname{Tr}\big(\partial_w X^+\mathcal{O}\partial_{\bar{w}}X^+ + 2\partial_w X^+(V_{\bar{w}} - \hbar\partial_{\bar{w}}X^-) + 2\partial_{\bar{w}}X^+(V_w - \hbar\partial_w X^-)\big)\,, \end{aligned} \tag{4.19}$$

where in the second Lagrangian we have locally solved the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ by setting

$$\Phi = \frac{1}{2}\partial_w X^+\,, \qquad \bar{\Phi} = \frac{1}{2}\partial_{\bar{w}}X^+\,, \qquad X^+ \in \mathfrak{h}\,. \tag{4.20}$$

As mentioned above, the first approach can also be straightforwardly applied for non-abelian H. Generalising the second approach is more subtle. The constraint imposed by the Lagrange multiplier $\tilde{X}^+$ in the Lagrangian (4.5) implies that

$$D_w\bar{\Phi} - D_{\bar{w}}\Phi = 0\,. \tag{4.21}$$

Typically the full solution to this equation would be expressed in terms of path-ordered exponentials of B_w and $B_{\bar{w}}$. To avoid non-local expressions, we can restrict Φ and $\bar{\Phi}$ to be valued in the centre of $\mathfrak{h}$, denoted $\mathcal{Z}(\mathfrak{h})$. Note that this is not a restriction if H is abelian. With this restriction, the Lagrangian (4.5) again simplifies to (4.17), and the constraint (4.21) becomes $\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi = 0$, which we can again locally solve by (4.20) now with $X^+ \in \mathcal{Z}(\mathfrak{h})$, similarly leading to the Lagrangian (4.19).

JHEP12(2024)202

Relation to PCM plus WZ term. Taking H to be abelian, we can relate the Lagrangian (4.17) to that of the PCM plus WZ term for $G \times H$ through a combination of T-dualities and field redefinitions. We start by parametrising

$$g = e^{\frac{1}{2}\tau} g e^{\frac{1}{2}\tau}, \qquad \tau \in \mathfrak{h}, \tag{4.22}$$

and setting $\partial_{w,\bar{w}}\tau \to 2C_{w,\bar{w}}$. We also integrate by parts and set $\partial_w X^- \to 2\Psi$ and $\partial_{\bar{w}} X^- \to 2\bar{\Psi}$. To maintain equivalence with the Lagrangian we started with, we add $\mathrm{Tr}\big(\tilde{\tau}(\partial_w C_{\bar{w}} - \partial_{\bar{w}} C_w)\big) + \mathrm{Tr}\big(\tilde{X}^-(\partial_w \bar{\Psi} - \partial_{\bar{w}}\Psi)\big)$, i.e. the Lagrange multipliers $\tilde{\tau}$ and $\tilde{X}^-$ locally impose $C_{w,\bar{w}} = \frac{1}{2}\partial_{\bar{w},w}\tau$, $\Psi = \frac{1}{2}\partial_w X^-$ and $\bar{\Psi} = \frac{1}{2}\partial_{\bar{w}} X^-$. We can then redefine the fields as[19]

$$\begin{aligned}
&B_w \to B_w - \frac{\hbar}{\hbar'}\Phi\,, && C_w \to C_w - \frac{1}{\hbar'}\Phi\,, && \Psi \to \Psi + \frac{\hbar}{\hbar'^2}\Phi\,, \\
&B_{\bar{w}} \to B_{\bar{w}} + \frac{\hbar}{\hbar'}\bar{\Phi}\,, && C_{\bar{w}} \to C_{\bar{w}} - \frac{1}{\hbar'}\bar{\Phi}\,, && \bar{\Psi} \to \bar{\Psi} + \frac{\hbar}{\hbar'^2}\bar{\Phi}\,, \\
&\tilde{X}^+ \to \frac{1}{\hbar'}\tilde{X}^+ - \frac{\hbar}{\hbar'}\tilde{X}^- + \frac{1}{\hbar'}\tilde{\tau}\,, && \tilde{X}^- \to \hbar'\tilde{X}^-\,, && \tilde{\tau} \to \tilde{\tau}\,.
\end{aligned} \tag{4.23}$$

Doing so, we arrive at the following Lagrangian

$$\begin{aligned}
L^{\mathrm{ab}}_{\mathrm{IFT}_2} = {}& \frac{1}{2}\mathrm{Tr}\big(g^{-1}\partial_w g g^{-1}\partial_{\bar{w}} g\big) + \frac{\hbar}{2} L_{\mathrm{WZ}}(g) \\
&+ \frac{1-\hbar}{2}\mathrm{Tr}\big(g^{-1}\partial_w g(C_{\bar{w}} - B_{\bar{w}}) + \partial_{\bar{w}} g g^{-1}(C_w + B_w) + (C_w + B_w)\mathrm{Ad}_g(C_{\bar{w}} - B_{\bar{w}})\big) \\
&+ \frac{1+\hbar}{2}\mathrm{Tr}\big(g^{-1}\partial_{\bar{w}} g(C_w - B_w) + \partial_w g g^{-1}(C_{\bar{w}} + B_{\bar{w}}) + (C_w - B_w)\mathrm{Ad}_g^{-1}(C_{\bar{w}} + B_{\bar{w}})\big) \\
&+ \mathrm{Tr}\big(B_w B_{\bar{w}} + C_w C_{\bar{w}} + \hbar C_w B_{\bar{w}} - \hbar B_w C_{\bar{w}}\big) \\
&+ \mathrm{Tr}\big(\tilde{\tau}(\partial_w C_{\bar{w}} - \partial_{\bar{w}} C_w)\big) + \hbar'\mathrm{Tr}\big(\tilde{X}^-(\partial_w\bar{\Psi} - \partial_{\bar{w}}\Psi)\big) + 2\hbar'\mathrm{Tr}\big(\Psi B_{\bar{w}} - B_w\bar{\Psi}\big) \\
&+ \frac{1}{\hbar'}\mathrm{Tr}\big(\tilde{X}^+(\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi)\big) - \frac{2}{\hbar'^2}\mathrm{Tr}\big(\Phi\bar{\Phi}\big)\,.
\end{aligned} \tag{4.24}$$

The final steps are to integrate out $\tilde{\tau}, \Psi, \bar{\Psi}$, and $\Phi, \bar{\Phi}$, leading us to set

$$C_{w,\bar{w}} = \frac{1}{2}\partial_{w,\bar{w}}\tau\,, \qquad B_{w,\bar{w}} = -\frac{1}{2}\partial_{w,\bar{w}}\tilde{X}^-\,, \qquad \Phi = -\frac{\hbar'}{2}\partial_w\tilde{X}^+\,, \quad \bar{\Phi} = \frac{\hbar'}{2}\partial_{\bar{w}}\tilde{X}^+\,. \tag{4.25}$$

Redefining $g \to e^{-\frac{1}{2}(\tau+\tilde{X}^-)} g e^{-\frac{1}{2}(\tau-\tilde{X}^-)}$, we find the difference of PCM plus WZ term Lagrangians for G and H

$$L_{\mathrm{PCM}+\hbar\mathrm{WZ}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}\partial_w g g^{-1}\partial_{\bar{w}} g\big) + \frac{\hbar}{2}L_{\mathrm{WZ}}(g) - \frac{1}{2}\mathrm{Tr}\big(\partial_w\tilde{X}^+\partial_{\bar{w}}\tilde{X}^+\big)\,, \tag{4.26}$$

where we recall that for abelian H the WZ term vanishes.

[19] To arrive at this field redefinition, we first look for the shifts of $B_{w,\bar{w}}$, $C_{w,\bar{w}}$, Ψ and $\bar{\Psi}$ that decouple Φ and $\bar{\Phi}$ from all other fields apart from $\tilde{X}^+$. Since both C_w and $C_{\bar{w}}$ transform in the same way, as do Ψ and $\bar{\Psi}$, we can then easily compute the transformation of $\tilde{\tau}$, $\tilde{X}^-$ and $\tilde{X}^+$ by demanding that the triplet of terms $\mathrm{Tr}\big(\tilde{\tau}F_{w\bar{w}}(C) + \tilde{X}^- F_{w\bar{w}}(\Psi) + \tilde{X}^+ F_{w\bar{w}}(\Phi)\big)$ is invariant up to a simple rescaling, i.e. it becomes $\mathrm{Tr}\big(\tilde{\tau}F_{w\bar{w}}(C) + \hbar'\tilde{X}^- F_{w\bar{w}}(\Psi) + \frac{1}{\hbar'}\tilde{X}^+ F_{w\bar{w}}(\Phi)\big)$.

JHEP12(2024)202

To summarise, starting from the sigma model (4.19) we T-dualise in τ, X^+ and X^-, we then perform a GL(3) transformation on the dual coordinates, and finally T-dualise back in $\tilde{\tau}$ to recover (4.26), the difference of the PCM plus WZ term Lagrangians for G and H. This relation through dualities may have been anticipated since this is the model we would expect to find starting from the ghCS$_6$ action (3.14) and instead imposing the boundary conditions $\mathscr{A}|_{\alpha,\beta} = \mathscr{B}|_{\alpha,\beta} = 0$.

$\hbar \to 1$ limit. As we have seen, the $\hbar \to 1$ limit is special since if we first constrain and then reduce we recover the gauged WZW coset CFT. By first reducing and then constraining, we can recover massive integrable perturbations of these theories. We consider the setup where Φ and $\bar{\Phi}$ are restricted to lie in $\mathscr{Z}(\mathfrak{h})$ and solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ by (4.20). Taking $\hbar \to 1$ the Lagrangian (4.19) simplifies further to

$$\begin{aligned} L_{\mathrm{IFT}_2} = &\frac{1}{2}\operatorname{Tr}(g^{-1}D_w g g^{-1}D_{\bar{w}}g) + \frac{1}{2}L_{\mathrm{gWZ}} \\ &+ \frac{1}{4}\operatorname{Tr}\big(\partial_w X^+(1-\mathrm{Ad}_g)\partial_{\bar{w}}X^+ - 2\partial_w X^+\partial_{\bar{w}}X^- - 2\partial_{\bar{w}}X^+\partial_w X^-\big)\,, \end{aligned} \tag{4.27}$$

This is reminiscent of a sigma model for a pp-wave background, with the kinetic terms for the transverse fields described by the gauged WZW model for the coset G/H, except that the would-be light-cone coordinates X^+ and X^- have $\dim \mathscr{Z}(\mathfrak{h})$ components. Nevertheless, we still have the key property that the equation of motion for X^- is $\partial_w\partial_{\bar{w}}X^+ = 0$, whose general solution is $X^+ = Y(w) + \bar{Y}(\bar{w})$. Substituting into the Lagrangian (4.27) we find

$$L_{\mathrm{IFT}_2} = \frac{1}{2}\operatorname{Tr}(g^{-1}D_w g g^{-1}D_{\bar{w}}g) + \frac{1}{2}L_{\mathrm{gWZ}} + \frac{1}{4}\operatorname{Tr}\big(\partial_w Y\partial_{\bar{w}}\bar{Y} - \partial_w Y\mathrm{Ad}_g\bar{\partial}_{\bar{w}}\bar{Y}\big)\,. \tag{4.28}$$

In the special case that $Y = w\Lambda$ and $\bar{Y} = \bar{w}\bar{\Lambda}$, which is the most general solution preserving the translational invariance of the action, this is the gauged WZW model for the coset G/H perturbed by a massive integrable potential $V = \operatorname{Tr}(\Lambda\mathrm{Ad}_g\bar{\Lambda}) - \operatorname{Tr}(\Lambda\bar{\Lambda})$ as studied in [40]. Taking the limit $\hbar \to 1$ directly at the level of the Lax connection given in eq. (4.7), keeping track of the definitions of the currents $K_w, K_{\bar{w}}$, which depend on $\hbar$, we find

$$\mathscr{L}_w \to \partial_w + B_w - D_w g g^{-1} + \frac{1}{2\zeta}\Lambda\,, \qquad \mathscr{L}_{\bar{w}} = \partial_{\bar{w}} + B_{\bar{w}} - \frac{\zeta}{2}\mathrm{Ad}_g\bar{\Lambda}\,, \tag{4.29}$$

recovering the Lax given in [32, 40].

When G is compact and $H = \mathrm{U}(1)^{\mathrm{rk}_G}$, Λ and $\bar{\Lambda}$ can be chosen such that these models have a positive-definite kinetic term and a mass gap. These are known as the homogeneous sine-Gordon models [32]. For $G = \mathrm{SU}(2)$ and $H = \mathrm{U}(1)$ the homogeneous sine-Gordon model becomes the complex sine-Gordon model after integrating out $B_w, B_{\bar{w}}$. Note that if $\mathscr{Z}(\mathfrak{h})$ is one-dimensional and $Y(w)$ and $\bar{Y}(\bar{w})$ are both non-constant then we can always use the classical conformal symmetry of the sigma model to reach $Y = w\Lambda$ and $\bar{Y} = \bar{w}\Lambda$, hence recovering a constant potential. This is not the case for higher-dimensional $\mathscr{Z}(\mathfrak{h})$.

4.3 Example: $\mathrm{SL}(2)/\mathrm{U}(1)_V$

To illustrate the features of this construction, let us consider the example of $\mathrm{SL}(2)/\mathrm{U}(1)_V$ for which the 2d gauged WZW describes the trumpet CFT. To be explicit we use $\mathfrak{sl}(2)$ generators

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.30}$$

and parametrise the group element as

$$g = \begin{pmatrix} \cos(\theta)\sinh(\rho) + \cosh(\rho)\cos(\tau) & \sin(\theta)\sinh(\rho) + \cosh(\rho)\sin(\tau) \\ \sin(\theta)\sinh(\rho) - \cosh(\rho)\sin(\tau) & \cosh(\rho)\cos(\tau) - \cos(\theta)\sinh(\rho) \end{pmatrix} . \tag{4.31}$$

We choose the U(1) vector action generated by T_3 such that

$$\delta g = \epsilon[g, T_3] \quad \Rightarrow \quad \delta\rho = \delta\tau = 0 , \quad \delta\theta = \epsilon , \tag{4.32}$$

hence we gauge fix by setting $\theta = 0$. The analysis here is simplified by the observation that there is no WZ term since there are no 3-forms on the two-dimensional target space.

The CFT point. For orientation, we first work at the CFT point corresponding to $\hbar = 1$. Recall from the discussion in section 3 that first constraining in 4d and then reducing enforces $\bar{\Phi} = \Phi = 0$ and the Lagrange multiplier sector vanishes. This gives the conventional gauged WZW model described by the target space geometry

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 . \tag{4.33}$$

Let us now consider the IFT$_2$ that results from taking the same reduction that would lead to the CFT, but now in our reduction ansatz set $\Phi = \frac{m}{2}T_3$ and $\bar{\Phi} = -\frac{m}{2}T_3$. The Lagrangian that follows is

$$L_{\mathrm{CsG}} = \partial_w\rho\partial_{\bar{w}}\rho + \coth^2\rho\,\partial_w\tau\partial_{\bar{w}}\tau - m^2\sinh^2\rho . \tag{4.34}$$

This theory is well known as the complex sinh-Gordon model, a special case of the integrable massive perturbations of G/H gauged WZW models known as the homogeneous sine-Gordon models [32, 40].

Unconstrained reduction: integrating out Φ, $\bar{\Phi}$ and $B_{w,\bar{w}}$. We now turn to the more general story, away from the CFT point, by considering the reduction without first imposing constraints. Taking the IFT$_2$ (4.17) and integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$ while retaining X^- and $\tilde{X}^+$, results in the sigma model with target space metric and B-field

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 + \mathrm{csch}^2\rho\left(\mathrm{d}\tilde{X}^{+2} - \mathrm{d}X^{-2}\right) , \\ B_2 &= \mathscr{V}\wedge\mathrm{d}\tilde{X}^+ , \qquad \mathscr{V} = \hbar\,\mathrm{csch}^2\rho\,\mathrm{d}X^- + \hbar'\coth^2\rho\,\mathrm{d}\tau . \end{aligned} \tag{4.35}$$

Unconstrained reduction: the dual. On the other hand, if we solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ setting $\Phi = \frac{1}{2}\partial_w X^+$ and $\bar{\Phi} = \frac{1}{2}\partial_{\bar{w}}X^+$, we find the sigma model with target space geometry

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 - \mathrm{csch}^2\rho\,\mathrm{d}X^{-2} + \sinh^2\rho\,(\mathrm{d}X^+ + \mathscr{V})^2 , \\ B_2 &= 0 . \end{aligned} \tag{4.36}$$

This can of course be recognised as the T-dual of (4.35) along $\tilde{X}^+$. In the limit $\hbar \to 1$ (4.36) becomes the pp-wave background

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 + \sinh^2\rho\,\mathrm{d}X^{+2} + 2\mathrm{d}X^+\mathrm{d}X^- , \\ B_2 &= 0 , \end{aligned} \tag{4.37}$$

and if we light-cone gauge fix, $X^+ = m(w - \bar{w})$, in the associated sigma model we recover the complex sinh-Gordon Lagrangian (4.34) as expected.

Relation to PCM plus WZ term. Finally we demonstrate a relation between the models above and the PCM plus WZ term. Let us start with the metric and B-field for the PCM plus WZ term for $G = \mathrm{GL}(2)$

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\tilde{X}^{+2} + \mathrm{d}\rho^2 - \cosh^2\rho\,\mathrm{d}\tau^2 + \sinh^2\rho\,\mathrm{d}\tilde{X}^{-2}\,, \\
B &= \text{ꝁ} \cosh^2\rho\,\mathrm{d}\tau \wedge \mathrm{d}\tilde{X}^{-}\,.
\end{aligned}
\tag{4.38}
$$

Note that $\mathrm{d}B = \text{ꝁ}\sinh 2\rho\,\mathrm{d}\rho \wedge \mathrm{d}\tau \wedge \mathrm{d}\tilde{X}^{-}$, which is proportional to the volume for SL(2). We first T-dualise $\tau \to \tilde{\tau}$, and then perform the following field redefinition

$$
\tilde{X}^{+} \to \text{ꝁ}'\tilde{X}^{+} + \frac{\text{ꝁ}}{\text{ꝁ}'}\tilde{X}^{-} - \tilde{\tau}\,, \qquad \tilde{X}^{-} \to \frac{1}{\text{ꝁ}'}\tilde{X}^{-}\,.
\tag{4.39}
$$

It is straightforward to check that this is the inverse transformation to (4.23). Finally, T-dualising back, $\tilde{X}^{+} \to X^{+}$, $\tilde{X}^{-} \to X^{-}$ and $\tilde{\tau} \to \tau$, we precisely recover the background (4.36), demonstrating that it can be understood as a generalised TsT transformation of the PCM plus WZ term.

4.4 The LMP limit

The PCM plus WZ term admits a limit in which it becomes the 2d analogue of the LMP model, otherwise known as the pseudodual of the PCM [41], see, e.g. [42]. It is possible to generalise this limit to the gauged model (4.5) by setting $g = \exp(\varepsilon U)$, $\text{ꝁ} = \varepsilon^{-1}\ell$, $\tilde{X}^{+} \to \varepsilon^{2}\tilde{X}^{+}$, $X^{-} \to \varepsilon^{3}X^{-} - \varepsilon P_{\mathfrak{h}}U$, rescaling the Lagrangian by ε^{-2}, and taking $\varepsilon \to 0$. Implementing this limit in (4.5) we find

$$
\begin{aligned}
L^{\mathrm{LMP}}_{\mathrm{IFT}_2} = {}& \frac{1}{2}\mathrm{Tr}\big(D_w U D_{\bar{w}} U + [\Phi, U][\bar{\Phi}, U]\big) - \frac{\ell}{6}\mathrm{Tr}\big((D_w U + [\Phi, U][U, (D_{\bar{w}} U - [\bar{\Phi}, U]]\big) \\
&+ \ell\mathrm{Tr}\big(X^{-}(F_{\bar{w}w} - [\bar{\Phi}, \Phi] + D_w\bar{\Phi} + D_{\bar{w}}\Phi)\big) + \mathrm{Tr}\big((\tilde{X}^{+}(D_w\bar{\Phi} - D_{\bar{w}}\Phi)\big) \\
&+ \frac{1}{2\ell}\mathrm{Tr}\big(\mathrm{U}(F_{\bar{w}w} - [\bar{\Phi}, \Phi] - D_w\bar{\Phi} - D_{\bar{w}}\Phi)\big)\,.
\end{aligned}
\tag{4.40}
$$

Similarly we can take the limit in the Lax connections (4.7) and (4.8). The B-Lax (4.8) is unchanged, while the A-Lax (4.7) becomes

$$
\begin{aligned}
\mathscr{L}^{(A)}_{w} &= \partial_w + B_w - \frac{\ell}{2}K^{\mathrm{LMP}}_{w} - \frac{1}{\zeta}\big(\Phi + \frac{\ell}{2}K^{\mathrm{LMP}}_{w}\big)\,, \\
\mathscr{L}^{(A)}_{\bar{w}} &= \partial_{\bar{w}} + B_{\bar{w}} + \frac{\ell}{2}K^{\mathrm{LMP}}_{\bar{w}} + \zeta\big(\bar{\Phi} + \frac{\ell}{2}K^{\mathrm{LMP}}_{\bar{w}}\big)\,,
\end{aligned}
\tag{4.41}
$$

where

$$
K^{\mathrm{LMP}}_{w} = D_w U + [\Phi, U]\,, \qquad K^{\mathrm{LMP}}_{\bar{w}} = D_{\bar{w}} U - [\bar{\Phi}, U]\,.
\tag{4.42}
$$

As we will see in section 6 this model can also be found directly from gauged hCS_6 and CS_4 by considering a twist function with a single fourth-order pole.

As for the gauged WZW case, we can again find an integrable sigma model from (4.40) by integrating out $\Phi, \bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$. For abelian H we can also construct the dual model by solving the constraint imposed by the Lagrange multiplier $\tilde{X}^{+}$ and integrating out

JHEP12(2024)202

$B_w, B_{\bar{w}}$. For SL(2)/U(1)$_V$ the resulting backgrounds can be found by taking the LMP limit

$$\begin{aligned} &\rho \to \varepsilon\rho - \frac{1}{6}\varepsilon^3\rho\tau^2\,, \qquad \tau \to \varepsilon\tau - \frac{1}{3}\varepsilon^3\rho^2\tau\,, \qquad (\mathrm{d}s^2, B_2) \to \varepsilon^{-2}(\mathrm{d}s^2, B_2)\,, \qquad \hbar \to \varepsilon^{-1}\ell\,, \\ &X^- \to \varepsilon^3 X^- - \varepsilon\tau\,, \qquad \tilde{X}^+ \to \varepsilon^2\tilde{X}^+\,, \qquad X^+ \to X^+\,, \qquad \varepsilon \to 0\,, \end{aligned} \tag{4.43}$$

in eqs. (4.35) and (4.36). This limit breaks the manifest global symmetry given by shifts of the coordinate τ. This is in agreement with the fact that the Lagrangian (4.40) is not invariant under $U \to U + H_0$ ($H_0 \in \mathfrak{h}$), while its gauged WZW counterpart (4.5) is invariant under $g \to h_0 g h_0$ ($h_0 \in H$) for abelian H.

Curiously, we can actually take a simplified LMP limit

$$\begin{aligned} &\rho \to \varepsilon\rho\,, \qquad \tau \to \varepsilon\tau\,, \qquad (\mathrm{d}s^2, B_2) \to \varepsilon^{-2}(\mathrm{d}s^2, B_2)\,, \qquad \hbar \to \varepsilon^{-1}\ell\,, \\ &X^- \to \varepsilon^3 X^- - \varepsilon\tau\,, \qquad \tilde{X}^+ \to \varepsilon^2\tilde{X}^+\,, \qquad X^+ \to X^+\,, \qquad \varepsilon \to 0\,, \end{aligned} \tag{4.44}$$

in the backgrounds (4.35) and (4.36) that preserves this global symmetry. Taking this limit in eq. (4.35) we find

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \mathrm{d}\tau^2 + \frac{1}{\rho^2}\mathrm{d}\tilde{X}^{+2} + \frac{2}{\rho^2}\mathrm{d}X^-\mathrm{d}\tau\,, \\ B_2 &= \mathscr{V} \wedge \mathrm{d}\tilde{X}^+\,, \qquad \mathscr{V} = \frac{\ell}{\rho^2}\mathrm{d}X^- + \left(\ell - \frac{1}{2\ell\rho^2}\right)\mathrm{d}\tau\,, \end{aligned} \tag{4.45}$$

while the limit of eq. (4.36) is

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \mathrm{d}\tau^2 + \rho^2(\mathrm{d}\tilde{X}^+ + \mathscr{V})^2 + \frac{2}{\rho^2}\mathrm{d}X^-\mathrm{d}\tau\,, \\ B_2 &= 0\,. \end{aligned} \tag{4.46}$$

As for the gauged WZW case these two backgrounds can also be constructed as a generalised TsT transformation of the background for the LMP model on GL(2)

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\tilde{X}^{+2} + \mathrm{d}\rho^2 - \mathrm{d}\tau^2 + \rho^2\mathrm{d}\tilde{X}^{-2}\,, \\ B_2 &= \ell\rho^2 d\tau \wedge \mathrm{d}\tilde{X}^-\,. \end{aligned} \tag{4.47}$$

Explicitly, if we first T-dualise $\tau \to \tilde{\tau}$, then perform the following field redefinition

$$\tilde{X}^+ \to \ell\tilde{X}^+ + \frac{1}{2\ell^2}\tilde{X}^- - \tilde{\tau}\,, \qquad \tilde{X}^- \to \frac{1}{\ell}\tilde{X}^-\,, \qquad \tilde{\tau} \to \tilde{\tau} - \frac{1}{2\ell^2}\tilde{X}^-\,, \tag{4.48}$$

and finally T-dualise back,[20] $\tilde{X}^+ \to X^+$, $\tilde{X}^- \to X^-$ and $\tilde{\tau} \to \tau$, we recover the background (4.45).

5 Reduction to gCS$_4$ and localisation

Having discussed the right hand side of the diamond, we briefly describe the left hand side that follows from first reducing to obtain a gauged 4d Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{CP}^1$ and then integrating over $\mathbb{CP}^1$ to localise to a two-dimensional field theory on $\mathbb{R}^2$. We show that the resulting IFT$_2$ matches (4.3).

[20] Note that here the order of T-dualities matters. In particular, we cannot first T-dualise $\tilde{\tau}$ after the coordinate redefinition since it turns out to be a null coordinate.

We recall the six-dimensional coupled action

$$S_{\mathrm{ghCS}_6}[\mathscr{A},\mathscr{B}] = S_{\mathrm{hCS}_6}[\mathscr{A}] - S_{\mathrm{hCS}_6}[\mathscr{B}] - \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\,(\mathscr{A}\wedge\mathscr{B})\,, \tag{5.1}$$

and note that the three terms in the action are invariant under the transformations $\mathscr{A} \mapsto \hat{\mathscr{A}} = \mathscr{A} + \rho^{\mathscr{A}}_{\dot{a}} e^{\dot{a}} + \rho^{\mathscr{A}}_0 e^0$ and $\mathscr{B} \mapsto \hat{\mathscr{B}} = \mathscr{B} + \rho^{\mathscr{B}}_{\dot{a}} e^{\dot{a}} + \rho^{\mathscr{B}}_0 e^0$, given that both Ω and $\bar{\partial}\Omega$ are top forms in the holomorphic directions. By choosing $\rho^{\mathscr{A}}$ and $\rho^{\mathscr{B}}$ appropriately, we can ensure that neither $\hat{\mathscr{A}}$ nor $\hat{\mathscr{B}}$ have $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs, so

$$\hat{\mathscr{A}} = \hat{\mathscr{A}}_w \mathrm{d}w + \hat{\mathscr{A}}_{\bar{w}} \mathrm{d}\bar{w} + \mathscr{A}_0 \bar{e}^0 \quad \text{with} \quad \hat{\mathscr{A}}_w = -\frac{[\mathscr{A}\kappa]}{\langle \pi\gamma\rangle}\,,\ \hat{\mathscr{A}}_{\bar{w}} = -\frac{[\mathscr{A}\hat{\kappa}]}{\langle \pi\hat{\gamma}\rangle}\,, \tag{5.2}$$

$$\hat{\mathscr{B}} = \hat{\mathscr{B}}_w \mathrm{d}w + \hat{\mathscr{B}}_{\bar{w}} \mathrm{d}\bar{w} + \mathscr{B}_0 \bar{e}^0 \quad \text{with} \quad \hat{\mathscr{B}}_w = -\frac{[\mathscr{B}\kappa]}{\langle \pi\gamma\rangle}\,,\ \hat{\mathscr{B}}_{\bar{w}} = -\frac{[\mathscr{B}\hat{\kappa}]}{\langle \pi\hat{\gamma}\rangle}\,. \tag{5.3}$$

To perform the reduction we follow the procedure outlined in section 2.4. Namely, we contract the six-dimensional Lagrangian of (5.1) with the vector fields ∂_z and $\partial_{\bar{z}}$, and restrict to gauge connections that are invariant under the flow of these vector fields. Thus, since the shifted gauge fields $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ manifestly have no $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs, and we are restricting to field configurations satisfying $L_{\partial_z}\hat{\mathscr{A}} = L_{\partial_z}\hat{\mathscr{B}} = L_{\partial_{\bar{z}}}\hat{\mathscr{A}} = L_{\partial_{\bar{z}}}\hat{\mathscr{B}} = 0$, the contraction by ∂_z and $\partial_{\bar{z}}$ only hits Ω in the first two terms and $\bar{\partial}\Omega$ in the third. In particular, we find

$$(\partial_z \wedge \partial_{\bar{z}}) \vee \Omega = \frac{\langle\alpha\beta\rangle^2}{2} \frac{\langle\pi\gamma\rangle\langle\pi\hat{\gamma}\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}\, e^0\,, \qquad (\partial_z \wedge \partial_{\bar{z}}) \vee \bar{\partial}\Omega = -\frac{\langle\alpha\beta\rangle^2}{2}\, \bar{\partial}_0 \left(\frac{\langle\pi\gamma\rangle\langle\pi\hat{\gamma}\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2} \right) e^0 \wedge \bar{e}^0\,. \tag{5.4}$$

Hence the six-dimensional action reduces to a four-dimensional coupled Chern-Simons action

$$S_{\mathrm{gCS}_4}[\hat{A},\hat{B}] = \int_X \omega \wedge \mathrm{CS}[\hat{A}] - \int_X \omega \wedge \mathrm{CS}[\hat{B}] - \frac{1}{2\pi i}\int_X \bar{\partial}\omega \wedge \mathrm{Tr}(\hat{A}\hat{B})\,, \tag{5.5}$$

where $X = \mathbb{CP}^1 \times \mathbb{R}^2$,

$$\omega = \frac{\langle\alpha\beta\rangle^2}{2} \frac{\langle\pi\gamma\rangle\langle\pi\hat{\gamma}\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}\, e^0\,, \tag{5.6}$$

and $\hat{A}$ and $\hat{B}$ are the restrictions of $\hat{\mathscr{A}}$ and $\hat{\mathscr{B}}$ to X. Similarly, the boundary conditions (3.19) descend to analogous boundary conditions on $\hat{A}$ and $\hat{B}$. The action (5.5) has been considered before in [30], albeit not with the choice of ω discussed here.

With the gauged 4d Chern-Simons action at hand, we may now localise. The procedure is entirely analogous to the one described in section 3.4 so we shall omit some of the details. We begin by reparametrising our four-dimensional gauge fields $\hat{A}$ and $\hat{B}$ in terms of a new pair of connections $\hat{A}', \hat{B}'$ and smooth functions $\hat{g} \in C^\infty(X,G)$ and $\hat{h} \in C^\infty(X,H)$. We use the redundancy in the reparametrisation to fix $\hat{A}'_0 = \hat{B}'_0 = 0$. The boundary degrees of freedom of the resulting IFT_2 will a priori be given by the evaluation of $\hat{g}$, $\hat{h}$, $\hat{u}$ and $\hat{v}$ at α and β. However, as in the 6d setting, we have some residual symmetry we can use to fix $\hat{g}|_\beta = \mathrm{id}$, $\hat{h}|_{\alpha,\beta} = \mathrm{id}$, and similarly, $\hat{v}|_{\alpha,\beta} = 0$. We are thus left with

$$\hat{g}|_\alpha \coloneqq g\,, \quad \hat{u}|_\alpha \coloneqq u\,, \quad \hat{u}|_\beta = \tilde{u}\,. \tag{5.7}$$

In terms of these variables, the bulk equations of motion of gCS_4 imply

$$\bar{\partial}_0 \hat{A}'_i = 0\,, \qquad \bar{\partial}_0 \hat{B}'_i = 0\,, \tag{5.8}$$

away from the zeroes of ω, namely γ and $\hat{\gamma}$. The on-shell gCS_4 action can be thus written as

$$\begin{aligned} S_{\mathrm{gCS}_4}[\hat{A}', \hat{B}'] = \frac{1}{2\pi \mathrm{i}} \int_X \bar{\partial}\omega \wedge \mathrm{Tr}\big(\hat{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - (\hat{g}^{-1}\hat{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}) \wedge \hat{B}'\big) \\ - \frac{1}{6\pi \mathrm{i}} \int_{X\times[0,1]} \bar{\partial}\omega \wedge \mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g} \wedge \hat{g}^{-1}\mathrm{d}\hat{g})\,. \end{aligned} \tag{5.9}$$

To obtain the IFT_2 we begin by looking at the bulk equations of motion (5.8). Liouville's theorem shows that the only bounded, holomorphic functions on $\mathbb{CP}^1$ are constant functions. However, we are after something more general than this since we do not require the components of our gauge field to be bounded at the zeroes of ω. Indeed, we allow the w-component to have a pole at $\pi \sim \gamma$ and the $\bar{w}$-component to have a pole at $\pi \sim \hat{\gamma}$. With this analytic structure in mind, we can parametrise the solution of the bulk equation for $\hat{B}'$ as

$$\hat{B}'_w = B_w + \frac{\langle \pi\hat{\gamma}\rangle}{\langle \pi\gamma\rangle}\,\Phi\,, \qquad \hat{B}'_{\bar{w}} = B_{\bar{w}} - \frac{\langle \pi\gamma\rangle}{\langle \pi\hat{\gamma}\rangle}\,\bar{\Phi}\,, \tag{5.10}$$

where we have conveniently used the field variables introduced in (4.1) to ease comparison with (4.3) after localisation to the IFT_2. In particular, under π-independent gauge transformations $B_w, B_{\bar{w}}$ transforms as the components of a 2d gauge field, while Φ and $\bar{\Phi}$ transform as adjoint scalars.

Note that in the singular part of these solutions, we have chosen to align the zero of each with the pole of the other. This choice is completely general since moving the zeroes in the singular parts amounts to field redefinitions relating B_w and Φ or $B_{\bar{w}}$ and $\bar{\Phi}$. However, it is a convenient choice since the flatness condition on $\hat{B}'$ immediately reproduces Hitchin's equations,

$$F_{w\bar{w}}[\hat{B}'] = F_{w\bar{w}}[B] - [\Phi, \bar{\Phi}] - \frac{\langle \pi\gamma\rangle}{\langle \pi\hat{\gamma}\rangle}\,D_w\bar{\Phi} - \frac{\langle \pi\hat{\gamma}\rangle}{\langle \pi\gamma\rangle}\,D_w\Phi\,. \tag{5.11}$$

On the other hand, for the $\hat{A}'$ gauge field a convenient choice of parametrisation when solving the bulk equation of motion (5.8) is

$$\hat{A}'_i = \frac{\langle \pi\alpha\rangle}{\langle \pi\gamma\rangle}\,\frac{\langle \beta\gamma\rangle}{\langle \beta\alpha\rangle}\,U_i + \frac{\langle \pi\beta\rangle}{\langle \pi\gamma\rangle}\,\frac{\langle \alpha\gamma\rangle}{\langle \alpha\beta\rangle}\,V_i\,, \quad i = w, \bar{w}\,. \tag{5.12}$$

This parametrisation, in which we have chosen the coefficients such that one term vanishes at $\pi \sim \alpha$ while the other vanishes at $\pi \sim \beta$, is adapted to the boundary conditions, which can be solved for U_i and V_i to yield

$$\begin{aligned} \hat{A}'_w &= \hat{B}'_w - \frac{\langle \pi\beta\rangle}{\langle \pi\gamma\rangle}\,\frac{\langle \alpha\gamma\rangle}{\langle \alpha\beta\rangle}\left(D_w g g^{-1} + \frac{\langle \alpha\hat{\gamma}\rangle}{\langle \alpha\gamma\rangle}\,(1 - \mathrm{Ad}_g)\Phi\right), \\ \hat{A}'_{\bar{w}} &= \hat{B}'_{\bar{w}} - \frac{\langle \pi\beta\rangle}{\langle \pi\hat{\gamma}\rangle}\,\frac{\langle \alpha\hat{\gamma}\rangle}{\langle \alpha\beta\rangle}\left(D_{\bar{w}} g g^{-1} - \frac{\langle \alpha\gamma\rangle}{\langle \alpha\hat{\gamma}\rangle}\,(1 - \mathrm{Ad}_g)\bar{\Phi}\right). \end{aligned} \tag{5.13}$$

Replacing (5.10) and (5.13) in (5.9) and integrating[21] along $\mathbb{CP}^1$ we recover the IFT_2 given in (4.3).

[21]To do so, we use the localisation formula in homogeneous coordinates

$$\frac{1}{2\pi \mathrm{i}} \int_X \bar{\partial}\omega \wedge Q = -\frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{\langle \alpha\gamma\rangle\langle \beta\hat{\gamma}\rangle + \langle \alpha\hat{\gamma}\rangle\langle \beta\gamma\rangle}{\langle \alpha\beta\rangle}\,Q|_\alpha + \langle \alpha\gamma\rangle\langle \alpha\hat{\gamma}\rangle(\partial_0 Q)|_\alpha \right] \quad + \quad \alpha \leftrightarrow \beta\,,$$

for any $Q \in \Omega^2(X)$.

JHEP12(2024)202

6 Gauged LMP action

In the previous sections, we analysed the ghCS$_6$ action (3.14) where the meromorphic $(3,0)$-form Ω had two double poles, showing that such a theory leads to a gauged WZW$_4$ upon localisation to $\mathbb{R}^4$. To highlight some of the universal features of this procedure, we will now focus on another example in which the meromorphic $(3,0)$-form has a single fourth-order pole. For the ungauged hCS$_6$, such a configuration was shown in [43] to lead to the LMP action for ASDYM [8, 9].

6.1 LMP action from hCS$_6$

Let us first review the localisation of hCS$_6$ with a fourth-order pole. We start with the action and $(3,0)$-form defined by

$$S_{\text{hCS}_6}[\mathscr{A}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \text{CS}(\mathscr{A}) , \qquad \Omega = k\, \frac{e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}}}{\langle \pi\alpha \rangle^4} . \tag{6.1}$$

As is usual in hCS$_6$, we impose boundary conditions on the gauge field $\mathscr{A}$ to ensure the vanishing of the boundary variation

$$\delta S_{\text{hCS}_6}\big|_{\text{bdry}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar\partial\Omega \wedge \text{Tr}(\delta\mathscr{A} \wedge \mathscr{A}) . \tag{6.2}$$

Evaluating this integral is achieved by making use of the localisation formula (see appendix D)

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar\partial\Omega \wedge Q = \frac{k}{6} \int_{\mathbb{R}^4} \alpha_a \alpha_b \Sigma^{ab} \wedge \partial_0^3 Q\big|_\alpha . \tag{6.3}$$

Then, we find that the boundary variation vanishes if we impose the boundary conditions

$$\mathscr{A}|_{\pi=\alpha} = 0 \quad \text{and} \quad \partial_0 \mathscr{A}|_{\pi=\alpha} = 0 . \tag{6.4}$$

Admissible gauge transformations. We now check which residual gauge symmetries survive once we impose our choice of boundary conditions. We proceed in the familiar fashion, introducing a new parametrisation of our gauge field $\mathscr{A}$ as

$$\mathscr{A} = \hat{g}^{-1}\mathscr{A}'\hat{g} + \hat{g}^{-1}\bar\partial\hat{g} , \qquad \mathscr{A}'_0 = 0 . \tag{6.5}$$

This parametrisation has both external and internal gauge symmetries, which act as

$$\begin{aligned} &\text{External} \quad \hat\gamma : \qquad \mathscr{A} \mapsto \mathscr{A}^{\hat\gamma} , \qquad \mathscr{A}' \mapsto \mathscr{A}' , \qquad \hat{g} \mapsto \hat{g}\hat\gamma , \\ &\text{Internal} \quad \check\gamma : \qquad \mathscr{A} \mapsto \mathscr{A} , \qquad \mathscr{A}' \mapsto \mathscr{A}'^{\check\gamma} , \qquad \hat{g} \mapsto \check\gamma^{-1}\hat{g} . \end{aligned} \tag{6.6}$$

The internal gauge transformations must satisfy $\bar\partial_0\check\gamma = 0$ to preserve the condition $\mathscr{A}'_0 = 0$. These transformations leave $\mathscr{A}$ invariant and as such they are fully compatible with the boundary conditions. We use the internal gauge symmetry to fix $\hat{g}|_{\pi=\alpha} = \text{id}$. The story for the external gauge symmetries is slightly different; under external gauge transformations $\mathscr{A} \mapsto \mathscr{A}^{\hat\gamma}$ and so the value of $\mathscr{A}$ at the poles is not necessarily invariant. Requiring our boundary conditions to be invariant under external gauge transformations imposes constraints

JHEP12(2024)202

on the admissible symmetries at $\pi = \alpha$. This limits the amount of symmetry available for gauge fixing. The gauge transformation of the first boundary condition reads

$$0 = \mathscr{A}^{\hat{\gamma}}|_{\pi=\alpha} = \left(\hat{\gamma}^{-1}\mathscr{A}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\right)|_{\pi=\alpha} \quad \Longrightarrow \quad \gamma^{-1}\alpha^a\partial_{a\dot{a}}\gamma = 0\,, \tag{6.7}$$

where we have defined

$$\hat{\gamma}|_{\pi=\alpha} = \gamma\,.$$

Here, we have shown that at $\pi = \alpha$ the gauge transformations are restricted such that they are holomorphic on $\mathbb{R}^4$ with respect to the complex structure given by the point $\pi = \alpha$. Another way of stating this is that our admissible external gauge symmetries on $\mathbb{PT}$ localise to semi-local symmetries in the effective theory on $\mathbb{R}^4$. However, this restriction is derived from only one half of the boundary conditions. Introducing the notation

$$\hat{\mathbf{\Gamma}} := \hat{\gamma}^{-1}\partial_0\hat{\gamma}\,,$$

the gauge transformation of the second boundary condition reads

$$\begin{aligned} 0 = \partial_0\mathscr{A}^{\hat{\gamma}}|_{\pi=\alpha} &= \partial_0\left(\hat{\gamma}^{-1}\mathscr{A}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\right)|_{\pi=\alpha} \\ &= \left(\left[\hat{\gamma}^{-1}\mathscr{A}\,\hat{\gamma},\,\hat{\mathbf{\Gamma}}\right] + \hat{\gamma}^{-1}\partial_0\mathscr{A}\hat{\gamma} + \bar{\partial}\hat{\mathbf{\Gamma}} + \left[\hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}, \hat{\mathbf{\Gamma}}\right] + \hat{\gamma}^{-1}\partial_{\dot{a}}\hat{\gamma}\,\bar{e}^{\dot{a}}\right)|_{\pi=\alpha}\,. \end{aligned} \tag{6.8}$$

Imposing the original boundary conditions we arrive at the constraint equation

$$\alpha^a\partial_{a\dot{a}}\mathbf{\Gamma} + \gamma^{-1}\hat{\alpha}^a\partial_{a\dot{a}}\gamma = 0\,, \tag{6.9}$$

where we have used $\langle\alpha\hat{\alpha}\rangle = 1$ and defined

$$\hat{\mathbf{\Gamma}}|_{\pi=\alpha} = \mathbf{\Gamma}\,.$$

One solution is that the external gauge transformations are global symmetries of the localised effective theory $\mathrm{d}_{\mathbb{R}^4}\gamma = 0$, and $\mathbf{\Gamma}$ is holomorphic on $\mathbb{R}^4$ with respect to the choice of complex structure given by the point $\alpha \in \mathbb{CP}^1$.

Tentatively, our localised theory should have 4 degrees of freedom, known as 'edge modes'

$$\underline{\mathbf{u}} := (g, \mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3)\,, \tag{6.10}$$

where

$$g = \hat{g}|_{\pi=\alpha}\,, \quad \mathbf{u}^1 := \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\alpha}\,, \quad \mathbf{u}^2 := \hat{g}^{-1}\partial_0^2\hat{g}|_{\pi=\alpha}\,, \quad \mathbf{u}^3 := \hat{g}^{-1}\partial_0^3\hat{g}|_{\pi=\alpha}\,. \tag{6.11}$$

However, some of these fields are spurious and can be gauged fixed away using the admissible gauge symmetries. We have already used the internal gauge symmetry to fix $g = \mathrm{id}$. Furthermore, the second and third ∂_0-derivatives of the external gauge transformations are unconstrained by the boundary conditions, so they can be used to gauge fix $\mathbf{u}^2 = \mathbf{u}^3 = 0$. This leaves us with one dynamical degree of freedom in the localised theory on $\mathbb{R}^4$, namely $\mathbf{u}^1 : \mathbb{R}^4 \to \mathfrak{g}$, which we will now denote by $\mathbf{u}$ for brevity. In conclusion, after gauge fixing we have

$$\underline{\mathbf{u}} = (\mathrm{id}, \mathbf{u}, 0, 0)\,. \tag{6.12}$$

JHEP12(2024)202

Solving the boundary conditions. Using the boundary conditions, we will solve for $\mathscr{A}'$ in the parametrisation (6.5) in terms of the edge modes. The first boundary condition tells us

$$\mathscr{A}'|_{\pi=\alpha} = 0 \quad \Rightarrow \quad \alpha^a A_{a\dot{a}} = 0 \quad \Rightarrow \quad A_{a\dot{a}} = \alpha_a C_{\dot{a}}\,. \tag{6.13}$$

The second boundary condition equation is then written as

$$(\partial_0 \mathscr{A}' + \bar{\partial}(\hat{g}^{-1}\partial_0\hat{g}))|_{\pi=\alpha} = 0 \quad \Rightarrow \quad \frac{\hat{\alpha}^a}{\langle\alpha\hat{\alpha}\rangle} A_{a\dot{a}} + \alpha^a \partial_{a\dot{a}}\mathbf{u} = 0\,, \tag{6.14}$$

which allows us to conclude that

$$C_{\dot{\alpha}} = \alpha^a \partial_{\alpha\dot{\alpha}}\mathbf{u}\,. \tag{6.15}$$

We now have all the ingredients to localise the hCS_6 action to $\mathbb{R}^4$.

Localisation to $\mathbb{R}^4$. We can write the action (6.1) in the new variables as

$$S = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\mathscr{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}) - \frac{1}{6\pi \mathrm{i}}\int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\left((\hat{g}^{-1}\mathrm{d}\hat{g})^3\right)\,, \tag{6.16}$$

where in the second term we have extended $\mathbb{PT}$ to the 7-manifold $\mathbb{PT}\times[0,1]$, whose boundary is a disjoint union of two copies of $\mathbb{PT}$. We have also extended our fields via a smooth homotopy $\hat{g} \to \hat{g}(t)$ so that $\hat{g}(0) = \mathrm{id}$ and $\hat{g}(1) = \hat{g}$. Applying the localisation formula (6.3) and the choice of gauge fixing (6.12), we arrive at the spacetime action

$$S_{\mathrm{LMP}}[\mathbf{u}] = \frac{k}{3}\int_{\mathbb{R}^4} \frac{1}{2}\mathrm{Tr}(\mathrm{d}\mathbf{u} \wedge \star\mathrm{d}\mathbf{u}) + \frac{1}{3}\alpha_a\alpha_b\Sigma^{ab} \wedge \mathrm{Tr}(\mathbf{u}\,[\mathrm{d}\mathbf{u},\mathrm{d}\mathbf{u}])\,. \tag{6.17}$$

The action (6.17) is the LMP model for ASDYM, which upon reduction to $\mathbb{R}^2$ becomes the pseudo-dual of the PCM [44].

6.2 Gauged LMP action from ghCS_6

In the previous section, we derived the LMP action from hCS_6. Next, we consider the same fourth-order pole structure for gauged hCS_6. The starting point is to compute the boundary variation and choose boundary conditions to ensure that it vanishes.

Boundary conditions. Starting from the action

$$S_{\mathrm{ghCS}_6}[\mathscr{A},\mathscr{B}] = S_{\mathrm{hCS}_6}[\mathscr{A}] - S_{\mathrm{hCS}_6}[\mathscr{B}] - \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\mathscr{A}\wedge\mathscr{B})\,, \tag{6.18}$$

the boundary variation is given by

$$\delta S_{\mathrm{ghCS}_6}|_{\mathrm{bdry}} = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\delta\mathscr{A}\wedge(\mathscr{A}-\mathscr{B}) - \delta\mathscr{B}\wedge(\mathscr{B}-\mathscr{A}))\,. \tag{6.19}$$

Following in a parallel fashion to the hCS_6 case, we find that a suitable choice of boundary conditions is given by

$$\mathscr{A}|_{\pi=\alpha} = \mathscr{B}|_{\pi=\alpha}\,, \quad \partial_0\mathscr{A}|_{\pi=\alpha} = \partial_0\mathscr{B}|_{\pi=\alpha}\,, \quad \partial_0^2\mathscr{A}^{\mathfrak{h}}|_{\pi=\alpha} = \partial_0^2\mathscr{B}|_{\pi=\alpha}\,, \quad \partial_0^3\mathscr{A}^{\mathfrak{h}}|_{\pi=\alpha} = \partial_0^3\mathscr{B}|_{\pi=\alpha}\,. \tag{6.20}$$

Gauge fixing. Gauge fixing will once again prove helpful to proceed with the localisation calculation. As such, we will consider the set of admissible gauge transformations respecting our boundary conditions. Performing a gauge transformation on the first boundary condition gives

$$\left(\hat{\gamma}^{-1}\mathscr{A}\hat{\gamma}+\hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\right)|_{\pi=\alpha}=\left(\hat{\eta}^{-1}\mathscr{B}\hat{\eta}+\hat{\eta}^{-1}\bar{\partial}\hat{\eta}\right)|_{\pi=\alpha}\,, \tag{6.21}$$

from which we conclude that the admissible gauge transformations should obey $\hat{\gamma}|_{\alpha}=\hat{\eta}|_{\alpha}$. Running through systematically, the second boundary condition requires

$$\begin{aligned}&\left(\left[\hat{\gamma}^{-1}\mathscr{A}\,\hat{\gamma}+\hat{\gamma}^{-1}\bar{\partial}\hat{\gamma},\,\hat{\mathbf{\Gamma}}\right]+\hat{\gamma}^{-1}\partial_0\mathscr{A}\hat{\gamma}+\bar{\partial}\hat{\mathbf{\Gamma}}+\hat{\gamma}^{-1}\partial_{\dot{a}}\hat{\gamma}\,\bar{e}^{\dot{a}}\right)|_{\pi=\alpha}=\\&=\left(\left[\hat{\eta}^{-1}\mathscr{B}\,\hat{\eta}+\hat{\eta}^{-1}\bar{\partial}\hat{\eta},\,\hat{\mathbf{N}}\right]+\hat{\eta}^{-1}\partial_0\mathscr{B}\hat{\eta}+\bar{\partial}\hat{\mathbf{N}}+\hat{\eta}^{-1}\partial_{\dot{a}}\hat{\eta}\,\bar{e}^{\dot{a}}\right)|_{\pi=\alpha}\,,\end{aligned} \tag{6.22}$$

where we have denoted $\hat{\mathbf{\Gamma}}=\hat{\gamma}^{-1}\partial_0\hat{\gamma}$ and $\hat{\mathbf{N}}=\hat{\eta}^{-1}\partial_0\hat{\eta}$. Making use of the original boundary condition and the constraint $\hat{\gamma}|_{\alpha}=\hat{\eta}|_{\alpha}$, we conclude that admissible gauge transformations should also obey $\hat{\mathbf{\Gamma}}|_{\pi=\alpha}=\hat{\mathbf{N}}|_{\pi=\alpha}$. In a similar fashion, from the third boundary condition we conclude that $\hat{\mathbf{\Gamma}}^{(2)}_{\mathfrak{h}}|_{\alpha}=\hat{\mathbf{N}}^{(2)}|_{\alpha}$ where $\hat{\mathbf{\Gamma}}^{(2)}:=\hat{\gamma}^{-1}\partial_0^2\hat{\gamma}$ and $\hat{\mathbf{N}}^{(2)}:=\hat{\eta}^{-1}\partial_0^2\hat{\eta}$. Finally, from the fourth boundary condition we find $\hat{\mathbf{\Gamma}}^{(3)}_{\mathfrak{h}}|_{\alpha}=\hat{\mathbf{N}}^{(3)}|_{\alpha}$ where $\hat{\mathbf{\Gamma}}^{(3)}:=\hat{\gamma}^{-1}\partial_0^3\hat{\gamma}$ and $\hat{\mathbf{N}}^{(3)}:=\hat{\eta}^{-1}\partial_0^3\hat{\eta}$.

Now we know the admissible gauge symmetries of our theory, we can gauge fix the degrees of freedom. Initially, there are 8 degrees of freedom in our theory,

$$\begin{aligned}\underline{\mathbf{u}}&:=\left(g,\mathbf{u}^1,\mathbf{u}^2,\mathbf{u}^3\right)\,,\\ \underline{\mathbf{v}}&:=\left(h,\mathbf{v}^1,\mathbf{v}^2,\mathbf{v}^3\right)\,.\end{aligned} \tag{6.23}$$

We first consider the internal gauge symmetries of $\mathscr{A}$ and $\mathscr{B}$, which we can use to set both g and h to the identity. Next, we note that the H-valued external gauge transformations of $\mathscr{B}$ parametrised by $\hat{\eta}$ are unconstrained at the point $\pi=\alpha$. As such, we can gauge fix $\mathbf{v}^i=0$ for $i=1,2,3$. Now, since the external gauge transformations of $\mathscr{A}$ parametrised by $\hat{\gamma}$ are constrained to coincide with $\hat{\eta}$ at $\pi=\alpha$, and we have used these symmetries in our choice of gauge fixing, we find that we are unable to gauge fix $\mathbf{u}^i$. As such, each of these degrees of freedom will appear as fields in our effective theory on $\mathbb{R}^4$. In summary, renaming $\mathbf{u}^1$ as $\mathbf{u}$, after gauge fixing we have

$$\begin{aligned}\underline{\mathbf{u}}&=\left(\mathrm{id},\mathbf{u},\mathbf{u}^2,\mathbf{u}^3\right)\,,\\ \underline{\mathbf{v}}&=(\mathrm{id},0,0,0)\,.\end{aligned} \tag{6.24}$$

Solving the boundary conditions. The first boundary condition reads

$$\left(\hat{g}^{-1}\mathscr{A}'\hat{g}+\hat{g}^{-1}\bar{\partial}\hat{g}\right)|_{\pi=\alpha}=(\hat{h}^{-1}\mathscr{B}'\hat{h}+\hat{h}^{-1}\bar{\partial}\hat{h})|_{\pi=\alpha}\,. \tag{6.25}$$

Given our choice of gauge fixing (6.24) and the bulk solutions $\mathscr{A}'_{\dot{a}}=\pi^a A_{a\dot{a}}$ and $\mathscr{B}'_{\dot{a}}=\pi^a B_{a\dot{a}}$, this implies

$$\mathscr{A}'|_{\pi=\alpha}=\mathscr{B}'|_{\pi=\alpha}\quad\Rightarrow\quad\alpha^a A_{a\dot{a}}=\alpha^a B_{a\dot{a}}\quad\Rightarrow\quad A_{a\dot{a}}=B_{a\dot{a}}-\alpha_a Q_{\dot{a}}\,. \tag{6.26}$$

We can then use the second boundary condition to solve for $Q_{\dot{a}}$,

$$\partial_0\mathscr{A}|_{\pi=\alpha}=\partial_0\mathscr{B}|_{\pi=\alpha}\,,\qquad\Longrightarrow\qquad Q_{\dot{a}}=-\alpha^a\left([B_{a\dot{a}},\mathbf{u}]+\partial_{a\dot{a}}\mathbf{u}\right)=-\alpha^a\nabla_{a\dot{a}}\mathbf{u}\,. \tag{6.27}$$

JHEP12(2024)202

These two boundary conditions are sufficient to solve for $A_{a\dot{a}}$ in terms of the other degrees of freedom,

$$A_{a\dot{a}} = B_{a\dot{a}} + \alpha_a \alpha^b \nabla_{b\dot{a}} \mathbf{u} \,. \tag{6.28}$$

Localisation to $\mathbb{R}^4$. Writing the action (6.18) in terms of the new field variables, the only terms that contribute to the effective action given our choice of gauge (6.24) will be

$$S_{\text{ghCS}_6} = \frac{1}{2\pi i}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \text{Tr}(\mathscr{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - \hat{g}^{-1}\mathscr{A}'\hat{g}\wedge\mathscr{B}' - \hat{g}^{-1}\bar{\partial}\hat{g}\wedge\mathscr{B}') - \frac{1}{6\pi i}\int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \text{Tr}\left((\hat{g}^{-1}\mathrm{d}\hat{g})^3\right). \tag{6.29}$$

The localisation calculation of the gauged model is slightly more involved than the ungauged case due to the additional degrees of freedom appearing. However, in analogy with calculations in previous sections, we expect $\mathbf{u}^2$ and $\mathbf{u}^3$ to appear only as Lagrange multipliers, in particular, imposing self-duality type constraints for our gauge field B. With this in mind, we can show that the 4d theory is given by

$$\begin{aligned} S_{\text{gLMP}}[\mathbf{u},B] = k\int_{\mathbb{R}^4} \text{vol}_4 \frac{1}{2}\text{Tr}(\nabla^{a\dot{a}}\mathbf{u}\nabla_{a\dot{a}}\mathbf{u}) + \frac{1}{3}\epsilon^{\dot{a}\dot{b}}\text{Tr}\left(\mathbf{u}\left[\alpha^a\nabla_{a\dot{a}}\mathbf{u}, \alpha^b\nabla_{b\dot{b}}\mathbf{u}\right]\right) + \mathbf{u}\,\epsilon^{\dot{a}\dot{b}}\hat{\alpha}^a\hat{\alpha}^b F_{a\dot{a}b\dot{b}}(B) \\ + \frac{1}{2}\mathbf{u}^2\epsilon^{\dot{a}\dot{b}}\left(\alpha^a\hat{\alpha}^b + \hat{\alpha}^a\alpha^b\right)F_{a\dot{a}b\dot{b}}(B) + \tilde{\mathbf{u}}^3\epsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}(B), \end{aligned} \tag{6.30}$$

where we have performed a field redefinition $\mathbf{u}^3 \to \tilde{\mathbf{u}}^3 := \frac{1}{6}\left(\mathbf{u}^3 + 2\left[\mathbf{u}, \mathbf{u}^2\right]\right)$. Reducing along a particular $\mathbb{R}^2$, and appropriately redefining fields and parameters, we find that the gauged LMP action gives the IFT_2 (4.40).

Implementing the Lagrange multipliers. In section 2.3 we reviewed how solutions to the ASDYM can be formulated in terms of Yang's matrix after a partial gauge fixing of the ASD connection. We conclude this section by integrating out the Lagrange multiplier fields present in the action (6.30) by solving the self duality constraints they impose in a similar fashion. Indeed, the LMP equations of motion can be understood as the remaining ASDYM equation once these two constraints have been solved. This is analogous to the statement that the WZW_4 equations of motion are the remaining ASDYM equation for Yang's matrix.

The equation of motion found by varying $\tilde{\mathbf{u}}^3$ is an integrability condition along the 2-plane defined by α^a, and it can be solved by

$$\epsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}(B) = 0 \qquad \Longrightarrow \qquad \alpha^a B_{a\dot{a}} = h^{-1}\alpha^a\partial_{a\dot{a}}h\,, \tag{6.31}$$

where $h \in C^\infty(\mathbb{R}^4)\otimes H$. It is helpful to parametrise the remaining degrees of freedom in $B_{a\dot{a}}$ in terms of a new field $C_{\dot{a}}$, defined by the relation

$$B_{a\dot{a}} = h^{-1}\partial_{a\dot{a}}h - \alpha_a\, h^{-1}C_{\dot{a}}h\,. \tag{6.32}$$

Then, the $\mathbf{u}^2$ equation of motion becomes

$$\epsilon^{\dot{a}\dot{b}}\left(\alpha^a\hat{\alpha}^b + \hat{\alpha}^a\alpha^b\right)F_{a\dot{a}b\dot{b}}(B) = 0 \qquad \Longleftrightarrow \qquad \epsilon^{\dot{a}\dot{b}}\alpha^a\partial_{a\dot{a}}C_{\dot{b}} = 0\,. \tag{6.33}$$

This may be solved explicitly by $C_{\dot{a}} = \alpha^a\partial_{a\dot{a}}f$ for $f \in C^\infty(\mathbb{R}^4)\otimes\mathfrak{h}$, such that the gauge field B is given by

$$B_{a\dot{a}} = h^{-1}\partial_{a\dot{a}}h + h^{-1}X_{a\dot{a}}h \quad \text{where} \quad X_{a\dot{a}} = -\alpha_a\alpha^b\partial_{b\dot{a}}f\,. \tag{6.34}$$

Reinserting this expression into the action (6.30), the resulting theory may be written as a difference of two LMP actions. This follows after performing a field redefinition $h\mathbf{u}h^{-1} = v - f$, for $v \in C^\infty(\mathbb{R}^4) \otimes \mathfrak{g}$, such that we arrive at the action

$$\begin{aligned} S_{\text{gLMP}}[\mathbf{u}, B] = &\hbar \int_{\mathbb{R}^4} \frac{1}{2} \operatorname{Tr}(\mathrm{d}v \wedge \star \mathrm{d}v) + \frac{1}{3} \alpha_a \alpha_b \Sigma^{ab} \wedge \operatorname{Tr}(v\,[\mathrm{d}v, \mathrm{d}v]) \\ &- \hbar \int_{\mathbb{R}^4} \frac{1}{2} \operatorname{Tr}(\mathrm{d}f \wedge \star \mathrm{d}f) + \frac{1}{3} \alpha_a \alpha_b \Sigma^{ab} \wedge \operatorname{Tr}(f\,[\mathrm{d}f, \mathrm{d}f])\,. \end{aligned} \tag{6.35}$$

This demonstrates the conclusion

$$S_{\text{gLMP}}[\mathbf{u}, B] = S_{\text{LMP}}[v] - S_{\text{LMP}}[f]\,. \tag{6.36}$$

7 Outlook

The construction presented in this work has led us to new integrable field theories in both four and two dimensions. We conclude by highlighting a number of interesting future directions prompted by these results.

Motivated by the observation that the gauged WZW model on the coset G/H in two dimensions can be written as the difference of WZW models for the groups G and H, we took the difference of two hCS_6 theories as our starting point. The boundary conditions (3.19) led us to add a boundary term resulting in the action (3.14). It is worth highlighting that the boundary variation vanishes on the boundary conditions (3.19) whether or not the boundary term is included, and the contribution of the boundary term to the IFT_4 vanishes if we invoke all the boundary conditions. However, while the algebraic boundary conditions, $\mathscr{A}^{\mathfrak{k}}|_{\alpha,\beta} = 0$ and $\mathscr{A}^{\mathfrak{h}}|_{\alpha,\beta} = \mathscr{B}|_{\alpha,\beta}$ can be straightforwardly solved, this is not the case for the differential one $\partial_0 \mathscr{A}^{\mathfrak{h}}|_{\alpha,\beta} = \partial_0 \mathscr{B}|_{\alpha,\beta}$. Therefore, we relaxed this condition such that the contribution of the boundary term no longer vanishes. Importantly, for the specific boundary term added in (3.14), the constraints implied by the differential boundary condition now follow as on-shell equations of motion, leading to fully consistent IFT_4 and IFT_2.

There are compelling reasons to follow this strategy, including that the symplectic potential becomes tautological upon including the boundary term. However, a systematic interpretation of when boundary conditions can be consistently dropped for particular choices of boundary term is an open question. To address this, it would be appropriate to pursue a more formal study, complementing a homotopic analysis (along the lines done for CS_4 in [37]) with a symplectic/Hamiltonian study of the 6d holomorphic Chern-Simons theory (similar to [45] in the context of CS_4).

A second arena for formal development is the connection between 6d holomorphic Chern-Simons and five-dimensional Kähler Chern-Simons (KCS_5) theory [5, 6]. This should mirror the relationship between CS_4 and CS_3 theories described by Yamazaki [39]. To make this suggestion precise in the present context one may consider a Kaluza-Klein expansion around the U(1) rotation in the $\mathbb{CP}^1$ that leaves the location of the double poles fixed, retaining the transverse coordinate as part of the bulk five-manifold of KCS_5. The details of this are left for future study.

It would also be interesting to explore the new integrable IFT_4 and IFT_2 that we have constructed. G/H coset CFTs in two dimensions have a rich spectrum of parafermionic operators [28, 46]. It would be very interesting to establish the lift or analogue of these objects in the context of the IFT_4. The natural framework for this is likely to involve the study of co-dimension one defects and associated higher-form symmetries.

For abelian H we find IFT_2 that, in the $\not{k} \to 1$ limit, are related to massive integrable perturbations of the G/H gauged WZW models known as homogeneous sine-Gordon models [32, 40]. These include the sine-Gordon and complex sine-Gordon models as special cases, two of the most well-understood IFT_2. There is nothing in our construction that prohibits non-abelian H and it would be interesting to study the resulting models in more detail. The homogeneous sine-Gordon models before gauging are closely related to the non-abelian Toda equations [47, 48], for which an alternative derivation from CS_4 involving both order and disorder defects was presented in [49]. It would be instructive to understand the relationship between the two approaches.

An important class of IFT_2 are the symmetric space sigma models. These can be constructed either by restricting fields to parametrise G/H directly or by gauging a left action of H in the PCM. These theories have been realised in CS_4 through branch cut defects [16] and recently in hCS_6 [50]. One might explore the realisation of the gauging construction of such models within the current framework, and generalise to $\mathbb{Z}_4$-graded semi-symmetric spaces (relevant for applications of CS_4 to string worldsheet theories [51, 52]).

When G/H is a symmetric space, an alternative class of massive integrable perturbations of the G/H gauged WZW model are known as the symmetric space sine-Gordon models [32, 53]. In the landscape of IFT_2 these are related to the $\lambda \to 0$ limit [54, 55] of the λ-deformation of the symmetric space sigma model [56]. Note that $\not{k} \to 1$ and $\lambda \to 0$ both correspond to conformal limits and it would be instructive to explore the relation between the two constructions. More generally, it would be interesting to generalise the construction in this work to deformed models, in particular, splitting one or both double poles in the meromorphic (3,0)-form Ω into simple poles, or dual models, for example, considering the alternative boundary conditions (3.30).

Finally, recently novel approaches to constructing IFT_3 using higher Chern-Simons theory in 5d have been explored in [57, 58]. Given that there is an overlap between the models that can be obtained from these constructions and from hCS_6, or more precisely its reduction to five dimensions, CS_5 on the mini-twistor correspondence space $\mathbb{PN}$ [18], it would be exciting to understand the link between the two, and investigate the existence of categorical generalisations of hCS_6.

Acknowledgments

The authors would like to thank Alex Arvanitakis, Roland Bittleston, Falk Hassler, Tim Hollowood, Sylvain Lacroix, Marc Magro and Benoit Vicedo for interesting discussions. This work was first presented by D.C.T. at the Workshop on Integrable Sigma-Models held in ETH Zurich in May 2024. B.H and D.C.T. would also like to thank the organisers and participants for numerous stimulating and lively discussions. B.H. is supported by a UKRI Future Leaders Fellowship (grant number MR/T018909/1). The work of J.L. is supported by CONICET. D.C.T. is supported by the STFC grant ST/X000648/1, and by the Royal Society through a University Research Fellowship 'Generalised Dualities in String Theory and Holography' URF 150185. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising. No new data were generated for this manuscript.

A Spinor and differential form conventions

We work on $\mathbb{R}^4$ and define coordinates in bispinor notation as

$$x^{a\dot{a}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_0 + \mathrm{i}x_1 & x_2 + \mathrm{i}x_3 \\ -x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix} . \tag{A.1}$$

We fix orientation such that $\star 1 = \mathrm{vol}_4 = \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. For 1-forms $\sigma = \sigma_{a\dot{a}}\mathrm{d}x^{a\dot{a}}$ and $\tau = \tau_{a\dot{a}}\mathrm{d}x^{a\dot{a}}$ we have

$$\star^2\sigma = -\sigma , \quad \sigma \wedge \star\tau = - \star \sigma \wedge \tau = \mathrm{vol}_4 \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma_{a\dot{a}}\tau_{b\dot{b}} , \quad \mathrm{d} \star \sigma = \mathrm{vol}_4 \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\partial_{a\dot{a}}\sigma_{b\dot{b}} . \tag{A.2}$$

Contraction of spinors is given by

$$\langle \alpha\beta \rangle = \alpha_1\beta_2 - \alpha_2\beta_1 = \alpha^a\beta_a , \tag{A.3}$$

and spinor indices are raised as

$$\alpha^a = \epsilon^{ab}\alpha_b , \quad \epsilon^{12} = -\epsilon^{21} = -1 . \tag{A.4}$$

We define $\epsilon_{12} = +1$ such that $\epsilon^{ab}\epsilon_{bc} = \delta^a_c$. The (quaternionic) conjugation of a spinor $\alpha_a = (\alpha_1, \alpha_2)$ is defined to be $\hat{\alpha}_a = (-\bar{\alpha}_2, \bar{\alpha}_1)$. Identical definitions hold for the anti-chiral spinors with dotted indices and contraction denoted with square brackets though these do not play a role in this work.

A basis for self-dual 2-forms is given by

$$\Sigma^{ab} = \epsilon_{\dot{a}\dot{b}}\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} , \tag{A.5}$$

from which, given any two spinors, we can define self-dual forms

$$\Sigma_{\alpha,\beta} = \alpha_a\beta_b\Sigma^{ab} = \alpha_a\beta_b\epsilon_{\dot{a}\dot{b}}\mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} , \qquad \star\Sigma_{\alpha,\beta} = \Sigma_{\alpha,\beta} . \tag{A.6}$$

As they will play key roles, we denote

$$\omega_{\alpha,\beta} = \frac{1}{\langle\alpha\beta\rangle}\Sigma_{\alpha,\beta} , \quad \mu_\alpha = \Sigma_{\alpha,\alpha} , \quad \mu_\beta = \Sigma_{\beta,\beta} . \tag{A.7}$$

$\mathbb{R}^4$ is equipped with a hyper-Kähler structure and has a $\mathbb{CP}^1$s worth of complex structures. We can compactly express the complex structure corresponding to a spinor γ_a as

$$\mathcal{J}_\gamma = -i(\gamma^a\partial_{a\dot{a}}) \otimes (\hat{\gamma}_b\mathrm{d}x^{b\dot{a}}) - i(\hat{\gamma}^a\partial_{a\dot{a}}) \otimes (\gamma_b\mathrm{d}x^{b\dot{a}}) , \tag{A.8}$$

for which adapted complex coordinates are given by

$$\mathrm{d}z = \gamma_a\kappa_{\dot{a}}\mathrm{d}x^{a\dot{a}} , \qquad \mathrm{d}\bar{z} = \hat{\gamma}_a\hat{\kappa}_{\dot{a}}\mathrm{d}x^{a\dot{a}} , \qquad \mathrm{d}w = \gamma_a\hat{\kappa}_{\dot{a}}\mathrm{d}x^{a\dot{a}} , \qquad \mathrm{d}\bar{w} = -\hat{\gamma}_a\kappa_{\dot{a}}\mathrm{d}x^{a\dot{a}} . \tag{A.9}$$

JHEP12(2024)202

With these coordinates we have that

$$\mu_\alpha = -2\langle\alpha\gamma\rangle^2 \mathrm{d}\bar{w}\wedge\mathrm{d}\bar{z} - 2\langle\alpha\gamma\rangle\langle\alpha\hat{\gamma}\rangle(\mathrm{d}z\wedge\mathrm{d}\bar{z}+\mathrm{d}w\wedge\mathrm{d}\bar{w}) - 2\langle\alpha\hat{\gamma}\rangle^2\mathrm{d}w\wedge\mathrm{d}z, \tag{A.10}$$

$$\omega_{\alpha,\beta} = -2\,\frac{\langle\alpha\hat{\gamma}\rangle\langle\beta\hat{\gamma}\rangle}{\langle\alpha\beta\rangle}\,\mathrm{d}w\wedge\mathrm{d}z - 2\,\frac{\langle\alpha\gamma\rangle\langle\beta\gamma\rangle}{\langle\alpha\beta\rangle}\,\mathrm{d}\bar{w}\wedge\mathrm{d}\bar{z} - \frac{\langle\alpha\gamma\rangle\langle\beta\hat{\gamma}\rangle+\langle\alpha\hat{\gamma}\rangle\langle\beta\gamma\rangle}{\langle\alpha\beta\rangle}\,(\mathrm{d}z\wedge\mathrm{d}\bar{z}+\mathrm{d}w\wedge\mathrm{d}\bar{w})\,. \tag{A.11}$$

Notice that, if we align the spinor α to γ and β to $\hat{\gamma}$, then $\omega_{\gamma,\hat{\gamma}}$ is (proportional to) the corresponding Kähler form ϖ of type $(1,1)$ and μ_γ is a holomorphic $(2,0)$-form and $\mu_{\hat{\gamma}}$ is $(0,2)$-form.

B Twistor space

We work on the Euclidean slice of Penrose's twistor space, $\mathbb{PT}_{\mathbb{E}}$. Starting from the twistor space of complexified Minkowski space,

$$\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1 = \{Z^\alpha = (\omega^{\dot{a}}, \pi_a)\} | \pi_a \neq 0\,, Z^\alpha \sim rZ^\alpha \quad r \in \mathbb{C}^\times\}\,, \tag{B.1}$$

we obtain $\mathbb{PT}_{\mathbb{E}}$ by making a choice of reality conditions, in particular, by selecting the slice of $\mathbb{PT}$ invariant under the anti-holomorphic (quartic) involution acting on the holomorphic coordinates as $Z^\alpha \mapsto \hat{Z}^\alpha = (\hat{\omega}^{\dot{a}}, \hat{\pi}_a)$. This choice of reality conditions induces a double fibration and we find that Euclidean twistor space can be viewed as the holomorphic vector bundle $\mathbb{PT}_{\mathbb{E}} \cong \mathcal{O}(1)\oplus\mathcal{O}(1) \to \mathbb{CP}^1$, where the holomorphic coordinates along the fibre direction are given by the incidence relations $\omega^{\dot{a}} = x^{a\dot{a}}\pi_a$. With this we choose a basis of $(1,0)$-forms and $(0,1)$-forms

$$\begin{aligned} e^0 &= \langle\pi\mathrm{d}\pi\rangle\,, & e^{\dot{a}} &= \pi_a \mathrm{d}x^{a\dot{a}}\,, \\ \bar{e}^0 &= \frac{\langle\hat{\pi}\mathrm{d}\hat{\pi}\rangle}{\langle\pi\hat{\pi}\rangle^2}\,, & \bar{e}^{\dot{a}} &= \frac{\hat{\pi}_a\mathrm{d}x^{\dot{a}a}}{\langle\pi\hat{\pi}\rangle}\,, \end{aligned} \tag{B.2}$$

and their dual vector fields

$$\begin{aligned} \partial_0 &= \frac{\hat{\pi}_a}{\langle\pi\hat{\pi}\rangle}\frac{\partial}{\partial\pi_a}\,, & \partial_{\dot{a}} &= -\frac{\hat{\pi}^a\partial_{a\dot{a}}}{\langle\pi\hat{\pi}\rangle}\,, \\ \bar{\partial}_0 &= -\langle\pi\hat{\pi}\rangle\,\pi_a\frac{\partial}{\partial\hat{\pi}_a}\,, & \bar{\partial}_{\dot{a}} &= \pi^a\partial_{a\dot{a}}\,. \end{aligned} \tag{B.3}$$

It is important to note that this basis of 1-forms, and their duals, satisfy the structure equations,

$$\begin{aligned} \bar{\partial}e^{\dot{a}} &= e^0\wedge\bar{e}^{\dot{a}}\,, & \partial\bar{e}^{\dot{a}} &= e^{\dot{a}}\wedge\bar{e}^0\,, \\ [\bar{\partial}_0\,,\partial_{\dot{a}}] &= \bar{\partial}_{\dot{a}}\,, & [\bar{\partial}_{\dot{a}}\,,\partial_0] &= \partial_{\dot{a}}\,. \end{aligned} \tag{B.4}$$

B.1 Homogeneous and inhomogeneous coordinates

Homogeneous coordinates on $\mathbb{CP}^1$ will be denoted by $\pi_a = (\pi_1, \pi_2)$, which are defined up to the equivalence relation $\pi_a \sim s\,\pi_a$ for any non-zero $s \in \mathbb{C}^*$. These have the advantage of being globally defined on $\mathbb{CP}^1$ but can lead to technical complications in certain calculations.

It can also be useful to work with inhomogeneous coordinates on two patches covering $\mathbb{CP}^1 \cong S^2$. Introducing an arbitrary spinor γ_a that satisfies $\langle \gamma \hat{\gamma} \rangle = 1$, the two patches covering $\mathbb{CP}^1$ will be defined as

$$U_1 = \{\pi_a \mid \langle \pi \hat{\gamma} \rangle \neq 0\}, \qquad U_2 = \{\pi_a \mid \langle \pi \gamma \rangle \neq 0\}. \tag{B.5}$$

Inhomogeneous coordinates may be defined on each patch by

$$\zeta = \frac{\langle \gamma \pi \rangle}{\langle \pi \hat{\gamma} \rangle}, \qquad \xi = \frac{\langle \pi \hat{\gamma} \rangle}{\langle \gamma \pi \rangle}, \qquad \xi = \zeta^{-1}. \tag{B.6}$$

In this section, we restrict our attention to U_1 and the inhomogeneous coordinate ζ, knowing that an analogous discussion holds for the other patch. The complex conjugate of the inhomogeneous coordinate ζ is

$$\bar{\zeta} = -\frac{\langle \hat{\pi} \hat{\gamma} \rangle}{\langle \gamma \hat{\pi} \rangle}. \tag{B.7}$$

Forms and vector fields on $\mathbb{CP}^1$ written in these coordinates are related to one another by

$$\begin{aligned} \mathrm{d}\zeta &= \frac{e^0}{\langle \pi \hat{\gamma} \rangle^2}, & \mathrm{d}\bar{\zeta} &= \frac{\langle \pi \hat{\pi} \rangle^2}{\langle \gamma \hat{\pi} \rangle^2} \bar{e}^0, \\ \partial_\zeta &= \langle \pi \hat{\gamma} \rangle^2 \partial_0, & \partial_{\bar{\zeta}} &= \frac{\langle \gamma \hat{\pi} \rangle^2}{\langle \pi \hat{\pi} \rangle^2} \bar{\partial}_0. \end{aligned} \tag{B.8}$$

It is also helpful to define a weight zero basis of $(1,0)$-forms on $\mathbb{R}^4 \subset \mathbb{PT}$ by

$$\theta^{\dot{a}} = \frac{e^{\dot{a}}}{\langle \pi \hat{\gamma} \rangle} = \mathrm{d}x^{a\dot{a}} \gamma_a + \zeta \, \mathrm{d}x^{a\dot{a}} \hat{\gamma}_a. \tag{B.9}$$

Likewise, the weight zero basis of $(0,1)$-forms on $\mathbb{R}^4 \subset \mathbb{PT}$ are defined by

$$\bar{\theta}^{\dot{a}} = \langle \pi \hat{\gamma} \rangle \, \bar{e}^{\dot{a}} = \frac{1}{1 + \zeta \bar{\zeta}} \left(dx^{a\dot{a}} \hat{\gamma}_a - \bar{\zeta} \, dx^{a\dot{a}} \gamma_a \right). \tag{B.10}$$

Given a point on $\mathbb{CP}^1$ defined by α_a in homogeneous coordinates, we denote the corresponding point in the inhomogeneous coordinate ζ by

$$\alpha = \frac{\langle \gamma \alpha \rangle}{\langle \alpha \hat{\gamma} \rangle} = \zeta|_{\pi_a \sim \alpha_a}. \tag{B.11}$$

We also have the relation

$$\frac{\langle \pi \alpha \rangle}{\langle \pi \hat{\gamma} \rangle \langle \hat{\gamma} \alpha \rangle} = (\zeta - \alpha). \tag{B.12}$$

C Projector technology

We consider the operator on 1-forms on $\mathbb{R}^4$ given by

$$J_{\alpha,\beta}(\sigma) = -\mathrm{i} \star (\omega_{\alpha,\beta} \wedge \sigma), \quad J_{\alpha,\beta}^2 = -\mathrm{id}, \tag{C.1}$$

which allows us to define projectors

$$P = \frac{1}{2}(\mathrm{id} - \mathrm{i}J) \qquad \bar{P} = \frac{1}{2}(\mathrm{id} + \mathrm{i}J)\,. \tag{C.2}$$

For this to define a complex structure on the real Euclidean slice of $\mathbb{R}^4 \subset \mathbb{C}^4$ we require that J maps Euclidean-real 1-forms to Euclidean-real 1-forms. While not true for general α and β, this is the case if we take $\alpha = \gamma$ and $\beta = \hat{\gamma}$. Then $J_{\gamma,\hat{\gamma}}$ is the complex structure J_γ, see eq. (A.8). The projectors P and $\bar{P}$ project onto the $(1,0)$ and $(0,1)$ components thus realising the Dolbeault complex.

These projectors satisfy a range of useful identities:

$$\bar{P}(\star(\mu_\alpha \wedge \sigma)) = 0\,, \qquad P(\star(\mu_\beta \wedge \sigma)) = 0\,,\ \mu_\beta \wedge \bar{P}(\sigma) = 0\,, \qquad \mu_\alpha \wedge P(\sigma) = 0\,, \tag{C.3}$$

$$\omega_{\alpha,\beta} \wedge \bar{P}(\sigma) = -\star \bar{P}(\sigma)\,, \qquad \omega_{\alpha,\beta} \wedge P(\sigma) = \star P(\sigma)\,, \tag{C.4}$$

$$\omega_{\alpha,\beta} \wedge \bar{P}(\sigma) \wedge \tau = \omega_{\alpha,\beta} \wedge \sigma \wedge P(\tau)\,, \qquad \omega_{\alpha,\beta} \wedge \bar{P}(\sigma) \wedge \bar{P}(\tau) = 0\,. \tag{C.5}$$

To move between form and component notation is useful to observe that

$$P(\sigma)_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\alpha_a\beta^b\sigma_{b\dot{a}}\,, \qquad \bar{P}(\sigma)_{a\dot{a}} = \frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\sigma_{b\dot{a}}\,. \tag{C.6}$$

Further relations, useful for analysing the $\mathbb{CP}^1$-derivative boundary conditions, are

$$2\alpha^a\sigma_{a\dot{a}}e^{\dot{a}}|_\alpha = \star(\mu_\alpha \wedge \sigma)\,, \qquad \beta^a\tau_{a\dot{a}}\bar{e}^{\dot{a}}|_\alpha = -\langle\alpha\beta\rangle P(\tau)\,, \tag{C.7}$$

$$2\beta^a\sigma_{a\dot{a}}e^{\dot{a}}|_\beta = \star(\mu_\beta \wedge \sigma)\,, \qquad \alpha^a\tau_{a\dot{a}}\bar{e}^{\dot{a}}|_\beta = \langle\alpha\beta\rangle \bar{P}(\tau)\,. \tag{C.8}$$

As an application of this projector technology let us consider the (ungauged) WZW_4 model, for which the equations of motion can be cast in terms of the right-invariant Maurer-Cartan form $R = \mathrm{d}gg^{-1}$, which obeys $\mathrm{d}R = R \wedge R$, as

$$\mathrm{d} \star \bar{P}(R) = \frac{1}{2}\mathrm{d}\left(\star - \omega_{\alpha,\beta}\wedge\right)\mathrm{d}gg^{-1} = 0\,. \tag{C.9}$$

We now consider a Yang-Mills connection $A = -\bar{P}(X)$. The equations for this to be anti-self dual are

$$\mu_\beta \wedge F[A] = 0\,, \qquad \mu_\alpha \wedge F[A] = 0\,, \qquad \omega_{\alpha,\beta} \wedge F[A] = 0\,. \tag{C.10}$$

The first of these vanishes identically by virtue of the fact that $\mu_\beta \wedge A = 0$. Since $\mu_\alpha \wedge A = -\mu_\alpha \wedge X$, the second yields a Bianchi identity

$$\mu_\alpha \wedge F[A] = -\mu_\alpha \wedge (\mathrm{d}X - X \wedge X)\,, \tag{C.11}$$

hence is solved by $X = R$. The final equation returns the equations of motion as

$$\omega_{\alpha,\beta} \wedge F[A] = -\mathrm{d}(\omega_{\alpha,\beta} \wedge \bar{P}(R)) + \omega_{\alpha,\beta} \wedge \bar{P}(R) \wedge \bar{P}(R) = \mathrm{d} \star \bar{P}(R)\,. \tag{C.12}$$

At the Kähler point $\beta = \hat{\alpha} = \hat{\gamma}$, we can simply write the ASDYM equations as

$$F^{2,0} = 0\,, \qquad F^{0,2} = 0\,, \qquad \varpi \wedge F^{1,1} = 0\,. \tag{C.13}$$

In this case, the connection given by $A = -\bar{\partial}gg^{-1}$ is of type $(0,1)$, hence $F^{2,0} = 0$ automatically, $F^{0,2} = 0$ is zero by the Bianchi identity and the equations of motion of WZW_4 are

$$\varpi \wedge \partial(\bar{\partial}gg^{-1}) = 0\,. \tag{C.14}$$

D Derivation of localisation formulae

In this work we are required to evaluate integrals of the form

$$I = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge Q \,, \qquad Q \in \Omega^{0,2}(\mathbb{PT}) \,. \tag{D.1}$$

In this appendix, we will derive general formulae for these integrals for the cases in which Ω has either two double poles or a single fourth-order pole. To compute these integrals efficiently we will work in inhomogeneous coordinates and make use of the identities

$$\partial_{\bar{\zeta}} \left(\frac{1}{\zeta - \alpha} \right) = -2\pi \mathrm{i}\, \delta^2(\zeta - \alpha) \,, \qquad \int_{\mathbb{CP}^1} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\, \delta^2(\zeta - \alpha)\, f(\zeta) = f(\alpha) \,. \tag{D.2}$$

D.1 Two double poles

We consider the $(3, 0)$-form given by

$$\Omega = \frac{1}{2} \frac{\langle \alpha\beta \rangle^2}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2}\, e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}} = \frac{1}{2} \frac{(\alpha - \beta)^2}{(\zeta - \alpha)^2 (\zeta - \beta)^2}\, \mathrm{d}\zeta \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}} \,. \tag{D.3}$$

Substituting this into the integral gives

$$I = -\frac{1}{2} \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\, \partial_{\bar{\zeta}} \left(\frac{(\alpha - \beta)^2}{(\zeta - \alpha)^2 (\zeta - \beta)^2} \right) \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q \,. \tag{D.4}$$

Then, using the identity (D.2) gives

$$I = -\frac{(\alpha - \beta)^2}{2} \int_{\mathbb{PT}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \left[\frac{\partial_\zeta \delta(\zeta - \alpha)}{(\zeta - \beta)^2} + \frac{\partial_\zeta \delta(\zeta - \beta)}{(\zeta - \alpha)^2} \right] \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q \,. \tag{D.5}$$

Since the integral is symmetric under $\alpha \leftrightarrow \beta$ we will only compute the first term explicitly. Integrating by parts and evaluating the integral over $\mathbb{CP}^1$ gives

$$I = \frac{(\alpha - \beta)^2}{2} \int_{\mathbb{R}^4} \partial_\zeta \left(\frac{\theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q}{(\zeta - \beta)^2} \right) \bigg|_\alpha \quad + \quad \alpha \leftrightarrow \beta \,. \tag{D.6}$$

We first distribute the ∂_ζ derivative, leaving the 2-form Q completely general, resulting in

$$\begin{aligned} I = \frac{(\alpha - \beta)^2}{2} \int_{\mathbb{R}^4} \bigg[&\frac{-2}{(\zeta - \beta)^3}\, \theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q + \frac{2}{(\zeta - \beta)^2}\, \hat{\gamma}_a \mathrm{d}x^{a\dot{a}} \wedge \theta_{\dot{a}} \wedge Q \\ &+ \frac{\theta^{\dot{a}} \wedge \theta_{\dot{a}}}{(\zeta - \beta)^2} \wedge \partial_\zeta Q \bigg] \bigg|_\alpha \quad + \quad \alpha \leftrightarrow \beta \,. \end{aligned} \tag{D.7}$$

The overall factor of $(\alpha - \beta)^2$ outside the integral cancels with the denominators in the integrand. We now make use of (B.12) to return to spinor notation and introduce self-dual 2-forms defined by $\Sigma^{ab} = \varepsilon_{\dot{a}\dot{b}} \mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}}$ to write

$$\begin{aligned} I = \frac{1}{2} \int_{\mathbb{R}^4} \bigg[&\frac{-2\langle \hat{\gamma}\beta \rangle}{\langle \alpha\beta \rangle \langle \alpha\hat{\gamma} \rangle}\, \alpha_a \alpha_b \Sigma^{ab} \wedge Q|_\alpha + \frac{2}{\langle \alpha\hat{\gamma} \rangle}\, \hat{\gamma}_a \alpha_b \Sigma^{ab} \wedge Q|_\alpha \\ &+ \alpha_a \alpha_b \Sigma^{ab} \wedge \frac{\partial_\zeta Q}{\langle \pi\hat{\gamma} \rangle^2} \bigg|_\alpha \bigg] \quad + \quad \alpha \leftrightarrow \beta \,. \end{aligned} \tag{D.8}$$

Expanding α_a in the basis formed by $\hat{\gamma}_a$ and β_a, we see that one component of the first term cancels the second term, and only a term proportional to $\alpha_a\beta_b\Sigma^{ab}$ survives. In the third term of the integral, we recognise the combination ∂_0 acting on Q and make this replacement. In conclusion, we have the general formula

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle}\wedge Q|_\alpha+\frac{1}{2}\,\alpha_a\alpha_b\Sigma^{ab}\wedge(\partial_0 Q)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\,, \tag{D.9}$$

or in differential form notation

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\omega_{\alpha,\beta}\wedge Q|_\alpha+\frac{1}{2}\,\mu_\alpha\wedge(\partial_0 Q)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\,. \tag{D.10}$$

It is also helpful to specialise to 2-forms of the form $Q=\pi^a\pi^b Q_{a\dot{a}b\dot{b}}\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}$, which we will often encounter. In this case, we may make use of the identity

$$e^{\dot{c}}\wedge e_{\dot{c}}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\varepsilon^{\dot{a}\dot{b}}\,, \tag{D.11}$$

and its generalisation valid for any spinors α_a and β_a

$$\alpha_a\beta_b\Sigma^{ab}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\frac{\langle\alpha\hat{\pi}\rangle\langle\beta\hat{\pi}\rangle}{\langle\pi\hat{\pi}\rangle^2}\,\varepsilon^{\dot{a}\dot{b}}\,. \tag{D.12}$$

Using these identities on the above formula for $Q=\pi^a\pi^b Q_{a\dot{a}b\dot{b}}\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}$ gives

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=-\int_{\mathbb{R}^4}\mathrm{vol}_4\left[\frac{\varepsilon^{\dot{a}\dot{b}}(\alpha^a\beta^b+\beta^a\alpha^b)}{\langle\alpha\beta\rangle}\,Q_{a\dot{a}b\dot{b}}|_\alpha+\varepsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b(\partial_0 Q_{a\dot{a}b\dot{b}})|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\,. \tag{D.13}$$

Finally, we specialise to the case when $Q_{a\dot{a}b\dot{b}}=X_{a\dot{a}}Y_{b\dot{b}}$, for which the answer can again be recast in differential form notation as

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\omega_{\alpha,\beta}\wedge X\wedge Y|_\alpha+\frac{1}{2}\,\mu_\alpha\partial_0\wedge(X\wedge Y)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\,. \tag{D.14}$$

To apply these formulae we also need the following $\mathbb{CP}^1$-derivatives:

$$\partial_0(\mathrm{d}\hat{g}\hat{g}^{-1})=\hat{g}\mathrm{d}\hat{u}\hat{g}^{-1}\,, \tag{D.15}$$
$$\partial_0(\hat{g}^{-1}\mathrm{d}\hat{g})=\mathrm{d}\hat{u}+[\hat{g}^{-1}\mathrm{d}\hat{g},\hat{u}]\,, \tag{D.16}$$
$$\partial_0(A)=\partial_0(B)=0\,, \tag{D.17}$$
$$\partial_0(\hat{g}^{-1}A\hat{g})=[\hat{g}^{-1}A\hat{g},\hat{u}]\,, \tag{D.18}$$
$$\partial_0\,\frac{1}{3}\,\mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g})^3=\mathrm{d}\ \mathrm{Tr}(\hat{u}(\hat{g}^{-1}\mathrm{d}\hat{g})^2)\,, \tag{D.19}$$

where we have defined $\hat{u}=\hat{g}^{-1}\partial_0\hat{g}$.

D.2 Fourth-order pole

In section 6, we consider a different $(3,0)$-form given by

$$\Omega=k\,\frac{e^0\wedge e^{\dot{a}}\wedge e_{\dot{a}}}{\langle\pi\alpha\rangle^4}=\frac{k'}{\langle\hat{\gamma}\alpha\rangle^4}\,\frac{d\zeta\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}}{(\zeta-\alpha)^4}\,. \tag{D.20}$$

Substituting this into the general integral expression above gives

$$I=-\frac{k}{\langle\hat{\gamma}\alpha\rangle^4}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\,\partial_{\bar{\zeta}}\left(\frac{1}{(\zeta-\alpha)^4}\right)\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\,. \tag{D.21}$$

Then, using the identity (D.2), we find

$$I=-\frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{PT}}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\left(\partial_\zeta^3\delta(\zeta-\alpha)\right)\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\,. \tag{D.22}$$

Integrating by parts and evaluating the integral over $\mathbb{CP}^1$ gives

$$I=\frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{R}^4}\partial_\zeta^3\left(\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\right)\Big|_\alpha\,. \tag{D.23}$$

In order to distribute the ∂_ζ derivatives, it is helpful to use the identities

$$\theta^a\big|_\alpha=\frac{\alpha_a\mathrm{d}x^{a\dot{a}}}{\langle\hat{\gamma}\alpha\rangle}\,,\qquad\partial_\zeta\theta^a\big|_\alpha=\hat{\gamma}_a\mathrm{d}x^{a\dot{a}}\,,\qquad\partial_\zeta^2\theta^a\big|_\alpha=0\,. \tag{D.24}$$

Distributing the three ∂_ζ derivatives gives

$$I=\frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{R}^4}\left[\frac{\alpha_a\alpha_b\Sigma^{ab}}{\langle\alpha\hat{\gamma}\rangle^2}\wedge\partial_\zeta^3 Q\big|_\alpha+6\,\frac{\alpha_a\hat{\gamma}_b\Sigma^{ab}}{\langle\alpha\hat{\gamma}\rangle}\wedge\partial_\zeta^2 Q\big|_\alpha+6\,\hat{\gamma}_a\hat{\gamma}_b\Sigma^{ab}\wedge\partial_\zeta Q\big|_\alpha\right]. \tag{D.25}$$

Converting this expression back into homogeneous coordinates (and using the fact that Q is a $(0,2)$-form on twistor space hence $\hat{\alpha}_a\mathrm{d}x^{a\dot{a}}\wedge Q|_\alpha=0$) this integral becomes

$$I=\frac{k}{6}\int_{\mathbb{R}^4}\alpha_a\alpha_b\Sigma^{ab}\wedge\partial_0^3 Q\big|_\alpha\,. \tag{D.26}$$

E Localisation derivation with general gaugings

In this appendix we describe in more detail the derivation of the gauged WZW_4 model from the gauged hCS_6 theory and the application of the localisation formulae in appendix (D.1). We will do this in a more general manner, allowing the gauging of an H subgroup that acts as

$$g\mapsto\rho_\beta(\ell)g\rho_\alpha(\ell^{-1})\,,\quad B\mapsto\ell B\ell^{-1}-\mathrm{d}\ell\ell^{-1}\,,\quad\ell\in H\subset G\,, \tag{E.1}$$

where $\rho_i:H\to G$ are group homomorphisms (algebra homomorphisms will be denoted by the same symbol). The covariant derivative is then given by

$$\nabla gg^{-1}=\mathrm{d}gg^{-1}+B_\beta-gB_\alpha g^{-1}\mapsto\rho_\beta(\ell)(\nabla gg^{-1})\rho_\beta(\ell^{-1})\,, \tag{E.2}$$

in which we ease the notation by setting $B_i=\rho_i(B)$.

The starting point is the six-dimensional theory

$$S_{\mathrm{ghCS}_6}[\mathscr{A},\mathscr{B}]=S_{\mathrm{hCS}_6}[\mathscr{A}]-S_{\mathrm{hCS}_6}[\mathscr{B}]+S_{\mathrm{bdry}}[\mathscr{A},\mathscr{B}]\,, \tag{E.3}$$

where we take the boundary interaction term to be

$$S_{\mathrm{bdry}}[\mathscr{A},\mathscr{B}]=-\frac{q}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge\mathrm{Tr}_{\mathfrak{g}}\left(\mathscr{A}\wedge\rho(\mathscr{B})\right)\,. \tag{E.4}$$

Here we have introduced a parameter q, which will ultimately be set to one, to keep track of the contributions from this boundary term. To specify this term we include an algebra homomorphism ρ that only needs to be defined piecewise on the components of the support of $\bar{\partial}\Omega$. We could choose to dispense with the higher-dimensional covariance and simply add different boundary terms specified only at the location of the poles, but it is convenient to formally consider ρ to be a defined as a piecewise map that takes values $\rho|_{\pi=\alpha,\beta} = \rho_{\alpha,\beta}$.

To define a six-dimensional theory requires imposing conditions that ensure the vanishing of the boundary variation

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left(\mathrm{Tr}_{\mathfrak{g}}\left(\delta\mathscr{A} \wedge (\mathscr{A} - q\rho(\mathscr{B})) + q\rho(\delta\mathscr{B}) \wedge \mathscr{A}\right) - \mathrm{Tr}_{\mathfrak{h}}(\delta\mathscr{B} \wedge \mathscr{B})\right) . \tag{E.5}$$

We are required to cancel a term involving the inner product on the algebra $\mathfrak{h}$ with one on $\mathfrak{g}$, which can be achieved by demanding

$$\mathrm{Tr}_{\mathfrak{g}}(\rho(x)\rho(y))|_{\alpha,\beta} = \mathrm{Tr}_{\mathfrak{h}}(xy) \quad \forall x, y \in \mathfrak{h} . \tag{E.6}$$

Note that as a consequence this implies

$$\mathrm{Tr}_{\mathfrak{g}}(\rho_\alpha(x)\rho_\alpha(y)) = \mathrm{Tr}_{\mathfrak{h}}(xy) = \mathrm{Tr}_{\mathfrak{g}}(\rho_\beta(x)\rho_\beta(y)) , \tag{E.7}$$

which is the familiar anomaly-free condition required to construct a gauge-invariant extension to the WZW model with the gauge symmetry (E.1). With this condition satisfied, the boundary term produced by variation is given by

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \Big(\mathrm{Tr}_{\mathfrak{g}}\Big(\delta\mathscr{A} \wedge (\mathscr{A} - q\rho(\mathscr{B})) + q\rho(\delta\mathscr{B}) \wedge (\mathscr{A} - q^{-1} \wedge \rho(\mathscr{B}))\Big)\Big) , \tag{E.8}$$

and is set to zero by the conditions

$$\mathscr{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0 , \qquad \mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathscr{B})\big|_{\alpha,\beta} , \qquad \partial_0\mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\partial_0\mathscr{B})\big|_{\alpha,\beta} . \tag{E.9}$$

If we impose all of these conditions from the outset, the contribution from the explicit boundary term $S_{\mathrm{bdry}}[\mathscr{A}, \mathscr{B}]$ would vanish. However, from a four-dimensional perspective the $\mathbb{CP}^1$-derivative boundary conditions lead to constraints relating derivatives of the fundamental fields to the 4-dimensional gauge field B that comes from $\mathscr{B}$. While these can be formally solved for B, our aim is to construct a gauged IFT_4 with a gauge field. Therefore, we only impose the conditions $\mathscr{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0$ and $\mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathscr{B})\big|_{\alpha,\beta}$, which can be solved for the 4-dimensional gauge field A that comes from $\mathscr{A}$ and substituted into the Lagrangian without concern. Doing this, we find that $S_{\mathrm{bdry}}[\mathscr{A}, \mathscr{B}]$ does contribute, and when $q = 1$ in particular, it provides a gauge invariant completion of the action. Importantly, the $\mathbb{CP}^1$-derivative boundary conditions that we have not imposed have not been forgotten, instead when $q = 1$ they are recovered as on-shell equations in this four-dimensional theory. This provides an alternative view of the procedure; when $q = 1$ the explicit boundary term (E.4) is serving to implement the constraints arising from $\partial_0\mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\partial_0\mathscr{B})\big|_{\alpha,\beta}$ at the Lagrangian level. We can see this explicitly by observing that if we just impose $\mathscr{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0$ and $\mathscr{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathscr{B})\big|_{\alpha,\beta}$ then

$$\begin{aligned} &\Big(\delta\mathscr{A} \wedge (\mathscr{A} - q\rho(\mathscr{B})) + q\rho(\delta\mathscr{B}) \wedge \mathscr{A} - \rho(\delta\mathscr{B}) \wedge \rho(\mathscr{B})\Big)|_{\alpha,\beta} = 0 , \\ &\partial_0\Big(\delta\mathscr{A} \wedge (\mathscr{A} - q\rho(\mathscr{B})) + q\rho(\delta\mathscr{B}) \wedge \mathscr{A} - \rho(\delta\mathscr{B}) \wedge \rho(\mathscr{B})\Big)|_{\alpha,\beta} \\ &\quad = (1-q)\delta(\partial_0\mathscr{A} - \rho(\partial_0\mathscr{B})) \wedge \rho(\mathscr{B})|_{\alpha,\beta} + (1+q)\rho(\delta\mathscr{B}) \wedge (\partial_0\mathscr{A} - \rho(\partial_0\mathscr{B}))|_{\alpha,\beta} . \end{aligned} \tag{E.10}$$

Therefore, for $q = 1$ we see that the boundary equations of motion for $\mathscr{B}$ are precisely the $\mathbb{CP}^1$-derivative boundary conditions $\partial_0 \mathscr{A}^{\mathfrak{h}}|_{\alpha,\beta} = \rho(\partial_0 \mathscr{B})|_{\alpha,\beta}$.

The localisation proceeds as follows. First, we change parametrisation $\mathscr{A} = \mathscr{A}'^{\hat{g}}$ and $\mathscr{B} = \mathscr{B}'^{\hat{h}}$ fixing some of the redundancy by demanding that $\mathscr{A}'$ and $\mathscr{B}'$ have no $\mathbb{CP}^1$ legs. Second, we fix some of the residual symmetry preserved by the boundary conditions to set $\hat{g}|_\beta = \hat{h}|_{\alpha,\beta} = \mathrm{id}$ and $\partial_0 \hat{h}|_{\alpha,\beta} = 0$. The remaining fields are $\hat{g}|_\alpha = g$, $\hat{g}^{-1}\partial_0 \hat{g}|_\alpha = u$, $\hat{g}^{-1}\partial_0\hat{g}|_\beta = \tilde{u}$ and the four-dimensional gauge fields A and B that arise from $\mathscr{A}'$ and $\mathscr{B}'$ once their holomorphicity is imposed.

We may now directly apply the localisation formulae (D.14) to show that the hCS_6 terms localise, before imposing boundary conditions, to give

$$
\begin{aligned}
S_{\mathrm{hCS}_6}[\mathscr{A}] \simeq \int_{\mathbb{R}^4} \omega_{\alpha,\beta} \wedge \mathrm{Tr}_{\mathfrak{g}}(A^g \wedge g^{-1}\mathrm{d}g) - \omega_{\alpha,\beta} \wedge \mathscr{L}_{\mathrm{WZ}}[g] \\
+ \frac{1}{2}\,\mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}(A^g \wedge \mathrm{d}u) + \frac{1}{2}\,\mu_\beta \wedge \mathrm{Tr}_{\mathfrak{g}}(A \wedge \mathrm{d}\tilde{u})\,,
\end{aligned}
\tag{E.11}
$$

while $S_{\mathrm{hCS}_6}[\mathscr{B}]$ yields zero in this gauge. Let us first consider the terms involving $\omega_{\alpha,\beta}$. Since the gauge completion of the WZ term is

$$
\mathscr{L}_{\mathrm{gWZ}}[g,B] = \mathscr{L}_{\mathrm{WZ}}[g] + \mathrm{Tr}_{\mathfrak{g}}\left(g^{-1}\mathrm{d}g \wedge B_\alpha + \mathrm{d}g g^{-1} \wedge B_\beta + g^{-1}B_\beta g B_\alpha\right)\,,
\tag{E.12}
$$

we may express them (trace implicit) as

$$
\begin{aligned}
&\omega_{\alpha,\beta} \wedge \left(A^g \wedge g^{-1}\mathrm{d}g - \mathscr{L}_{\mathrm{WZ}}[g]\right) \\
&= \omega_{\alpha,\beta} \wedge \left(A^g \wedge g^{-1}\mathrm{d}g - \mathscr{L}_{\mathrm{gWZ}}[g,B] + g^{-1}\mathrm{d}g \wedge B_\alpha + \mathrm{d}g g^{-1} B_\beta + g^{-1}B_\beta g B_\alpha\right) \\
&= \omega_{\alpha,\beta} \wedge \left((A^g - B_\alpha) \wedge g^{-1}\nabla g - \mathscr{L}_{\mathrm{gWZ}}[g,B] + A^g \wedge B_\alpha - A \wedge B_\beta\right)\,.
\end{aligned}
\tag{E.13}
$$

In differential form notation, the algebraic boundary conditions of eq. (E.9) become

$$
A = B_\beta - \bar{P}(\nabla g g^{-1})\,, \qquad A^g = P(g^{-1}\nabla g) + B_\alpha\,.
\tag{E.14}
$$

It follows that

$$
\begin{aligned}
&\omega_{\alpha,\beta} \wedge \left(A^g \wedge g^{-1}\mathrm{d}g - \mathscr{L}_{\mathrm{WZ}}[g]\right) \\
&= \omega_{\alpha,\beta} \wedge \left(P(g^{-1}\nabla g) \wedge g^{-1}\nabla g - \mathscr{L}_{\mathrm{gWZ}}[g,B] + A^g \wedge B_\alpha - A \wedge B_\beta\right) \\
&= -\frac{1}{2}\, g^{-1}\nabla g \wedge \star(g^{-1}\nabla g) - \omega_{\alpha,\beta} \wedge \left(\mathscr{L}_{\mathrm{gWZ}}[g,B] - A^g \wedge B_\alpha + A \wedge B_\beta\right)\,.
\end{aligned}
\tag{E.15}
$$

Here, in the last line, we made use of the identity $\omega \wedge P(\sigma) \wedge \sigma = -\frac{1}{2}\,\sigma \wedge \star\sigma$ for a 1-form σ. To treat the terms involving μ_α and μ_β we combine the algebraic boundary conditions (E.14) with the properties $\mu_\alpha \wedge P(X) = \mu_\beta \wedge \bar{P}(X) = 0$ such that $\mu_\alpha \wedge A^g = \mu_\alpha B_\alpha$ and $\mu_\beta \wedge A = \mu_\beta B_\beta$. In summary, we find

$$
\begin{aligned}
S_{\mathrm{hCS}_6}[\mathscr{A}] \simeq \int_{\mathbb{R}^4} -\frac{1}{2}\,\mathrm{Tr}_{\mathfrak{g}}\left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g\right) - \omega_{\alpha,\beta} \wedge \left(\mathscr{L}_{\mathrm{gWZ}}[g,B] + \mathrm{Tr}_{\mathfrak{g}}(A \wedge B_\beta - A^g B_\alpha)\right) \\
+ \frac{1}{2}\,\mu_\alpha \wedge \mathrm{Tr}(B_\alpha \wedge \mathrm{d}u) + \frac{1}{2}\,\mu_\beta \wedge \mathrm{Tr}(B_\beta \wedge \mathrm{d}\tilde{u})\,.
\end{aligned}
\tag{E.16}
$$

The localisation of the explicit boundary term yields, after using $\mu_a \wedge A^g = \mu_\alpha B_\alpha$,

$$S_{\rm bdry}[\mathscr{A},\mathscr{B}] \simeq -q \int_{\mathbb{R}^4} \omega_{\alpha,\beta} \wedge \mathrm{Tr}_{\mathfrak{g}}(A^g B_\alpha - AB_\beta) \\ + \frac{1}{2}\mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}((\mathrm{d}u + [B_\alpha, u])B_\alpha) + \frac{1}{2}\mu_\beta \wedge \mathrm{Tr}_{\mathfrak{g}}((\mathrm{d}\tilde{u} + [B_\beta, \tilde{u}])B_\beta)\,. \tag{E.17}$$

The significance of the boundary term now becomes clear. It serves to ensure manifest gauge invariance when we do not impose the $\mathbb{CP}^1$-derivative boundary conditions. When $q = 1$ the terms $\omega_{\alpha,\beta} \wedge \mathrm{Tr}(A^g B_\alpha - AB_\beta)$ directly cancel. The contributions of the entire localised action that are wedged against μ_α sum to

$$\mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}\left((1-q)\ \mathrm{d}u \wedge B_\alpha + 2q\ uF[B]_\alpha - 2q\ \mathrm{d}(B_\alpha u)\right)\,. \tag{E.18}$$

We see that for $q = 1$ we find a gauge-invariant field strength together with a total derivative term that we discard. The terms wedged against μ_β give a similar contribution. Hence the fully localised action becomes

$$S \simeq \int_{\mathbb{R}^4} -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}\left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g\right) - \omega_{\alpha,\beta} \wedge \mathscr{L}_{\rm gWZ}[g,B] \\ + \mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}(uF[B]_\alpha) + \mu_\beta \wedge \mathrm{Tr}_{\mathfrak{g}}(\tilde{u}F[B]_\beta)\,. \tag{E.19}$$

Noting that the components of u and $\tilde{u}$ in the complement of $\mathfrak{h}$ decouple, we can view u and $\tilde{u}$ as $\mathfrak{h}$-valued and write

$$S \simeq \int_{\mathbb{R}^4} -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}\left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g\right) - \omega_{\alpha,\beta} \wedge \mathscr{L}_{\rm gWZ}[g,B] \\ + \mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{h}}(uF[B]) + \mu_\beta \wedge \mathrm{Tr}_{\mathfrak{h}}(\tilde{u}F[B])\,. \tag{E.20}$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

References

[1] R.S. Ward, *Integrable and solvable systems, and relations among them*, *Phil. Trans. Roy. Soc. Lond. A* **315** (1985) 451 [INSPIRE].

[2] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of Instantons*, *Phys. Lett. A* **65** (1978) 185 [INSPIRE].

[3] L.J. Mason and N.M.J. Woodhouse, *Integrability, self-duality, and twistor theory*, Oxford University Press (1991) [DOI:10.1093/oso/9780198534983.001.0001].

[4] S.K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, *Proc. Lond. Math. Soc.* **50** (1985) 1 [INSPIRE].

[5] V.P. Nair and J. Schiff, *A Kähler-Chern-Simons Theory and Quantization of Instanton Moduli Spaces*, Phys. Lett. B **246** (1990) 423 [INSPIRE].

[6] V.P. Nair and J. Schiff, *Kähler Chern-Simons theory and symmetries of antiselfdual gauge fields*, Nucl. Phys. B **371** (1992) 329 [INSPIRE].

[7] A. Losev, G.W. Moore, N. Nekrasov and S. Shatashvili, *Four-dimensional avatars of two-dimensional RCFT*, Nucl. Phys. B Proc. Suppl. **46** (1996) 130 [hep-th/9509151] [INSPIRE].

[8] A.N. Leznov and M.A. Mukhtarov, *Deformation of Algebras and Solution of Selfduality Equation*, J. Math. Phys. **28** (1987) 2574 [INSPIRE].

[9] A. Parkes, *A cubic action for selfdual Yang-Mills*, Phys. Lett. B **286** (1992) 265 [hep-th/9203074] [INSPIRE].

[10] E. Witten, *Gauge Theories and Integrable Lattice Models*, Nucl. Phys. B **322** (1989) 629 [INSPIRE].

[11] K. Costello, *Supersymmetric gauge theory and the Yangian*, arXiv:1303.2632 [INSPIRE].

[12] K. Costello, *Integrable lattice models from four-dimensional field theories*, Proc. Symp. Pure Math. **88** (2014) 3 [arXiv:1308.0370] [INSPIRE].

[13] K. Budzik et al., *Feynman diagrams in four-dimensional holomorphic theories and the Operatope*, JHEP **07** (2023) 127 [arXiv:2207.14321] [INSPIRE].

[14] K. Costello, E. Witten and M. Yamazaki, *Gauge Theory and Integrability, I*, ICCM Not. **06** (2018) 46 [arXiv:1709.09993] [INSPIRE].

[15] K. Costello, E. Witten and M. Yamazaki, *Gauge Theory and Integrability, II*, ICCM Not. **06** (2018) 120 [arXiv:1802.01579] [INSPIRE].

[16] K. Costello and M. Yamazaki, *Gauge Theory And Integrability, III*, arXiv:1908.02289 [INSPIRE].

[17] S. Lacroix, *Four-dimensional Chern-Simons theory and integrable field theories*, J. Phys. A **55** (2022) 083001 [arXiv:2109.14278] [INSPIRE].

[18] R. Bittleston and D. Skinner, *Twistors, the ASD Yang-Mills equations and 4d Chern-Simons theory*, JHEP **02** (2023) 227 [arXiv:2011.04638] [INSPIRE].

[19] K. Costello, *Topological strings, twistors and skyrmions*, talk at *Western Hemisphere Colloquium on Geometry and Physics (WHCGP)* April 27 (2020) [https://youtu.be/ZlDNpPHvA8A].

[20] K.J. Costello, *Quantizing local holomorphic field theories on twistor space*, arXiv:2111.08879 [INSPIRE].

[21] R. Penrose, *Twistor algebra*, J. Math. Phys. **8** (1967) 345 [INSPIRE].

[22] R. Bittleston, D. Skinner and A. Sharma, *Quantizing the Non-linear Graviton*, Commun. Math. Phys. **403** (2023) 1543 [arXiv:2208.12701] [INSPIRE].

[23] R.F. Penna, *Twistor Actions for Integrable Systems*, JHEP **09** (2021) 140 [arXiv:2011.05831] [INSPIRE].

[24] L.T. Cole et al., *Integrable deformations from twistor space*, SciPost Phys. **17** (2024) 008 [arXiv:2311.17551] [INSPIRE].

[25] P. Goddard, A. Kent and D.I. Olive, *Virasoro Algebras and Coset Space Models*, Phys. Lett. B **152** (1985) 88 [INSPIRE].

JHEP12(2024)202

[26] K. Gawedzki and A. Kupiainen, *G/h Conformal Field Theory from Gauged WZW Model*, *Phys. Lett. B* **215** (1988) 119 [INSPIRE].

[27] D. Karabali, Q.-H. Park, H.J. Schnitzer and Z. Yang, *A GKO Construction Based on a Path Integral Formulation of Gauged Wess-Zumino-Witten Actions*, *Phys. Lett. B* **216** (1989) 307 [INSPIRE].

[28] K. Bardakci, M.J. Crescimanno and E. Rabinovici, *Parafermions From Coset Models*, *Nucl. Phys. B* **344** (1990) 344 [INSPIRE].

[29] E. Witten, *On holomorphic factorization of WZW and coset models*, *Commun. Math. Phys.* **144** (1992) 189 [INSPIRE].

[30] J. Stedman, *Four-Dimensional Chern-Simons and Gauged Sigma Models*, arXiv:2109.08101 [INSPIRE].

[31] N.J. Hitchin, *The selfduality equations on a Riemann surface*, *Proc. Lond. Math. Soc.* **55** (1987) 59 [INSPIRE].

[32] C.R. Fernandez-Pousa, M.V. Gallas, T.J. Hollowood and J.L. Miramontes, *The symmetric space and homogeneous sine-Gordon theories*, *Nucl. Phys. B* **484** (1997) 609 [hep-th/9606032] [INSPIRE].

[33] J.M. Figueroa-O'Farrill and S. Stanciu, *Equivariant cohomology and gauged bosonic sigma models*, hep-th/9407149 [INSPIRE].

[34] J.M. Figueroa-O'Farrill and S. Stanciu, *Gauged Wess-Zumino terms and equivariant cohomology*, *Phys. Lett. B* **341** (1994) 153 [hep-th/9407196] [INSPIRE].

[35] J.M. Figueroa-O'Farrill and N. Mohammedi, *Gauging the Wess-Zumino term of a sigma model with boundary*, *JHEP* **08** (2005) 086 [hep-th/0506049] [INSPIRE].

[36] A.M. Polyakov and P.B. Wiegmann, *Theory of Nonabelian Goldstone Bosons*, *Phys. Lett. B* **131** (1983) 121 [INSPIRE].

[37] M. Benini, A. Schenkel and B. Vicedo, *Homotopical Analysis of 4d Chern-Simons Theory and Integrable Field Theories*, *Commun. Math. Phys.* **389** (2022) 1417 [arXiv:2008.01829] [INSPIRE].

[38] S. Lacroix and B. Vicedo, *Integrable $\mathcal{E}$-Models, 4d Chern-Simons Theory and Affine Gaudin Models. I. Lagrangian Aspects*, *SIGMA* **17** (2021) 058 [arXiv:2011.13809] [INSPIRE].

[39] M. Yamazaki, *New T-duality for Chern-Simons Theory*, *JHEP* **12** (2019) 090 [arXiv:1904.04976] [INSPIRE].

[40] Q.-H. Park, *Deformed coset models from gauged WZW actions*, *Phys. Lett. B* **328** (1994) 329 [hep-th/9402038] [INSPIRE].

[41] V.E. Zakharov and A.V. Mikhailov, *Relativistically Invariant Two-Dimensional Models in Field Theory Integrable by the Inverse Problem Technique* (in Russian), *Sov. Phys. JETP* **47** (1978) 1017 [INSPIRE].

[42] B. Hoare, N. Levine and A.A. Tseytlin, *On the massless tree-level S-matrix in 2d sigma models*, *J. Phys. A* **52** (2019) 144005 [arXiv:1812.02549] [INSPIRE].

[43] R. Bittleston, *Integrability from Chern-Simons theories*, Ph.D. thesis, University of Cambridge, Cambridge CB3 0WA, U.K. (2022) [DOI:10.17863/CAM.80090] [INSPIRE].

[44] J. Liniado and B. Vicedo, *Integrable Degenerate $\mathcal{E}$-Models from 4d Chern-Simons Theory*, *Annales Henri Poincare* **24** (2023) 3421 [arXiv:2301.09583] [INSPIRE].

[45] B. Vicedo, *4D Chern-Simons theory and affine Gaudin models*, *Lett. Math. Phys.* **111** (2021) 24 [arXiv:1908.07511] [INSPIRE].

[46] K. Bardakci, M.J. Crescimanno and S. Hotes, *Parafermions from nonabelian coset models*, *Nucl. Phys. B* **349** (1991) 439 [INSPIRE].

[47] A.N. Leznov and M.V. Savelev, *Two-dimensional Exactly and Completely Integrable Dynamical Systems (Monopoles, Instantons, Dual Models, Relativistic Strings, Lund Regge Model, Generalized Toda Lattice, Etc.)*, *Commun. Math. Phys.* **89** (1983) 59 [INSPIRE].

[48] T.J. Hollowood, J.L. Miramontes and Q.-H. Park, *Massive integrable soliton theories*, *Nucl. Phys. B* **445** (1995) 451 [hep-th/9412062] [INSPIRE].

[49] O. Fukushima, J.-I. Sakamoto and K. Yoshida, *Non-Abelian Toda field theories from a 4D Chern-Simons theory*, *JHEP* **03** (2022) 158 [arXiv:2112.11276] [INSPIRE].

[50] L.T. Cole and P. Weck, *Integrability in gravity from Chern-Simons theory*, *JHEP* **10** (2024) 080 [arXiv:2407.08782] [INSPIRE].

[51] K. Costello and B. Stefański, *Chern-Simons Origin of Superstring Integrability*, *Phys. Rev. Lett.* **125** (2020) 121602 [arXiv:2005.03064] [INSPIRE].

[52] N. Berkovits and R.S. Pitombo, *4D Chern-Simons and the pure spinor $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **109** (2024) 106015 [arXiv:2401.03976] [INSPIRE].

[53] I. Bakas, Q.-H. Park and H.-J. Shin, *Lagrangian formulation of symmetric space sine-Gordon models*, *Phys. Lett. B* **372** (1996) 45 [hep-th/9512030] [INSPIRE].

[54] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *Integrable Deformations of Strings on Symmetric Spaces*, *JHEP* **11** (2014) 009 [arXiv:1407.2840] [INSPIRE].

[55] B. Hoare and A.A. Tseytlin, *On integrable deformations of superstring sigma models related to $AdS_n \times S^n$ supercosets*, *Nucl. Phys. B* **897** (2015) 448 [arXiv:1504.07213] [INSPIRE].

[56] K. Sfetsos, *Integrable interpolations: From exact CFTs to non-Abelian T-duals*, *Nucl. Phys. B* **880** (2014) 225 [arXiv:1312.4560] [INSPIRE].

[57] A. Schenkel and B. Vicedo, *5d 2-Chern-Simons Theory and 3d Integrable Field Theories*, *Commun. Math. Phys.* **405** (2024) 293 [arXiv:2405.08083] [INSPIRE].

[58] H. Chen and J. Liniado, *Higher gauge theory and integrability*, *Phys. Rev. D* **110** (2024) 086017 [arXiv:2405.18625] [INSPIRE].

JHEP12(2024)202