A stochastic threshold to predict extinction and persistence of an epidemic SIRS system with a general incidence rate

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Abstract

This work aims to give a detailed analysis of a stochastic epidemic model with a general incidence rate $g(S)I$. We introduce the generalized stochastic threshold $\mathcal{R}_s(g)$ that will be used as a threshold condition of extinction, persistence and existence of an ergodic stationary distribution. We also investigate the critical case when $\mathcal{R}_s(g) = 1$. Numerical illustrations of the findings are given via different types of function g.

Key words: Stochastic epidemic model, Stochastic stability, Persistence, Stationary distribution.

1 Introduction

Mathematical models, with the aid of computer simulations, are useful for building and testing theories about complex biological systems involving disease, assessing quantitative conjectures, determining sensitivities to changes in parameter values, estimating key parameters from data, and also applying optimal control on some parameters [43, 46, 47, 44]. Modeling is especially crucial in epidemiology since, in most cases, we not aware of the level of complexity that leads to the spread of the disease and cannot do experiments. Many authors have recently proposed and investigated various types of epidemic models to understand disease transmission mechanisms such as tuberculosis, measles, influenza (see, for instance, [7, 8, 10, 15, 13, 16] and the references cited therein). A key role in mathematical models of infectious diseases is played by the so-called incidence rate, namely a function describing the mechanism of transmission of the disease. In many epidemiological models, the corresponding incidence rate is bilinear with respect to the numbers of susceptible and infective individuals. More specifically, if $S(t)$ and $I(t)$ are the fractions of susceptible and infective individuals in the population, and if β is the per capita contact rate, then the principle of mass action implies that the infection spreads with the rate βSI. A rigorous mathematical model with a bilinear incidence rate was

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first introduced by Kermack and Mckendrick in 1927 [11]. Their model, known as the SIR model, has been based on all later epidemic models[9, 34, 8, 36]. One of the derived models is the SIRS epidemic model, where it is assumed that a recovered person does not necessarily acquire permanent immunity and may become susceptible again. The SIRS model can apply to many infectious diseases such as polio, tetanus, diphtheria, measles, hepatitis, chickenpox, influenza, measles, mumps, rubella, AIDS, and others [35, 42, 2]. Since the first deterministic SIRS model with constant recruitment and disease-induced death was developed by Anderson and May [1], there are many applications of such models. For instance, the host-vector SIRS models have been proposed to study spread of Japanese Encephalitis malaria [41]. The following differential system describes a simple SIRS epidemic model (see,e.g., [9, 31] and the references cited therein).

$$\begin{cases} dS(t) = [\mu - \mu S(t) - \beta S(t)I(t) + \gamma R(t)]\, dt, \\ dI(t) = [-(\mu+\lambda)I(t) + \beta S(t)I(t)]\, dt, \\ dR(t) = [-(\mu+\gamma)R(t) + \lambda I(t)]\, dt, \end{cases} \tag{1}$$

where $R(t)$ represents the fractions of individuals who have been removed from the possibility of infection. The constants μ, λ, and γ are positive constants that stand for birth and death rates, recovery rate of the infective individuals, and losing immunity rate, respectively.

However, the bilinear contact law is more appropriate for communicable diseases such as influenza, but not for sexually transmitted diseases. There are several reasons why this standard bilinear incidence rate may require modification. For instance, the underlying assumption of homogeneous mixing and homogeneous environment may be invalid. In this case the necessary population structure and heterogeneous mixing may be incorporated into a model with a specific form of nonlinear transmission. For example, Capasso and Serio [27] proposed a saturated incidence rate $g(I)S$ into epidemic models where g is a nonlinear function. Ruan and Wang [28] investigated an epidemic model with a nonlinear incidence rate $g(I)S = \frac{\beta I^2 S}{1+\rho I^2}$. Liu et al. [32] introduced a general incidence rate $g(S,I)$ where they treated special cases like $\beta S^p I^q$ and also $S^q g(I)$, their goal was to prove the existence of periodic solutions when the incidence rate is similar to the previous forms. Lahrouz et al. [31] and recently Liu and Qingmei [29] proposed a more generalized and more realistic incidence rate $g(I)S = \frac{SI}{f(I)}$. Abta et al. [33] also took the number of the susceptible $S(t)$ in the nonlinearity of the incidence rate by considering the bilinear function rate $g(S,I) = \frac{\beta IS}{1+aI+bS}$. Recently, Gan and Wei [49] studied an stochastic epidemic model with delay using a general incidence rate $g(S)f(I)$. Motivated by the previous works on nonlinear incidence rate and to make system (1) more realistic and interesting with respect to the number of the susceptible individuals $S(t)$, we adopt the nonlinear incidence rate $\beta g(S)I$, so system (1) becomes

$$\begin{cases} dS(t) = (\mu - \mu S(t) - \beta g(S(t))I(t) + \gamma R(t))\, dt, \\ dI(t) = (-(\mu+\lambda)I(t) + \beta g(S(t))I(t))\, dt, \\ dR(t) = (-(\mu+\gamma)R(t) + \lambda I(t))\, dt, \end{cases} \tag{2}$$

where $g(S) \geq 0$ is a continuously differentiable function with $g(0) = 0$. The positive aspect of this choice is to avoid a constant rate of change for the derivative of $g(S)$. Unlike the bilinear incidence rate βSI, where the partial derivative with respect to S of g is always equal to 1, the nonlinear incidence rate $g(S)I$ offers more intersting dyanmics. On the other hand, epidemic models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism compared to their deterministic counterparts. Many scholars have studied the effect of stochasticity on epidemic models (see, e.g., [3, 4, 5, 6, 12, 13, 14, 15, 16, 17, 18, 20] and the references cited therein). Lu [14] introduced stochasticity into the SIRS model (2) via the technique of parameter perturbation. He replaced the infection coefficient β by $\beta + \sigma\frac{dB}{dt}$, where B is a Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and σ is the intensity of the noise. So, the stochastic SIRS epidemic model with nonlinear incidence rate takes the following form

$$\begin{cases} dS(t) = [\mu - \mu S(t) - \beta g(S(t))I(t) + \gamma R(t)]\,dt - \sigma g(S(t))I(t)dB_t, \\ dI(t) = [-(\mu+\lambda)I(t) + \beta g(S(t))I(t)]\,dt + \sigma g(S(t))I(t)dB_t, \\ dR(t) = [-(\mu+\gamma)R(t) + \lambda I(t)]\,dt, \end{cases} \tag{3}$$

where $B(t)$ is a Brownian motion and $\sigma > 0$ (small compared with β) is the intensity of the environmental noise on the infection coefficient β. Recently, Lahrouz and Settati [30] have considered the stochastic model (3) with $g(S) = S$. They have proved the conjecture of Tornatore et al.[12] and improved all conditions in [12, 14, 16, 17] by extending the deterministic threshold $\mathcal{R}_0$ to the stochastic threshold one

$$\mathcal{R}_s \triangleq \frac{\beta}{\mu + \lambda + \frac{1}{2}\sigma^2}. \tag{4}$$

That is, under condition

$$\mathcal{R}_s < 1, \quad \beta \geq \sigma^2, \tag{5}$$

the infection-free equilibrium state E_0 is globally asymptotically stable in probability. While, if condition $\mathcal{R}_s > 1$ is verified, each component of the solutions $(S(t), I(t), R(t))$ rises to or above certain positive level infinitely often with probability one. In addition, they have established the persistence in mean and the existence of a unique ergodic stationary distribution under condition $\mathcal{R}_s > 1$. Motivated by the previous works, in this paper, we shall extend the threshold (4) to our more general one

$$\mathcal{R}_s(g) \triangleq \frac{\beta g(1)}{\mu + \lambda + \frac{1}{2}(g(1))^2\sigma^2}. \tag{6}$$

Our conjecture is that, the threshold (6) is the natural generalization of (4). That is so for its corresponding conditions. Specifically, the asymptotic stability conditions (5) will become

$$\mathcal{R}_s(g) < 1, \quad \beta \geq g(1)\sigma^2,$$

and if $\mathcal{R}_s(g) > 1$ then system (3) is persistent in mean and admits a unique ergodic stationary distribution. We also shall investigate the critical case, when $\mathcal{R}_s(g) = 1$, that is to say, if $\mathcal{R}_s(g) = 1$ and $\beta \geq \sigma^2(1 \vee g(1))$ then system (3) is extinctive. The rest of this article is organized as follows: In Section 2, the global stability of the disease-free equilibrium is proven. In Section 3, we show that the disease persists in mean. Section 4 covers the convergence of the stochastic model towards an endemic stationary distribution. In Section 5,we provide some numerical simulations to support our findings. In the last section, Section 6, we provide a brief discussion and the summary of the main results.

2 The global stability of the disease-free equilibrium state

In this section, we will discuss the extinction of SDE system (3) in order to provide the threshold condition for disease control or eradication. We introduce the notation

$$\mathbb{R}^3_+ = \{(x_1, x_2, x_3)|x_i > 0,\ i = 1, 2, 3\}.$$

To begin the analysis of the model, define the subset

$$\Delta = \left\{x \in \mathbb{R}^3_+;\ x_1 + x_2 + x_3 = 1\right\}.$$

System (3) can be written as the following form:

$$dX(t) = f(X(t))dt + h(X(t))dB_t, \tag{7}$$

where $X(t) = (S(t), I(t), R(t))$, B is a Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The diffusion matrix and the generator $\mathcal{L}$ associated with (7) are defined respectively by

$$G(X) = h(X)h^T(X), \quad \mathcal{L}V(X) = f^T(X).\nabla V(X) + \frac{1}{2}Tr\left(h^T(X).\nabla^2 V(X).h(X)\right), \tag{8}$$

for any twice continuously differentiable $V(.)$. To ensure that the model is well posed and thus biologically meaningful, we need to prove that the solution remains in Δ. By the similar proof of Theorem 2 in [16], we have the following theorem.

Theorem 2.1. *The set Δ is almost surely positively invariant by the system (3), that is, for any initial values $(S_0, I_0, R_0) \in \Delta$, we have $\mathbb{P}((S(t), I(t), R(t)) \in \Delta) = 1$ for all $t \geq 0$.*

Before investigating the critical case $\mathcal{R}_s(g) = 1$, we put forward the following Lemma which is a special case of Proposition 5.1 in Settati et al. [19].

Lemma 2.2. *For any initial values $(S_0, I_0, R_0) \in \Delta$, it holds that*

$$\frac{\lambda}{\mu+\gamma}\liminf_{t\to\infty} I(t) \leq \liminf_{t\to\infty} R(t) \leq \limsup_{t\to\infty} R(t) \leq \frac{\lambda}{\mu+\gamma}\limsup_{t\to\infty} I(t).$$

Theorem 2.3. *For any initial values $(S_0, I_0, R_0) \in \Delta$, if $\mathcal{R}_s(g) = 1$ and $\beta \geq \sigma^2(1 \vee g(1))$ then the solution of SDE (3) obeys*

$$\lim_{t\to\infty} S(t) = 1, \quad \lim_{t\to\infty} I(t) = 0, \quad \lim_{t\to\infty} R(t) = 0 \quad a.s.$$

Proof. Let $\varepsilon < I_0$. Define the stopping time

$$\tau_\epsilon = \inf\{t > 0,\ I(t) \leq \varepsilon\}, \qquad \tau'_\epsilon = \inf\{t > \tau_\epsilon,\ I(t) \geq \varepsilon\}.$$

We claim that $\mathbb{E}(\tau_\varepsilon) < \infty$. For all $T > 0$ and $t \leq T \wedge \tau_\varepsilon$, we have $I(t) \geq \varepsilon$. Applying Itô's formula on $t \in (0, T \wedge \tau_\varepsilon)$, gives if $\sigma^2 \leq \beta$,

$$\begin{aligned} d(\log(I)) &= \left(-(\mu+\lambda) + \beta g(S) - \frac{1}{2}\sigma^2 (g(S))^2\right) dt + \sigma g(S) dB, \\ &\leq \left(-(\mu+\lambda) + \beta g(1-I) - \frac{1}{2}\sigma^2 (g(1-I))^2\right) dt + \sigma g(S) dB, \\ &\leq \left(-(\mu+\lambda) + \beta g(1-\varepsilon) - \frac{1}{2}\sigma^2 (g(1-\varepsilon))^2\right) dt + \sigma g(S) dB. \end{aligned} \tag{9}$$

where the last inequality is obtained by studying the function $\Phi(x) \triangleq -\frac{1}{2}\sigma^2 x^2 + \beta x - (\mu+\lambda)$. So, $\Phi(x)$ is increasing on $(0,1)$ and then $\Phi(g(S)) < \Phi(g(1-I)) < \Phi(g(1-\varepsilon))$. One can see that there exists $l_1 \in (1-\varepsilon, 1)$ such that

$$g(1-\varepsilon) = g(1) - \varepsilon g'(l_1)$$

and then (9) implies

$$\begin{aligned} d\left(\log(I)\right) &\leq \left(\beta g(1)\left(1 - \frac{1}{\mathcal{R}_s(g)}\right) + \left(-\beta + \sigma^2 g(1)\right)\left(\varepsilon g'(l_1)\right) - \frac{1}{2}\sigma^2\left(\varepsilon g'(l_1)\right)^2\right) dt \\ &\quad + \sigma g(S) dB. \end{aligned} \tag{10}$$

Integrating the above inequality between 0 and $T \wedge \tau_\varepsilon$, and taking expectation in both sides, we have if $\mathcal{R}_s(g) = 1$

$$\left(\left(\beta - \sigma^2 g(1)\right)\left(\varepsilon g'(l_1)\right) + \frac{1}{2}\sigma^2\left(\varepsilon g'(l_1)\right)^2\right) E(T \wedge \tau_\varepsilon) \leq -E(\log I(T \wedge \tau_\varepsilon)) + E(\log I_0). \tag{11}$$

By letting $T \to \infty$ and using Fatou's lemma, if $\beta \geq \sigma^2 g(1)$, (11) yields

$$E(\tau_\varepsilon) \leq -\frac{\log \varepsilon}{\left(\beta - \sigma^2 g(1)\right)\left(\varepsilon g'(l_1)\right) + \frac{1}{2}\sigma^2\left(\varepsilon g'(l_1)\right)^2}.$$

Thus

$$\mathbb{P}\left(\tau_\epsilon < \infty\right) = 1. \tag{12}$$

Second, we claim that

$$\mathbb{P}\left(\tau'_\epsilon = \infty\right) = 1. \tag{13}$$

Assume that (13) is not true. That is, $\mathbb{P}\left(\tau'_\epsilon < \infty\right) > 0$. Then, define the stopping time

$$\tau''_\epsilon = \inf\left\{t > \tau'_\epsilon,\ I(t) < \varepsilon\right\}. \tag{14}$$

For all $T' > 0$ and $T' \wedge \tau'_\epsilon \leq t \leq T \wedge \tau''_\epsilon$, we have $I(t) \geq \varepsilon$. Integrating (10) between $T' \wedge \tau'_\epsilon$ and $T' \wedge \tau''_\epsilon$ and taking expectation, gives

$$\begin{aligned} \mathbb{E}\left[\log I\left(T' \wedge \tau''_\epsilon\right) - \log I\left(T' \wedge \tau'_\epsilon\right)\right] &\leq \mathbb{E}\left(T' \wedge \tau''_\epsilon - T' \wedge \tau'_\epsilon\right) \\ &\quad \times \left(\left(-\beta + \sigma^2 g(1)\right)\left(\varepsilon g'(l_1)\right) - \frac{1}{2}\sigma^2\left(\varepsilon g'(l_1)\right)^2\right). \end{aligned}$$

Since

$$\mathbb{E}\left[\log I\left(T' \wedge \tau''_\epsilon\right) \mathcal{X}_{(\tau'_\epsilon = \infty)}\right] = \mathbb{E}\left[\log I\left(T' \wedge \tau'_\epsilon\right) \mathcal{X}_{(\tau'_\epsilon = \infty)}\right],$$

and

$$\mathbb{E}\left[\mathcal{X}_{(\tau'_\epsilon = \infty)}\left(T' \wedge \tau''_\epsilon - T' \wedge \tau'_\epsilon\right)\right] = 0,$$

we deduce that

$$\begin{aligned} \mathbb{E}\left[\log I\left(T' \wedge \tau''_\epsilon\right) \mathcal{X}_{(\tau'_\epsilon < \infty)}\right] - \mathbb{E}\left[\log I\left(T' \wedge \tau'_\epsilon\right) \mathcal{X}_{(\tau'_\epsilon < \infty)}\right] &\leq \mathbb{E}\left[\mathcal{X}_{(\tau'_\epsilon < \infty)}\left(T' \wedge \tau''_\epsilon - T' \wedge \tau'_\epsilon\right)\right] \\ &\quad \times \left(\left(-\beta + \sigma^2 g(1)\right)\left(\varepsilon g'(l_1)\right) - \frac{1}{2}\sigma^2\left(\varepsilon g'(l_1)\right)^2\right). \end{aligned} \tag{15}$$

By the monotone convergence theorem, we get

$$\lim_{T'\longrightarrow\infty}\mathbb{E}\left[\mathcal{X}_{(\tau'_\epsilon<\infty)}T'\wedge\tau''_\epsilon\right]=\mathbb{E}\left[\mathcal{X}_{(\tau'_\epsilon<\infty)}\tau''_\epsilon\right],\quad \lim_{T'\longrightarrow\infty}\mathbb{E}\left[\mathcal{X}_{(\tau'_\epsilon<\infty)}T'\wedge\tau'_\epsilon\right]=\mathbb{E}\left[\mathcal{X}_{(\tau'_\epsilon<\infty)}\tau'_\epsilon\right].\quad(16)$$

One can easily verify that I is decreasing on the left of τ''_ϵ. Hence, the negative function

$$F(T')\triangleq\log I\left(T'\wedge\tau''_\epsilon\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}$$

is also decreasing for T' sufficiently large and

$$\lim_{T'\longrightarrow\infty}F(T')=\log\epsilon\times\mathcal{X}_{(\tau'_\epsilon<\infty)}.$$

Applying the monotone convergence theorem to the non-negative function $-F(T')$, yields

$$\lim_{T'\longrightarrow\infty}\mathbb{E}\left[\log I\left(T'\wedge\tau''_\epsilon\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}\right]=\log\epsilon\times\mathbb{P}\left(\tau'_\epsilon<\infty\right).\quad(17)$$

On the other hand, for $T'>\tau_\epsilon$ we can easily have

$$\log I\left(T'\wedge\tau'_\epsilon\right)\leq\log\epsilon,$$

and

$$\lim_{T'\longrightarrow\infty}\log I\left(T'\wedge\tau'_\epsilon\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}=\log\epsilon\mathcal{X}_{(\tau'_\epsilon<\infty)}.\quad(18)$$

So, by the Fatou's lemma applying to the non-negative function

$$G(T')\triangleq\left(\log\epsilon-\log I\left(T'\wedge\tau'_\epsilon\right)\right)\mathcal{X}_{(\tau'_\epsilon<\infty)},$$

we get

$$\mathbb{E}\left[\liminf_{T'\longrightarrow\infty}\left(\log\epsilon-\log I\left(T'\wedge\tau'_\epsilon\right)\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}\right]\leq\liminf_{T'\longrightarrow\infty}\mathbb{E}\left[\left(\log\epsilon-\log I\left(T'\wedge\tau'_\epsilon\right)\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}\right].\quad(19)$$

From (18) and (19) we obtain

$$\limsup_{T'\longrightarrow\infty}\mathbb{E}\left[\log I\left(T'\wedge\tau'_\epsilon\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}\right]\leq\epsilon\times\mathbb{P}\left(\tau'_\epsilon<\infty\right).\quad(20)$$

By (15), (16), (17) and (20) one can easily deduce

$$0\leq-\left((\beta-\sigma^2g(1))\left(\varepsilon g'(l_1)\right)+\frac{1}{2}\sigma^2\left(\varepsilon g'(l_1)\right)^2\right)\mathbb{E}\left[\left(\tau''_\epsilon-\tau'_\epsilon\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}\right].$$

By $g'(l_1)\geq 0$ and (14), if $\beta\geq\sigma^2g(1))$, then

$$\mathbb{E}\left[\left(\tau''_\epsilon-\tau'_\epsilon\right)\mathcal{X}_{(\tau'_\epsilon<\infty)}\right]=0.$$

Consequently

$$\tau''_\epsilon-\tau'_\epsilon=0\quad\text{for almost } w\in\left(\tau'_\epsilon<\infty\right).$$

This contradicts the definition of τ'_ϵ and τ''_ϵ. Thus, our claim is true. Now, combining (12) and (13), one can write that for every $\epsilon>0$, and for almost all $w\in\Omega$, there exists $\tau_\epsilon(w)>0$ such that

$$I(t,w)<\epsilon\quad\forall t\geq\tau_\epsilon(w).$$

This means that $\lim_{t\to\infty}I(t)=0$ as.. Combining it with Lemma 2.2 yields $\lim_{t\to\infty}R(t)=0$ and then $\lim_{t\to\infty}S(t)=1$. □

With reference to Khasminskii et al. [21, 22] and Yuan and Mao [23], we have the following lemma giving sufficient condition for asymptotical stability in probability in term of Lyapunov fuctions. We refer to Khasminskii et al. [22] for the precise meaning of asymptotical stability in probability.

Lemma 2.4. *Assume that there are functions $V \in \mathcal{C}^2\left(\mathbb{R}^3;\mathbb{R}^+\right)$ and $w \in \left(\mathbb{R}^3;\mathbb{R}^+\right)$ vanishing only at $E_0(1,0,0)$ such that $\underset{|x|\to\infty}{V}(x) = \infty$ and*

$$\mathcal{L}V(X(t)) \leq -w(X(t)) \quad \forall t > 0, \tag{21}$$

then the equilibrium $E_0(1,0,0)$ of the system (3) is globally asymptotically stable in probability.

Thereafter, we suppose that g is a non-negative $\mathcal{C}^1$ function with $g(0) = 0$ and $g'(S) \geq 0$ on $(0,1)$.

Theorem 2.5. *For any initial values $(S_0, I_0, R_0) \in \Delta$. If $\beta \geq g(1)\sigma^2$ and $\mathcal{R}_s(g) < 1$ then the disease-free E_0 of system (3) is globally asymptotically stable in probability.*

Proof. Let $(S(0), I(0), R(0)) \in \Delta$. Let us define the Lyapunov functions

$$V(S,I,R) = \theta_1(1-S)^2 + \kappa I^{\frac{1}{\kappa}} + \theta_2 R^2, \tag{22}$$

where θ_1, κ and θ_2 are real positive constants to be chosen in the following. We have

$$\begin{aligned}\mathcal{L}V = &-2\theta_1\mu(1-S)^2 + 2\theta_1\beta g(S)I(1-S) - 2\theta_1\gamma R(1-S) + \theta_1\sigma^2(g(S))^2 I^2 - (\mu+\lambda)I^{\frac{1}{\kappa}} \\ &+\beta g(S)I^{\frac{1}{\kappa}} + \frac{1}{2}\left(\frac{1}{\kappa}-1\right)\sigma^2(g(S))^2 I^{\frac{1}{\kappa}} - 2\theta_2(\mu+\gamma)R^2 + 2\theta_2\lambda IR.\end{aligned}$$

Since $S, I \in (0,1)$ and $I \leq 1-S$, we have, for all $\kappa \geq 1$,

$$\begin{aligned}\mathcal{L}V \leq &-2\theta_1\mu(1-S)^2 + 2\theta_1 g(1)\beta I^{\frac{1}{\kappa}} + \theta_1\sigma^2(g(1))^2 I^{\frac{1}{\kappa}} - (\mu+\lambda)I^{\frac{1}{\kappa}} + \beta g(S)I^{\frac{1}{\kappa}} \\ &+\frac{1}{2}\left(\frac{1}{\kappa}-1\right)\sigma^2(g(S))^2 I^{\frac{1}{\kappa}} - 2\theta_2(\mu+\gamma)R^2 + 2(\theta_2\lambda - \theta_1\gamma)IR.\end{aligned}$$

Hence, by choosing $\theta_2 < \frac{\theta_1\gamma}{\lambda}$, we get, for $\kappa \geq 1$,

$$\begin{aligned}\mathcal{L}V \leq &-2\theta_1\mu(1-S)^2 - 2\theta_2(\mu+\gamma)R^2 + I^{\frac{1}{\kappa}}\Bigg(\theta_1 g(1)(2\beta + g(1)\sigma^2) + \frac{1}{2\kappa}\sigma^2(g(1))^2 \\ &-(\mu+\lambda) + \beta g(S) - \frac{1}{2}\sigma^2(g(S))^2\Bigg).\end{aligned} \tag{23}$$

By the mean value theorem one can see that there exists $c \in (0,1)$ such that

$$g(S) = g(1) - (1-S)g'(c)$$

and then (23) implies

$$\begin{aligned}\mathcal{L}V \leq &-2\theta_1\mu(1-S)^2 - 2\theta_2(\mu+\gamma)R^2 + I^{\frac{1}{\kappa}}\Bigg(\theta_1 g(1)(2\beta + g(1)\sigma^2) + \frac{1}{2\kappa}\sigma^2(g(1))^2 \\ &-(\mu+\lambda) + \beta g(1) - \frac{1}{2}\sigma^2(g(1))^2 - \beta(1-S)g'(c) + \sigma^2(1-S)g(1)g'(c) \\ &-\frac{1}{2}\sigma^2(1-S)^2(g'(c))^2\Bigg).\end{aligned}$$

From (6) we have

$$\mathcal{L}V \leq -2\theta_1\mu(1-S)^2 - 2\theta_2(\mu+\gamma)R^2 + I^{\frac{1}{\kappa}}\Big(\theta_1 g(1)(2\beta + g(1)\sigma^2) + \frac{1}{2\kappa}\sigma^2(g(1))^2 + \Gamma\left((1-S)g'(c)\right)\Big), \tag{24}$$

where

$$\Gamma(z) = \beta g(1)\left(1 - \frac{1}{\mathcal{R}_s(g)}\right) + (-\beta + \sigma^2 g(1))z - \frac{1}{2}\sigma^2 z^2. \tag{25}$$

One can easily show, that if $\beta \geq \sigma^2 g(1)$ and $g'(S) \geq 0$ on $(0,1)$ then the function $\Gamma(.)$ is decreasing on $\mathbb{R}_+$ and $\Gamma\left((1-S)g'(c)\right) \leq \Gamma(0)$. So, (24) implies

$$\mathcal{L}V \leq -2\theta_1\mu(1-S)^2 - 2\theta_2(\mu+\gamma)R^2 + I^{\frac{1}{\kappa}}\left(\theta_1 g(1)(2\beta + g(1)\sigma^2) + \frac{1}{2\kappa}\sigma^2(g(1))^2 + \beta g(1)\left(1 - \frac{1}{\mathcal{R}_s(g)}\right)\right). \tag{26}$$

By $\mathcal{R}_s(g) < 1$, we can choose a sufficiently large κ and a sufficiently small θ_1 such that

$$\theta_1 g(1)(2\beta + g(1)\sigma^2) + \frac{1}{2\kappa}\sigma^2(g(1))^2 + \beta g(1)\left(1 - \frac{1}{\mathcal{R}_s(g)}\right) < 0,$$

which means that the coefficients of $(1-S)^2, I^{\frac{1}{\kappa}}$ and R^2 in (27) are all negatives. According to Lemma 2.4 the proof is completed. □

3 Disease persistence

In the epidemic models, persistence is an important property because it implies that the disease continues to exist for any initial conditions. To study the persistence of SDE model (3) we need the following lemma (see Lemma 17 in [25] or Lemma 4 in [26]).

Lemma 3.1. *Suppose $X \in \mathcal{C}(\mathbb{R}_+ \times \Omega, \mathbb{R}_+)$ and $Y \in \mathcal{C}(\mathbb{R}_+ \times \Omega, \mathbb{R})$. If there exist positive constants ν_0 and ν such that for all $t \geq 0$*

$$\log X(t) \geq \nu_0 t - \nu \int_0^t X(u)du + Y(t) \quad \textit{and} \quad \lim_{t\to\infty} \frac{Y(t)}{t} = 0 \ \textit{a.s., then}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t X(u)du \geq \frac{\nu_0}{\nu} \ \textit{a.s.}$$

To begin with, let us put

$$M = \sup_{S\in(0,1)} g'(S). \tag{27}$$

Theorem 3.2. *For any initial values $(S(0), I(0), R(0)) \in \Delta$, if $\mathcal{R}_s(g) > 1$ then the solution of the stochastic differential equation (3) obeys*

$$(i) \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t S(u)du \geq \max\left\{\frac{\mu}{\mu+\beta M}, \frac{\mu-\beta g(1)}{\mu}\right\},$$

$$(ii) \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t I(u)du \geq \frac{g(1)(\mu+\gamma)}{M(\beta g(1)+\gamma)}\left(1-\frac{1}{\mathcal{R}_s(g)}\right),$$

$$(iii) \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t R(u)du \geq \frac{\lambda g(1)}{M(\beta g(1)+\gamma)}\left(1-\frac{1}{\mathcal{R}_s(g)}\right),$$

and then the solutions of stochastic model system (3) starting from any point in Δ are strongly persistent in mean.

Proof. (i) From the first equation of system (3) and the fact that $(S, I, R) \in \Delta$ and $g(S) \leq g(1)$, we have

$$dS \geq (\mu - \beta g(1) - \mu S)\, dt - \sigma g(S(t))I(t)dB(t).$$

Integrating and dividing both sides by t, one can easily get

$$\frac{\mu}{t}\int_0^t S(u)du \geq \mu - \beta g(1) - \frac{S(t)-S(0)}{t} - \frac{\sigma}{t}\int_0^t g(S(t))I(u)dB(u). \tag{28}$$

On the other hand, by the mean value theorem and (27) one can see that $g(S) \leq MS$, then from the first equation of system (3) we have

$$dS \geq (\mu - \mu S - \beta MS(t))\, dt - \sigma g(S(t))I(t)dB(t),$$

hence

$$\frac{\mu+\beta M}{t}\int_0^t S(u)du \geq \mu - \frac{S(t)-S(0)}{t} - \frac{\sigma}{t}\int_0^t g(S(t))I(u)dB(u). \tag{29}$$

From (28), (29), $S(t) \in (0, 1)$ and the large number theorem for martingales, we obtain the desired result (i).

(ii) By Itô's formula, we get from the second equation of system (3),

$$\begin{aligned}
\log(I(t)) &= \log(I(0)) + \int_0^t \left(-\frac{1}{2}\sigma^2(g(S(s)))^2 + \beta g(S(s)) - (\mu+\lambda)\right) ds + \int_0^t \sigma g(S(s))dB(s), \\
&= \log(I(0)) + \left(-\frac{1}{2}\sigma^2(g(1))^2 + \beta g(1) - (\mu+\lambda)\right) t \\
&\quad + \int_0^t \frac{1}{2}\sigma^2\left((g(1))^2 - g(S(s))^2\right) - \beta\left(g(1) - g(S(s))\right) ds + \int_0^t \sigma g(S(s))dB(s).
\end{aligned}$$

Using $g(S) \leq g(1)$, yields

$$\log(I(t)) \geq \log(I(0)) + \beta g(1)\left(1 - \frac{1}{\mathcal{R}_s(g)}\right)t - \beta\int_0^t (g(1) - g(S(s)))\, ds + \int_0^t \sigma g(S(s))dB(s). \tag{30}$$

On the other hand, from the first equation of SDE (3) and $R = 1 - S - I$, we get

$$\begin{aligned}
dS(t) &= (\mu + \gamma - (\mu+\gamma) S(t) - \beta g(S(t))I(t) - \gamma I(t))\, dt - \sigma g(S(t))I(t)dB(t) \\
&\geq ((\mu+\gamma)(1 - S(t)) - (\beta g(1) + \gamma) I(t))\, dt - \sigma g(S(t))I(t)dB(t).
\end{aligned} \tag{31}$$

By the mean value theorem one can see that

$$(1-S) \geq \frac{g(1)-g(S)}{M}. \tag{32}$$

From (31) and (32) we deduce

$$S(t)-S(0) \geq \frac{\mu+\gamma}{M}\int_0^t (g(1)-g(S(s)))\,ds - (\beta g(1)+\gamma)\int_0^t I(s)ds - \sigma\int_0^t g(S(s))I(s)dB(s).$$

Combining it with (30) and rearranging to get

$$\log(I(t)) \geq \beta g(1)\left(1-\frac{1}{\mathcal{R}_s(g)}\right)t - \frac{\beta M\,(\beta g(1)+\gamma)}{(\mu+\gamma)}\int_0^t I(s)ds + Y(t),$$

where

$$Y(t) = \log(I(0)) - \frac{\beta M}{\mu+\gamma}\,(S(t)-S(0)) + \int_0^t \sigma S(s)\left(1-\frac{\beta M}{\mu+\gamma}I(s)\right)dB(s).$$

Moreover, by the fact that $S(t) \in (0,1)$ and the large number theorem for martingales, we have $\lim\limits_{t\to\infty}\frac{Y(t)}{t} = 0$ $a.s.$ Applying Lemma 3.1, we get the desired estimation (ii).
(iii) Integrating the third equation of system (3) and dividing both sides by t yields

$$\frac{1}{t}\int_0^t R(u)du = \frac{\lambda}{(\mu+\gamma)\,t}\int_0^t I(u)du + \frac{R(t)-R(0)}{t}. \tag{33}$$

Since $\lim\limits_{t\to\infty}\frac{R(t)-R(0)}{t} = 0$ $a.s.$ Then, the assertion (iii) follows immediately from (33) and (ii). □

4 Stationary distribution

To get more information about the asymptotic behavior of the diseases governed by the stochastic system (3) in the case when $\mathcal{R}_s(g) > 1$, we show that distribution of process $(S(t), I(t), R(t))$ converges weakly to a unique invariant or stationary distribution. Let $X(t) = (X_1(t), X_2(t), ..., X_n(t))$ be a homogeneous Markov process described by the following stochastic differential equation:

$$dX_j(t) = a_j(X)dt + \sigma_j(X)dB, \quad j = 1, 2..., n. \tag{34}$$

Let $\mathcal{Z}$ be an invariant set by system (34), that is if $X(0) \in \mathcal{Z}$ then $\mathbb{P}(X(t) \in \mathcal{Z}) = 1$. Consider the space $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}), m)$ where $\mathcal{B}(\mathcal{Z})$ is the σ-algebra of Borel subsets of $\mathcal{Z}$ and m is the Lebesgue measure on $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$. The probability low of the solution $X(t)$ statrting from $X(0) = x_0 \in \mathcal{Z}$ is defined by the transition probability, i.e.

$$P_X(t, x_0, C) = \mathbb{P}\left(X(t) \in C | X(0) = x_0\right), \quad C \in \mathcal{B}(\mathcal{Z}).$$

or also by

$$P_X(t)f(x_0) = \int_{\mathcal{Z}} f(y)dP_X(t, x_0, dy), \quad f \in \mathcal{C}.$$

where $\mathcal{C}$ is the subset of all positive measurable functions. Let us require some definitions and results about the stability in distribution. We refer to Rudnicki [37] for the precise meaning of asymptotical stability in distribution.

Definition 4.1. The solution process X of system (34) is said to be
(i) absolutely continuous if there exists a measurable function $\mathcal{K} : [0,\infty) \times \mathcal{Z} \times \mathcal{Z} \to [0,\infty)$ such that for any $f \in \mathcal{C}$ and $x_0 \in \mathcal{Z}$

$$P_X(t)f(x_0) = \int_{\mathcal{Z}} \mathcal{K}(t,x_0,y)f(y)m(dy) \quad \text{and} \quad \int_{\mathcal{Z}} \mathcal{K}(t,x_0,y)m(dx_0) = 1 \quad \text{for any} \quad y \in \mathcal{Z}.$$

(ii) asymptotically stable in indistribution if there exists a stationary distribution ν such that for any bounded continuous function f and $x_0 \in \mathcal{Z}$

$$P_X(t)f(x_0) = \int_{\mathcal{Z}} f(y)dP_X(t,x_0,dy) \longrightarrow \int_{\mathcal{Z}} f(y)\nu(dy).$$

With reference to Rudnicki [37], Rudnicki and Pichór [38] and Lin et al. [39, 40], we have the following lemma giving sufficient condition for asymptotical stability in distribution. To begin with, consider the Stratonovitch's version equation of (34)

$$dX_j(t) = A_j(X)dt + \Sigma_j(X) \circ dB, \quad j = 1,2...,n, \tag{35}$$

where

$$A_j(X) = a_j(X) - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial}{\partial x_k}\sigma_j(X)\right)\sigma_k(X), \quad \Sigma_j(X) = \sigma_j(X), \quad j = 1,2...,n.$$

Define the Lie bracket $[A,\Sigma](x)$ of the vectors $A(x)$ and $\Sigma(x)$ by

$$[A,\Sigma]_j(x) = \sum_{k=1}^{n} A_k \frac{\partial \Sigma_j}{\partial x_k}(x) - \Sigma_k \frac{\partial A_j}{\partial x_k}(x), \qquad j = 1,...,n.$$

Lemma 4.2. *Assume that*
i) The family of vectors generated by the Lie brackets of $\xi_0(X) \triangleq A(X)$ *and* $\xi_1(X) \triangleq \Sigma(X)$, *that is*

$$\xi_1(X), \quad [\xi_0(X),\xi_1(X)], \ [\xi_1(X),\xi_0(X)], \quad [\xi_i(X),[\xi_j(X),\xi_k(X)]]_{0\le i,j,k\le 1}.....$$

span the space $\mathbb{R}^n$ *for every solution* X *of (34).*
ii) For a fixed point $x_0 \in \mathcal{Z}$ *and a function* $\phi \in \mathcal{C}([0,T],\mathbb{R})$*, the derivative* $D_{x_0,\phi}$ *of the function* $h \to X_{\phi+h}(T)$ *from* $\mathcal{C}([0,T],\mathbb{R})$ *to* $\mathbb{R}^n$*, where* X_ϕ *is the solution of the differential equations*

$$d(X_\phi)_j(t) = A_j(X_\phi)dt + \Sigma_j(X_\phi)\phi(t), \quad j = 1,2...,n, \quad X_\phi(0) = x_0, \tag{36}$$

has rank 2.
iii) For any x_1, $x_2 \in \mathcal{Z}$ *there exists a control function* ϕ *and* $T > 0$ *such that the solution of system (39) satisfies* $X_\phi(0) = x_1$*, and* $X_\phi(T) = x_2$*.*
iv) There is a bounded open subset $\mathcal{D}$ *of* $\mathbb{R}^n$ *and a nonnegative function* $V : \mathcal{D}^c \to \mathbb{R}$ *such that* V *is twice continuously differentiable and that for some* $\varrho > 0$,

$$\mathcal{L}V(x) \le -\varrho, \quad \textit{for any} \quad x \in \mathcal{D}^c.$$

Then system (34) is absolutely continuous with density $\mathcal{K} : [0,\infty) \times \mathcal{Z} \times \mathcal{Z} \to [0,\infty)$*, asymptotically stable in distribution and admits a unique stationary distribution with density* $\mathcal{K}^*(y)$*, furthermore,*

$$\lim_{t\to\infty} \int_{\mathcal{Z}} |\mathcal{K}(t,x_0,y) - \mathcal{K}^*(y)|dy = 0 \quad \textit{for all} \ x_0 \in \mathcal{Z}.$$

Theorem 4.3. *Consider the stochastic system (3) with initial condition in Δ. Assume that $\mathcal{R}_s(g) > 1$. Then $(I(t), S(t), R(t))$ is positive recurrent and admits a unique ergodic stationary distribution.*

Proof. Given that $S(t) + I(t) + R(t) = 1$, it is sufficient to study the SDE for $(S(t), I(t))$ in the feasible region

$$\mathcal{Z} = \left\{x \in (0,1)^2;\ x_1 + x_2 < 1\right\}.$$

But for a measurable set $C \in \mathcal{B}(\mathcal{Z})$, we have

$$P_{(S,I)}(t, (s(0), i(0)), C) = P_{(R,I)}(t, (1 - s(0) - i(0), i(0)), C'),$$

where $C' = \{(1 - s - i, i), (s, i) \in C\}$, so, it suffices to verify conditions i)-iii) for (R, I) and condition iv) for (S, I).
i) Rewrite the Itô SDE (3) as SDE in the Stratonovitch sense:

$$\begin{aligned} dI &= A_1(I,R)dt + \sigma g(1 - I(t) - R(t))I \circ dB \\ dR &= A_2(I,R)dt, \end{aligned} \tag{37}$$

where

$$\begin{aligned} A_1(i,r) &= \beta g\left(1-i-r\right)i - (\mu+\lambda)i - \frac{1}{2}\sigma^2 i g\left(1-i-r\right)\left(g\left(1-i-r\right) - ig'\left(1-i-r\right)\right) \\ A_2(i,r) &= \lambda i - (\mu+\gamma)r. \end{aligned}$$

Let

$$A(i,r) = \begin{pmatrix} A_1(i,r) \\ A_2(i,r) \end{pmatrix} \quad \text{and} \quad \Sigma(i,r) = \begin{pmatrix} \sigma g\left(1-i-r\right)i \\ 0 \end{pmatrix}. \tag{38}$$

By direct calculation we get that

$$[A,\Sigma](i,r) = \begin{pmatrix} \sigma\left(-g'(1-i-r)i + g(1-i-r)\right)A_1(i,r) - \sigma g(1-i-r)i\frac{\partial A_1}{\partial i}(i,r) - \sigma g'(1-i-r)iA_2(i,r) \\ -\sigma\lambda g(1-i-r)i \end{pmatrix}.$$

Thus, $det\left([A,\Sigma](i,r), \Sigma(i,r)\right) = \lambda\sigma^2(g(1-i-r))^2 i^2 > 0$ for any $(i,r) \in \mathcal{Z}$. Consequently, the vectors $[A,\Sigma](i,r)$ and $\Sigma(i,r)$ span the space $\mathbb{R}^n$ for any $(i,r) \in \mathcal{Z}$.

ii) Let $(i_0, r_0) \in \mathcal{Z}$ and $\phi \in \mathcal{C}\left([0,T], \mathbb{R}\right)$. Consider the following system of differential equations:

$$\begin{cases} \dot{I}_\phi(t) = A_1\left(I_\phi(t), R_\phi(t)\right) + \sigma g\left(1 - I_\phi(t) - R_\phi(t)\right)I_\phi(t)\phi(t), \\ \\ \dot{R}_\phi(t) = A_2\left(I_\phi(t), R_\phi(t)\right), \end{cases} \tag{39}$$

with the initial condition $I_\phi(0) = i_0$, $R_\phi(0) = r_0$ and let $D_{i_0,r_0,\phi}$ be the derivative of the function $h \to \begin{pmatrix} I_{\phi+h}(T) \\ R_{\phi+h}(T) \end{pmatrix}$ from $\mathcal{C}\left([0,T], \mathbb{R}\right)$ to $\mathbb{R}^2$. Using the perturbation method for ordinary differential equations, the derivative $D_{i_0,r_0,\phi}$ can be calculated as follows. Let

$$\zeta(t) = A'\left(I_\phi(t), R_\phi(t)\right) + \phi(t)\Sigma'\left(I_\phi(t), R_\phi(t)\right)$$

where A' and Σ' are the Jacobian of $A(i,r)$ and $\Sigma(i,r)$ defined by (38), respectively. Let $Q(t, t_0)$, for $0 \le t_0 \le t \le T$, be the matrix function such that

$$Q(t_0, t_0) = I, \quad \frac{\partial Q}{\partial t}(t, t_0) = \zeta(t)Q(t, t_0).$$

Then

$$D_{i_0,r_0,\phi}h = \int_0^T Q(T,s)\Sigma\left(I_\phi(s),R_\phi(s)\right)h(s)ds. \tag{40}$$

Let $\varepsilon \in (0,T)$ and

$$h(t) = \frac{1}{\sigma g\left(1-I_\phi(t)-R_\phi(t)\right)I_\phi(t)}\mathbf{1}_{[T-\varepsilon,T]}(t).$$

By Taylor formula, one can write

$$Q(T,s) = I - \zeta(T)(T-s) + o(T-s) \quad \text{as } s \to T.$$

Then, from (40) we get

$$D_{i_0,r_0,\phi}h = \varepsilon \mathbf{e}_1 - \frac{1}{2}\varepsilon^2\zeta(T)\mathbf{e}_1 + o(\varepsilon^2),$$

where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. By a direct calculation one can obtain

$$\zeta(T)\mathbf{e}_1 = \begin{pmatrix} \frac{\partial A_1}{\partial i}\left(I_\phi(T),R_\phi(T)\right) + \phi(T)\frac{\partial \Sigma_1}{\partial i}\left(I_\phi(T),R_\phi(T)\right) \\ \lambda \end{pmatrix}.$$

Hence, $\mathbf{e}_1$ and $\zeta(T)\mathbf{e}_1$ are linearly independent. Thus $D_{i_0,r_0,\phi}$ has rank 2.

iii) Fixed (i_1,r_1), $(i_2,r_2) \in \Gamma$. Without lose of generality, we suppose that $r_2 < r_1$. We need to prove that there exist a control function ϕ and $T > 0$ such that the solution of system (39) satisfies $I_\phi(0) = i_1$, $R_\phi(0) = r_1$, $I_\phi(T) = i_2$, $R_\phi(T) = r_2$. From the third equation of system (3), it suffies to construct a $\mathcal{C}^1$ function which will posses the following properties:

$$R_\phi(0) = r_1, \quad R_\phi(T) = r_2, \tag{41}$$

$$\dot{R}_\phi(0) = \lambda i_1 - (\mu+\gamma)r_1, \quad \dot{R}_\phi(T) = \lambda i_2 - (\mu+\gamma)r_2, \tag{42}$$

$$-(\mu+\gamma)R_\phi(t) < \dot{R}_\phi(t) < \lambda - (\mu+\gamma+\lambda)R_\phi(t), \tag{43}$$

so, the functions $I_\phi(t)$, $S_\phi(t)$ and $\phi(t)$ will be immediately given by system (3) as follows

$$I_\phi(t) = \frac{1}{\lambda}\left(\dot{R}_\phi(t) + (\mu+\gamma)R_\phi(t)\right),\ S_\phi(t) = 1 - I_\phi(t) + R_\phi(t),\ \phi(t) = \frac{\dot{I}_\phi(t) - A_1\left(I_\phi(t),R_\phi(t)\right)}{\sigma g\left(1-I_\phi(t)-R_\phi(t)\right)I_\phi(t)}.$$

To this end, first we consider (X_1,Y_1) and (X_2,Y_2) the solutions of the following ODEs starting respectively from (r_1,r_1) and (r_2,r_2)

$$\begin{cases} \dot{X} = \lambda - (\mu+\gamma+\lambda)X, \\ \dot{Y} = -(\mu+\gamma)Y. \end{cases} \tag{44}$$

Consider the function $R_1(t)$ and $Z_1(t)$ defined as follows.

$$R_1(t) = Y_1(t) + \frac{i_1}{1-r_1}\left(X_1(t)-Y_1(t)\right),\quad Z(t) = Y_2(t) + \frac{i_2}{1-r_2}\left(X_2(t)-Y_2(t)\right).$$

Hence, it is easy to see that

$$R_1(0) = r_1,\ Z(0) = r_2,\ \dot{R}_1(0) = \lambda i_1 - (\mu+\gamma)r_1,\ \dot{Z}(0) = \lambda i_2 - (\mu+\gamma)r_2. \tag{45}$$

Furthermore

$$\begin{aligned} \dot{R}_1(t) &= -(\mu+\gamma)R_1(t) + \frac{i_1}{1-r_1}\left(1-X_1(t)\right), \quad (46)\\ &= \lambda - (\mu+\gamma+\lambda)R_1(t) + \frac{\lambda(1-i_1-r_1)}{1-r_1}\left(Y_1(t)-1\right), \quad (47) \end{aligned}$$

and

$$\dot{Z}(t) = -(\mu+\gamma)Z(t) + \frac{i_2}{1-r_2}\left(1 - X_2(t)\right), \tag{48}$$

$$= \lambda - (\mu+\gamma+\lambda)Z(t) + \frac{\lambda(1-i_2-r_2)}{1-r_2}\left(Y_2(t) - 1\right). \tag{49}$$

Since $X_1(t)$, $Y_1(t)$, $X_2(t)$, $Y_2(t) \in (0,1)$ for all $t \geq 0$. Then the equalities (46), (47), (48) and (49) imply

$$-(\mu+\gamma)R_1(t) < \dot{R}_1(t) < \lambda - (\mu+\gamma+\lambda)R_1(t) \quad t \geq 0, \tag{50}$$

and

$$-(\mu+\gamma)Z(t) < Z(t) < \lambda - (\mu+\gamma+\lambda)Z(t) \quad t \geq 0. \tag{51}$$

Let T be a sufficiently large number and ϵ be a sufficiently small positive number and consider the function $R_3(t) = Z(t-T)$. Therefore, it follows from (45) and (51) that for all $t \in [T-\epsilon, T]$, $R_3(t)$ verifies

$$-(\mu+\gamma+)R_3(t) < \dot{R}_3(t) < \lambda - (\mu+\gamma+\lambda)R_3(t), \tag{52}$$

$$R_3(T) = r_2, \quad \dot{R}_3(T) = \lambda i_2 - (\mu+\gamma)r_2. \tag{53}$$

Finally, consider a function $R_2(t)$ defined on $[\epsilon, T-\epsilon]$ such that $Y_1(t) \leq R_2(t) \leq X_1(t)$, which implies that

$$-(\mu+\gamma)R_2(t) < \dot{R}_2(t) < \lambda - (\mu+\gamma+\lambda)R_2(t). \tag{54}$$

In addition, we choose $R_2(t)$ such that the function

$$R_\phi(t) = \begin{cases} R_1(t), & 0 \leq t \leq \epsilon, \\ R_2(t), & \epsilon \leq t \leq T-\epsilon, \\ R_3(t), & T-\epsilon \leq t \leq T, \end{cases}$$

be a $\mathcal{C}^1$ function. Thus, in view of (45), (50), (52), (53) and (54), the function $R_\phi(t)$ constructed in this way posses the properties (41), (42) and (43).

iv) Let α be a sufficiently large number and

$$\mathcal{D} = \left\{ x \in \mathcal{Z}; \quad \frac{1}{\alpha} < x_1 < 1 - \frac{1}{\alpha} \quad \text{and} \quad \frac{1}{\alpha} < x_2 < 1 - \frac{1}{\alpha} \right\}.$$

Consider the positive function defined by

$$\psi(x_1, x_2) = \frac{1}{\eta} x_1^{-\sqrt{\eta}} x_2^{-\eta},$$

where η is a positive number to be chosen later. The differential operator $\mathcal{L}$ acting on the Lyapunov function ψ gives

$$\mathcal{L}\psi(S,I) = S^{-\sqrt{\eta}} I^{-\eta} \Bigg((\mu+\lambda) - \beta g(S) + \frac{1}{2}(1+\eta)\sigma^2 (g(S))^2 + \frac{1}{\sqrt{\eta}S}\left(-\mu + \mu S + \beta g(S) I - \gamma R + \frac{1}{2}(1+\sqrt{\eta})\frac{\sigma^2 (g(S))^2 I^2}{S} \right) + \sqrt{\eta}\sigma^2 g(S) I \Bigg). \tag{55}$$

Let $S, I \in \mathcal{D}^c$, which implies by $S + I < 1$ that either $S < \frac{1}{\alpha}$ or $I < \frac{1}{\alpha}$. Firstly, if $S < \frac{1}{\alpha}$ then from I, $S \in (0,1)$, and (55), we have for $\eta < 1$

$$\mathcal{L}\psi(S,I) \leq S^{-\sqrt{\eta}} I^{-\eta} \left(\mu + \lambda + \frac{1}{2}\sigma^2 (g(S))^2 + \frac{1}{\sqrt{\eta}S}\left(-\mu + \mu S + \beta g(S) + \frac{\sigma^2 (g(S))^2}{S} \right) + \mathcal{O}(\sqrt{\eta}) \right),$$

and by the fact that $g(0)=0$ we have $g(S)/S \le M$ with $M=\sup_{S\in(0,1)}|g'(S)|$, so we can easily verify

$$\mu+\lambda+\frac{1}{2}\sigma^2(g(S))^2=\mu+\lambda+\mathcal{O}\left(\frac{1}{\alpha}\right),\quad -\mu+\mu S+\beta g(S)+\frac{\sigma^2(g(S))^2}{S}=-\mu+\mathcal{O}\left(\frac{1}{\alpha}\right).$$

Hence, for α sufficiently large and η sufficiently small, we have

$$\begin{aligned}\mathcal{L}\psi((S,I),j)\le&\alpha^{\sqrt{\eta}}\left[\mu+\lambda+\frac{1}{2}\sigma^2(g(1))^2+\frac{\alpha}{\sqrt{\eta}}\left(-\mu+\mathcal{O}\left(\frac{1}{\alpha}\right)\right)+\mathcal{O}\left(\frac{1}{\alpha}\right)+\mathcal{O}(\sqrt{\eta})\right]\\ \le&-1.\end{aligned}\tag{56}$$

Secondly, if $I<\frac{1}{\alpha}$, then, from $I+R+S=1$, $-\mu+\mu S<0$ and (55), we have

$$\begin{aligned}\mathcal{L}\psi(S,I)\;=\;&S^{-\sqrt{\eta}}I^{-\eta}\Bigg((\mu+\lambda)-\beta g(S)+\frac{1}{2}\sigma^2(g(S))^2\\ &+\frac{1}{\sqrt{\eta}S}\left(\beta g(S)I-\gamma(1-S-I)+\frac{1}{2}(1+\sqrt{\eta})\frac{\sigma^2(g(S))^2I^2}{S}\right)+\mathcal{O}(\sqrt{\eta})\Bigg).\end{aligned}\tag{57}$$

One can easly have that there exists $c_1\in(0,S)$, $c_2\in(1-S-I,1-I)$ and $c_3\in(1-I,1)$ such taht

$$g(S)=Sg'(c_1),\quad g(S)=g(1-I)-(1-S-I)g'(c_2),\quad g(1-I)=g(1)-Ig'(c_3).\tag{58}$$

Injecting (58) in (57) we get

$$\begin{aligned}\mathcal{L}\psi(S,I)\;=\;&S^{-\sqrt{\eta}}I^{-\eta}\big((\mu+\lambda)-\beta(g(1)-Ig'(c_3)-(1-S-I)g'(c_2))\\ &+\frac{1}{2}\sigma^2\left(g(1)-Ig'(c_3)-(1-S-I)g'(c_2)\right)^2\\ &+\frac{\beta g'(c_1)}{\sqrt{\eta}}I-\frac{\gamma}{\sqrt{\eta}S}(1-S-I)+\frac{1}{2\sqrt{\eta}}(1+\sqrt{\eta})\sigma^2(g'(c_1))^2I^2+\mathcal{O}(\sqrt{\eta})\Big),\\ =\;&S^{-\sqrt{\eta}}I^{-\eta}\Big(\mu+\lambda-\beta g(1)+\frac{1}{2}\sigma^2g(1)+\beta Ig'(c_3)+\beta(1-S-I)g'(c_2)\\ &+\frac{1}{2}\sigma^2I^2(g'(c_3))^2+\frac{1}{2}\sigma^2(1-S-I)^2(g'(c_2))^2-\sigma^2Ig(1)g'(c_3)\\ &-\sigma^2(1-S-I)g(1)g'(c_2)+\sigma^2I(1-S-I)g'(c_3)g'(c_2)\\ &+\frac{\beta g'(c_1)}{\sqrt{\eta}}I-\frac{\gamma}{\sqrt{\eta}S}(1-S-I)+\frac{1}{2\sqrt{\eta}}(1+\sqrt{\eta})\sigma^2(g'(c_1))^2I^2+\mathcal{O}(\sqrt{\eta})\Big).\end{aligned}$$

Using $|g'(c_i)|\le M$, i=1,2,3, yields

$$\begin{aligned}\mathcal{L}\psi(S,I)\;\le\;&S^{-\sqrt{\eta}}I^{-\eta}\Big(\mu+\lambda-\beta g(1)+\frac{1}{2}\sigma^2g(1)+I\left(\beta M+\sigma^2M^2I+\sigma^2g(1)M+\sigma^2M^2(1-S-I)\right)\\ &(1-S-I)\left(\beta M+\frac{1}{2}\sigma^2M^2(1-S-I)+\sigma^2g(1)M-\frac{\gamma}{\sqrt{\eta}S}\right)+\\ &+\frac{1}{\sqrt{\eta}}I\left(\beta M+\frac{1}{2}\sigma^2M^2I\right)+\mathcal{O}(\sqrt{\eta})\Big).\end{aligned}\tag{59}$$

From $1-S-I<1$ and $S<1$ we have for $\eta<\left(\frac{\gamma}{\beta M+\frac{1}{2}\sigma^2M^2+\sigma^2g(1)M}\right)^2$

$$\beta M+\frac{1}{2}\sigma^2M^2(1-S-I)+\sigma^2g(1)M-\frac{\gamma}{\sqrt{\eta}S}<0.\tag{60}$$

Combining (59) and (60) and using $I<1$ gives

$$\mathcal{L}\psi(S,I) \leq S^{-\sqrt{\eta}}I^{-\eta}\left(\mu+\lambda-\beta g(1)+\frac{1}{2}\sigma^2 g(1)+\mathcal{O}\left(\frac{1}{\alpha\sqrt{\eta}}\right)+\mathcal{O}\left(\frac{1}{\alpha}\right)+\mathcal{O}(\sqrt{\eta})\right). \quad (61)$$

Let us choose α sufficiently large and η sufficiently small such that

$$\mu+\lambda-\beta g(1)+\frac{1}{2}\sigma^2 g(1)+\mathcal{O}\left(\frac{1}{\alpha\sqrt{\eta}}\right)+\mathcal{O}\left(\frac{1}{\alpha}\right)+\mathcal{O}(\sqrt{\eta})<0, \quad (62)$$

which is allowed by the condition $\mathcal{R}_s(g) > 1$. Hence, From (61) and (62) we have easily that there exists α sufficiently large and η sufficiently small such that

$$\begin{aligned}\mathcal{L}\psi(S,I) &\leq \alpha^{-\eta}\left(\mu+\lambda-\beta g(1)+\frac{1}{2}\sigma^2 g(1)+\mathcal{O}\left(\frac{1}{\alpha\sqrt{\eta}}\right)+\mathcal{O}\left(\frac{1}{\alpha}\right)+\mathcal{O}(\sqrt{\eta})\right).\\ &\leq -1. \end{aligned} \quad (63)$$

From (56) and (63), we have for α sufficiently large and η sufficiently small,

$$\mathcal{L}\psi((S,I)) \leq -1, \qquad \text{for all } (S,I)\in\mathcal{D}^c.$$

The proof is completed according to Lemma 4.2. □

5 Numerical simulations

In the framwork of numerical stochastics, the system (3) is equivalent to the following

$$\begin{cases} dS_t = F^1(S,I,R)dt + G^1(S,I,R)dB_t,\\ dI_t = F^2(S,I,R)dt + G^2(S,I,R)dB_t,\\ dR_t = F^3(S,I,R)dt. \end{cases} \quad (64)$$

We approximate the solution of the system above using the first order Itô-Taylor scheme (also called Milstein scheme). Since the system (64) is driven by one noise dB_t, the double stochastic integrals are used explicitly, for more details see [45]. By discritizing the time interval into 200 equidistant time steps, we simulate the system (64) under the conditions of our theoretical results above. The corresponding mean simulations are results of 1000 realizations. We examine two type of tests.

Test 1. We consider the power function $g(x) = x^p$. Consequently, we have $g(1) = 1$ and $\mathcal{R}_s(g) = \mathcal{R}_s$. In this test, we simulate six different cases :

Table 1: List of parameters for test 1.

	Test 1.1	Test 1.2	Test 1.3	Test 1.4	Test 1.5	Test 1.6
p	1.5	1.5	0.95	0.95	0.9	1.9
$\mathcal{R}_s$	0.876712	1.03014	1.03014	0.876712	1	1
λ	0.5	0.5	0.5	0.5	0.65	0.65
μ	0.25	0.25	0.25	0.25	0.25	0.25
β	2	2.35	2.35	2	1.56125	1.56125
γ	0.005	0.005	0.005	0.005	0.15	0.15
ρ	0.10	0.10	0.1	0.1	0.25	0.25
σ	1.75	1.75	1.75	1.75	1.15	1.15

Test 2. In this test, we compare the solution behavior of (3) using two different functions. Namely, the exponential function $g_1(x) = 1 - e^{-ax}$ and the rational function $g_2(x) = \dfrac{1}{1+bx}$, where the parameters a and b are chosen as follows

$$a > -\ln\left(1 - \frac{2(\mu+\lambda)}{\sigma^2}\right) \quad \text{and} \quad b > \frac{2(\mu+\lambda)}{\sigma^2}$$

which is equivalent respectively to

$$\mathcal{R}_s > \mathcal{R}_s(g_1) \quad \text{and} \quad \mathcal{R}_s > \mathcal{R}_s(g_2).$$

Hence, we simulate and compare between the following cases respectively.

$$\mathcal{R}_s(g_1) < \mathcal{R}_s(g_2) < 1 < \mathcal{R}_s, \quad \mathcal{R}_s(g_2) < \mathcal{R}_s(g_1) < 1 < \mathcal{R}_s,$$
$$\mathcal{R}_s(g_1) < 1 < \mathcal{R}_s(g_2) < \mathcal{R}_s, \quad \mathcal{R}_s(g_2) < 1 < \mathcal{R}_s(g_1) < \mathcal{R}_s.$$

Table 2: List of parameters for Test 2.

	Test 2.1	Test 2.2	Test 2.3	Test 2.4
$\mathcal{R}_s$	1.014490	1.014490	1.22415	1.22415
$\mathcal{R}_s(g_1)$	0.889431	0.889431	0.98224	1.21295
$\mathcal{R}_s(g_2)$	0.992908	0.804598	1.11740	0.87938
a	0.5	0.5	0.5	1
b	1	2	1	2
λ	0.35	0.35	0.35	0.35
μ	0.25	0.25	0.25	0.25
β	1.75	1.75	1.85	1.85
γ	0.005	0.005	0.005	0.005
ρ	0.10	0.10	0.10	0.10
σ	1.5	1.5	1.35	1.35

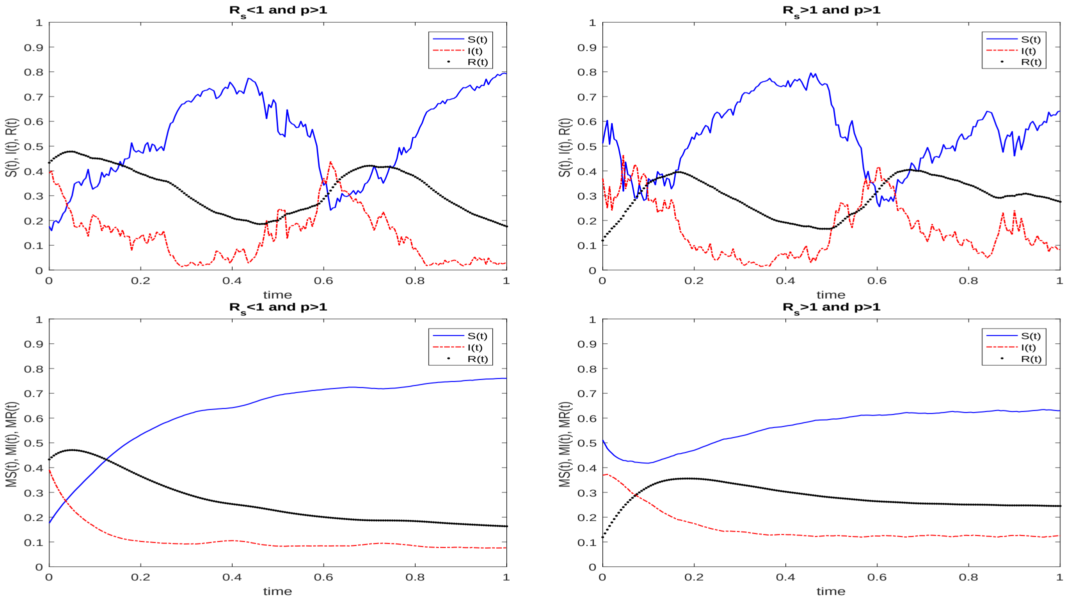

Figure 1: Test 1.1 and 1.2: One realization and the corresponding mean of 1000 solutions.

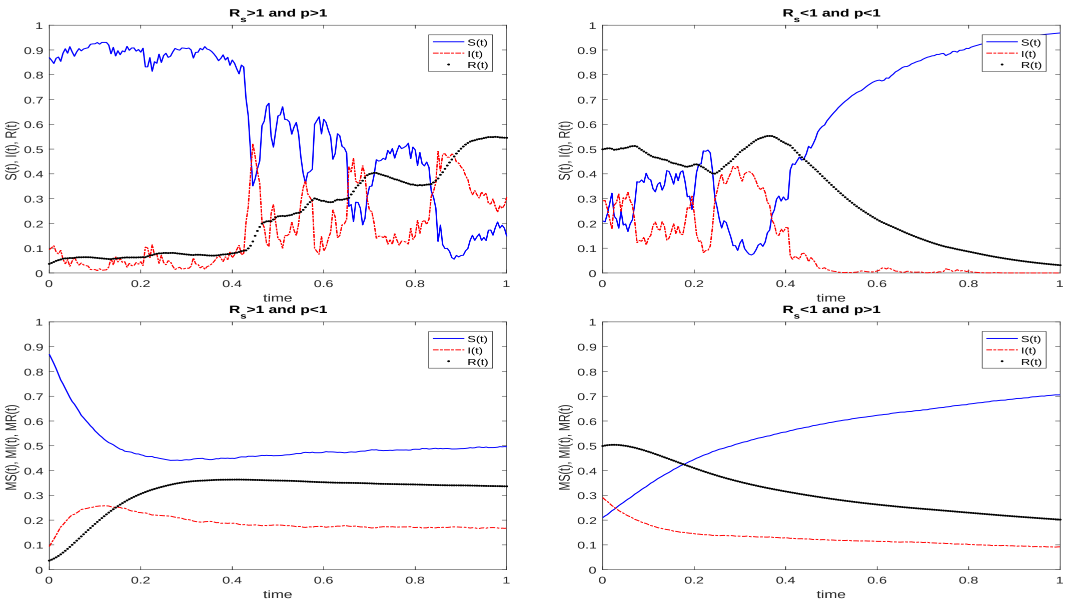

Figure 2: Test 1.3 and 1.4: One realization and the corresponding mean of 1000 solutions.

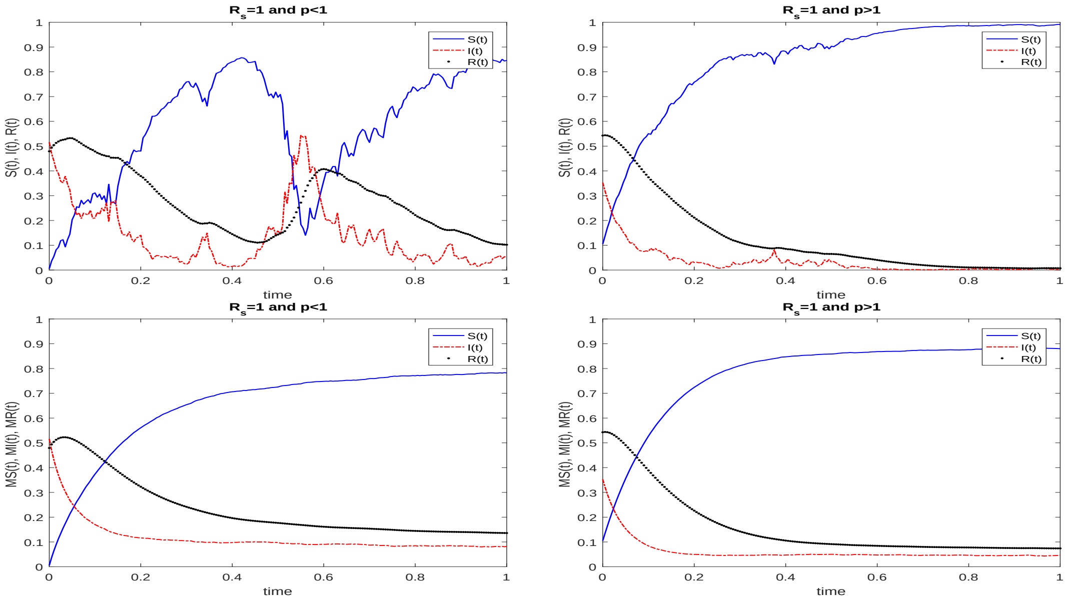

Figure 3: Test 1.5 and 1.6, one realization and the corresponding mean of 1000 solutions.

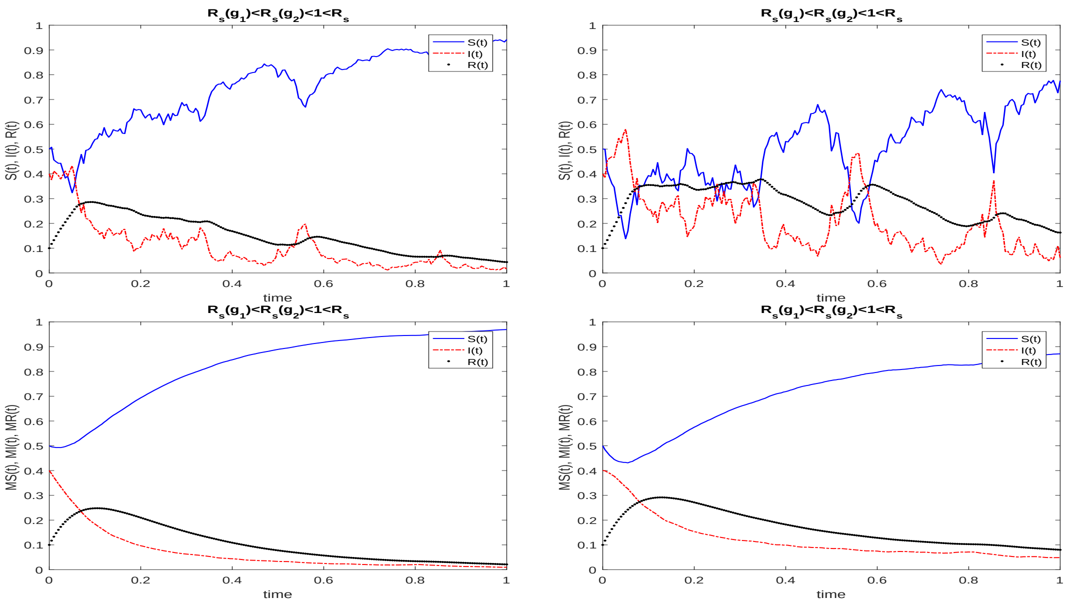

Figure 4: Test 2.1, one realization and the mean of 1000 solutions, for g_1 left and for g_2 right.

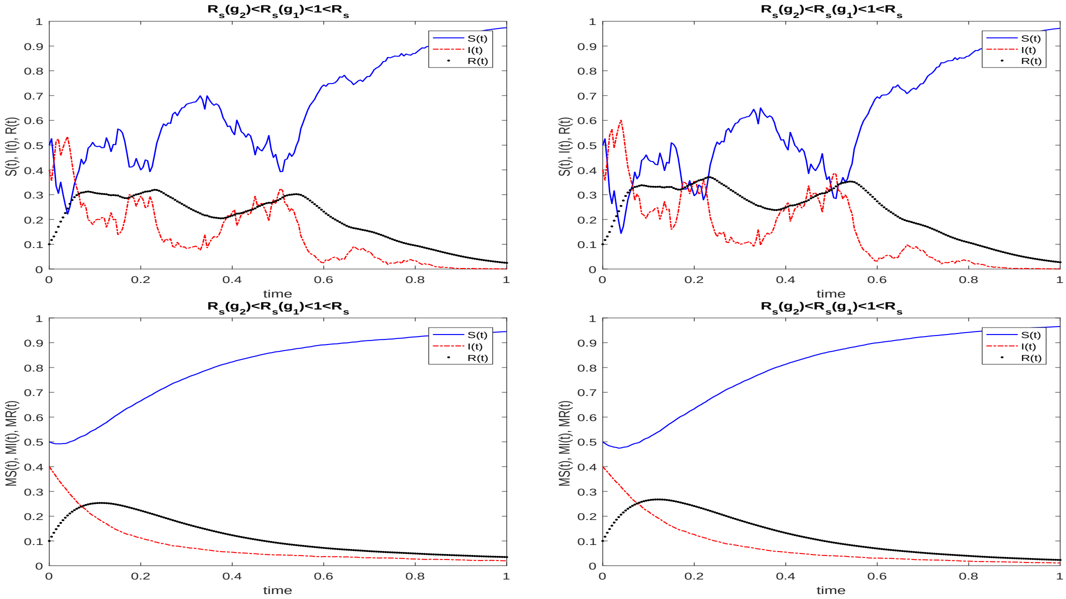

Figure 5: Test 2.2, one realization and the mean of 1000 solutions, for g_1 left and for g_2 right.

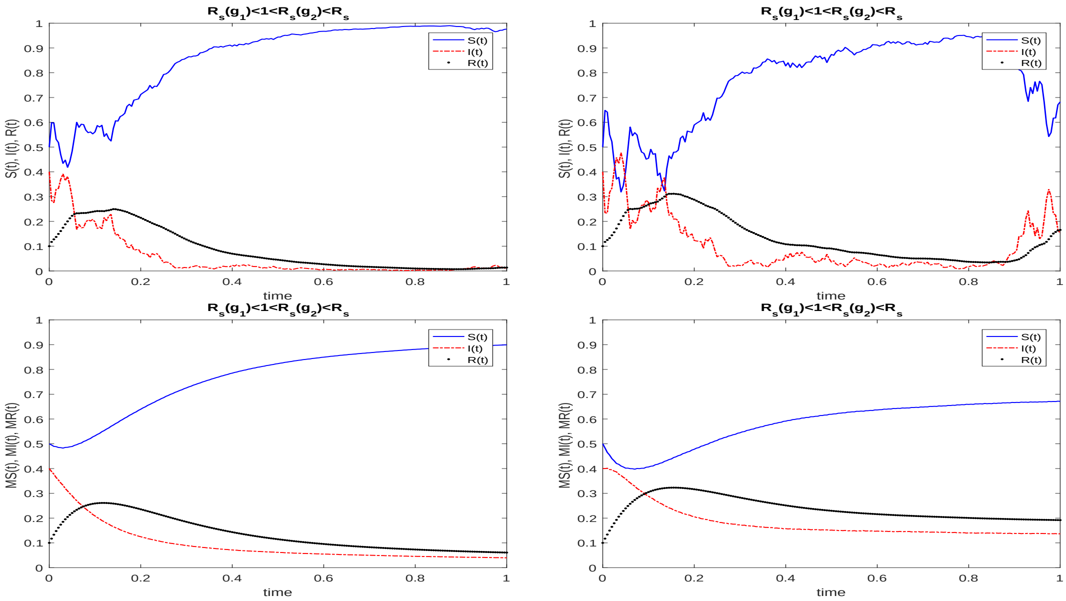

Figure 6: Test 2.3, one realization and the mean of 1000 solutions, for g_1 left and for g_2 right.

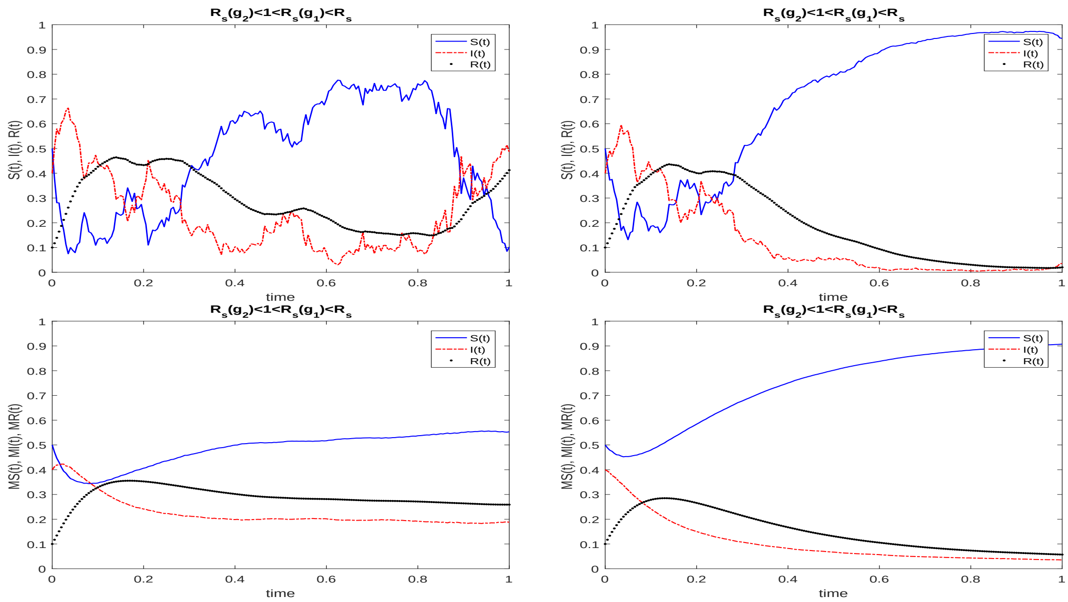

Figure 7: Test 2.4, one realization and the mean of 1000 solutions, for g_1 left and for g_2 right.

6 Concluding remarks

The current paper makes a useful contribution to understanding extinction and persistence of a stochastic SIRS epidemic model with a general incidence rate. From the analytical results established in this paper, the introduction of the nonlinearity rate $g(S)I$ modifies the classical threshold quantity $\mathcal{R}_s$ giving rise to a new threshold quantity $\mathcal{R}_s(g)$. This new threshold is the most natural way to extend the threshold previously established with the bilinear incidence rate. For example, if we consider the function $g(x) = x^p$, it is clearly a nonlinear one for $p \neq 1$. In this particular case, the classical threshold $\mathcal{R}_s$ coincides with $\mathcal{R}_s(g)$ because $g(1) = 1$. However, for $p > 1$ we get a significant effect on the rate of the extinction and the persistence of the epidemic as one can see from Figures 1-3 of Test 1.

We have conducted a second test also (See Test 2),where an exponential and ratinal functions were used : $g_1(x) = 1 - e^{-ax}$ and $g_2(x) = \dfrac{1}{1+bx}$ for $a \geq -\ln\left(1 - \dfrac{2(\mu+\lambda)}{\sigma^2}\right)$ and $b \geq \dfrac{2(\mu+\lambda)}{\sigma^2}$. Which implies an improvement on the stochastic threshold so $\mathcal{R}_s > \mathcal{R}_s(g_1)$ and $\mathcal{R}_s > \mathcal{R}_s(g_2)$ .The illustrations in Figures 4-7, confirm this improvement. They show that both models with nonlinearity rate $g_1(S)I$ and $g_2(S)I$ go towards extinction quicker than the model with the classical rate SI. It suggests that nonlinearity may radically change the system's asymptotic behavior by promoting the extinction of the epidemic.

Thus, we conclude that we can improve the threshold of such a stochastic SIRS system by including an adequate nonlinear incidence rate. This work has also shown that the stochastic solutions converge to a stationary distribution as a limit of a homogeneous markovian process.

Finally, we point out that some issues deserve further investigation. For instance, what is the long-time behavior of the epidemic model (3) in the two cases when $\mathcal{R}_s(g) < 1, \beta < \sigma^2$ and $\mathcal{R}_s(g) = 1, \beta < \sigma^2$? Another interesting continuation of this work might be introducing jump noise into some system parameters (3). We leave these for future investigation.

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