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RESEARCH ARTICLE

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The infinite Dyson Brownian motion with $\beta = 2$ does not have a spectral gap

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Abstract

We prove that the Dirichlet forms associated with the unlabelled infinite Dyson Brownian motion with the inverse temperature $\beta = 2$ do not have a spectral gap.

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The interacting particle system $\mathbb{X}_t = (X_t^i)_{i\in\mathbb{N}}$ discussed in this paper is *formally* described as the following stochastic differential equation of infinitely many particles in $\mathbb{R}$:

$$dX_t^i = \frac{\beta}{2}\sum_{j:j\neq i}\frac{1}{X_t^i - X_t^j}\,dt + dB_t^i, \quad i\in\mathbb{N}\,, \tag{DBM}$$

where $(B_t^i : i\in\mathbb{N})$ is the family of infinitely many independent Brownian motions in $\mathbb{R}$ and $\beta > 0$ is a positive constant called inverse temperature. The solution $\mathbb{X}$ to (DBM) is called *infinite Dyson Brownian motion with inverse temperature* β, named after Dyson [1], which has a particular importance in relation to random matrix theory. It can be thought of as a diffusion process in the space Υ of locally finite point measures (called *configuration space*) by dropping the labelling via the map $(x_i)_{i=1}^\infty \mapsto \sum_{i=1}^\infty \delta_{x_i}$, which is called *unlabelled solution* and denoted by X. Over these 35 years, the construction of weak/strong solutions and their uniqueness have been studied, for example, in [2, 4, 6–8, 10, 12].

A question that is addressed in this paper is a spectral gap of the unlabelled solution X in the case $\beta = 2$. In this case, a weaker property, what is called *irreducibility* (called also *ergodicity*, or *convergence to equilibrium*), has been recently settled affirmatively in [5, 9, 11]. In particular, the law of the time marginal X_t converges to an equilibrium measure as $t \to \infty$, which is the law of the sine_2 point process. Any aspect of quantitative rate of this convergence, however, remains uncharted so far. The objective of this paper is to provide a negative result for spectral gaps of the unlabelled solution X with $\beta = 2$.

List of notation

- $\mathcal{D} = C_c^\infty(\mathbb{R})$ for the space of real-valued compactly supported smooth functions in $\mathbb{R}$;
- $C_b^\infty(\mathbb{R}^k)$ for the space of real-valued smooth functions in $\mathbb{R}^k$ ($k \in \mathbb{N}$) whose arbitrary order of derivative is continuous and bounded;
- $\Gamma^{\mathbb{R}}(u, v)$ for the square field of functions $u, v : \mathbb{R} \to \mathbb{R}$ defined as $\Gamma^{\mathbb{R}}(u, v) := (\frac{d}{dx}u)(\frac{d}{dx}v)$;
- Υ for the *configuration space over* $\mathbb{R}$, that is, the space of Radon point measures on $\mathbb{R}$;
- μ for the law of the sine$_2$ *point process*. Namely, μ is the law of the determinantal point process whose n-point correlation function is

$$\rho^{(n)}(x_1, \dots, x_n) = \mathsf{det}\Big(K(x_i, x_j)_{i,j=1}^n\Big) , \quad K(x_i, x_j) = \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)} ;$$

- $L^p(\Upsilon, \mu)$ ($1 \leqslant p < \infty$) for the μ-equivalence classes of functions $u : \Upsilon \to \mathbb{R}$ with $\|u\|_{L^p(\mu)}^p := \int_\Upsilon |u|^p \, d\mu < \infty$;
- $\mathsf{Var}(u)$ for the *variance* of $u \in L^2(\Upsilon, \mu)$ defined as $\mathsf{Var}(u) := \int_\Upsilon u^2 \, d\mu - \left(\int_\Upsilon u \, d\mu\right)^2$;
- $u^* : \Upsilon \to \mathbb{R}$ for *linear statistics of* $u : \mathbb{R} \to \mathbb{R}$ defined as

$$u^*(\gamma) := \int_{\mathbb{R}} u(x) \, d\gamma(x) , \quad \gamma \in \Upsilon ;$$

- $\mathcal{F}C_b^\infty(\mathcal{D})$ for the space of *cylinder functions* $U : \Upsilon \to \mathbb{R}$:

$$U = \Phi(u_1^*, \dots, u_k^*) , \ \{u_1, \dots, u_k\} \subset \mathcal{D} , \ \Phi \in C_b^\infty(\mathbb{R}^k) , \ k \in \mathbb{N} ;$$

- Γ^Υ for the *square field operator in* Υ:

$$\Gamma^\Upsilon(U) := \sum_{i,j=1}^k \partial_i \Phi\left(u_1^*, \dots, u_k^*\right) \partial_j \Phi\left(u_1^*, \dots, u_k^*\right) \Gamma^{\mathbb{R}}(u_i, u_j)^*, \quad U \in \mathcal{F}C_b^\infty(\mathcal{D}) ;$$

- $\mathcal{E}$ for the functional $\mathcal{E} : \mathcal{F}C_b^\infty(\mathcal{D}) \times \mathcal{F}C_b^\infty(\mathcal{D}) \to \mathbb{R}$ defined as

$$\mathcal{E}(u, v) := \frac{1}{2} \int_\Upsilon \Gamma^\Upsilon(u, v) \, d\mu , \quad u, v \in \mathcal{F}C_b^\infty(\mathcal{D}) , \tag{1}$$

  where $\Gamma^\Upsilon(u, v) := \frac{1}{4}\big(\Gamma^\Upsilon(u + v) - \Gamma^\Upsilon(u - v)\big)$. We write $\mathcal{E}(u, u) = \mathcal{E}(u)$.

Let $\mathcal{Q} : L^2(\Upsilon, \mu) \times L^2(\Upsilon, \mu) \to \mathbb{R} \cup \{\infty\}$ be a symmetric bilinear function with a dense domain $\mathcal{F} := \{u \in L^2(\Upsilon, \mu) : \mathcal{Q}(u, u) < \infty\}$ and $\mathcal{Q}(u, u) \geqslant 0$ for every $u \in \mathcal{F}$. It is called *closed* if the space $\mathcal{F}$ endowed with the norm $\| \cdot \|_{\mathcal{F}}$ defined as $\| \cdot \|_{\mathcal{F}}^2 := \mathcal{Q}(\cdot) + \| \cdot \|_{L^2(\mu)}^2$ is a real Hilbert space. A pair $(\mathcal{Q}, \mathcal{F})$ is a *closed extension of* $(\mathcal{E}, \mathcal{F}C_b^\infty(\mathcal{D}))$ if $(\mathcal{Q}, \mathcal{F})$ is closed and

$$\mathcal{F}C_b^\infty(\mathcal{D}) \subset \mathcal{F} , \quad \mathcal{Q} = \mathcal{E} \quad \text{on} \quad \mathcal{F}C_b^\infty(\mathcal{D}) \times \mathcal{F}C_b^\infty(\mathcal{D}) .$$

In the rest of the paper, we keep the same symbol $\mathcal{E}$ for extensions and simply say that $(\mathcal{E}, \mathcal{F})$ is a closed extension of $(\mathcal{E}, \mathcal{F}C_b^\infty(\mathcal{D}))$. The unlabelled solution X of (DBM) with $\beta = 2$ starting at a particular class of admissible initial conditions has been identified with the diffusion process properly associated with a closed extension of $(\mathcal{E}, \mathcal{F}C_b^\infty(\mathcal{D}))$, see [7, Thm. 24].

Theorem. *Any closed extension $(\mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, \mathcal{F}C_b^\infty(\mathcal{D}))$ does not have a spectral gap, that is,*

$$\inf_{\substack{u\in\mathcal{F}\\ u\neq 0}} \frac{\mathcal{E}(u)}{\mathrm{Var}(u)} = 0 \ .$$

1 | PROOF OF THE THEOREM

In the rest of the arguments, we often use the following facts about the sine_2 point process.

- **(Intensity measure)** If $u \in L^1(\mathbb{R})$, then $u^* \in L^1(\boldsymbol{\Upsilon}, \mu)$ and

$$\int_{\boldsymbol{\Upsilon}} u^* \, d\mu = \int_{\mathbb{R}} u \, dx \ . \tag{2}$$

- **(Two-point correlation)** For every Borel measurable $u : \mathbb{R} \to \mathbb{R}$,

$$\int_{\boldsymbol{\Upsilon}} \sum_{\substack{x,y\in\gamma\\ x\neq y}} u(x)u(y) d\mu(\gamma) = \int_{\mathbb{R}^2} u(x)u(y)\rho^{(2)}(x,y) dx dy \quad (\leqslant \infty), \tag{3}$$

  where $\rho^{(2)}$ is the two-point correlation function given by

$$\rho^{(2)}(x,y) = 1 - \frac{\sin^2\big(\pi(x-y)\big)}{\pi^2(x-y)^2} \ .$$

Proof of the theorem. Take $u_\sigma(x) := xe^{-\frac{x^2}{2\sigma^2}}$ and define the associated linear statistics $U_\sigma : \boldsymbol{\Upsilon} \to \mathbb{R}$ as $U_\sigma(\gamma) := u_\sigma^*(\gamma) = \int_{\mathbb{R}} u_\sigma(x) \, d\gamma$. In the following argument, we fix σ and simply write u and U.

We first compute the variance of U. By the intensity formula (2) and the mean zero property $\int_{\mathbb{R}} u \, dx = 0$,

$$\mathsf{Var}(U) = \int_{\boldsymbol{\Upsilon}} U^2 \, d\mu - \left(\int_{\boldsymbol{\Upsilon}} U \, d\mu\right)^2 = \int_{\boldsymbol{\Upsilon}} U^2 \, d\mu - \left(\int_{\mathbb{R}} u \, dx\right)^2 = \int_{\boldsymbol{\Upsilon}} U^2 \, d\mu \ .$$

The right-hand side can be further deduced to

$$\begin{aligned}\int_{\boldsymbol{\Upsilon}} \left(\sum_{x\in\gamma} u(x)\right)^2 d\mu(\gamma) &= \int_{\boldsymbol{\Upsilon}} \left(\sum_{x\in\gamma} u(x)^2 + \sum_{\substack{x,y\in\gamma\\ x\neq y}} u(x)u(y)\right) d\mu(\gamma) \\ &= \int_{\mathbb{R}} u(x)^2 \, dx + \int_{\mathbb{R}^2} u(x)u(y)\rho^{(2)}(x,y) \, dx \, dy \\ &= \frac{\sqrt{\pi}\sigma^3}{2} + \int_{\mathbb{R}^2} u(x)u(y)\rho^{(2)}(x,y) \, dx \, dy \ .\end{aligned} \tag{4}$$

We now give the evaluation on the second term $\int_{\mathbb{R}^2} u(x)u(y)\rho^{(2)}(x,y)\,dx\,dy$. Recalling $\rho^{(2)}(x,y) = 1 - \frac{\sin^2(\pi(x-y))}{\pi^2(x-y)^2}$ and $\int_{\mathbb{R}^2} u(x)u(y)\,dx\,dy = 0$, we focus only on the evaluation of

$$-\int_{\mathbb{R}^2} u(x)u(y)\frac{\sin^2(\pi(x-y))}{\pi^2(x-y)^2}\,dx\,dy\ . \tag{5}$$

By the change of variables (the rotation by 45°)

$$x = \frac{1}{\sqrt{2}}(u+v)\ , \qquad y = \frac{1}{\sqrt{2}}(v-u)\ ,$$

the integral (5) comes down to

$$\begin{aligned}
&-\frac{1}{4}\int_{\mathbb{R}^2}(v^2-u^2)e^{-\frac{u^2+v^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2u^2}\,du\,dv\\
&=\frac{1}{4}\int_{\mathbb{R}^2}e^{-\frac{u^2+v^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2}\,du\,dv-\frac{1}{4}\int_{\mathbb{R}^2}v^2e^{-\frac{u^2+v^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2u^2}\,du\,dv\\
&=:\ (\mathrm{I})+(\mathrm{II})\ .
\end{aligned}$$

By using the following formula (a proof will be given later):

$$\int_{\mathbb{R}} e^{-\frac{u^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2}\,du = \frac{\sigma e^{-4\pi^2\sigma^2}(e^{4\pi^2\sigma^2}-1)}{\sqrt{2}\pi^{3/2}}\ , \tag{6}$$

the first term can be further computed as

$$\begin{aligned}
(\mathrm{I}) &= \frac{1}{4}\int_{\mathbb{R}}e^{-\frac{v^2}{2\sigma^2}}\,dv\int_{\mathbb{R}}e^{-\frac{u^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2}\,du = \frac{\sqrt{2\pi}\sigma}{4}\times\frac{\sigma e^{-4\pi^2\sigma^2}(e^{4\pi^2\sigma^2}-1)}{\sqrt{2}\pi^{3/2}}\\
&= \frac{\sigma^2}{4\pi}(1-e^{-4\pi^2\sigma^2})\ .
\end{aligned}$$

For the second term, we first note that $\int_{\mathbb{R}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2u^2}\,du = \sqrt{2}$, which can be immediately seen by the formula $\int_{\mathbb{R}}\frac{\sin^2(u)}{u^2}\,du = \pi$ and the change of variable. Thus, we have

$$\begin{aligned}
(\mathrm{II}) &= -\frac{1}{4}\int_{\mathbb{R}}e^{-\frac{u^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2u^2}\,du\int_{\mathbb{R}}v^2e^{-\frac{v^2}{2\sigma^2}}\,dv = -\int_{\mathbb{R}}e^{-\frac{u^2}{2\sigma^2}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2u^2}\,du\times\frac{\sqrt{2\pi}\sigma^3}{4}\\
&\geqslant -\int_{\mathbb{R}}\frac{\sin^2(\sqrt{2}\pi u)}{\pi^2u^2}\,du\times\frac{\sqrt{2\pi}\sigma^3}{4} = -\frac{\sqrt{\pi}\sigma^3}{2}\ .
\end{aligned}$$

By plugging these estimates into Equation (4),

$$\mathsf{Var}(U) \geqslant \frac{\sqrt{\pi}\sigma^3}{2}-\frac{\sqrt{\pi}\sigma^3}{2}+\frac{\sigma^2}{4\pi}(1-e^{-4\pi^2\sigma^2}) = \frac{\sigma^2}{4\pi}(1-e^{-4\pi^2\sigma^2})\ . \tag{7}$$

We next compute $\mathcal{E}(U) = (1/2)\int_\Upsilon \Gamma^\Upsilon(U)\,d\mu$. By, for example, [3, Prop. 4.6], $U = U_\sigma \in \mathcal{F}$ for every $\sigma > 0$ and $\Gamma^\Upsilon(U) = \Gamma^\mathbb{R}(u)^*$ μ-a.e. Thus, for μ-a.e. γ,

$$\Gamma^\Upsilon(U)(\gamma) = \sum_{x\in\gamma} \Gamma^\mathbb{R}(u)(x) \leqslant 2\sum_{x\in\gamma} \mathrm{e}^{-\frac{x^2}{\sigma^2}}(x) + \frac{2}{\sigma^4}\sum_{x\in\gamma} x^4 \mathrm{e}^{-\frac{x^2}{\sigma^2}}(x)\ .$$

By the intensity formula (2), we have

$$\begin{aligned}\mathcal{E}(U) = \frac{1}{2}\int_\Upsilon \Gamma^\Upsilon(U)\,d\mu &\leqslant \int_\Upsilon \left(\sum_{x\in\gamma} \mathrm{e}^{-\frac{x^2}{\sigma^2}}(x) + \frac{1}{\sigma^4}\sum_{x\in\gamma} x^4\mathrm{e}^{-\frac{x^2}{\sigma^2}}(x)\right) d\mu(\gamma)\\ &= \int_\mathbb{R}\left(\mathrm{e}^{-\frac{x^2}{\sigma^2}} + \frac{1}{\sigma^4}x^4\mathrm{e}^{-\frac{x^2}{\sigma^2}}\right) dx\\ &= \sqrt{\pi}\sigma + \frac{3\sqrt{\pi}\sigma}{4}\ .\end{aligned}$$

Therefore, we conclude

$$\inf_{\substack{F\in\mathcal{F}\\F\neq 0}} \frac{\mathcal{E}(F)}{\mathsf{Var}(F)} \leqslant \inf_{\sigma>0} \frac{\mathcal{E}(U_\sigma)}{\mathsf{Var}(U_\sigma)} \leqslant \frac{\sqrt{\pi}\sigma + \frac{3\sqrt{\pi}\sigma}{4}}{\frac{\sigma^2}{4\pi}(1-\mathrm{e}^{-4\pi^2\sigma^2})} \xrightarrow{\sigma\to\infty} 0\ .$$

□

Proof of Equation (6). By the formula $\sin^2 u = (1-\cos(2u))/2$,

$$\begin{aligned}\int_\mathbb{R} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\sin^2(\sqrt{2}\pi u)}{\pi^2}\,du &= \frac{1}{2\pi^2}\int_\mathbb{R} \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\left(1-\cos(2\sqrt{2}\pi u)\right) du\\ &= \frac{1}{2\pi^2}\sqrt{2\pi}\sigma - \frac{1}{2\pi^2}\int_\mathbb{R} \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\cos(2\sqrt{2}\pi u)\,du\ .\end{aligned}$$

Let $h(a) := \int_0^\infty \mathrm{e}^{-bu^2}\cos(au)\,du$ for $a, b > 0$. As the integrant $\mathrm{e}^{-bu^2}\cos(au)$ is symmetric in variable u, we have $2h(a) = \int_\mathbb{R} \mathrm{e}^{-bu^2}\cos(au)\,du$. It is straightforward to check

$$\partial_a h = -\frac{a}{2b}h\ , \qquad h(0) = \int_0^\infty \mathrm{e}^{-bu^2}\,du = \frac{\sqrt{\pi}}{2\sqrt{b}}\ .$$

Solving this first-order ordinary differential equation, we get $2h(a) = \frac{\sqrt{\pi}}{\sqrt{b}}\mathrm{e}^{-\frac{a^2}{4b}}$. Plugging $a = 2\sqrt{2}\pi$ and $b = 1/(2\sigma^2)$, the sought formula is obtained. □

2 | CONCLUDING REMARK

Remark 2.1 (Growth of variance and spectral gap). Let μ be the law of a point process in $\mathbb{R}$ with constant intensity, Var_μ be the variance with respect to μ, and $\mathcal{E}_\mu$ be the functional defined in Equation (1) with μ. The proof of the theorem shows that as long as

$$\frac{\sigma}{\mathrm{Var}_\mu(u^*_\sigma)} \xrightarrow{\sigma\to\infty} 0\ , \tag{8}$$

any closed extension of $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(\mathcal{D}))$ (if it exists) does not have a spectral gap. In the case where μ is the law π of the Poisson point process in $\mathbb{R}$ with intensity 1, Equation (8) holds true as we have

$$\mathrm{Var}_\pi(u^*_\sigma) = \frac{\sqrt{\pi}\sigma^3}{2}\ .$$

To find an example μ with which $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(\mathcal{D}))$ has a spectral gap, we need to look into point processes having a considerably slower growth of the variance than that of the Poisson point process so that Equation (8) does not hold. This might be related to the property called *hyperuniformity*.

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JOURNAL INFORMATION

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