

On Equal Consecutive Values of Multiplicative Functions

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Abstract: Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function for which

$$\sum_{p:|f(p)|\neq 1} \frac{1}{p} = \infty.$$

We show under this condition alone that for any integer $h \neq 0$ the set

$$\{n \in \mathbb{N} : f(n) = f(n+h) \neq 0\}$$

has logarithmic density 0. We also prove a converse result, along with an application to the Fourier coefficients of holomorphic cusp forms.

The proof involves analysing the value distribution of f using the compositions $|f|^{\#}$, relying crucially on various applications of Tao's theorem on logarithmically-averaged correlations of non-pretentious multiplicative functions. Further key inputs arise from the inverse theory of sumsets in continuous additive combinatorics.

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1 Introduction

There is an expected general phenomenon in analytic number theory that the factorisations of additively coupled positive integers n and $n+h$, $h \geq 1$, are *statistically independent*, in the sense that for *very few* $n \in \mathbb{N}$ do we expect the features (number of primes, size of prime factors etc.) of the prime factorisation of n to influence the corresponding features of the factorisation of $n+h$. To investigate this phenomenon we are motivated to study the joint value distribution, as n varies, of pairs $(f(n), f(n+h))$, where $f : \mathbb{N} \rightarrow \mathbb{C}$ is a multiplicative function.

Bounding the size of the set $\{n \leq x : f(n) = f(n+h)\}$ as $x \rightarrow \infty$, for $h \in \mathbb{N}$ and f a multiplicative function, is a natural problem, for which some precedent in the literature already exists. For example, when $f = d$, the divisor function, Erdős, Pomerance and Sarkőzy [4] proved using sieve theoretic arguments that

$$|\{n \leq x : d(n) = d(n+1)\}| \ll \frac{x}{\sqrt{\log \log x}},$$

and thus $d(n) = d(n+1)$ occurs on a set of natural¹ density 0. In a separate paper [3], the same authors obtain stronger upper bounds for the number of solutions $n \leq x$ to $\sigma(n) = \sigma(n+1)$ and to $\phi(n) = \phi(n+1)$, where σ and ϕ are, respectively, the sum-of-divisors function and Euler's phi function. The explicit nature of these three examples is significant in the arguments of [4] and [3].

In a recent paper [8], Klurman and the author considered a different example of interest, when $(f(n))_n$ corresponds to the sequence of Fourier coefficients of a holomorphic cusp form. More precisely, for $k \geq 2$ let ϕ be a weight k , arithmetically normalised, holomorphic cuspidal eigenform without complex multiplication for $\mathrm{SL}_2(\mathbb{Z})$. Writing ϕ in its Fourier expansion

$$\phi(z) = \sum_{n \geq 1} a_\phi(n) e(nz), \quad \mathrm{Im}(z) > 0,$$

normalised so that $a_\phi(1) = 1$, it was shown among other things that each of the sets

$$\{n \in \mathbb{N} : |a_\phi(n)| < |a_\phi(n+h)|\}, \quad \{m \in \mathbb{N} : |a_\phi(m+h)| < |a_\phi(m)|\},$$

has positive density, and in fact it follows from [8, Thm. 1.7] that for every $h \geq 1$ the set of *non-vanishing coincidences*²

$$\{n \in \mathbb{N} : a_\phi(n) = a_\phi(n+h) \neq 0\}$$

has natural density 0. This can be understood heuristically from the fact that if ϕ has weight $k \geq 2$ then $a_\phi(n)$ generically grows like $n^{(k-1)/2+o(1)}$ (due to work of Deligne), and thus the likelihood of coincidences among non-zero values in such a large set is rather small.

More generally, we might speculate (as is done in some form on [8, p. 2]) that the same conclusion about the paucity of coincidences $f(n) = f(n+h)$ ought to be true for *any* unbounded multiplicative function $f : \mathbb{N} \rightarrow \mathbb{C}$ such that $|f(p)| \neq 1$ for most primes p (if, conversely, $|f(p)| = 1$ for many p this would account for numerous coincidences among non-zero values in $(f(m))_m$, as indicated by Proposition 1.5 below).

In this paper, we consider the general problem of bounding the number of solutions in positive integers n to the equation

$$f(n) = f(n+h) \neq 0,$$

¹The *density* of a set $A \subseteq \mathbb{N}$ is defined by $\lim_{x \rightarrow \infty} |A \cap [1, x]|/x$, provided the limit exists. The *upper density* of f is given by the same definition with \limsup in place of \lim .

²The non-vanishing assumption here is necessary given our current knowledge about the vanishing of Fourier coefficients of cusp forms. In particular, if there exists even a single *odd* integer d_0 for which $a_\phi(d_0) = 0$ then a positive density sequence of coincidences $a_\phi(n+d_0) = a_\phi(n) = 0$ can be found. Indeed, it suffices to choose the set $\{md_0 : m \in \mathbb{N}, (m(m+1), d_0) = 1\}$. Ruling out the existence of such d_0 for the Ramanujan τ -function, for example, would require proving a well-known conjecture due to Lehmer.

where $f : \mathbb{N} \rightarrow \mathbb{C}$ is an *arbitrary* multiplicative function, provided only that $|f(p)| \neq 1$ sufficiently often (in a precise sense). While we are unable to establish that this set has *natural* density 0, we do prove that it is a thin set in a weaker sense.

Our main result is the following.

Theorem 1.1. *Let $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{C}$ be multiplicative functions and let $h \geq 1$. Assume that*

$$\sum_{p:|f_1(p)| \neq 1} \frac{1}{p} = \infty. \tag{1}$$

Let $a, b \in \mathbb{N}$, $c, d \in \mathbb{Z}$ with $ad - bc \neq 0$ and $(a, b) = (c, d) = 1$. Let also $C \in \mathbb{C} \setminus \{0\}$. Then the set

$$\{n \in \mathbb{N} : f_1(an + b) = Cf_2(cn + d) \neq 0\}$$

has logarithmic density 0, i.e., we have

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{1_{f_1(an+b)=Cf_2(cn+d)}}{n} = o(1). \tag{2}$$

Remark 1.2. Ideally, we would like to prove the same result with the notion of “logarithmic density” replaced by “natural density”. Unfortunately, we are (unconditionally) limited in this matter by the fact that the linchpin of our proof, Tao’s theorem on logarithmically-averaged binary correlations of multiplicative functions (see Theorem 2.2 below) has not been proven for Cesàro-averaged binary correlations. The corresponding result is, however, expected to hold by a conjecture due to Elliott (see [17] for a discussion), and assuming Elliott’s conjecture our result would immediately be upgraded in this way.

While a Cesàro-averaged result of this kind is known at *almost every scale* x (see [18, Cor. 1.13]), this variant would be incompatible with our objectives, since in order to prove Theorem 1.1 we already need to restrict our attention to an arbitrary infinite subsequence of scales which may not intersect with the “good” scales of such a theorem.

Remark 1.3. The non-vanishing condition $f_1(an + b)f_2(cn + d) \neq 0$ is necessary³ in this theorem. Indeed, take any complex-valued multiplicative function f and define $\tilde{f} := \mu^2 f$. Then \tilde{f} is supported on squarefree integers. For any fixed, distinct primes p_1, p_2 not dividing $ac(ad - bc)$ the set

$$\{n \in \mathbb{N} : \tilde{f}(an + b) = \tilde{f}(cn + d)\}$$

contains the positive density set

$$\{n \in \mathbb{N} : p_1^2 | (an + b), \quad p_2^2 | (cn + d)\}.$$

For any n in this subset both $an + b$ and $cn + d$ are *not* squarefree, and hence $\tilde{f}(an + b) = 0 = \tilde{f}(cn + d)$.

Remark 1.4. When f_1, f_2 are completely multiplicative, the condition $(a, b) = (c, d) = 1$ may be dropped, as we may always reduce to the coprime case (by adjusting the value of C).

³We thank the anonymous referee for pointing out this simple construction.

Theorem 1.1 can be compared with the main result of [2]. There, Elliott and Kish prove that if $f : \mathbb{N} \rightarrow \mathbb{C}$ is completely multiplicative and

$$f(an + b) = Cf(cn + d) \neq 0 \text{ for all } n \geq n_0,$$

where $ad - bc \neq 0$ and $C \in \mathbb{C} \setminus \{0\}$, then there is a Dirichlet character χ of some modulus $m = m(a, b, c, d)$ such that

$$f(p) = \chi(p) \text{ for all } p \nmid m.$$

When $f_1 = f_2 = f$ our result addresses the “1% world” version of their problem, showing that if the (logarithmic) proportion of n that satisfy the equation is $\geq \delta$, for $\delta > 0$ a small parameter, then

$$\sum_{p:|f(p)| \neq 1} \frac{1}{p} < \infty. \tag{3}$$

While this is consistent with the result in [2], in our setting no further structural information about f can possibly be gleaned, and in particular f need not behave like a Dirichlet character, or even be pretentious (see Section 2.1 for the relevant definition)! Indeed, it is known for instance [17, Cor. 1.7] that the Möbius function μ satisfies

$$\mu(n) = \mu(n + 1) \neq 0 \text{ on a set of logarithmic density } \frac{1}{2} \prod_p (1 - 2/p^2),$$

and it is well-known that μ is non-pretentious. In fact, we may obtain the following (elementary) converse for real-valued f ; for simplicity we constrain ourselves to a single case of affine maps $an + b$ and $cn + d$.

Proposition 1.5. *Let $f : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\}$ be a multiplicative function for which (3) holds. Then there is a $u \in \{-1, +1\}$ such that*

$$\limsup_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{1_{f(n)=uf(n+2)}}{n} > 0.$$

Theorem 1.1 allows us to recover the result

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ a_\phi(n)a_\phi(n+h) \neq 0}} \frac{1_{a_\phi(n)=a_\phi(n+h)}}{n} = 0 \tag{4}$$

for the sequence of Fourier coefficients $(a_\phi(n))_n$ of any holomorphic cusp form without CM for $\mathrm{SL}_2(\mathbb{Z})$. While this is weaker than what is proved in [8], (4) may be shown *without* appealing to an effective version of the Sato-Tate theorem such as that found in [19]. As such, we may avoid having to apply the deep theorems of Newton-Thorne [14] on the functoriality of symmetric power lifts of holomorphic cusp forms. Indeed, our proof (see Section 5) requires nothing more than:

- the prime number theorem for Rankin-Selberg L -functions, and
- Deligne’s bound for the prime-indexed Fourier coefficients $|a_\phi(p)|$,

both of which apply (at least conjecturally and in some cases unconditionally) to the sequence of coefficients of the standard L -function of a broader collection of cuspidal automorphic forms.

Corollary 1.6. *Let $h \geq 1$ be fixed and let ϕ_1, ϕ_2 be holomorphic cusp forms for the full modular group. Then the set*

$$\{n \in \mathbb{N} : a_{\phi_1}(n) = a_{\phi_2}(n+j) \neq 0 \text{ for some } 1 \leq |j| \leq h\}$$

has logarithmic density 0.

Remark 1.7. Note that the claim is trivial if, say, ϕ_1 is assumed to have CM. In this case, it is known [16] that

$$|\{n \leq X : a_{\phi_1}(n) \neq 0\}| \asymp \frac{X}{\sqrt{\log X}},$$

which necessarily implies that $\{n \in \mathbb{N} : a_{\phi_1}(n) = a_{\phi_2}(n+h) \neq 0\}$ has logarithmic density 0.

Remark 1.8. A result like Corollary 1.6 may be proved for the sequences of coefficients $(a_{\pi_j}(n))_n$ of the standard L -functions of fixed cuspidal automorphic representations π_j for GL_{m_j} , $m_j \geq 1$, with unitary central character ($j = 1, 2$), assuming that at least one of π_1, π_2 satisfies the Generalised Ramanujan Conjecture (see e.g. [13, Sec. 1.3.1] for relevant preliminaries). We leave this extension to the interested reader.

1.1 Proof Ideas

1.1.1 Proof strategy towards Proposition 1.5

We prove Proposition 1.5 in Section 4 by studying the equation $|f(n)| = |f(n+2)|$ for *squarefree* n and $n+2$, noting that since f is real-valued this implies that one of $f(n) = f(n+2)$ or $f(n) = -f(n+2)$ holds. Now, as f is multiplicative and n and $n+2$ are squarefree,

$$|f(n)| = |f(n+2)| \text{ whenever } |f(p)| = 1 \text{ for all } p|n(n+2).$$

Writing $\mathcal{P}_{\neq 1} := \{p : |f(p)| \neq 1\}$, it thus suffices to show that the set⁴

$$\{n \in \mathbb{N} : (n(n+2), \mathcal{P}_{\neq 1}) = 1\}$$

has positive density. This follows straightforwardly from a zero-dimensional sieve, using crucially the fact that $\sum_{p \in \mathcal{P}_{\neq 1}} p^{-1} < \infty$.

1.1.2 Proof strategy towards Theorem 1.1

The proof of Theorem 1.1 is the main goal of the paper. We will assume for the sake of contradiction the existence of a (small) parameter $\delta \in (0, 1)$ and an infinite increasing sequence $(x_j)_j$ such that

$$\frac{1}{\log x_j} \sum_{\substack{n \leq x_j \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{1_{f_1(an+b)=Cf_2(cn+d)}}{n} \geq \delta, \quad j \geq 1. \tag{5}$$

⁴For a set $S \subset \mathbb{N}$ we write $(n, S) = 1$ if $(n, m) = 1$ for all $m \in S$.

We seek a contradiction by obtaining from (5) precise information about the sequence $(|f(p)|)_p$.

In order to analyse the condition (5) we appeal to a deep theorem of Tao [17] (see Theorem 2.2 below) on logarithmically-averaged correlations of multiplicative functions g_1, g_2 taking values in \mathbb{U} , the closed unit disc. This result implies structural information about the prime-indexed sequences $(g_j(p))_p$, $j = 1, 2$, whenever

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \left| \sum_{n \leq x} \frac{g_1(an+b)g_2(cn+d)}{n} \right| \neq 0.$$

To apply this we “project” our multiplicative functions f_1, f_2 onto \mathbb{U} in a manner that retains their multiplicativity.

For ease of exposition, suppose that our affine linear forms are $(an+b, cn+d) = (n, n+h)$, $h \geq 1$, that $C = 1$ and that $f_1 = f_2 = f$. We observe that for $t \in \mathbb{R}$ the composition

$$|f|_t(n) := \begin{cases} |f(n)|^{it} & \text{if } f(n) \neq 0 \\ 0 & \text{if } f(n) = 0 \end{cases}$$

is a well-defined multiplicative function that takes values in \mathbb{U} .

Let $A \geq 1$ be a parameter to be chosen later. The function

$$g(n) := A \log |f(n)|$$

is well-defined on the set $\{n \in \mathbb{N} : f(n) \neq 0\}$, and is additive when restricted to that set. In particular, we have

$$f(n) = f(n+h) \neq 0 \Rightarrow g(n) = g(n+h),$$

and also⁵ $|f|_{2\pi A t}(n) = e(tg(n))$. Using a Fourier analytic identity, we thus show that if (5) holds then there is a bounded set $X \subset \mathbb{R}$ with positive Lebesgue measure such that for each $y \in X$ we have lower bounds

$$\frac{1}{\log x_k} \left| \sum_{n \leq x_k} \frac{|f|_{2\pi A y}(n) |f|_{-2\pi A y}(n+h)}{n} \right| \gg_{\delta} 1.$$

For each of these $y \in X$, Tao’s theorem implies the existence of a Dirichlet character χ_y of conductor depending only on δ , and $t_y \in \mathbb{R}$ of controlled size such that f_y pretends to be a twisted character $\chi_y(n)n^{it_y}$ (see Section 2.1 for a relevant definition). By passing to a positive measure subset X' of X if needed we may assume that χ_y is the same for all y , and so we focus on studying the dependence of t_y on y .

Using an idea going back to Halász [6] and subsequently refined by Ruzsa [15], we show that the mapping $y \mapsto t_y$ extends to an entire interval I and is uniformly approximated on I by a linear function $y \mapsto ry$ for some $r \in \mathbb{R}$ depending at most on δ . In this way we find that for most primes $p \leq x_k$ and for all $y \in I$,

$$|f(p)|^{2\pi i A y} \approx p^{2\pi i t_y} \approx p^{2\pi i r y},$$

and thus that $|f(p)|^A \approx p^r$ for “typical” primes p . Since A is arbitrarily large, we deduce that $|f(p)| = 1$ outside of a sparse set of primes, which contradicts our assumption (1).

⁵Here we use the standard notation $e(y) := e^{2\pi i y}$ for $y \in \mathbb{R}$.

Remark 1.9. In the recent work [11], we considered the variant problem of bounding the size of sets

$$\{n \leq x : f(n+a) = f(n) + b\}, \text{ where } ab \neq 0, x \geq 1,$$

for f an *integer-valued* multiplicative function. We used this together with the work in the present paper to obtain a partial classification of all unbounded multiplicative functions $f : \mathbb{N} \rightarrow \mathbb{N}$ for which the set

$$\{n \in \mathbb{N} : |f(n+1) - f(n)| \leq C\},$$

i.e., the set of n for which the gaps between consecutive values of $f(n)$ is bounded by some $C > 0$, has positive upper density.

Unlike in the *homogeneous* (i.e., $a = 0$) problem considered in this paper, examples do exist where the number of such solutions has positive density (e.g., take $a = b = 1$, then $f(n) = n$ is such an example). In addition to the “archimedean” projections $|f(n)|^u$ applied here, we also consider corresponding “non-archimedean” projections $\chi(f(n))$, for χ a Dirichlet character to a suitably chosen modulus $q \geq 1$. The application of these latter projections were developed in connection to the present paper, but thanks to a neat observation by the anonymous referee these turned out to be unnecessary here.

2 Background Results about Multiplicative Functions

In this section we collect some background notions and results about multiplicative functions and pretentious number theory to be used in the sequel.

2.1 Properties of the pretentious distance

A key rôle is played by the pretentious distances (in the sense of Granville and Soundararajan). Precisely, for arithmetic functions $f, g : \mathbb{N} \rightarrow \mathbb{U}$ and $x \geq 2$, define

$$\mathbb{D}(f, g; x) := \left(\sum_{p \leq x} \frac{1 - \operatorname{Re}(f(p)\bar{g}(p))}{p} \right)^{1/2}, \quad \mathbb{D}(f, g; \infty) := \lim_{x \rightarrow \infty} \mathbb{D}(f, g; x).$$

Clearly, $0 \leq \mathbb{D}(f, g; x)^2 \leq 2 \log \log x + O(1)$ by Mertens’ theorem. When $\mathbb{D}(f, g; \infty) < \infty$ (so that the upper bound $2 \log \log x + O(1)$ is far from the truth) we say that f *pretends to be* g (symmetrically, g *pretends to be* f), or that f is *g -pretentious*.

The pretentious distance functions satisfy the *pretentious triangle inequality*, which we will use in the following two forms [5, Lem. 3.1]. For functions $f, g, h, f_1, f_2, g_1, g_2 : \mathbb{N} \rightarrow \mathbb{U}$, we have

$$\mathbb{D}(f, h; x) \leq \mathbb{D}(f, g; x) + \mathbb{D}(g, h; x), \tag{6}$$

$$\mathbb{D}(f_1 f_2, g_1 g_2; x) \leq \mathbb{D}(f_1, g_1; x) + \mathbb{D}(f_2, g_2; x). \tag{7}$$

Applying (7) inductively, we can show that for $f, g : \mathbb{N} \rightarrow \mathbb{U}$ and $m \geq 1$,

$$\mathbb{D}(f^m, g^m; x) \leq m \mathbb{D}(f, g; x). \tag{8}$$

This result implies, for instance, that if f pretends to be g then f^m pretends to be g^m . The latter is especially helpful when, say, g takes its non-zero values in roots of unity of some order (as is the case for instance for Dirichlet characters), in which case $g^m = |g|$, and thus f^m pretends to be a function taking values 0 or 1 (and thus is, in a precise sense, “close” to being non-negative).

Note that when $f(p), g(p) \in S^1$ we may rewrite the numerators of the summands in the definition of $\mathbb{D}(f, g; x)$ using the relation

$$|f(p) - g(p)|^2 = 2(1 - \operatorname{Re}(f(p)\bar{g}(p))).$$

In particular, it follows that if f is g -pretentious and $\eta > 0$ then the set

$$\mathcal{S}_\eta := \{p : |f(p) - g(p)| > \eta\}$$

satisfies

$$\sum_{\substack{p \leq x \\ p \in \mathcal{S}_\eta}} \frac{1}{p} \leq \eta^{-2} \sum_{p \leq x} \frac{|f(p) - g(p)|^2}{p} = \eta^{-2} \mathbb{D}(f, g; x)^2 \ll_\eta 1.$$

For convenience, in the sequel we will refer to a subset of primes S as being *sparse* (respectively *C-sparse*) if

$$\sum_{\substack{p \leq x \\ p \in S}} \frac{1}{p} \ll 1,$$

(respectively, $\ll_C 1$) as $x \rightarrow \infty$.

The Archimedean characters $n \mapsto n^{it}$, $t \in \mathbb{R}$, form an important class of examples of multiplicative functions with non-zero mean values and correlations (see Theorem 2.2 below, for example). In many results in the sequel we will need control of the distance between n^{it} and 1. This is provided by the following standard result.

Lemma 2.1. *Let $\varepsilon > 0$ and let $x \geq x_0(\varepsilon)$. If $10 \leq |t| \leq x^2$ then*

$$\mathbb{D}(n^{it}, 1; x)^2 \geq (1/3 - \varepsilon) \log \log x. \tag{9}$$

Moreover, if $|t| \leq 10$ then

$$\mathbb{D}(n^{it}, 1; x)^2 = \log(1 + |t| \log x) + O(1). \tag{10}$$

Proof. Since we could not locate a proof of this precise statement in the literature, we give one here.

Set $\sigma := 1 + 1/\log x$. Note that for any $t \in \mathbb{R}$, Mertens’ theorem implies that

$$\mathbb{D}(1, n^{it}; x)^2 = \log \log x - \sum_{p \leq x} \frac{\operatorname{Re}(p^{it})}{p} + O(1) = \log \log x - \log |\zeta(\sigma + it)| + O(1),$$

When $|t| \leq 10$ the Laurent expansion of ζ near $s = 1$ yields

$$\zeta(\sigma + it) = \frac{1}{\sigma - 1 + it} + O(1 + |\sigma + it - 1|) = \frac{\log x}{1 + it \log x} + O(1),$$

so that since $(1 + |t| \log x)^2 \leq 2(1 + t^2 \log^2 x) \leq 2(1 + |t| \log x)^2$ we find

$$\log |\zeta(\sigma + it)| = \log \log x - \frac{1}{2} \log(1 + t^2 \log^2 x) + O(1) = \log \log x - \log(1 + |t| \log x) + O(1).$$

This proves (10) in the case $|t| \leq 10$.

Next, suppose that $10 \leq |t| \leq x^2$. Set $y := V_{x^2} \geq V_t$, where

$$V_t := \exp((\log^{2/3}(2 + |t|)) \log \log^{1/3}(2 + |t|)), \quad t \in \mathbb{R}.$$

For $\operatorname{Re}(s) > 1$ define

$$\zeta_y(s) := \prod_{p > y} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Combining [9, Lem. 3.2] and [9, Lem. 4.1] (with χ the trivial character), we find that

$$\operatorname{Re} \left(\sum_{y < p \leq x} \frac{1}{p^{1+it}} \right) = \log |\zeta_y(\sigma + it)| + O(1) = O(1).$$

Therefore, we find

$$\mathbb{D}(1, n^{it}; x)^2 \geq \sum_{y < p \leq x} \frac{1 - \operatorname{Re}(p^{-it})}{p} = \log \left(\frac{\log x}{\log y} \right) + O(1) = \frac{1}{3} (\log \log x - \log \log \log x) + O(1),$$

and the claim follows. □

2.2 Tao's theorem on logarithmically-averaged correlations

As described in the previous section, a key tool in our analysis of the distribution of pairs $(f(an + b), f(cn + d))$ is the following theorem of Tao [17], which prescribes necessary conditions for two multiplicative function $g_j : \mathbb{N} \rightarrow \mathbb{U}$, $j = 1, 2$, to have large, logarithmically-averaged binary correlations.

Theorem 2.2 (Tao's theorem on binary correlations). *Let $\eta > 0$, and let $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ with $ad - bc \neq 0$. Let $g_1, g_2 : \mathbb{N} \rightarrow \mathbb{U}$ be 1-bounded multiplicative functions with the property that for some $x \geq x_0(\eta)$ we have*

$$\frac{1}{\log x} \left| \sum_{n \leq x} \frac{g_1(an + b)g_2(cn + d)}{n} \right| \geq \eta.$$

Then for each $j = 1, 2$ there is a real number $t_j = t_{j,x}$ with $t_j = O_\eta(x)$ and a primitive Dirichlet character $\psi_j = \psi_{j,x}$ with conductor $O_\eta(1)$ such that

$$\sum_{p \leq x} \frac{1 - \operatorname{Re}(g_j(p)\bar{\psi}_j(p)p^{-it_j})}{p} = O_\eta(1).$$

3 The archimedean projections and the proof of Theorem 1.1

Using Tao's theorem as a principal tool, we will prove the following.

Proposition 3.1. *Let $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{C}$ be multiplicative functions, and suppose f_1 satisfies*

$$\sum_{p:f_1(p)=0} \frac{1}{p} < \infty, \quad \sum_{p:|f_1(p)| \neq 1} \frac{1}{p} = \infty. \quad (11)$$

Let $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ and $C \in \mathbb{C}$, with $C(ad - bc) \neq 0$. Suppose there is a $\delta > 0$ and an infinite increasing sequence $(x_j)_j$ such that for each $j \geq 1$,

$$\frac{1}{\log x_j} \sum_{\substack{n \leq x_j \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{1_{f_1(an+b)=Cf_2(cn+d)}}{n} \geq \delta. \quad (12)$$

Then if $j \geq j_0(\delta)$ and $A \geq 1$ there is a parameter $r_j = r_j(A) \in [-x_j^2, x_j^2]$ such that if $\delta^{-2} \leq B \ll_\delta 1$ then

$$\sum_{\substack{p \leq x_j \\ |f_1(p)|^A / p^{r_j} \notin [e^{-2/B}, e^{2/B}]}} \frac{1}{p} \ll_\delta 1, \quad (13)$$

the bound (13) being independent of A .

Remark 3.2. As is suggested by the hypotheses of Proposition 3.1, we will be able to assume in the sequel that

$$\sum_{p:f_1(p)=0} \frac{1}{p} < \infty, \quad (14)$$

as otherwise Theorem 1.1 is essentially trivial.

Indeed, as the function $1_{f_1(n) \neq 0}$ is multiplicative, it follows that

$$\sum_{n \leq x} \frac{1_{f_1(n) \neq 0}}{n} \ll \prod_{p \leq x} \left(1 + \frac{1_{f_1(p) \neq 0}}{p} \right) \ll (\log x) \exp \left(\sum_{p \leq x} \frac{1_{f_1(p) \neq 0} - 1}{p} \right) = o(\log x)$$

whenever $\sum_{p:f_1(p)=0} p^{-1} = \infty$. This implies that $\{n : f_1(an + b) = Cf_2(cn + d) \neq 0\}$ has logarithmic density 0 for any fixed a, b, c, d, C as in the theorem.

Theorem 1.1 follows swiftly from Proposition 3.1.

Proof of Theorem 1.1. Let $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{C}$ be multiplicative functions such that (1) holds. Assume for the sake of contradiction that

$$\{n \in \mathbb{N} : f_1(an + b) = Cf_2(cn + d) \neq 0\}$$

does not have logarithmic density 0, for some $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ with $ad \neq bc$, $(a, b) = (c, d) = 1$, and $C \in \mathbb{C} \setminus \{0\}$. Then there is a $\delta > 0$ and a sequence of scales $(x_j)_j$ such that for each $j \geq 1$,

$$\frac{1}{\log x_j} \sum_{\substack{n \leq x_j \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{1_{f_1(an+b)=Cf_2(cn+d)}}{n} \geq \delta,$$

i.e., (12) holds. By Remark 3.2, it suffices to assume also that f_1 satisfies (14), and thus the conditions in (11) both hold by assumption.

Now, let $A \geq 1$. By Proposition 3.1, for each $j \geq j_0(\delta)$ there exists $r_j = r_j(A)$ satisfying $|r_j| \leq x_j^2$ such that if we define

$$I_{A,j} := [e^{-2/(AB)}, e^{2/(AB)} x_j^{2/A}]$$

then it follows that

$$\Sigma_{A,j} := \sum_{\substack{p \leq x_j \\ |f_1(p)| \notin I_{A,j}}} \frac{1}{p} \leq \sum_{\substack{p \leq x_j \\ |f_1(p)|/|p^{r_j/A}| \notin [e^{-2/(AB)}, e^{2/(AB)}]}} \frac{1}{p} \ll_\delta 1,$$

where the bound is independent of A . Since $I_{A',j} \subseteq I_{A,j}$ whenever $A' \geq A$, it follows that $A \mapsto \Sigma_{A,j}$ is non-decreasing and uniformly bounded in A , for each j . Thus, the limit $\Sigma_j := \lim_{A \rightarrow \infty} \Sigma_{A,j}$ exists for each $j \geq j_0(\delta)$, and since moreover

$$\bigcap_{A \geq 1} I_{A,j} = \{1\}$$

we deduce that

$$\Sigma_j = \sum_{\substack{p \leq x_j \\ |f_1(p)| \neq 1}} \frac{1}{p} \ll_\delta 1.$$

Since $(\Sigma_j)_j$ is also bounded and non-decreasing, it converges as $j \rightarrow \infty$, and thus

$$\sum_{p: |f_1(p)| \neq 1} \frac{1}{p} = \lim_{j \rightarrow \infty} \Sigma_j < \infty,$$

which contradicts (1). The claim of the theorem thus follows. □

3.1 Reduction to correlations of 1-bounded functions

We incorporate the maps $|f|_t := |f|^t$ in proving the following, which brings into play lower bounds for correlations of multiplicative functions taking values in \mathbb{U} .

Lemma 3.3. *Let $A \geq 1$, $\delta \in (0, 1)$ and let $B \geq \delta^{-2}$. Let x be a positive real number, sufficiently large with respect to δ , for which*

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{1_{f_1(an+b)=Cf_2(cn+d)}}{n} \geq \delta. \tag{15}$$

Define $F_j := f_j^A$ for $j = 1, 2$. Then the set

$$X = X_{x, \delta, A, B} := \left\{ \alpha \in [-B, B] : \frac{1}{\log x} \left| \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{|F_1|_\alpha(an+b)|F_2|_{-\alpha}(cn+d)}{n} \right| \geq \frac{\delta}{16\pi} \right\} \quad (16)$$

has Lebesgue measure $\geq \delta/(16\pi)$.

Proof. Let X denote the set in (16). Clearly, X is symmetric and contains 0. The equality $f_1(an+b) = Cf_2(cn+d)$ implies that $|F_1|_\alpha(an+b)| = C'|F_2|_{-\alpha}(cn+d)|$, where $C' := |C|^A$. We next relax this equality to an inequality. Borrowing an idea from Davenport and Heilbronn [1], we observe that for $t \in \mathbb{R}$,

$$\max\{1 - |t|, 0\} = \int_{-\infty}^{\infty} e(t\alpha) \left(\frac{\sin(\pi\alpha)}{\pi\alpha} \right)^2 d\alpha. \quad (17)$$

Crucially, this latter expression is absolutely integrable, with integral ≤ 1 . Writing $g_j(n) := \log |F_j(n)|$, well-defined whenever $f_j(n) \neq 0$, and $\rho := \log |C'| \in \mathbb{R}$, we have the trivial implication

$$f_1(an+b) = Cf_2(cn+d) \neq 0 \Rightarrow |g_1(an+b) - g_2(cn+d) - \rho| \leq 1/2,$$

from which it follows immediately that

$$\mathbf{1}_{f_1(an+b)=Cf_2(cn+d) \neq 0} \leq 2 \max\{1 - |g_1(an+b) - g_2(cn+d) - \rho|, 0\}. \quad (18)$$

Combining (18) with (17) and inserting the result into (15), we obtain

$$\begin{aligned} \delta &\leq \frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{\mathbf{1}_{f_1(an+b)=Cf_2(cn+d)}}{n} \\ &\leq 2 \int_{-\infty}^{\infty} \left(\frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{e(g_1(an+b)\alpha)e(-g_2(cn+d)\alpha)}{n} \right) e(-\rho\alpha) \left(\frac{\sin(\pi\alpha)}{\pi\alpha} \right)^2 d\alpha \\ &= 2 \int_{-\infty}^{\infty} \left(\frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{|F_1|_{2\pi\alpha}(an+b)|F_2|_{-2\pi\alpha}(cn+d)}{n} \right) e(-\rho\alpha) \left(\frac{\sin(\pi\alpha)}{\pi\alpha} \right)^2 d\alpha. \end{aligned}$$

Bounding the range $|\alpha| > B$ trivially, we obtain after a change of variables $\alpha \mapsto 2\pi\alpha$ that

$$\delta \leq 4\pi \int_{-B}^B \left(\frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{|F_1|_\alpha(an+b)|F_2|_{-\alpha}(cn+d)}{n} \right) e^{-i\rho\alpha} \left(\frac{\sin(\alpha/2)}{\alpha/2} \right)^2 d\alpha + O\left(\frac{1}{B}\right). \quad (19)$$

Recall that $B \geq \delta^{-2}$ and note that the integrand above is bounded by 1, From the definition (16) of X and (19) we derive when δ is small enough that

$$\frac{\delta}{2} \leq 4\pi \int_X \left| \frac{1}{\log x} \sum_{\substack{n \leq x \\ f_1(an+b)f_2(cn+d) \neq 0}} \frac{|F_1|_\alpha(an+b)|F_2|_{-\alpha}(cn+d)|}{n} \right| \left(\frac{\sin(\alpha/2)}{\alpha/2} \right)^2 d\alpha + \frac{\delta}{4} \leq 4\pi\lambda(X) + \frac{\delta}{4},$$

where $\lambda(X)$ denotes the Lebesgue measure of X . Rearranging, we obtain $\lambda(X) \geq \delta/(16\pi)$, as claimed. \square

As a corollary of this and Tao’s theorem, we deduce the following.

Corollary 3.4. *Assume the notation of Lemma 3.3, and let $f : \mathbb{N} \rightarrow \mathbb{C}$ be the multiplicative function given at prime powers by $f(p^k) = 1$ if $f_1(p^k) = 0$, otherwise $f(p^k) = F_1(p^k)$. Then there are positive integers $m, L = O_\delta(1)$, independent of A , such that for each $\beta \in X$ there is a real number $t_\beta \in [-Lx, Lx]$ such that*

$$\mathbb{D}(|f^m|_\beta, n^{imt_\beta}; x) = O_\delta(1),$$

and the upper bound is independent of A .

Proof. Write $\mathcal{P}_{f_1} := \{p^k : f_1(p^k) = 0\}$, so that $f(p^k) = F_1(p^k)$ for all $p^k \notin \mathcal{P}_{f_1}$. Let $h(n)$ be the multiplicative function given at prime powers by $h(p^k) := 1 - 1_{\mathcal{P}_{f_1}}(p^k)$. Combining Lemma 3.3 with Theorem 2.2, applied to the multiplicative function $g = |f|_\alpha h = |F_1|_\alpha h$, we deduce that there is $L = O_\delta(1)$ such that for each $\alpha \in X$ there is a primitive character ψ_α of conductor $O_\delta(1)$ and a $t_\alpha \in [-Lx, Lx]$ such that

$$\begin{aligned} \mathbb{D}(|f|_\alpha, \psi_\alpha(n)n^{it_\alpha})^2 &= \sum_{p \leq x} \frac{1 - \operatorname{Re}(|f(p)|^{i\alpha} \overline{\psi}_\alpha(p)p^{-it_\alpha})}{p} \\ &= \sum_{p \leq x} \frac{1 - \operatorname{Re}(|F_1(p)|^{i\alpha} h(p) \overline{\psi}_\alpha(p)p^{-it_\alpha})}{p} + O(1) \ll_\delta 1. \end{aligned}$$

Now let $v = O_\delta(1)$ be the maximum of the conductors of all the ψ_α , and set $m := v!$. Then $\psi_\alpha^m(p) = 1$ for all $p \nmid m$. By the pretentious triangle inequality (8), uniformly over $\alpha \in X$ we have

$$\mathbb{D}(|f^m|^{i\alpha}, n^{imt_\alpha}; x)^2 \leq m^2 \mathbb{D}(|f|_\alpha, \psi_\alpha(n)n^{it_\alpha}; x)^2 + O_\delta(1) \ll_\delta 1.$$

Note that the above bound depended only on the validity of (15) and (11), both of which are independent of A , as required. \square

3.2 An application of the approximate Cauchy functional equation

We seek to understand the structure of the map $\beta \mapsto t_\beta$ arising in Corollary 3.4. In this direction we will use an idea of Halász [6], improved upon in a paper by Ruzsa [6], which will allow us, roughly speaking, to approximate this map by a linear map $\beta \mapsto c\beta$, $c \in \mathbb{R}$, on some appropriate *sumset* of $\tilde{X} := m \cdot X'$. The

proof is based on a few lemmas. Here and in the sequel, given subsets T_1, T_2 of an abelian group $(G, +)$, we define the sumset

$$T_1 + T_2 := \{t_1 + t_2 : t_1 \in T_1, t_2 \in T_2\}$$

and for $m \geq 2$ the iterated sumset

$$mT := \{t_1 + \cdots + t_m : t_i \in A \text{ for all } 1 \leq i \leq m\}.$$

In the sequel, for $S \subseteq \mathbb{R}$ and $\lambda > 0$ we will also denote by $\lambda \cdot S$ the dilation

$$\lambda \cdot S := \{\lambda t : t \in S\}.$$

Lemma 3.5. *Let $D > 0$ and let $Y \subseteq [-D, D]$ be a symmetric⁶ set containing 0, and denote its Lebesgue measure by $\lambda(Y)$. Then if $\ell \geq \lfloor 12D/\lambda(Y) \rfloor$ the sumset ℓY contains $[-D, D]$.*

Proof. Setting $Y' := \frac{1}{D} \cdot Y \subseteq [-1, 1]$, we have $\lambda(Y') = \lambda(Y)/D$. Applying [15, Lemma (5.3)] to Y' , we get that $\ell Y'$ contains $[-1, 1]$ as soon as $\ell \geq \lfloor 12/\lambda(Y') \rfloor = \lfloor 12D/\lambda(Y) \rfloor$. The claim then follows, as $y' = y'_1 + \cdots + y'_\ell$ holds with $y'_j \in Y'$ if and only if $Dy' = y_1 + \cdots + y_\ell$ with $y_j \in Y = D \cdot Y'$. \square

Lemma 3.6. *Let $K \geq 0$. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function, and suppose that whenever $\alpha_1, \alpha_2, \alpha_1 + \alpha_2 \in [-B, B]$ we have*

$$|\phi(\alpha_1 + \alpha_2) - \phi(\alpha_1) - \phi(\alpha_2)| \leq K.$$

Then there is a $c \in \mathbb{R}$, specifically $c = \phi(B)/B$, with

$$\sup_{\alpha \in [-B, B]} |\phi(\alpha) - c\alpha| \leq 3K.$$

Proof. Defining $\psi(\beta) := \phi(B\beta)/B$ for $\beta \in [-1, 1]$, this follows directly from applying [15, Lem. 5.11] to ψ and rescaling by B . \square

Proof of Proposition 3.1. Let $\delta \in (0, 1)$ and suppose $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies (12) for $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ and $C \in \mathbb{C}$ and $C(ad - bc) \neq 0$. Thus, there is a sequence $(x_j)_j$ such that for $j \geq 1$ and $x = x_j$, (15) holds. Fix j large for the moment. By Corollary 3.4, there is a set $X \subseteq [-B, B]$ and an integer $m = O_\delta(1)$ such that for each $\beta \in X$ there is $t_\beta = t_\beta(x_j)$ with

$$\mathbb{D}(|f^m|_\beta, n^{im_\beta}; x_j) = O_\delta(1), \tag{20}$$

where we recall that $f(p^k) = F_1(p^k) = f_1(p^k)^A$ whenever $f_1(p^k) \neq 0$. The bound here is independent of A .

Lemma 3.5 shows that when $k = \lceil 100B/\lambda(X) \rceil \asymp_\delta 1$ the sumset kX contains $[-B, B]$. In particular, every $\beta \in [-B, B]$ may be written as

$$\beta = \alpha_1 + \cdots + \alpha_k, \quad \alpha_\ell \in X. \tag{21}$$

⁶We say that a set $T \subseteq \mathbb{R}$ is *symmetric* if $t \in T$ if and only if $-t \in T$.

Define $t_\beta := t_{\alpha_1} + \dots + t_{\alpha_k}$ in this case (if more than one such decomposition exists, pick any of them). This defines a mapping $\beta \mapsto t_\beta$ on all of $[-B, B]$, such that $|t_\beta| \leq kALx_j < x_j^{3/2}$ uniformly, provided x_j is sufficiently large with respect to δ . By the pretentious triangle inequality,

$$\mathbb{D}(|f^m|_\beta, n^{imt_\beta}; x_j) = \mathbb{D}(|f^m|^{i(\alpha_1 + \dots + \alpha_k)}, n^{im(t_{\alpha_1} + \dots + t_{\alpha_k})}; x_j) \leq \sum_{\ell=1}^k \mathbb{D}(|f^m|_{\alpha_\ell}, n^{imt_{\alpha_\ell}}; x_j) \ll_\delta 1.$$

Moreover, if $\beta_1, \beta_2, \beta_1 + \beta_2 \in [-B, B]$ then

$$\begin{aligned} \mathbb{D}(n^{imt_{\beta_1 + \beta_2}}, n^{im(t_{\beta_1} + t_{\beta_2})}; x_j) &\leq \mathbb{D}(|f^m|_{\beta_1 + \beta_2}, n^{imt_{\beta_1 + \beta_2}}; x_j) + \mathbb{D}(|f^m|_{\beta_1} n^{imt_{\beta_2}}; x_j) + \mathbb{D}(|f^m|_{\beta_2}, n^{imt_{\beta_2}}; x_j) \\ &\ll_\delta 1. \end{aligned}$$

Since $m|t_{\beta_1 + \beta_2} - t_{\beta_1} - t_{\beta_2}| = O_\delta(x_j^{3/2}) < x_j^2$ for j sufficiently large, Lemma 2.1 yields

$$\log(1 + m|t_{\beta_1 + \beta_2} - t_{\beta_1} - t_{\beta_2}| \log x_j) \ll_\delta 1.$$

It follows that there is $K = O_\delta(1)$ such that for any $\beta_1, \beta_2 \in [-B, B]$ with $\beta_1 + \beta_2 \in [-B, B]$,

$$|t_{\beta_1 + \beta_2} - t_{\beta_1} - t_{\beta_2}| \leq \frac{K}{\log x_j}.$$

By Lemma 3.6 there is $r_j = r_j(A) \in \mathbb{R}$ such that uniformly over $\beta \in [-B, B]$,

$$t_\beta = r_j \beta + O_\delta\left(\frac{1}{\log x_j}\right);$$

since $|t_\beta| < x_j^{3/2}$, say, for j large enough, we see that $|r_j| < x_j^2$. It follows that

$$p^{im(t_\beta - r_j \beta)} = 1 + O_\delta\left(\frac{mK \log p}{\log x_j}\right) \text{ for all } p \leq x_j^{1/(3mK)}.$$

Applying (8), we find that, uniformly over $\beta \in [-B, B]$,

$$\mathbb{D}(|f^m|_\beta, n^{imr_j \beta}; x_j)^2 = \sum_{p \leq x_j^{1/(3mK)}} \frac{1 - \operatorname{Re}(|f(p)|^{im\beta} p^{-imr_j \beta})}{p} + O\left(\sum_{x_j^{1/(3mK)} < p \leq x_j} \frac{1}{p} + \frac{mK}{\log x_j} \sum_{p \leq x_j^{1/(3mK)}} \frac{\log p}{p}\right) \ll_\delta 1.$$

Equivalently, we may write this as

$$\max_{\beta \in [-B, B]} \sum_{p \leq x_j} \frac{1 - \operatorname{Re}(|f(p)|/p^{r_j})^{im\beta}}{p} \ll_\delta 1, \tag{22}$$

emphasising once again that this bound is independent of A . We may now complete the proof of the proposition. Given (22), after the change of variables $\beta \mapsto m\beta$ we observe that

$$\frac{1}{2mB} \int_{-mB}^{mB} \sum_{p \leq x_j} \frac{1 - \operatorname{Re}(|f(p)|/p^{r_j})^{i\beta}}{p} d\beta = O_\delta(1).$$

Since $\frac{1}{2R} \int_{-R}^R y^{it} dt = \frac{\sin(R \log y)}{R \log y} =: \text{sinc}(R \log y)$, we deduce that

$$\sum_{p \leq x_j} \frac{1 - \text{sinc}(mB \log(|f(p)|/p^{f_j}))}{p} = O_\delta(1).$$

As $1 - \text{sinc}(t) \gg 1$ whenever $|t| \geq 2$, recalling the notation of the proposition and that $f(p) \neq f_1(p)$ on a sparse set, we deduce using $m \geq 1$ that

$$\sum_{\substack{p \leq x_j \\ |F_1(p)|/p^{f_j} \notin [e^{-2/B}, e^{2/B}]} \frac{1}{p} \leq \sum_{\substack{p \leq X \\ |\log(|f(p)|/p^{f_j})| \geq 2/(mB)}} \frac{1}{p} + O(1) = O_\delta(1).$$

Recalling that $F_1 = f_1^A$, the claim follows. □

4 Proof of the Converse Result

Proof of Proposition 1.5. Let $f : \mathbb{N} \rightarrow \mathbb{R} \setminus \{0\}$ be multiplicative, satisfying the condition

$$\sum_{p: |f(p)| \neq 1} \frac{1}{p} < \infty. \tag{23}$$

Observe that if there is $c'_f > 0$ such that

$$\frac{1}{\log x} \sum_{n \leq x} \frac{1_{|f(n)|=|f(n+2)|}}{n} \geq c'_f + o(1), \tag{24}$$

then by the pigeonhole principle there is a $u_x \in \{-1, +1\}$ such that if $c_f := c'_f/2$ then

$$\frac{1}{\log x} \sum_{n \leq x} \frac{1_{f(n)=u_x f(n+2)}}{n} \geq c_f + o(1).$$

Pigeonholing once again, we can choose an infinite sequence $(x_j)_j$ on which $u = u_{x_j} \in \{-1, +1\}$ is constant, in which case we obtain

$$\limsup_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{1_{f(n)=u f(n+2)}}{n} > 0,$$

as claimed.

Moreover, the condition (23) is unchanged when passing from f to $|f|$. Thus, in the sequel, we replace f by $|f|$ and assume that $f \geq 0$. We will show that (24) holds whenever $x \geq x_0(f)$.

By partial summation, it will suffice to show that

$$\sum_{n \leq x} 1_{f(n)=f(n+2)} \geq (c'_f + o(1))x$$

for all $x \geq x_0(f)$. To this end, we restrict to $n \equiv 1 \pmod{2}$, so that n and $n+2$ are coprime, and also such that $n, n+2$ are both squarefree. Observe that for such n , if $f(p) = 1$ for all $p|n$ and all $p|n+2$ then $f(n) = 1 = f(n+2)$. Thus, we obtain

$$\sum_{n \leq x} 1_{f(n)=f(n+2)} \geq \sum_{\substack{n \leq x \\ p|n(n+2) \Rightarrow f(p)=1}} \mu^2(n)\mu^2(n+2)1_{2 \nmid n}. \tag{25}$$

Take $z := (\log x)^{0.99}$. As $\sum_{p:f(p) \neq 1} p^{-1} < \infty$, we find

$$\sum_{\substack{n \leq x \\ \exists p > z: \\ p|n(n+2), f(p) \neq 1}} 1 \ll \sum_{\substack{z < p \leq x \\ f(p) \neq 1}} \left(\frac{2x}{p} + 1 \right) \ll x \left(\sum_{\substack{p > z \\ f(p) \neq 1}} \frac{1}{p} + \frac{1}{\log x} \right) = o(x).$$

It follows that

$$\sum_{\substack{n \leq x \\ p|n(n+2) \Rightarrow f(p)=1}} \mu^2(n)\mu^2(n+2)1_{2 \nmid n} = \sum_{\substack{n \leq x \\ p|n(n+2), p \leq z \Rightarrow f(p)=1}} \mu^2(n)\mu^2(n+2)1_{2 \nmid n} + o(x).$$

If we set

$$P(z) := \prod_{\substack{3 \leq p \leq z \\ f(p) \neq 1}} p,$$

then the prime number theorem trivially gives $P(z) \leq x^{0.01}$ for large enough x . Thus, for any divisors $d_1, d_2 | P(z)$ we have $d_1 d_2 \leq x^{0.02}$. Since $2 \nmid P(z)$, by Möbius inversion we obtain

$$\sum_{\substack{n \leq x \\ p|n(n+2), p \leq z \Rightarrow f(p)=1}} \mu^2(n)\mu^2(n+2)1_{2 \nmid n} = \sum_{\substack{d_1, d_2 | P(z) \\ (d_1, d_2)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d_1} \\ n+2 \equiv 0 \pmod{d_2}}} \mu^2(n)\mu^2(n+2)1_{2 \nmid n}.$$

By e.g. [12, Lem. 3.3] (taking $a = 2$) the inner sum in this last expression is, for each pair of coprime divisors d_1, d_2 of $P(z)$,

$$x \frac{\phi(d_1)\phi(d_2)}{2(d_1 d_2)^2} \prod_{p|d_1 d_2} \left(1 - \frac{2}{p^2}\right) + O_\varepsilon(x^{2/3+\varepsilon}).$$

Summing over all such d_1, d_2 , we thus obtain

$$Cx \sum_{\substack{d_1, d_2 | P(z) \\ (d_1, d_2)=1}} \frac{\mu(d_1)\mu(d_2)\phi(d_1)\phi(d_2)}{(d_1 d_2)^2} \prod_{p|d_1 d_2} \left(1 - \frac{2}{p^2}\right)^{-1} + O(x^{0.7}),$$

where $C := \frac{1}{2} \prod_p (1 - 2p^{-2})$. As

$$\sum_{\substack{d \geq 1 \\ p|d \Rightarrow f(p) \neq 1}} \frac{1}{d} \ll \exp \left(\sum_{p:f(p) \neq 1} \frac{1}{p} \right) \ll 1,$$

due to the trivial bound $\phi(d_1) \leq d_1$ the double series in $d_1, d_2 | P(z)$ converges absolutely as $z \rightarrow \infty$, and therefore

$$\begin{aligned} & C \sum_{\substack{d_1, d_2 | P(z) \\ (d_1, d_2) = 1}} \frac{\mu(d_1)\mu(d_2)\phi(d_1)\phi(d_2)}{(d_1 d_2)^2} \prod_{p|d_1 d_2} \left(1 - \frac{2}{p^2}\right)^{-1} \\ &= C \sum_{\substack{d_1, d_2 \geq 1 \\ p|d_1 d_2 \Rightarrow f(p) \neq 1 \\ (d_1, d_2) = 1 \\ 2 \nmid d_1 d_2}} \frac{\mu(d_1)\mu(d_2)\phi(d_1)\phi(d_2)}{(d_1 d_2)^2} \prod_{p|d_1 d_2} \left(1 - \frac{2}{p^2}\right)^{-1} + o(1). \end{aligned}$$

Upon computing Euler products we find the asymptotic estimate

$$\sum_{\substack{n \leq x \\ p|n(n+2) \Rightarrow f(p) = 1}} \mu^2(n)\mu^2(n+2)1_{2 \nmid n} = (c'_f + o(1))x,$$

where c'_f is defined by the (convergent) product

$$c'_f := \prod_{\substack{p \geq 3 \\ f(p) \neq 1}} \left(1 - \frac{2(p-1)}{p^2 - 2}\right) > 0.$$

Combined with (25), this leads to the lower bound

$$\sum_{n \leq x} 1_{f(n) = f(n+2) \neq 0} \geq (c'_f + o(1))x,$$

as claimed. □

5 Proof of Corollary 1.6

Let ϕ be a holomorphic cuspidal eigenform without CM for $\mathrm{SL}_2(\mathbb{Z})$, of weight $k \geq 2$, normalised so its sequence of Fourier coefficients $(a_\phi(n))_n$ satisfies $a_\phi(1) = 1$. It is well-known that the map $n \mapsto a_\phi(n)$ is multiplicative, so that $f = a_\phi$ fits the setup of Theorem 1.1.

To prove Corollary 1.6 it suffices to check that (1) holds for $a_\phi(n)$, after which the claim then follows from Theorem 1.1 (applied to each of the equations $a_\phi(n) = a_\phi(n+j)$, $1 \leq |j| \leq h$).

Set $\lambda_\phi(n) := a_\phi(n)/n^{(k-1)/2}$. By the prime number theorem for Rankin-Selberg L -functions (e.g. combining [10, Cor. 1.2 and Lem. 5.1] with [7, Exer. 6]),

$$\sum_{p \leq X} |\lambda_\phi(p)|^2 \log p = X + O(Xe^{-c\sqrt{\log X}}), \tag{26}$$

for some $c = c_\phi > 0$. We claim that if y_0 is sufficiently large and $y \geq y_0$ then

$$\sum_{\substack{y \leq p \leq 2y \\ |a_\phi(p)| \neq 1}} \log p \geq y/5. \tag{27}$$

Indeed, suppose (27) fails for some $y \geq y_0$, with y_0 large enough. From (26) and Deligne’s bound $|a_\phi(p)| \leq 2p^{(k-1)/2}$ we obtain

$$\sum_{\substack{y \leq p \leq 2y \\ |a_\phi(p)|=1}} |a_\phi(p)p^{-(k-1)/2}|^2 \log p \geq 9y/10 - 4 \sum_{\substack{y \leq p \leq 2y \\ |a_\phi(p)| \neq 1}} \log p > y/10.$$

But by the prime number theorem, the left-hand side of this last expression is

$$\leq y^{1-k} \sum_{y \leq p \leq 2y} \log p \ll y^{2-k},$$

which is impossible for y_0 large enough and $k \geq 2$.

By a dyadic decomposition and (27), we find

$$\sum_{\substack{p \leq x \\ |a_\phi(p)| \neq 1}} \frac{1}{p} \geq \sum_{y_0 \leq 2^j \leq x/2} \frac{1}{2^{j+1} \log(2^{j+1})} \sum_{\substack{2^j \leq p \leq 2^{j+1} \\ |a_\phi(p)| \neq 1}} \log p \geq \frac{1}{10 \log 2} \sum_{y_0 \leq 2^j \leq x/2} \frac{1}{j+1} = c \log \log x + O(1),$$

with $c = 1/(10 \log 2)$, so (1) clearly holds as well.

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