

Analytic amplitudes for a pair of Higgs bosons in association with three partons

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ABSTRACT: The pair production of Higgs bosons at the LHC can give information about the triple Higgs boson coupling. We perform an analytic one-loop calculation of the amplitudes for a pair of Higgs bosons in association with three partons, retaining the exact dependence on the quark mass circulating in the loop. These amplitudes constitute the real radiation corrections in the calculation of Higgs boson pair production at next-to-leading order in the strong coupling. The results of an analytic generalised-unitarity computation are simplified via analytic reconstruction in spinor variables. Compact ansätze for kinematic pole residues are iteratively fitted via p -adic evaluations near said poles and subtracted until no pole remains. A new ansatz construction is introduced to minimally parametrise coefficients of amplitudes with multiple massive external legs. The simplified expressions are faster to evaluate than automatic codes and can lead to more stable results near singular regions.

KEYWORDS: Higgs Production, Higher-Order Perturbative Calculations, Scattering Amplitudes

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Contents

1	Introduction	2
2	One-loop amplitude for $ggHH$	3
2.1	General decompositions of one-loop amplitudes	5
3	Advancements in analytic reconstruction techniques	6
3.1	Spinor variables subject to additional constraints	6
3.2	Minimal spinor ansatz for an arbitrary number of massive scalars	9
3.3	Projective space in m	12
4	Results for the process $0 \rightarrow q\bar{q}gHH$	14
4.1	Process $0 \rightarrow q\bar{q}gH^*$	14
4.2	Process $0 \rightarrow q\bar{q}gHH$	14
5	Results for the process $0 \rightarrow gggH^*(\rightarrow HH)$	17
6	Calculation methods for the process $0 \rightarrow gggHH$	18
6.1	Definition of colour amplitudes	19
6.2	Decomposition into scalar integrals	19
6.3	Basis integrals	19
6.4	Strategy for integral coefficients	20
7	Analytic results for the process $0 \rightarrow gggHH$	22
7.1	Scalar pentagons reduced to boxes	22
7.2	$g^-g^-g^-HH$	25
7.3	$g^-g^-g^+HH$	27
7.4	$g^-g^+g^-HH$	31
7.5	$g^+g^-g^-HH$	34
8	Implementation of NLO $pp \rightarrow HH$ calculation	36
8.1	Validation	37
8.2	Phenomenology	38
9	Conclusions	39
A	Spinor algebra	40
B	Loop integral definitions	41
C	Spinor decompositions	42
D	Box manipulations and spurious singularities at infinity	43

1 Introduction

The exploration of the details of the Higgs phenomenon is one of the primary goals of the high-luminosity LHC and its successor machines. The exploration of the Higgs potential is an important part of that endeavour. Within the context of the standard model, where the Lagrangian is limited to terms of mass dimension $d \leq 4$, the Higgs potential is fully determined,

$$V(h) = \frac{1}{2}M_H^2 H^2 + \lambda v H^3 + \frac{1}{4}\lambda H^4, \quad (1.1)$$

in terms of the Higgs boson mass, M_H , and the Fermi constant, G_F , where $M_H = \sqrt{2}\lambda v$, and $G_F/\sqrt{2} = 1/(2v^2)$. Beyond the standard model one can introduce operators of dimension higher than four, and hence deviations from the simple form for the triple Higgs boson coupling given in eq. (1.1). For a complete review we refer the reader to ref. [1].

Non-perturbative analyses [2] indicate that, at the observed mass of the Higgs boson, the electroweak phase transition is a rapid crossover, but extensions of the standard model can make it first order. A first order phase transition would be required, *inter alia*, for an electroweak baryogenesis explanation of the baryon asymmetry of the universe. This explains the intense interest in extending our knowledge of the Higgs potential, beyond the limited information about the shape of the potential derived from quadratic excursions about the minimum of the potential, governed by the Higgs boson mass.

Constraints on the triple Higgs boson coupling derive from measurements of both single Higgs [3] and double Higgs boson production [4–7]. The production of Higgs boson pairs gives direct access to the self-coupling of the Higgs boson. Deviations from the standard model are most easily assessed in the kappa framework [8, 9], in which the standard model triple Higgs boson coupling is allowed to float by an overall factor κ_λ . In ref. [10] the ATLAS collaboration find limits on the triple Higgs boson coupling modification of $-6.3 < \kappa_\lambda < 11.6$ at 95% CL, using final states with leptons (including taus), and photons. A combination limit from ATLAS in ref. [11] using final states including b -quarks, determines that κ_λ lies in the range $-1.2 < \kappa_\lambda < 7.2$ at 95% CL. In ref. [12] the CMS collaboration exclude values of the coupling modifier outside the range $-1.2 < \kappa_\lambda < 7.5$ at 95% CL, using results from both Higgs boson pair production and single Higgs boson production.

These limits on the triple Higgs boson coupling rely on theoretical calculations of Higgs boson processes at next-to-leading order (NLO) and beyond. NLO corrections to Higgs boson pair production including full top quark mass dependence have been calculated in refs. [13–16]. These calculations have been used to introduce top quark mass dependence into NNLO calculations performed in the $m \rightarrow \infty$ limit in refs. [17–19]. The theoretical uncertainties due to renormalization and factorization scale choice, with a special focus on the renormalization scheme for the top quark mass have been discussed in ref. [20]. The effects of matching with a parton shower are described in refs. [21, 22]. The calculation of double Higgs boson production at NLO has been further improved by combining the numerical fit result, based on events with limited coverage of the high-energy region, with the high-energy expansion, yielding a NLO result valid in the low, medium and high-energy regions [23]. For a short recent review including Higgs boson pair production we refer the reader to ref. [24].

In this paper we re-examine part of the NLO calculation of Higgs boson pair production, namely the one-loop amplitudes for the processes,

$$0 \rightarrow g(p_1) + g(p_2) + g(p_3) + H(p_4) + H(p_5), \quad (1.2)$$

$$0 \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3) + H(p_4) + H(p_5), \quad (1.3)$$

retaining all the dependence on the top quark mass, m . All the results presented assume that the two Higgs bosons are both on their mass shell. These processes represent the real radiation contribution to the NLO Higgs boson pair production process. The motivation for doing this is twofold. First, we hope to produce expressions which are faster to evaluate than expressions generated with automatic tools.¹ Second, in NLO calculations the real radiation is probed in regions where the emitted parton is either soft or collinear with respect to the initial partons and the one-loop expressions can be quite unstable. We achieve these goals by simplifying the spinor expressions for our results using the techniques of refs. [25, 26]. This has benefits for the evaluation time, since the expressions for the coefficients are shorter, and also for the stability of the results. This is the case since rational functions are for the most part reduced to least common denominator form, and partial fraction decompositions are used to reveal their underlying divergence structure.

In addition to presenting the analytic forms for the amplitudes, we perform a comparison with matrix elements calculated using automatic procedures. By reusing the prior calculation of the $ggHH$ 2-loop contribution [21], we also present a new public implementation of the NLO Higgs-pair production process in MCFM [27–29]. Additionally, this calculation is used to provide a result that is accurate to NLO+NNLL at small transverse momentum of the Higgs boson pair, which may be of interest at a future high-energy pp collider.

2 One-loop amplitude for $ggHH$

We first review the amplitude for the lowest order Higgs boson pair production process, which at order α_s in the strong coupling, occurs through the one-loop process,

$$0 \rightarrow g(p_1) + g(p_2) + H(p_3) + H(p_4). \quad (2.1)$$

We present these results here for completeness and to introduce our notation. The full one-loop amplitude, known for many years, is given in ref. [30]. This result supersedes an earlier partial result in ref. [31]. Since much of the interest in this process derives from its sensitivity to the trilinear coupling of the Higgs boson, we modify that term in the Lagrangian by introducing a rescaling κ_λ ,

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2} \partial_\mu H \partial^\mu H - \frac{1}{2} M_H^2 H^2 - \kappa_\lambda \lambda v H^3 + \dots \quad (2.2)$$

We have dropped the quartic coupling of the Higgs boson since it plays no part in the present calculation. The full amplitude for the process in eq. (2.1) is given by,

$$-i\mathcal{A}^{C_1 C_2} = -\frac{1}{2} \delta^{C_1 C_2} \alpha_s \frac{\alpha_W}{4M_W^2} A(1_g, 2_g, 3_H, 4_H), \quad (2.3)$$

¹In NLO corrections the two-loop virtual correction is represented by a fit [21], which is quite fast to evaluate. In previous calculations the one-loop real contribution is responsible for a sizable part of the computation time.

where,

$$M_W = \frac{1}{2}g_W v, \quad \alpha_s = \frac{g_s^2}{4\pi}, \quad \alpha_W = \frac{g_W^2}{4\pi}. \quad (2.4)$$

C_1 and C_2 are the colour indices of the gluons, $v \simeq 246$ GeV is the vacuum expectation value of the Higgs field, M_W is the mass of the W -boson, g_W is the gauge coupling of the $SU(2)_W$ weak gauge group, and g_s is the gauge coupling of the $SU(3)$ strong gauge group. There are two independent helicity amplitudes,

$$\begin{aligned} A(1_g^+, 2_g^+, 3_H, 4_H) &= (\kappa_\lambda g_1^\Delta + g_1^\square) \frac{[12]^2}{s_{12}}, \\ A(1_g^+, 2_g^-, 3_H, 4_H) &= g_2^\square \frac{[1\mathbf{3}|2]^2}{s_{12}p_T^2}, \end{aligned} \quad (2.5)$$

where we have defined,

$$\begin{aligned} s_{12} &= (p_1 + p_2)^2, \quad M_H^2 = p_3^2 = p_4^2 \equiv (p_1 + p_2 + p_3)^2, \\ \frac{2p_1 \cdot p_3 p_2 \cdot p_3}{p_1 \cdot p_2} &= p_T^2 + M_H^2, \end{aligned} \quad (2.6)$$

and M_H is the Higgs boson mass. The remaining two helicity combinations are obtained from eq. (2.5) by interchange. g_1^Δ denotes the triangle-graph pieces of the amplitude, contributing via the triple Higgs boson coupling. The determination of the triple Higgs boson coupling provides much of the motivation for the experimental measurement of Higgs boson pair production. In this equation $[ij]$ and $\langle i|\mathbf{k}|j \rangle$ are Lorentz invariant contractions of the spinors with momentum p_i, p_j and p_k . The momenta p_i and p_j are lightlike, whereas $p_k^2 = M_H^2$. As a reminder of its non-zero mass, in spinor products massive four-vectors are written in boldface.

$$\langle ij \rangle = \bar{u}_-(p_i)u_+(p_j), \quad [ij] = \bar{u}_+(p_i)u_-(p_j), \quad \langle i|\mathbf{k}|j \rangle = \bar{u}_-(p_i)\not{k}u_-(p_j). \quad (2.7)$$

Full details of the spinor notation and more complicated spinor strings, such as $\langle 1|4|5|2 \rangle$ are given in appendix A.

The results for the coefficients $g_1^\Delta, g_1^\square$ and $g_2^\square$ are [30],

$$g_1^\Delta = \frac{12m^2 M_H^2}{s_{12} - M_H^2} \left[2 + (4m^2 - s_{12})C_0(p_1, p_2) \right], \quad (2.8)$$

$$\begin{aligned} g_1^\square &= 4m^2 \left\{ m^2(8m^2 - s_{12} - 2M_H^2)(D_0(p_1, p_2, p_3; m) + D_0(p_2, p_1, p_3; m) + D_0(p_1, p_3, p_2; m)) \right. \\ &\quad + \frac{(s_{13}s_{23} - M_H^4)}{s_{12}}(4m^2 - M_H^2)D_0(p_1, p_3, p_2; m) + 2 + 4m^2 C_0(p_1, p_2; m) \\ &\quad \left. + \frac{2}{s_{12}}(M_H^2 - 4m^2)((s_{13} - M_H^2)C_0(p_1, p_3; m) + (s_{23} - M_H^2)C_0(p_2, p_3; m)) \right\}, \end{aligned} \quad (2.9)$$

$$\begin{aligned} g_2^\square &= 2m^2 \left\{ 2(8m^2 + s_{12} - 2M_H^2) \right. \\ &\quad \times \{ m^2[D_0(p_1, p_2, p_3; m) + D_0(p_2, p_1, p_3; m) + D_0(p_1, p_3, p_2; m)] - C_0(p_3, p_4; m) \} \\ &\quad \left. - 2\{ s_{12}C_0(p_1, p_2; m) + (s_{13} - M_H^2)C_0(p_1, p_3; m) + (s_{23} - M_H^2)C_0(p_2, p_3; m) \} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{(s_{13}s_{23} - M_H^4)} \left[s_{12}s_{23}(8s_{23}m^2 - s_{23}^2 - M_H^4)D_0(p_1, p_2, p_3; m) \right. \\
 & + s_{12}s_{13}(8s_{13}m^2 - s_{13}^2 - M_H^4)D_0(p_2, p_1, p_3; m) \\
 & + (8m^2 + s_{12} - 2M_H^2) \{ s_{12}(s_{12} - 2M_H^2)C_0(p_1, p_2; m) + s_{12}(s_{12} - 4M_H^2)C_0(p_3, p_4; m) \\
 & \left. + 2s_{13}(M_H^2 - s_{13})C_0(p_1, p_3; m) + 2s_{23}(M_H^2 - s_{23})C_0(p_2, p_3; m) \} \right], \quad (2.10)
 \end{aligned}$$

with

$$s_{13} = (p_1 + p_3)^2, \quad s_{23} = (p_2 + p_3)^2, \quad (2.11)$$

and where m is the (top) quark mass. B_0, C_0 and D_0 are bubble, triangle and box scalar integrals respectively in a more-or-less standard notation [32]. Full details of the notation for scalar integrals are given in appendix B. To emphasize that the momenta of the Higgs bosons p_3 (and p_4) are not light-like, we denote their presence in scalar products in boldface, thus, for example, $s_{13} = (p_1 + p_3)^2$.

2.1 General decompositions of one-loop amplitudes

The one-loop amplitude for $ggHH$ was naturally expressed in terms of (tadpole), bubble, triangle and box scalar integrals, because the amplitude had four external lines. Scalar integrals are loop integrals with no powers of the loop momentum in the numerator. The tadpole integral is not needed since it can be eliminated in terms of a bubble integral and a rational term. However it is more generally true that one loop amplitudes can be expressed as a sum of bubble, triangle and box scalar integrals. Thus even in the case of higher point amplitudes with a larger number of external legs it is still true that we may write,

$$\begin{aligned}
 A &= \frac{\bar{\mu}^{4-n}}{r_\Gamma} \frac{1}{i\pi^{n/2}} \int d^n \ell \frac{\text{Num}(\ell)}{\prod_i d_i(\ell)} \\
 &= \sum_{i,j,k} d_{i \times j \times k}(1^{h_1}, 2^{h_2}, 3^{h_3}) D_0(p_i, p_j, p_k; m) \\
 &\quad + \sum_{i,j} c_{i \times j}(1^{h_1}, 2^{h_2}, 3^{h_3}) C_0(p_i, p_j; m) \\
 &\quad + \sum_i b_i(1^{h_1}, 2^{h_2}, 3^{h_3}) B_0(p_i; m) + r(1^{h_1}, 2^{h_2}, 3^{h_3}). \quad (2.12)
 \end{aligned}$$

The scalar bubble (B_0), triangle (C_0), box (D_0) integrals, and the constant r_Γ , are defined in appendix B. This decomposition has the merit that the number of loop integrals that need to be evaluated is minimized.

For the case of the amplitude $gggHH$ we have pentagon diagrams with 5 external legs, but the general decomposition in eq. (2.12) still holds. Indeed in four dimensions, the scalar pentagon integral, E_0 is expressible as a sum of 5 scalar boxes obtained by removing the denominators of E_0 one at a time,

$$\begin{aligned}
 E_0(p_1, p_2, p_3, p_4; m) &= \mathcal{C}_1^{1 \times 2 \times 3 \times 4} D_0(p_2, p_3, p_4; m) + \mathcal{C}_2^{1 \times 2 \times 3 \times 4} D_0(p_{12}, p_3, p_4; m) \\
 &\quad + \mathcal{C}_3^{1 \times 2 \times 3 \times 4} D_0(p_1, p_{23}, p_4; m) \\
 &\quad + \mathcal{C}_4^{1 \times 2 \times 3 \times 4} D_0(p_1, p_2, p_{34}; m) + \mathcal{C}_5^{1 \times 2 \times 3 \times 4} D_0(p_1, p_2, p_3; m). \quad (2.13)
 \end{aligned}$$

Rules for calculating the coefficients $\mathcal{C}$ are given in section 7.1. Even though eq. (2.12) is always true, we find that for the $gggHH$ process there is merit in using a more general decomposition that retains the pentagon integrals and yields more compact results for the coefficients of the integrals. In this basis we have,

$$\begin{aligned}
A &= \frac{\bar{\mu}^{4-n}}{r_\Gamma} \frac{1}{i\pi^{n/2}} \int d^n\ell \frac{\text{Num}(\ell)}{\prod_i d_i(\ell)} \\
&= \sum_{i,j,k,l} \hat{e}_{i \times j \times k \times l}(1^{h_1}, 2^{h_2}, 3^{h_3}) E_0(p_i, p_j, p_k, p_l; m) \\
&\quad + \sum_{i,j,k} \hat{d}_{i \times j \times k}(1^{h_1}, 2^{h_2}, 3^{h_3}) D_0(p_i, p_j, p_k; m) \\
&\quad + \sum_{i,j} c_{i \times j}(1^{h_1}, 2^{h_2}, 3^{h_3}) C_0(p_i, p_j; m) \\
&\quad + \sum_i b_i(1^{h_1}, 2^{h_2}, 3^{h_3}) B_0(p_i; m) + r(1^{h_1}, 2^{h_2}, 3^{h_3}). \tag{2.14}
\end{aligned}$$

In our final results, expressions for the box coefficients $d_{i \times j \times k}$ in eq. (2.12) are given in terms of the appropriate sum of effective pentagon and remainder box coefficients, $\hat{e}_{i \times j \times k \times l}$ and $\hat{d}_{i \times j \times k}$ respectively, in eq. (2.14). This method of effective pentagons has also been used in the description of the $Hgggg$ process in ref. [33].

3 Advancements in analytic reconstruction techniques

The one-loop coefficients contributing to the process with an additional parton in the final state, $pp \rightarrow HHj$, are presented here in the form obtained through analytic reconstruction. They are iteratively reconstructed one pole residue at a time, as described in ref. [25, section 3.3]. The use of algebraic geometry ensures control over the analytic structure of the coefficients, while p -adic numbers enable stable numerical evaluations [26]. See also related work in refs. [34, 35].

Three new features of this process affect the complexity of the reconstruction procedures:

- the calculation assumes that the two Higgs bosons have equal mass, therefore an extra equivalence relation, besides momentum conservation, has to be imposed;
- the presence of *two* massive particles complicates the construction of a minimal ansatz, since the dependence on both their four-momenta cannot be removed by momentum conservation;
- the dependence on the mass of the quark in the loop occurs, not just as a Taylor series, but also mixed with kinematic poles.

We address these points in this section, with further details given in appendices C and D. This was required to arrive at a form of the coefficients simple enough for them to be presented in this article.

3.1 Spinor variables subject to additional constraints

In the first instance, the massive five-point process under consideration can be embedded into a seven-point massless process. Thus, Lorentz-covariant polynomials can be taken to

belong to the following polynomial ring,

$$S_7 = \mathbb{F}[[1], [1], \dots, [7], [7]] . \quad (3.1)$$

Legs 4, 5 and 6, 7 can be thought of as fictitious massless scalar decay products of the two Higgs bosons.

To account for equivalence relations, such as momentum conservation $\sum_i |i\rangle[i]$, we introduce a polynomial quotient ring. The quotient ring construction of ref. [26] can be easily modified to account for the extra relation on the Higgs masses,

$$R_7 = S_7 / \left\langle \sum_{i=1}^7 |i\rangle[i], s_{45} - s_{67} \right\rangle . \quad (3.2)$$

When reconstructing the integral coefficients, it is crucial to work within the correct quotient ring. For example, consider the box coefficient $d_{4 \times 1 \times 23}$ (defined later in section 6). If we were to forget the $s_{45} - s_{67}$ constraint, the least common denominator (LCD) as obtained from the generalized unitarity computation would involve the following invariants,

$$\langle 13 \rangle, \langle 23 \rangle, [23], \langle 1|(4+5)|1], \langle 2|(4+5)|1], \langle 3|(4+5)|1], [1|(4+5)|(6+7)|1]^2, \quad (3.3)$$

as well as two more invariants that also involve m^2 , the mass the quark in the loop. However, once we impose the extra constraint $s_{45} = s_{67}$, we are left only with 3 out of the 7 singularities of eq. (3.3),

$$\langle 23 \rangle, [23], [1|(4+5)|(6+7)|1]^2 . \quad (3.4)$$

This is a drastic simplification. Furthermore, if we were to analytically reconstruct the numerators of these four extra poles, there is no guarantee that these would yield simple expressions, given that the extra four poles effectively have residues proportional to $(s_{45} - s_{67})$, meaning they could be arbitrarily complicated rewritings of zero.

3.1.1 Univariate interpolation with additional constraints

A key step in the reconstruction of multivariate rational functions is the determination of their irreducible denominator factors. This can be achieved by univariate Thiele interpolation [36] on a generic univariate slice in the multivariate space [37]. When spinor variables are involved, such a univariate slice can be constructed from a BCFW shift [38, 39] simultaneously applied to all holomorphic or anti-holomorphic spinors [40], i.e. an *all-line shift*,

$$|i\rangle \rightarrow |i\rangle + t c_i |\eta\rangle, \quad [i] \rightarrow [i], \quad (3.5)$$

for a given constant $|\eta\rangle$ and with generic c_i 's that satisfy momentum conservation,

$$\sum_i c_i |\eta\rangle = 0 . \quad (3.6)$$

A single such slice suffices to determine the denominators of functions of Mandelstam invariants and $\text{tr}_5 = \text{Tr} \{ \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4 \gamma_5 \}$ [41], while the holomorphic plus anti-holomorphic pair is required to obtain denominators of spinor-helicity functions [42], since we need the univariate slice to

intersect all codimension one varieties. The slice of eq. (3.5) does not intersect the purely anti-holomorphic varieties associated to ideals generated by just square brackets, $[ij]$.

Here we show that a single slice (instead of a pair) suffices, and that additional constraints can also be imposed. To this end, we construct a shift that involves all variables,

$$|i\rangle \rightarrow |i\rangle + t x_i |\eta\rangle, \quad [i] \rightarrow [i] + t y_i [\eta]. \quad (3.7)$$

Now momentum conservation takes the form,

$$\sum_i \left(|i\rangle [i] + t(x_i |\eta\rangle [i] + y_i |i\rangle [\eta]) + t^2 x_i y_i |\eta\rangle [\eta] \right) = 0, \quad (3.8)$$

where the t^0 coefficient is automatically zero, as the starting point is assumed to be in R_7 . Moreover, the extra constraint $s_{45} - s_{67}$ becomes,

$$\left[- \left(\langle 45 \rangle + t x_4 \langle \eta 5 \rangle + t x_5 \langle 4 \eta \rangle \right) \times \left(\langle \rangle \leftrightarrow [] \right) \right] - \left[\{4, 5\} \leftrightarrow \{6, 7\} \right] = 0, \quad (3.9)$$

which, once expanded, yields a quadratic polynomial in t , where, once again, the t^0 coefficient is zero.

Collecting all coefficients in the t -polynomials, which in this case means the t and t^2 coefficients, we obtain a system of equations in the variables x_i, y_i . A generic solution to this system ensures the shifted line of eq. (3.7) lies entirely within R_7 and crosses all relevant codimension-one varieties. This allows the determination of the LCD by matching irreducible denominator factors in t to a list of expected singularities.

Implementation. We implement the univariate slice using `lips` [43, 44], with the numeric part relying on `NumPy` [45], and the analytic part on `SymPy` [46]. The package `pyadic` [47] is used for its implementation of the number field $\mathbb{F}$, taken to be either finite fields ($\mathbb{F}_p$) or p -adic number, ($\mathbb{Q}_p$), and of the Thiele and Newton interpolation algorithms [36]. The generic solution to the system of equations in $\mathbb{F}[x_i, y_i]$ is obtained by a generalization of the algorithm presented in ref. [26, section 3] for arbitrary polynomial (quotient) rings, as implemented in `syngular` [48] (`Ideal.point_on_variety`). The required lexicographic Gröbner bases are obtained from `Singular` [49].

Arbitrary quotient ring. It appears clear that the same procedure could be applied when working in an arbitrary quotient ring,

$$\mathbb{F}[\underline{X}] / \langle q_1(\underline{X}), \dots, q_m(\underline{X}) \rangle \quad (3.10)$$

by performing a shift,

$$\underline{X} \rightarrow \underline{X} + t \underline{Y}, \quad (3.11)$$

and substituting it into the ideal $\langle q_1, \dots, q_m \rangle$. Then, the equations need to be expanded in t and each coefficient in the t -polynomials needs to be set to zero by choosing appropriate solutions for $\underline{Y}$, such that the line lies in the quotient ring for arbitrary values of t .

This procedure has its limits. Namely, if the polynomial quotient ring is not a unique factorization domain, i.e. if there exist irreducible polynomials that generate non-prime ideals, then one has to be careful with determining the LCD, as it is not unique. This is actually relevant for the determination of the effective pentagon coefficients discussed in section 2.1

3.2 Minimal spinor ansatz for an arbitrary number of massive scalars

Minimal ansätze for the analytic reconstruction of an n -point process with a single massive external scalar leg, such as single Higgs production in association with jets [33], can be constructed by considering an $(n - 1)$ -point process without momentum conservation. This amounts to replacing every occurrence of the massive four-momentum with a sum over all the massless ones. However, this is no longer possible in the presence of multiple massive lines, since removing a massive four-momentum causes the introduction of another one.

Since the rings of eq. (3.1) and eq. (3.2) over-parametrise the space of the $pp \rightarrow HHj$ coefficients, to construct a minimal ansatz, we need to consider the covariant ring without fictitious decays,

$$\mathbf{S}_5 = \mathbb{F}[|1\rangle, [1], |2\rangle, [2], |3\rangle, [3], \mathbf{4}, \mathbf{5}], \quad (3.12)$$

where the bold numbers denote rank two spinors, $\mathbf{4} = 4_{\alpha\dot{\alpha}}$ and $\mathbf{5} = 5_{\alpha\dot{\alpha}}$. The relation between the seven massless legs of eq. (3.1) and three massless plus two massive ones of eq. (3.12) is,

$$1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3, \mathbf{4} \rightarrow 4 + 5, \mathbf{5} \rightarrow 6 + 7. \quad (3.13)$$

Because the external massive particles are scalars, the rank-two spinors never appear decomposed as sums over two pairs of rank-one spinors, $\mathbf{p} = \sum_I |\mathbf{p}^I\rangle [\mathbf{p}^I]$. In fact, this would be equivalent to the seven-point space of eqs. (3.1) and (3.2). It introduces 8 spinor components in lieu of 4.

The four, four-momentum conservation equations read

$$J_{\text{mom. cons.}} = \langle |1\rangle [1] + |2\rangle [2] + |3\rangle [3] + \mathbf{4} + \mathbf{5} \rangle. \quad (3.14)$$

We further impose the constraint that $m_4 = m_5$. In terms of the ring variables, the masses can be written as $\text{tr}(\mathbf{4}|\mathbf{4}) = 2m_4^2$, or $\det(\mathbf{4}) = m_4^2$, and equivalently for $\mathbf{5}$. These are to be understood as polynomials in the four components of the rank-two spinors.

The quotient ring is then,

$$\mathbf{R}_5 = \mathbf{S}_5 / (J_{\text{mom. cons.}} + \langle \det(\mathbf{4}) - \det(\mathbf{5}) \rangle). \quad (3.15)$$

Since amplitudes are Lorentz invariant, we now need to convert to an invariant ring following the elimination algorithm of ref. [26, section 2]. However, the standard spinor brackets,

$$\langle ij \rangle \quad \text{and} \quad [ij], \quad (3.16)$$

are no longer sufficient, since i, j can only be in $\{1, 2, 3\}$. In addition, we must consider contractions involving the rank-two spinors $\mathbf{4}$ and $\mathbf{5}$, both with the rank-one spinors and among themselves. The former case reads,

$$\langle i|\mathbf{k}|i \rangle \quad \text{and} \quad \langle i|\mathbf{k}|j \rangle, \quad (3.17)$$

while the latter case requires the introduction of traces, now considered as variables in their own right (i.e. no longer as polynomials in the components),

$$\text{tr}(\mathbf{k}|\mathbf{k}) \quad \text{and} \quad \text{tr}(\mathbf{k}|\mathbf{l}), \quad (3.18)$$

with $\mathbf{k}, \mathbf{l}$ in $\{\mathbf{4}, \mathbf{5}\}$. Since we work with 2-component spinors, these are understood as traces of 2×2 matrices, explicitly $\text{tr}(\mathbf{k}|\mathbf{l}) = \mathbf{k}_{\alpha\dot{\alpha}} \mathbf{l}^{\dot{\alpha}\alpha}$. In terms of Mandelstam invariants and M_H they read $\text{tr}(\mathbf{4}|\mathbf{4}) = 2M_H^2$, $\text{tr}(\mathbf{4}|\mathbf{5}) = s_{123} - 2M_H^2$.

The invariants of eq. (3.16), eq. (3.17), and eq. (3.18) altogether can be used to define a Lorentz invariant polynomial ring analogous to $\mathcal{S}_n$ of ref. [26, section 2.2, eq. 2.59],

$$\mathcal{S}_5 = \mathbb{F}[\langle ij \rangle, [ij], \langle i|\mathbf{k}|i \rangle, \langle i|\mathbf{k}|j \rangle, \text{tr}(\mathbf{k}|\mathbf{k}), \text{tr}(\mathbf{k}|\mathbf{l})]. \quad (3.19)$$

Once again these variables are subject to equivalence relations, thus a polynomial quotient ring, $\mathcal{R}_5$, is needed. The equivalence relations now include Schouten identities, besides momentum conservation and the equality between the two Higgs masses.

We can obtain all these relations among Lorentz invariant spinor brackets from those among the Lorentz covariant spinors. To do this, we build an extended covariant plus invariant polynomial ring with the variables from both $\mathbf{S}_5$ and $\mathcal{S}_5$. Following the notation of ref. [26] we have,

$$\Sigma_5 = \mathbb{F}[|1\rangle, |1\rangle, |2\rangle, |2\rangle, |3\rangle, |3\rangle, \mathbf{4}, \mathbf{5}, \langle ij \rangle, [ij], \langle i|\mathbf{k}|i \rangle, \langle i|\mathbf{k}|j \rangle, \text{tr}(\mathbf{k}|\mathbf{k}), \text{tr}(\mathbf{k}|\mathbf{l})]. \quad (3.20)$$

In this extended ring, we consider the ideal

$$\begin{aligned} \kappa[J_{\text{mom. cons.}} + \langle \det(\mathbf{4}) - \det(\mathbf{5}) \rangle] = & \langle |1\rangle[1] + |2\rangle[2] + |3\rangle[3] + \mathbf{4} + \mathbf{5}, \\ & \det(\mathbf{4}) - \det(\mathbf{5}), \langle 12 \rangle - (\lambda_{1,0}\lambda_{2,1} - \lambda_{2,0}\lambda_{1,1}, \dots) \rangle, \end{aligned} \quad (3.21)$$

where the ellipsis contains all other equations defining the invariant spinor brackets as contractions of the components of the covariant spinors. For clarity, we have expanded the definition of the $\langle 12 \rangle$ bracket in terms of the components of the spinors $|1\rangle = (\lambda_{1,0}, \lambda_{1,1})$ and $|2\rangle = (\lambda_{2,0}, \lambda_{2,1})$. The minus sign arises from the Levi-Civita tensor, which is the metric in spinor space.

The invariant quotient ring, $\mathcal{R}_5$, is obtained as

$$\mathcal{R}_5 = \mathcal{S}_5 / (\kappa[J_{\text{mom. cons.}} + \langle \det(\mathbf{4}) - \det(\mathbf{5}) \rangle] \cap \mathcal{S}_5), \quad (3.22)$$

where the intersection of the ideal κ of Σ_5 with the subring $\mathcal{S}_5$ amounts to eliminating the spinor component variables. That is, within a suitably chosen block-order, one picks the subset of the Gröbner basis generated only by Lorentz invariant variables. This automatically generates all equivalence relations among invariants. They are: a) the invariant equivalent of $J_{\text{mom. cons.}}$, i.e. the ideal generated by all contractions of the covariant generator of eq. (3.14); b) $\text{tr}(\mathbf{4}|\mathbf{4}) = \text{tr}(\mathbf{5}|\mathbf{5})$, which comes from $\det(\mathbf{4}) = \det(\mathbf{5})$; c) all Schouten identities. These are no longer the usual ones,

$$\langle ij \rangle \langle kl \rangle + \langle ik \rangle \langle lj \rangle + \langle il \rangle \langle jk \rangle = 0, \quad (3.23)$$

since we have only three massless legs, instead we have ones of the form,

$$\langle jk \rangle \langle i|\mathbf{l}|k \rangle - \langle ik \rangle \langle j|\mathbf{l}|k \rangle + \langle ij \rangle \langle k|\mathbf{l}|k \rangle = 0. \quad (3.24)$$

mass dimension:	2	4	6	8	10	12
1. $0 \rightarrow ggggg$	5	16	40	85	161	280
2. $0 \rightarrow gggggg$	9	50	205	675	1886	4644
3. $0 \rightarrow gggggg$	14	120	735	3486	13566	45178
4. $0 \rightarrow ggggH$	6	22	62	147	308	588
5. $0 \rightarrow ggggH$	10	60	265	940	2826	7470
6. $0 \rightarrow gggHH$	6	22	62	147	308	588
7. $0 \rightarrow gggHH^*$	7	29	91	238	546	1134

Table 1. Ansatz dimensions at zero phase weights for various processes representative of external kinematic configurations. The first column corresponds to the number of independent Mandelstam invariants. Rows 1, 4, 6 and 7 involve a single independent tr_5 , thus their ansatz sizes saturate the upper bound of ref. [50, eq. 3.2]. Row 2 reproduces ref. [25, table 1] (implementations differ). Rows 4 and 6 are identical, but represent different ansätze. For instance, the former may have non-zero little-group weight associated with the fourth leg, but the latter cannot.

Constructing an ansatz then amounts to enumerating all independent monomials of $\mathcal{R}_5$ with a given mass dimension and little-group weight. Mass dimension and polynomial degree now are related by a weight vector assigning weight 1 to the two-particle spinor brackets, and weight 2 to the three-particle spinor brackets and traces. Polynomials are still homogeneous, given this weight vector.

It is clear that this construction could be generalized to construct ansätze for an arbitrary combination of m massless legs plus n massive scalar ones.

To conclude, we report the size of various ansätze in table 1. The ansatz sizes for the process of interest in this work is given in row 6, while row 7 shows the closely related process where the two Higgs bosons are not constrained to have the same mass. Comparing row 3 with row 6 demonstrates the importance of using the appropriate ansatz construction for the present calculation. In fact, while the ansatz of row 3, representing polynomials in $\mathcal{R}_7$, would have worked, it clearly becomes orders of magnitude more complex than the minimal one for this process in row 6.

Implementation. We implement the ansatz construction via the described adaptation of the algorithm of ref. [26, section 2.2] using `lips` for the spinor algebra [43, 44], `Singular` for the Gröbner bases and variable elimination [49], and the `CP-SAT` solver from `OR-Tools` [51] to enumerate the ansatz monomials.

Massive vector bosons. A similar construction, with appropriate modification, should be suitable to build minimal ansätze for processes involving multiple bosons, including vector ones. In a soon-to-appear publication [52], the two-loop amplitudes for the process $pp \rightarrow Wjj$ are reconstructed in spinor-helicity variables. To do so, the ansatz for a single massive scalar (row 4 of table 1) is modified to allow the spinors of the now non-fictitious decay products, e.g. a charged lepton plus neutrino pair, to appear with a degree bound of one (for more details see that article). The two constructions could be combined with suitable degree bounds on the leptonic decays of the vector bosons to construct ansätze for processes

involving e.g. WW , WZ , WH , ZZ or ZH . The computation of two-loop amplitudes for such processes in association with a jet is now within reach [53].

3.3 Projective space in m

Besides the kinematic variables discussed so far, the mass of the quark running in the loop also appears in the amplitude. The polynomial ring is thus,

$$\mathbf{S}_5[m] = \mathbb{F}[|1\rangle, |1|, |2\rangle, |2|, |3\rangle, |3|, \mathbf{4}, \mathbf{5}, m]. \quad (3.25)$$

The quotient ring construction is unchanged, since no equivalence relation involves m , so we simply have the covariant quotient ring $\mathbf{R}_5[m]$ and the invariant one $\mathbf{R}_5[m]$.

It is clear that m has a special role in $\mathbf{R}_5[m]$ and $\mathbf{R}_5[m]$: it is the only variable not subject to equivalence relations. It is then straightforward to consider this space as being projective in m , since m can be taken to infinity independently. On the other hand, we remain in an affine space for the spinors, since taking a spinor to infinity would require another one also being either large (if additive in the equivalence relation) or small (if multiplicative). That is, we include the point $m \rightarrow \infty$, but not any point at infinity for the spinor variables. In a projective space at the point at infinity the role of poles and zeros is inverted, a numerator factor is a pole, while a denominator factor is a zero.

The box coefficients in the $gggHH$ amplitude have a rich analytic structure in m . Since performing four unitarity cuts can result in a fifth propagator being evaluated on the cut, the kinematic and m dependence mix in the denominator. The large and small m limits provide useful information on the general m expression, but do not directly translate to individual terms of a Taylor expansion as in the case of the triangle coefficients. In appendix D we investigate how the loop quark mass dependence affects various rewritings of a box coefficient, especially in relation to the location of spurious singularities. In particular, we make the interesting observation that there can be a spurious pole at $m \rightarrow \infty$. This could provide an even more numerically efficient way to write the box coefficients, if the size of the expressions in that form can be contained. In fact, the mass m , when interpreted as the top-quark mass, is large, but never truly approaches infinity — unlike some kinematic variables that may need to approach zero. The compact form of the box coefficients that we present in section 7 is instead based on an effective pentagon decomposition, as discussed in section 2.1.

3.3.1 Effective pentagons as residues of mixed m -kinematic poles

There are two singularities (plus permutations) that mix the kinematic dependence with the m dependence. Their origin is explained in section 7.1. For the purpose of the present discussion, it is sufficient to anticipate the form of eq. (7.11) and eq. (7.20). For convenience, we repeat one here,

$$|S^{1 \times 2 \times 3 \times 4}| = -s_{12}s_{23} \langle 1|\mathbf{5}|\mathbf{4}|3\rangle [3|\mathbf{4}|\mathbf{5}|1] + m^2 (\text{tr}_5)^2. \quad (3.26)$$

An advantage of the decomposition in eq. (2.13) is that the entire dependence on such mixed mass-kinematic poles is captured by the coefficients $\mathcal{C}$, leaving the effective pentagons $\hat{e}_{i \times j \times k \times l}$ and effective boxes $\hat{d}_{i \times j \times k}$ free of any m dependence in the denominator.

The effective pentagons $\hat{e}_{i \times j \times k \times l}$ are defined as residues of these mixed mass-kinematic singularities (up to the numerator part of the $\mathcal{C}$'s). The residue of a simple pole in $|S^{1 \times 2 \times 3 \times 4}|$ is defined in the quotient ring,

$$\mathbf{S}_5 / (J_{\text{mom. cons.}} + \langle m_4^2 - m_5^2 \rangle + \langle |S^{1 \times 2 \times 3 \times 4}| \rangle). \quad (3.27)$$

This is not a unique factorization domain, because we can find an irreducible polynomial, e.g. tr_5 , that generates a non-prime ideal. For instance, in the quotient ring of eq. (3.27), we have

$$\langle \text{tr}_5 \rangle \ni m^2 \text{tr}_5^2 = s_{12} s_{23} \langle 1|5|4|3 \rangle [3|4|5|1]. \quad (3.28)$$

Since none of the factors $\langle 12 \rangle$, $[12]$, $\langle 23 \rangle$, $[23]$, $\langle 1|5|4|3 \rangle$, and $[3|4|5|1]$ belongs to $\langle \text{tr}_5 \rangle$ this proves that $\langle \text{tr}_5 \rangle$ is not prime (in the quotient ring of eq. (3.27)).

This implies that the LCD of an effective pentagon coefficient is not unique. In practice, we can shift its definition either additively by,

$$0 = -s_{12} s_{23} \langle 1|5|4|3 \rangle [3|4|5|1] + m^2 (\text{tr}_5)^2, \quad (3.29)$$

or multiplicatively by,

$$1 = \frac{m^2 (\text{tr}_5)^2}{s_{12} s_{23} \langle 1|5|4|3 \rangle [3|4|5|1]}, \quad (3.30)$$

without affecting its validity in relation to eq. (2.13) and eq. (2.14).

We use the redundancy of eq. (3.29) to show that the effective pentagons can be written without a double pole in tr_5 . This step was crucial to obtain expressions compact enough to be presented in this article. The redundancy of eq. (3.30) could be used to replace the spurious simple pole in tr_5 with a zero in tr_5 , but with extra spurious poles in s_{12} , s_{23} , $\langle 1|5|4|3 \rangle$, $[3|4|5|1]$. Such re-definitions affect the form of the effective box coefficients, as these are not defined as residues of the same pole.

3.3.2 Interpolation on leading p -adic digit

When reconstructing functions that depend on m , it is generally useful to consider the behaviour at $m = 0$ and $m \rightarrow \infty$. For functions where the m dependence is a Taylor series, this allows the isolation of individual terms in the series. For functions that have m dependence mixed with kinematic variables in the LCD, the small and large m limits still yield useful information, but cannot be directly used to obtain the general m expression.

It is interesting to note that the univariate interpolation of section 3.1.1, traditionally used with finite fields, can be equally well performed on a leading p -adic digit, by treating it as if it was a $\mathbb{F}_p$ number [35]. This allows one to isolate the $m = 0$ contribution by setting $m \propto p$, and $m \rightarrow \infty$ by setting $m \propto \frac{1}{p}$. If we were to work in $\mathbb{F}_p$ this would require interpolation on a bi-variate slice. Since the ideal $\langle m \rangle$ is prime, the LCDs at $m = 0$ and $m \rightarrow \infty$ are uniquely determined by the slicing procedure.

4 Results for the process $0 \rightarrow q\bar{q}gHH$

4.1 Process $0 \rightarrow q\bar{q}gH^*$

We first report on the contribution to Higgs boson pair production via the triple Higgs boson coupling, due to the process,

$$0 \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3) + H^*(p_4), \quad p_4 = -p_1 - p_2 - p_3, \quad p_4^2 \neq M_H^2. \quad (4.1)$$

The amplitude is given by,

$$-i\mathcal{A}_{i_1 i_2}^{C_3}(1_q, 2_{\bar{q}}, 3_g; 4_{H^*}) = -\frac{g_s^3}{16\pi^2} \frac{g_W}{2M_W} t_{i_1 i_2}^{C_3} A(1_q^{h_1}, 2_{\bar{q}}^{h_2}, 3_g^{h_3}; 4_{H^*}). \quad (4.2)$$

The t matrices are the $SU(N)$ matrices in the fundamental representation normalized such that,

$$\text{tr}(t^a t^b) = \delta^{ab}, \quad (4.3)$$

and $N = 3$. i_1, i_2 and C_3 are thus the $SU(3)$ indices of the quark, antiquark and gluon respectively. The helicities are denoted by, h_1, h_2 and h_3 for the outgoing quark, anti-quark and gluon respectively. The result for the colour-stripped amplitude is,

$$A(1_q^-, 2_{\bar{q}}^+, 3_g^+; 4_{H^*}) = -4 \frac{\langle 12 \rangle [23]^2 m^2}{(s_{123} - s_{12})^2} \left[\bar{B}_0(p_{12}) + \left[\frac{(s_{123} - s_{12})}{2s_{12}} - 2 \frac{m^2}{s_{12}} \right] \bar{C}_0(p_3, p_{12}) - \frac{(s_{123} - s_{12})}{s_{12}} \right]. \quad (4.4)$$

For conciseness we have introduced modified (dimensionless) forms of the scalar integrals,

$$\begin{aligned} \bar{C}_0(p_3, p_{12}) &= (2p_3 \cdot p_{12}) C_0(p_3, p_{12}; m) \\ \bar{B}_0(p_{12}) &= B_0(p_{12}; m) - B_0(p_{123}; m). \end{aligned} \quad (4.5)$$

The amplitudes for the Higgs boson pair production cross section due to the triple Higgs boson coupling are simply related to the above result,

$$\mathcal{A}_{i_1 i_2}^{C_3}(1_q, 2_{\bar{q}}, 3_g; H^* \rightarrow 4_H, 5_H) = \frac{3g_W}{2M_W} \kappa_\lambda \frac{M_H^2}{s_{123} - M_H^2} \mathcal{A}_{i_1 i_2}^{C_3}(1_q, 2_{\bar{q}}, 3_g; 4_{H^*}). \quad (4.6)$$

By combining eqs. (4.4) and (4.6) one arrives at the result for this contribution to the full amplitude, which is included in the expressions in the following section.

4.2 Process $0 \rightarrow q\bar{q}gHH$

The contribution to the full physical amplitude for the process,

$$0 \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3) + H(p_4) + H(p_5), \quad (4.7)$$

is given by,

$$-i\mathcal{A}_{i_1 i_2}^{C_3}(1_q^{h_1}, 2_{\bar{q}}^{h_2}, 3_g^{h_3}; 4_H, 5_H) = -\frac{g_s^3}{16\pi^2} \frac{m^2}{v^2} t_{i_1 i_2}^{C_3} A(1_q^{h_1}, 2_{\bar{q}}^{h_2}, 3_g^{h_3}; 4_H, 5_H). \quad (4.8)$$

The colour-ordered sub-amplitudes in eq. (4.8) can be expressed in terms of scalar integrals. For the $0 \rightarrow q\bar{q}gHH$ colour-stripped sub-amplitude we have,

$$\begin{aligned}
 A(1_q^{h_1}, 2_{\bar{q}}^{h_2}, 3_g^{h_3}; 4_H, 5_H) &= \frac{\bar{\mu}^{4-n}}{r_\Gamma} \frac{1}{i\pi^{n/2}} \int d^n\ell \frac{\text{Num}(\ell)}{\prod_i d_i(\ell)} \\
 &= \sum_{i,j,k} d_{i \times j \times k}(1^{h_1}, 2^{h_2}, 3^{h_3}) D_0(p_i, p_j, p_k; m) \\
 &\quad + \sum_{i,j} c_{i \times j}(1^{h_1}, 2^{h_2}, 3^{h_3}) C_0(p_i, p_j; m) \\
 &\quad + \sum_i b_i(1^{h_1}, 2^{h_2}, 3^{h_3}) B_0(p_i; m) + r(1^{h_1}, 2^{h_2}, 3^{h_3}). \quad (4.9)
 \end{aligned}$$

The scalar bubble (B_0), triangle (C_0), box (D_0) integrals, and the constant r_Γ , are defined in appendix B. $r(1^{h_1}, 2^{h_2}, 3^{h_3})$ is the rational contribution to the amplitude. We can further decompose the box and triangle coefficients according to the power of the quark mass running in the loop,

$$\begin{aligned}
 d_{i \times j \times k} &= d_{i \times j \times k}^{(0)} + m^2 d_{i \times j \times k}^{(2)} + m^4 d_{i \times j \times k}^{(4)}, \\
 c_{i \times j} &= c_{i \times j}^{(0)} + m^2 c_{i \times j}^{(2)}. \quad (4.10)
 \end{aligned}$$

We present results for the helicity choices $h_1 = -1, h_2 = +1, h_3 = +1$. The other helicity choices are obtained as follows,

$$\begin{aligned}
 A(1^+, 2^-, 3^+) &= A(2^-, 1^+, 3^+) \\
 A(1^+, 2^-, 3^-) &= A(1^-, 2^+, 3^+) \big|_{\langle \rangle \leftrightarrow []}. \quad (4.11)
 \end{aligned}$$

4.2.1 Boxes

There are two independent box coefficients. The first one reads,

$$d_{12 \times 4 \times 3}^{(0)}(1^-, 2^+, 3^+) = M_H^2 (s_{34}s_{35} - M_H^4) \frac{(\langle 3|4|2]^2 + \langle 3|5|2]^2)}{[1\,2] \langle 3|4|5|3]^2}, \quad (4.12)$$

$$d_{12 \times 4 \times 3}^{(2)}(1^-, 2^+, 3^+) = 8 \frac{\langle 3|4|2] \langle 3|5|2] (s_{34}s_{35} - M_H^4)}{[1\,2] \langle 3|4|5|3]^2} + \bar{d}^{(2)}(1^-, 2^+, 3^+), \quad (4.13)$$

$$\begin{aligned}
 \text{with } \bar{d}^{(2)}(1^-, 2^+, 3^+) &= 4 \frac{\langle 1|3|2] (\langle 1|4|(1+2)|5|1] - \langle 1|5|(1+2)|4|1])}{s_{12} \langle 3|4|5|3]}, \\
 &\quad - 2 \frac{(\langle 1|4|(1+2)|5|2] - \langle 1|5|(1+2)|4|2]) (s_{45} - 2s_{23} - 2M_H^2)}{s_{12} \langle 3|4|5|3]}, \quad (4.14)
 \end{aligned}$$

$$d_{12 \times 4 \times 3}^{(4)}(1^-, 2^+, 3^+) = -8 \frac{\langle 1\,3 \rangle \langle 1|(4-5)|3]}{\langle 1\,2 \rangle \langle 3|4|5|3]} - 8 \frac{[2\,3] \langle 3|(4-5)|2]}{[1\,2] \langle 3|4|5|3]}. \quad (4.15)$$

The second one reads,

$$d_{12 \times 3 \times 4}^{(0)}(1^-, 2^+, 3^+) = -(s_{34}s_{45} - M_H^2 s_{12}) \frac{(\langle 1|5|4|3]^2 + \langle 1\,3 \rangle^2 M_H^4)}{\langle 1\,2 \rangle \langle 3|4|5|3]^2}, \quad (4.16)$$

$$d_{12 \times 3 \times 4}^{(2)}(1^-, 2^+, 3^+) = -8 \frac{\langle 1\,3 \rangle \langle 1|5|4|3] (s_{34}s_{45} - M_H^2 s_{12})}{\langle 1\,2 \rangle \langle 3|4|5|3]^2} + \bar{d}^{(2)}(1^-, 2^+, 3^+), \quad (4.17)$$

$$d_{12 \times 3 \times 4}^{(4)}(1^-, 2^+, 3^+) = d_{12 \times 4 \times 3}^{(4)}(1^-, 2^+, 3^+). \quad (4.18)$$

The third box coefficient is not independent,

$$d_{3 \times 12 \times 4}(1^-, 2^+, 3^+) = d_{12 \times 3 \times 4}(1^-, 2^+, 3^+)|_{4 \leftrightarrow 5}. \quad (4.19)$$

4.2.2 Triangles

In the limit $m \rightarrow 0$ the box integrals, and a subset of the triangle integrals, develop poles in the limit $\epsilon \rightarrow 0$. Since the result for the amplitude must be finite, this yields constraints on the integral coefficients. The order m^0 triangle coefficients are determined by these IR relations,

$$\frac{c_{3 \times 4}^{(0)}}{s_{34} - M_H^2} = -\frac{d_{12 \times 3 \times 4}^{(0)}}{s_{34} s_{45} - M_H^2 s_{12}} - \frac{d_{12 \times 4 \times 3}^{(0)}}{s_{34} s_{35} - M_H^4}, \quad (4.20)$$

$$\frac{c_{3 \times 124}^{(0)}}{s_{35} - M_H^2} = -\frac{d_{3 \times 12 \times 4}^{(0)}}{s_{35} s_{45} - M_H^2 s_{12}} - \frac{d_{12 \times 4 \times 3}^{(0)}}{s_{34} s_{35} - M_H^4}, \quad (4.21)$$

$$\frac{\bar{c}_{3 \times 12}^{(0)}}{s_{45} - s_{12}} = -\frac{d_{3 \times 12 \times 4}^{(0)}}{s_{35} s_{45} - M_H^2 s_{12}} - \frac{d_{12 \times 3 \times 4}^{(0)}}{s_{34} s_{45} - M_H^2 s_{12}}, \quad (4.22)$$

where this last relation represents the triangle coefficient originating from diagrams that do not involve the triple Higgs coupling. The full result for this coefficient is,

$$c_{3 \times 12}^{(0)} = \bar{c}_{3 \times 12}^{(0)} + 6\kappa_\lambda \frac{[23]^2}{[12]} \frac{M_H^2}{(s_{45} - M_H^2)}. \quad (4.23)$$

The order m^2 pieces for the triangles $c_{3 \times 12}$, $c_{3 \times 4}$ and $c_{3 \times 124}$ with one light-like external line are,

$$c_{3 \times 12}^{(2)}(1^-, 2^+, 3^+) = \frac{8 \langle 13 \rangle^2 (s_{45} - s_{12})(s_{45} - 2M_H^2)}{\langle 12 \rangle \langle 3|4|5|3 \rangle^2} + \frac{8[23]^2}{[21] (s_{45} - s_{12})} \left[1 + \kappa_\lambda \frac{3M_H^2}{(s_{45} - M_H^2)} \right], \quad (4.24)$$

$$c_{3 \times 4}^{(2)}(1^-, 2^+, 3^+) = 8 \frac{\langle 3|4|3 \rangle}{\langle 3|4|5|3 \rangle^2} \left\{ \frac{\langle 13 \rangle \langle 1|5|4|3 \rangle}{\langle 12 \rangle} - \frac{\langle 3|4|2 \rangle \langle 3|5|2 \rangle}{[12]} \right\}, \quad (4.25)$$

$$c_{3 \times 124}^{(2)} = c_{3 \times 4}^{(2)}(1^-, 2^+, 3^+)|_{4 \leftrightarrow 5}. \quad (4.26)$$

The triangle coefficients for the triangles without a light-like external line are,

$$c_{4 \times 123}(1^-, 2^+, 3^+) = (s_{45} - 2M_H^2 + 8m^2) \times \left\{ \frac{\langle 1|(2+3)|(4-5)|3 \rangle (\langle 1|4|5|3 \rangle - \langle 1|5|4|3 \rangle)}{\langle 12 \rangle \langle 3|4|5|3 \rangle^2} - 2 \frac{\langle 1|4|5|1 \rangle}{\langle 12 \rangle \langle 3|4|5|3 \rangle} \right\}, \quad (4.27)$$

$$\begin{aligned} c_{4 \times 12}(1^-, 2^+, 3^+) = & (s_{45} - 2M_H^2 + 8m^2) \left\{ \frac{\langle 13 \rangle \langle 3|5|2 \rangle \Delta_{12|4|35}}{s_{12} \langle 3|4|5|3 \rangle^2} - \frac{\langle 1|4|2 \rangle (s_{35} + s_{12} - M_H^2)}{s_{12} \langle 3|4|5|3 \rangle} \right\} \\ & + \frac{[32] \left\{ \langle 2|4|2 \rangle (\langle 1|(1+2)|4|3 \rangle - \langle 1|4|(1+2)|3 \rangle) \right\}}{s_{12} \langle 3|5|4|3 \rangle} \\ & - \frac{[32] \left\{ \langle 1|4|2 \rangle (\langle 2|(1+2)|4|3 \rangle - \langle 2|4|(1+2)|3 \rangle) \right\}}{s_{12} \langle 3|5|4|3 \rangle} \\ & + \frac{(\langle 1|(1+2)|4|3 \rangle - \langle 1|4|(1+2)|3 \rangle) \langle 1|4|3 \rangle}{\langle 12 \rangle \langle 3|5|4|3 \rangle}, \end{aligned} \quad (4.28)$$

$$c_{12 \times 34}(1^-, 2^+, 3^+) = c_{4 \times 12}|_{4 \leftrightarrow 5}, \quad (4.29)$$

and the Källén function is,

$$\Delta_{12|4|35} = (s_{12} + M_H^2 - s_{35})^2 - 4s_{12}M_H^2. \quad (4.30)$$

4.2.3 Bubbles and rational pieces

The bubble coefficients and the rational terms are given by,

$$b_{123}(1^-, 2^+, 3^+) = 4 \frac{\langle 12 \rangle [23]^2}{(s_{45} - s_{12})^2} \left[1 + \kappa_\lambda \frac{3M_H^2}{(s_{45} - M_H^2)} \right], \quad b_{12}(1^-, 2^+, 3^+) = -b_{123}(1^-, 2^+, 3^+), \quad (4.31)$$

$$r(1^-, 2^+, 3^+) = \frac{4[23]^2}{[21](s_{45} - s_{12})} \left[1 + \kappa_\lambda \frac{3M_H^2}{(s_{45} - M_H^2)} \right]. \quad (4.32)$$

This concludes the discussion of the quark-antiquark-gluon contribution to the Higgs boson pair amplitudes.

5 Results for the process $0 \rightarrow gggH^*(\rightarrow HH)$

We first present results for the process involving a single Higgs boson [54, 55] which contribute to the Higgs boson pair production via the Higgs boson self-coupling,

$$i\mathcal{A}^{C_1 C_2 C_3}(gggH^*) = (i\sqrt{2}f^{C_1 C_2 C_3}) \frac{g_s^3}{16\pi^2} \frac{g_W}{M_W} s_{123}^2 A(1_g^{h_1}, 2_g^{h_2}, 3_g^{h_3}, 4_{H^*}), \quad (5.1)$$

where $p_4 = -p_1 - p_2 - p_3$ and $s_{12} = (p_1 + p_2)^2$, $s_{13} = (p_1 + p_3)^2$, $s_{23} = (p_2 + p_3)^2$, $s_{123} = (p_1 + p_2 + p_3)^2$. With these definitions we have,

$$A(1_g^+, 2_g^+, 3_g^+, 4_{H^*}) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} A_4(p_1, p_2, p_3), \quad (5.2)$$

$$A(1_g^+, 2_g^+, 3_g^-, 4_{H^*}) = \frac{[12]}{\langle 12 \rangle^2 [13] [23]} A_2(p_1, p_2, p_3). \quad (5.3)$$

The remaining amplitudes can be obtained by symmetry operations,

$$A(1_g^+, 2_g^-, 3_g^+, 4_{H^*}) = -A(1_g^+, 3_g^+, 2_g^-, 4_{H^*}), \quad (5.4)$$

$$A(1_g^-, 2_g^+, 3_g^+, 4_{H^*}) = -A(3_g^+, 2_g^+, 1_g^-, 4_{H^*}), \quad (5.5)$$

$$A(1_g^{-h_1}, 2_g^{-h_2}, 3_g^{-h_3}, 4_{H^*}) = - \left[A(1_g^{h_1}, 2_g^{h_2}, 3_g^{h_3}, 4_{H^*}) \right]_{\langle \rangle \leftrightarrow []}. \quad (5.6)$$

The two helicity amplitudes A_2, A_4 are given by [54],

$$\begin{aligned} A_4(p_1, p_2, p_3) = & \frac{m^2}{s_{123}} \left[-2 - \left(\frac{m^2}{s_{123}} - \frac{1}{4} \right) \left\{ \overline{D}_0(p_1, p_2, p_3) + \overline{D}_0(p_1, p_3, p_2) + \overline{D}_0(p_2, p_1, p_3) \right. \right. \\ & \left. \left. + 2\overline{C}_0(p_1, p_{23}) + 2\overline{C}_0(p_2, p_{13}) + 2\overline{C}_0(p_3, p_{12}) \right\} \right], \end{aligned} \quad (5.7)$$

$$\begin{aligned}
 A_2(p_1, p_2, p_3) = & \frac{m^2}{s_{123}^2} \left[\frac{s_{12}(s_{23}-s_{12})}{s_{12}+s_{23}} + \frac{s_{12}(s_{13}-s_{12})}{s_{12}+s_{13}} \right. \\
 & - 2s_{13}s_{23} \frac{(2s_{12}+s_{23})}{(s_{12}+s_{23})^2} \bar{B}_0(p_{13}) - 2s_{13}s_{23} \frac{(2s_{12}+s_{13})}{(s_{12}+s_{13})^2} \bar{B}_0(p_{23}) \\
 & + \left(m^2 - \frac{s_{12}}{4} \right) \left\{ 2\bar{C}_0(p_2, p_{13}) + 2\bar{C}_0(p_1, p_{23}) - 2\bar{C}_0(p_3, p_{12}) - \bar{D}_0(p_1, p_2, p_3) - \bar{D}_0(p_2, p_1, p_3) \right\} \\
 & - 2s_{12}^2 \left[\left(\frac{2m^2}{(s_{12}+s_{23})^2} - \frac{1}{2(s_{12}+s_{23})} \right) \bar{C}_0(p_2, p_{13}) + \left(\frac{2m^2}{(s_{12}+s_{13})^2} - \frac{1}{2(s_{12}+s_{13})} \right) \bar{C}_0(p_1, p_{23}) \right] \\
 & + \frac{s_{23}s_{13}}{s_{12}} \left(\bar{C}_0(p_2, p_{13}) + \bar{C}_0(p_1, p_{23}) - \bar{C}_0(p_1, p_3) - \bar{C}_0(p_2, p_3) \right) \\
 & \left. - \frac{1}{4} \left(s_{12} - 12m^2 - \frac{4s_{23}s_{13}}{s_{12}} \right) \bar{D}_0(p_1, p_3, p_2) \right], \tag{5.8}
 \end{aligned}$$

where the reduced scalar integrals $\bar{C}_0$ and $\bar{B}_0$ are defined in eq. (4.5) and

$$\bar{D}_0(p_1, p_2, p_3) = (4p_1 \cdot p_2 \, p_2 \cdot p_3) D_0(p_1, p_2, p_3; m). \tag{5.9}$$

We note that A_2 is symmetric under the exchange $p_1 \leftrightarrow p_2$, whereas A_4 is totally symmetric.

The amplitudes for the Higgs pair production cross section due to the triple Higgs boson coupling are simply related to the above result,

$$\mathcal{A}^{C_1 C_2 C_3}(gggHH) = \frac{3g_W}{2M_W} \kappa_\lambda \frac{M_H^2}{s_{123} - M_H^2} \mathcal{A}^{C_1 C_2 C_3}(gggH^*). \tag{5.10}$$

Putting this together we arrive at the final form,

$$\begin{aligned}
 i\mathcal{A}^{C_1 C_2 C_3}(gggHH) = & \frac{3g_W}{2M_W} \kappa_\lambda \frac{M_H^2}{s_{123} - M_H^2} (i\sqrt{2}f^{C_1 C_2 C_3}) \frac{g_s^3}{16\pi^2} \frac{g_W}{M_W} s_{123}^2 A(1_g^{h_1}, 2_g^{h_2}, 3_g^{h_3}, (4+5)_{H^*}) \\
 = & \frac{g_s^3}{4\pi^2} \frac{m^2}{v^2} [\text{tr}(t^{C_1} t^{C_2} t^{C_3}) - \text{tr}(t^{C_3} t^{C_2} t^{C_1})] H_\kappa(1^{h_1}, 2^{h_2}, 3^{h_3}; H, H), \tag{5.11}
 \end{aligned}$$

where we have extracted the overall Yukawa coupling factor from A for later convenience and defined the auxiliary amplitude,

$$H_\kappa(1^{h_1}, 2^{h_2}, 3^{h_3}; H, H) = \frac{3}{2} \kappa_\lambda \frac{M_H^2}{s_{123} - M_H^2} s_{123}^2 \frac{A(1_g^{h_1}, 2_g^{h_2}, 3_g^{h_3}, (4+5)_{H^*})}{m^2}. \tag{5.12}$$

6 Calculation methods for the process $0 \rightarrow gggHH$

In this section we introduce calculation details for the part of the process $0 \rightarrow gggHH$ which does not involve the triple Higgs coupling,

$$0 \rightarrow g(p_1) + g(p_2) + g(p_3) + H(p_4) + H(p_5). \tag{6.1}$$

Both Higgs bosons are radiated off the quark line, with $p_1^2 = p_2^2 = p_3^2 = 0$ and $p_4^2 = p_5^2 = M_H^2$. The analytic results will be presented in section 7.

6.1 Definition of colour amplitudes

The amplitude for the production of a pair of Higgs bosons and 3 gluons can be expressed as colour-stripped sub-amplitudes as follows,

$$i\mathcal{H}_n^{ggg}(\{p_i, h_i\}) = \frac{g_s^3}{4\pi^2} \frac{m^2}{v^2} [\text{tr}(t^{C_1} t^{C_2} t^{C_3}) - \text{tr}(t^{C_3} t^{C_2} t^{C_1})] H(1^{h_1}, 2^{h_2}, 3^{h_3}; H, H). \quad (6.2)$$

m is the mass of the quark circulating in the loop, and v is the vacuum expectation value. Squaring the amplitude and summing over colours we have,

$$\sum_{\text{colours}} |\mathcal{H}_n^{ggg}(\{p_i, h_i\})|^2 = 2VN \left(\frac{g_s^3}{4\pi^2} \frac{m^2}{v^2} \right)^2 \left| H(1^{h_1}, 2^{h_2}, 3^{h_3}; H, H) \right|^2, \quad (6.3)$$

where $V = N^2 - 1$. From eqs. (5.11) and (6.2) it is clear that we can account for all diagrams, including the triple Higgs boson interaction, by simple modification,

$$\sum_{\text{colours}} |\mathcal{H}_n^{ggg}(\{p_i, h_i\})|^2 = 2VN \left(\frac{g_s^3}{4\pi^2} \frac{m^2}{v^2} \right)^2 \left| H(1^{h_1}, 2^{h_2}, 3^{h_3}; H, H) + H_\kappa(1^{h_1}, 2^{h_2}, 3^{h_3}; H, H) \right|^2. \quad (6.4)$$

6.2 Decomposition into scalar integrals

The colour-ordered sub-amplitudes can be expressed in terms of scalar integrals. For the $0 \rightarrow gggHH$ sub-amplitude we have,

$$\begin{aligned} H(1^{h_1}, 2^{h_2}, 3^{h_3}; 4_H 5_H) &= \frac{\bar{\mu}^{4-n}}{r_\Gamma} \frac{1}{i\pi^{n/2}} \int d^n \ell \frac{\text{Num}(\ell)}{\prod_i d_i(\ell)} \\ &= \sum_{i,j,k} d_{i \times j \times k}(1^{h_1}, 2^{h_2}, 3^{h_3}) D_0(p_i, p_j, p_k; m) \\ &\quad + \sum_{i,j} c_{i \times j}(1^{h_1}, 2^{h_2}, 3^{h_3}) C_0(p_i, p_j; m) \\ &\quad + \sum_i b_i(1^{h_1}, 2^{h_2}, 3^{h_3}) B_0(p_i; m) + r(1^{h_1}, 2^{h_2}, 3^{h_3}). \end{aligned} \quad (6.5)$$

The scalar bubble (B_0), triangle (C_0), box (D_0) integrals, and the constant r_Γ , are defined in appendix B. $\bar{\mu}$ is an arbitrary mass scale, and r are the rational terms. All scalar integrals are well known and readily evaluated using existing libraries [56–58].

In order to obtain concise analytic expressions, we found that it was expedient to re-express the box coefficients in terms of scalar pentagon integrals and a remainder. In order to perform this separation we are forced to introduce a denominator factor of tr_5 , which is given by,

$$\text{tr}_5 = \text{Tr} \{ \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4 \gamma_5 \}. \quad (6.6)$$

In infrared configurations involving p_1 , p_2 and p_3 this factor vanishes, giving rise to potential numerical issues in these limits. However we have been able to eliminate all factors except for a single pole in tr_5 , which mitigates these issues to a large extent.

6.3 Basis integrals

In section 7 we will present results for a minimal set of integral coefficients. The remaining coefficients can be simply related to these by permutation of momentum labels. Here we summarize the basic set and the permutations required to generate all coefficients.

There are 2 independent bubbles:

- $B_0(p_{12}; m)$ ($\times 3$ perms);
- $B_0(p_{123}; m)$.

There are 5 independent triangles:

- $C_0(p_1, p_2; m)$ ($\times 3$ perms);
- $C_0(p_1, p_4; m)$ ($\times 3$ perms); with $C_0(p_1, p_{234}; m)$ obtained by $4 \leftrightarrow 5$ ($= 6$ perms);
- $C_0(p_3, p_{12}; m)$ ($\times 3$ perms);
- $C_0(p_4, p_{12}; m)$ ($\times 3$ perms); with $C_0(p_{12}, p_{34}; m)$ obtained by $4 \leftrightarrow 5$ ($= 6$ perms);
- $C_0(p_4, p_{123}; m)$.

There are 5 independent boxes:

- $D_0(p_1, p_2, p_3; m)$ ($\times 3$ perms);
- $D_0(p_1, p_2, p_4; m)$ ($\times 6$ perms); with $D_0(p_{34}, p_1, p_2; m)$ obtained by $4 \leftrightarrow 5$ ($= 12$ perms);
- $D_0(p_1, p_4, p_{23}; m)$ ($\times 3$ perms);
- $D_0(p_1, p_4, p_2; m)$ ($\times 3$ perms); with $D_0(p_2, p_{34}, p_1; m)$ obtained by $4 \leftrightarrow 5$ ($= 6$ perms);
- $D_0(p_4, p_1, p_{23}; m)$ ($\times 3$ perms); with $D_0(p_1, p_{23}, p_4; m)$ obtained by $4 \leftrightarrow 5$ ($= 6$ perms).

Additional permutations correspond to either the three cyclic choices of $(1, 2, 3)$, or to all six permutations. Some of these coefficients vanish for particular helicity choices.

Furthermore we can limit ourselves to the calculation of coefficients where no more than one gluon has positive helicity:

$$c(1^-, 2^-, 3^-; 4, 5), \quad c(1^-, 2^-, 3^+; 4, 5), \quad c(1^-, 2^+, 3^-; 4, 5) \quad \text{and} \quad c(1^+, 2^-, 3^-; 4, 5), \quad (6.7)$$

where c represents any of the coefficients $d_{i \times j \times k}$, $c_{i \times j}$ or b_i . This is because parity relates coefficients with opposite helicities,

$$c(1^{-h_1}, 2^{-h_2}, 3^{-h_3}; 4, 5) = \left[c(1^{h_1}, 2^{h_2}, 3^{h_3}; 4, 5) \right]^*. \quad (6.8)$$

6.4 Strategy for integral coefficients

6.4.1 Bubbles and rational terms

The bubble coefficient b_{12} and rational term R are computed by a direct calculation using Passarino-Veltman reduction. A generic tensor is constructed with free indices μ_1, μ_2, μ_3 corresponding to the currents for each of the gluons, eliminating terms that vanish due to gauge invariance, ($p_1^{\mu_1} = p_2^{\mu_2} = p_3^{\mu_3} = 0$) and employing a cyclic choice of gauge, ($p_2^{\mu_1} = p_3^{\mu_2} = p_1^{\mu_3} = 0$). The final result for the tensor is then simply contracted with the appropriate polarization vectors.

The coefficients of the bubbles b_{23} and b_{13} are obtained by cyclic permutation of $\{p_1, p_2, p_3\}$:

$$b_{23}(1^{h_1}, 2^{h_2}, 3^{h_3}) = b_{12}(2^{h_2}, 3^{h_3}, 1^{h_1}), \quad (6.9)$$

$$b_{13}(1^{h_1}, 2^{h_2}, 3^{h_3}) = b_{12}(3^{h_3}, 1^{h_1}, 2^{h_2}). \quad (6.10)$$

The remaining bubble coefficient is then determined by the ultra-violet finiteness of the amplitude,

$$b_{123}(1^{h_1}, 2^{h_2}, 3^{h_3}) = -b_{12}(1^{h_1}, 2^{h_2}, 3^{h_3}) - b_{23}(1^{h_1}, 2^{h_2}, 3^{h_3}) - b_{13}(1^{h_1}, 2^{h_2}, 3^{h_3}). \quad (6.11)$$

6.4.2 Triangles

The triangle coefficient $c_{1 \times 2}$ and the m^2 contributions to $c_{3 \times 12}$ and $c_{1 \times 4}$ are calculated in the same fashion as the bubble and rational contributions. The triangle coefficients $c_{4 \times 12}$ and $c_{4 \times 123}$ are obtained by the unitarity methods of Forde [59] and subsequently simplified.

The m^0 contribution to $c_{3 \times 12}$ and $c_{1 \times 4}$ coefficients are obtained through infrared relations. We perform a decomposition of the triangle coefficients,

$$c_{A \times B} = c_{A \times B}^{(0)} + m^2 c_{A \times B}^{(2)}, \quad (6.12)$$

such that the first term is the result obtained when setting the mass of the circulating fermion to zero, except in the Yukawa coupling to the Higgs bosons. We can again exploit the fact that the amplitude must be infra-red finite in this limit to constrain the coefficients of box and triangle integrals that develop poles as $\epsilon \rightarrow 0$. Specifically we find that $c_{1 \times 4}^{(0)}$ is given by a combination of box coefficients,

$$\begin{aligned} \frac{c_{1 \times 4}^{(0)}}{(s_{14} - M_H^2)} = & \frac{2d_{3 \times 14 \times 2}^{(0)}}{(s_{25} s_{35} - s_{14} M_H^2)} - \frac{2d_{1 \times 4 \times 3}^{(0)}}{(s_{34} s_{14} - s_{25} M_H^2)} - \frac{2d_{2 \times 4 \times 1}^{(0)}}{(s_{14} s_{24} - s_{35} M_H^2)} - \frac{d_{23 \times 4 \times 1}^{(0)}}{(s_{15} s_{14} - M_H^4)} \\ & - \frac{d_{23 \times 1 \times 4}^{(0)}}{(s_{14} s_{45} - M_H^2 s_{23})} + \frac{d_{14 \times 2 \times 3}^{(0)}}{s_{23} s_{35}} + \frac{d_{14 \times 3 \times 2}^{(0)}}{s_{23} s_{25}} - 2 \frac{d_{4 \times 1 \times 2}^{(0)}}{s_{12} s_{14}} - 2 \frac{d_{3 \times 1 \times 4}^{(0)}}{s_{13} s_{14}}. \end{aligned} \quad (6.13)$$

A second relation determines $c_{3 \times 12}^{(0)}$ in terms of box coefficients and the result for $c_{1 \times 2}^{(0)}$,

$$\begin{aligned} \frac{c_{3 \times 12}^{(0)}}{s_{13} + s_{23}} = & \frac{c_{1 \times 2}^{(0)}}{s_{12}} - \frac{d_{3 \times 12 \times 4}^{(0)}}{s_{35} s_{45} - M_H^2 s_{12}} - \frac{d_{12 \times 3 \times 4}^{(0)}}{s_{34} s_{45} - M_H^2 s_{12}} \\ & + \frac{d_{1 \times 2 \times 4}^{(0)}}{s_{12} s_{24}} + \frac{d_{4 \times 1 \times 2}^{(0)}}{s_{12} s_{14}} + \frac{d_{34 \times 1 \times 2}^{(0)}}{s_{12} s_{25}} + \frac{d_{34 \times 2 \times 1}^{(0)}}{s_{12} s_{15}} \\ & + 2 \frac{d_{1 \times 2 \times 3}^{(0)}}{s_{12} s_{23}} + 2 \frac{d_{3 \times 1 \times 2}^{(0)}}{s_{12} s_{13}}. \end{aligned} \quad (6.14)$$

6.4.3 Boxes

All the box coefficients are computed using unitarity cuts and subsequently simplified using the analytic reconstruction techniques discussed in section 3. Although one might expect all four helicities to be required for each of the five boxes, symmetry relations allow this number to be reduced. We choose to use the basic set shown in table 2. Other helicity combinations, or momentum configurations, can be obtained through permutations of these.

Configuration	Helicities			
$1 \times 2 \times 3$	$1^- 2^- 3^-$	$1^- 2^- 3^+$	$1^- 2^+ 3^-$	
$1 \times 2 \times 4$	$1^- 2^- 3^-$	$1^- 2^- 3^+$	$1^- 2^+ 3^-$	$1^+ 2^- 3^-$
$1 \times 4 \times 2$	$1^- 2^- 3^-$	$1^- 2^- 3^+$	$1^- 2^+ 3^-$	
$1 \times 4 \times 23$	$1^- 2^- 3^-$	$1^- 2^- 3^+$	$1^+ 2^- 3^-$	
$4 \times 1 \times 23$	$1^- 2^- 3^-$	$1^- 2^- 3^+$	$1^+ 2^- 3^-$	

Table 2. All box coefficient functions needed in the calculation of the $0 \rightarrow gggHH$ amplitude.

7 Analytic results for the process $0 \rightarrow gggHH$

We now give detailed analytic results for the contributions to the process $0 \rightarrow gggHH$ that do not involve the triple Higgs coupling.

7.1 Scalar pentagons reduced to boxes

In the process $0 \rightarrow gggHH$ we encounter for the first time pentagon integrals, so we now discuss the treatment of such scalar pentagon integrals, which will be useful in the following. In four dimensions the scalar pentagon integral can be reduced to a sum of the five box integrals obtained by removing one propagator [60–62]. This decomposition is detailed for the two pertinent cases below.

7.1.1 Case 1: adjacent Higgs bosons

For the pentagon scalar integral with two adjacent Higgs bosons we have,

$$\begin{aligned}
 E_0(p_1, p_2, p_3, p_4; m) &= \mathcal{C}_1^{1 \times 2 \times 3 \times 4} D_0(p_2, p_3, p_4; m) + \mathcal{C}_2^{1 \times 2 \times 3 \times 4} D_0(p_{12}, p_3, p_4; m) \\
 &\quad + \mathcal{C}_3^{1 \times 2 \times 3 \times 4} D_0(p_1, p_{23}, p_4; m) \\
 &\quad + \mathcal{C}_4^{1 \times 2 \times 3 \times 4} D_0(p_1, p_2, p_{34}; m) + \mathcal{C}_5^{1 \times 2 \times 3 \times 4} D_0(p_1, p_2, p_3; m). \quad (7.1)
 \end{aligned}$$

In terms of the Cayley matrix the reduction coefficients are given by

$$\mathcal{C}_i = -\frac{1}{2} \sum_j S_{ij}^{-1}. \quad (7.2)$$

The Cayley matrix, $[S^{1 \times 2 \times 3 \times 4}]_{ij} = [m^2 - \frac{1}{2}(q_{i-1} - q_{j-1})^2]$ where q_i is the offset (affine) momentum is given by

$$S^{1 \times 2 \times 3 \times 4} = \begin{pmatrix} m^2 & m^2 & m^2 - \frac{1}{2}s_{12} & m^2 - \frac{1}{2}s_{45} & m^2 - \frac{1}{2}M_H^2 \\ m^2 & m^2 & m^2 & m^2 - \frac{1}{2}s_{23} & m^2 - \frac{1}{2}s_{15} \\ m^2 - \frac{1}{2}s_{12} & m^2 & m^2 & m^2 & m^2 - \frac{1}{2}s_{34} \\ m^2 - \frac{1}{2}s_{45} & m^2 - \frac{1}{2}s_{23} & m^2 & m^2 & m^2 - \frac{1}{2}M_H^2 \\ m^2 - \frac{1}{2}M_H^2 & m^2 - \frac{1}{2}s_{15} & m^2 - \frac{1}{2}s_{34} & m^2 - \frac{1}{2}M_H^2 & m^2 \end{pmatrix}. \quad (7.3)$$

Explicit forms for the pentagon reduction coefficients, $\mathcal{C}_i^{1 \times 2 \times 3 \times 4}$ are (with $s_{ij} = (p_i + p_j)^2$),

$$\begin{aligned} \mathcal{C}_1^{1 \times 2 \times 3 \times 4} &= -\frac{1}{32 |S^{1 \times 2 \times 3 \times 4}|} \left[s_{23} (s_{34} (s_{15} s_{45} + s_{23} s_{34} - s_{34} s_{45} - s_{12} s_{23} + s_{12} s_{15})) \right. \\ &\quad \left. + M_H^2 (s_{12} s_{34} - s_{23} s_{34} - 2 s_{12} s_{15}) \right] \\ &\equiv -s_{23} \frac{2 [3|4|5|1] \langle 1|5|4|3 \rangle - s_{34} ([13] \langle 1|5|4|3 \rangle - \langle 13 \rangle [3|4|5|1])}{32 |S^{1 \times 2 \times 3 \times 4}|}, \end{aligned} \quad (7.4)$$

$$\begin{aligned} \mathcal{C}_2^{1 \times 2 \times 3 \times 4} &= -\frac{1}{32 |S^{1 \times 2 \times 3 \times 4}|} \left[s_{34} s_{45} (s_{34} s_{45} - s_{15} s_{45} - s_{23} s_{34} + s_{12} s_{23} + s_{12} s_{15}) \right. \\ &\quad + M_H^2 (s_{45} (s_{23} s_{34} + s_{12} s_{15} - 2 s_{12} s_{34}) - s_{12} (s_{23} s_{34} + s_{12} s_{23} + s_{12} s_{15})) \\ &\quad \left. + M_H^4 s_{12} (s_{23} + s_{12}) \right] \\ &\equiv -\frac{2 [3|(1+2)|3] \langle 1|5|4|3 \rangle [1|5|4|3] - \langle 3|4|5|(1+2)|3 \rangle ([13] \langle 1|5|4|3 \rangle - \langle 13 \rangle [3|4|5|1])}{32 |S^{1 \times 2 \times 3 \times 4}|}, \end{aligned} \quad (7.5)$$

$$\begin{aligned} \mathcal{C}_3^{1 \times 2 \times 3 \times 4} &= -\frac{1}{32 |S^{1 \times 2 \times 3 \times 4}|} \left[s_{15} s_{45} (s_{15} s_{45} + s_{23} s_{34} + s_{12} s_{23} - s_{34} s_{45} - s_{12} s_{15}) \right. \\ &\quad + M_H^2 (s_{23} (s_{34} s_{45} - s_{23} s_{34} - s_{12} s_{23}) + s_{15} (s_{12} s_{45} - s_{12} s_{23} - 2 s_{23} s_{45})) \\ &\quad \left. + M_H^4 s_{23} (s_{23} + s_{12}) \right] \\ &\equiv \mathcal{C}_2^{1 \times 2 \times 3 \times 4} (1 \leftrightarrow 3, 4 \leftrightarrow 5), \end{aligned} \quad (7.6)$$

$$\begin{aligned} \mathcal{C}_4^{1 \times 2 \times 3 \times 4} &= -\frac{1}{32 |S^{1 \times 2 \times 3 \times 4}|} \left[s_{12} (s_{15} (s_{34} s_{45} - s_{15} s_{45} + s_{23} s_{34} - s_{12} s_{23} + s_{12} s_{15})) \right. \\ &\quad \left. + M_H^2 (s_{15} s_{23} - s_{12} s_{15} - 2 s_{23} s_{34}) \right] \\ &\equiv \mathcal{C}_1^{1 \times 2 \times 3 \times 4} (1 \leftrightarrow 3, 4 \leftrightarrow 5), \end{aligned} \quad (7.7)$$

$$\begin{aligned} \mathcal{C}_5^{1 \times 2 \times 3 \times 4} &= -\frac{1}{32 |S^{1 \times 2 \times 3 \times 4}|} s_{12} s_{23} \left[s_{34} s_{45} + s_{15} s_{45} - s_{23} s_{34} + s_{12} s_{23} - s_{12} s_{15} \right. \\ &\quad \left. - M_H^2 (s_{12} + s_{23}) \right] \\ &\equiv -s_{12} s_{23} \frac{[13] \langle 1|5|4|3 \rangle - \langle 13 \rangle [3|4|5|1]}{32 |S^{1 \times 2 \times 3 \times 4}|}. \end{aligned} \quad (7.8)$$

$\mathcal{C}_5^{1 \times 2 \times 3 \times 4}$ is unchanged under $(1 \leftrightarrow 3, 4 \leftrightarrow 5)$ exchange. The alternative spinor expressions given in eq. (7.4)–(7.8) have the merit that partial cancellations which occur in the $m \rightarrow 0$ limit are made manifest, (see eq. (7.11) below).

The factor $|S^{1 \times 2 \times 3 \times 4}|$ is the determinant of the Cayley matrix, eq. (7.3). It can be written as,

$$16 |S^{1 \times 2 \times 3 \times 4}| = -s_{12} s_{23} \left(s_{15} s_{34} s_{45} - M_H^2 (s_{23} s_{34} + s_{12} s_{15}) \right) + 16 m^2 \Delta(p_1, p_2, p_3, p_4), \quad (7.9)$$

where

$$\begin{aligned} \Delta(p_1, p_2, p_3, p_4) &= (p_1 \cdot p_2 p_3 \cdot p_4 - p_1 \cdot p_3 p_2 \cdot p_4 - p_1 \cdot p_4 p_2 \cdot p_3)^2 + 2 p_1 \cdot p_3 p_2 \cdot p_3 (p_1 \cdot p_2 M_H^2 - 2 p_1 \cdot p_4 p_2 \cdot p_4) \\ &= \frac{1}{16} (\text{tr}_5)^2, \end{aligned} \quad (7.10)$$

where tr_5 has been defined in eq. (6.6). This can be written as another useful relation,

$$\begin{aligned} 16 |S^{1 \times 2 \times 3 \times 4}| &= -s_{12} \langle 1 | \not{p}_5 \not{p}_4 \not{p}_3 | 2 \rangle \langle 2 | \not{p}_3 \not{p}_4 \not{p}_5 | 1 \rangle + m^2 (\text{tr}_5)^2 \\ &= -s_{12} s_{23} \langle 1 | \mathbf{5} | \mathbf{4} | \mathbf{3} \rangle [3 | \mathbf{4} | \mathbf{5} | 1] + m^2 (\text{tr}_5)^2. \end{aligned} \quad (7.11)$$

7.1.2 Case 2: non-adjacent Higgs bosons

For the case of a scalar pentagon integral with non-adjacent Higgs bosons, we denote the coefficients for the reduction of the scalar pentagon to scalar boxes by $\bar{C}_i^{1 \times 2 \times 4 \times 3}$,

$$\begin{aligned} E_0(p_1, p_2, p_4, p_3; m) &= \bar{C}_1^{1 \times 2 \times 4 \times 3} D_0(p_2, p_4, p_3; m) + \bar{C}_2^{1 \times 2 \times 4 \times 3} D_0(p_{12}, p_4, p_3; m) \\ &\quad + \bar{C}_3^{1 \times 2 \times 4 \times 3} D_0(p_1, p_{24}, p_3; m) \\ &\quad + \bar{C}_4^{1 \times 2 \times 4 \times 3} D_0(p_1, p_2, p_{34}; m) + \bar{C}_5^{1 \times 2 \times 4 \times 3} D_0(p_1, p_2, p_4; m). \end{aligned} \quad (7.12)$$

For the case where the Higgs boson are not adjacent the Cayley matrix is given by

$$S^{1 \times 2 \times 4 \times 3} = \begin{pmatrix} m^2 & m^2 & m^2 - \frac{1}{2} s_{12} & m^2 - \frac{1}{2} s_{35} & m^2 - \frac{1}{2} M_H^2 \\ m^2 & m^2 & m^2 & m^2 - \frac{1}{2} s_{24} & m^2 - \frac{1}{2} s_{15} \\ m^2 - \frac{1}{2} s_{12} & m^2 & m^2 & m^2 - \frac{1}{2} M_H^2 & m^2 - \frac{1}{2} s_{34} \\ m^2 - \frac{1}{2} s_{35} & m^2 - \frac{1}{2} s_{24} & m^2 - \frac{1}{2} M_H^2 & m^2 & m^2 \\ m^2 - \frac{1}{2} M_H^2 & m^2 - \frac{1}{2} s_{15} & m^2 - \frac{1}{2} s_{34} & m^2 & m^2 \end{pmatrix}. \quad (7.13)$$

The reduction coefficients are given as before, using eq. (7.2),

$$\begin{aligned} \bar{C}_1^{1 \times 2 \times 4 \times 3} &= -\frac{1}{32 |S^{1 \times 2 \times 4 \times 3}|} \left[(s_{24} s_{34} - M_H^2 s_{15}) (s_{15} s_{35} + s_{12} s_{15} + s_{24} s_{34} - s_{12} s_{24} - s_{34} s_{35} \right. \\ &\quad \left. - M_H^2 (s_{15} + s_{24}) + M_H^4) \right] \\ &\equiv -\langle 2 | \mathbf{4} | \mathbf{3} \rangle \langle 3 | \mathbf{4} | \mathbf{2} \rangle \frac{2 \langle 1 | \mathbf{5} | \mathbf{3} \rangle \langle 3 | \mathbf{5} | \mathbf{1} \rangle + \langle 3 | \mathbf{4} | \mathbf{1} \rangle \langle 1 | \mathbf{5} | \mathbf{3} \rangle + \langle 1 | \mathbf{4} | \mathbf{3} \rangle \langle 3 | \mathbf{5} | \mathbf{1} \rangle}{32 |S^{1 \times 2 \times 4 \times 3}|}, \end{aligned} \quad (7.14)$$

$$\begin{aligned} \bar{C}_2^{1 \times 2 \times 4 \times 3} &= -\frac{1}{32 |S^{1 \times 2 \times 4 \times 3}|} \left[s_{34} s_{35} (s_{34} s_{35} - s_{15} s_{35} - s_{24} s_{34} + s_{12} s_{24} + s_{12} s_{15}) \right. \\ &\quad + M_H^2 (s_{24} s_{34} s_{35} + s_{15} s_{34} s_{35} - 2 s_{12} s_{15} s_{35} - 2 s_{12} s_{24} s_{34}) \\ &\quad + M_H^4 (s_{12} s_{15} + s_{15} s_{35} + s_{24} s_{34} + s_{12} s_{24} - 2 s_{34} s_{35}) - M_H^6 (s_{15} + s_{24}) + M_H^8 \left. \right] \\ &\equiv -\frac{-2 \langle 3 | \mathbf{5} | \mathbf{1} \rangle \langle 2 | \mathbf{4} | \mathbf{3} \rangle \langle 1 | \mathbf{5} | \mathbf{3} \rangle \langle 3 | \mathbf{4} | \mathbf{2} \rangle + (\langle 3 | \mathbf{5} | \mathbf{2} \rangle \langle 3 | \mathbf{4} | \mathbf{1} \rangle \langle 1 | \mathbf{5} | \mathbf{3} \rangle \langle 2 | \mathbf{4} | \mathbf{3} \rangle) + (\langle \rangle \leftrightarrow [])}{32 |S^{1 \times 2 \times 4 \times 3}|}, \end{aligned} \quad (7.15)$$

$$\begin{aligned} \bar{C}_3^{1 \times 2 \times 4 \times 3} &= -\frac{1}{32 |S^{1 \times 2 \times 4 \times 3}|} \left[(s_{15} s_{35} - M_H^2 s_{24}) (s_{15} s_{35} + s_{24} s_{34} + s_{12} s_{24} - s_{12} s_{15} - s_{34} s_{35} \right. \\ &\quad \left. - M_H^2 (s_{15} + s_{24}) + M_H^4) \right] \\ &\equiv \bar{C}_1^{1 \times 2 \times 4 \times 3} (1 \leftrightarrow 2, \mathbf{4} \leftrightarrow \mathbf{5}), \end{aligned} \quad (7.16)$$

$$\begin{aligned} \bar{C}_4^{1 \times 2 \times 4 \times 3} &= -\frac{1}{32 |S^{1 \times 2 \times 4 \times 3}|} \left[s_{12} (s_{15} (s_{34} s_{35} - s_{15} s_{35} + s_{24} s_{34} + s_{12} s_{15} - s_{12} s_{24}) \right. \\ &\quad \left. + M_H^2 (s_{15} s_{24} - s_{15}^2 - 2 s_{24} s_{34}) + M_H^4 s_{15}) \right] \\ &\equiv \frac{s_{12} (\langle 1 | \mathbf{5} | \mathbf{2} \rangle \langle 3 | \mathbf{5} | \mathbf{1} \rangle \langle 2 | \mathbf{4} | \mathbf{3} \rangle + \langle 2 | \mathbf{5} | \mathbf{1} \rangle \langle 1 | \mathbf{5} | \mathbf{3} \rangle \langle 3 | \mathbf{4} | \mathbf{2} \rangle)}{32 |S^{1 \times 2 \times 4 \times 3}|}, \end{aligned} \quad (7.17)$$

$$\begin{aligned}\bar{\mathcal{C}}_5^{1 \times 2 \times 4 \times 3} &= -\frac{1}{32 |S^{1 \times 2 \times 4 \times 3}|} \left[s_{12} (s_{24} (s_{34} s_{35} + s_{15} s_{35} - s_{12} s_{15} - s_{24} s_{34} + s_{12} s_{24}) \right. \\ &\quad \left. + M_H^2 (s_{15} s_{24} - s_{24}^2 - 2 s_{15} s_{35}) + M_H^4 s_{24} \right) \\ &\equiv \bar{\mathcal{C}}_4^{1 \times 2 \times 4 \times 3} (1 \leftrightarrow 2, 4 \leftrightarrow 5). \end{aligned} \quad (7.18)$$

$\bar{\mathcal{C}}_2^{1 \times 2 \times 4 \times 3}$ is symmetric under $(1 \leftrightarrow 2, 4 \leftrightarrow 5)$. For the non-adjacent Higgs boson case, the determinant of the Cayley matrix is,

$$16 |S^{1 \times 2 \times 4 \times 3}| = -s_{12} (s_{24} s_{34} - M_H^2 s_{15}) (s_{15} s_{35} - M_H^2 s_{24}) + 16 m^2 \Delta(p_1, p_2, p_3, p_4), \quad (7.19)$$

or equivalently,

$$\begin{aligned}16 |S^{1 \times 2 \times 4 \times 3}| &= -s_{12} \langle 1 | \not{p}_5 \not{p}_3 \not{p}_4 | 2 \rangle \langle 2 | \not{p}_4 \not{p}_3 \not{p}_5 | 1 \rangle + m^2 (\text{tr}_5)^2 \\ &= -s_{12} \langle 1 | 5 | 3 \rangle \langle 3 | 5 | 1 \rangle \langle 3 | 4 | 2 \rangle \langle 2 | 4 | 3 \rangle + m^2 (\text{tr}_5)^2. \end{aligned} \quad (7.20)$$

7.2 $g^- g^- g^- H H$

7.2.1 Effective pentagons

We will write the box coefficients (d) in terms of a combination of effective pentagon ($\hat{e}$) and box ($\hat{d}$) coefficients. We begin by specifying the effective pentagon coefficients.

The effective pentagon coefficient (for adjacent Higgs bosons) is given by,

$$\hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^-, 3^-) = \frac{m^2 s_{12} s_{23}}{4 \text{tr}_5} (8m^2 - s_{45} - 2M_H^2) [13] \langle 1 | 5 | 4 | 3 \rangle. \quad (7.21)$$

The effective pentagon coefficient $\hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^-, 3^-)$ (appropriate for the case where the Higgs bosons are not adjacent) is,

$$\begin{aligned}\hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^-, 3^-) &= \frac{m^2}{4} \langle 12 \rangle [13] [23] \\ &\quad \times \left\{ \frac{\langle 12 \rangle \langle 3 | 5 | 1 \rangle \langle 3 | 4 | 2 \rangle}{\text{tr}_5} (8m^2 - s_{45} - 2M_H^2) + \langle 3 | 4 | 5 | 3 \rangle \right\}. \end{aligned} \quad (7.22)$$

Note that $\hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^-, 3^-)$ is manifestly symmetric under $(1 \leftrightarrow 2, 4 \leftrightarrow 5)$.

7.2.2 Boxes

The box coefficient is written in terms of an effective pentagon coefficient ($\hat{e}$) plus a remainder term, $\hat{d}_{1 \times 2 \times 3}(1^-, 2^-, 3^-)$ as,

$$\begin{aligned}d_{1 \times 2 \times 3}(1^-, 2^-, 3^-) &= \left\{ \frac{1}{[12][23][31]} \left[\mathcal{C}_5^{1 \times 2 \times 3 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^-, 3^-) \right] \right\} \\ &\quad + \left\{ 4 \leftrightarrow 5 \right\} + \hat{d}_{1 \times 2 \times 3}(1^-, 2^-, 3^-), \end{aligned} \quad (7.23)$$

where the remainder term is

$$\hat{d}_{1 \times 2 \times 3}(1^-, 2^-, 3^-) = \frac{m^2 \langle 12 \rangle \langle 23 \rangle}{2 [31]}. \quad (7.24)$$

The reduction factor $\mathcal{C}_5^{1 \times 2 \times 3 \times 4}$ is given in section 7.1.

In a similar way we can write,

$$d_{1 \times 2 \times 4}(1^-, 2^-, 3^-) = \frac{1}{[12][23][31]} \left[\mathcal{C}_1^{3 \times 1 \times 2 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^-, 1^-, 2^-) + \bar{\mathcal{C}}_5^{1 \times 2 \times 4 \times 3} \hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^-, 3^-) \right], \quad (7.25)$$

where in this case there is no remainder term,

$$\hat{d}_{1 \times 2 \times 4}(1^-, 2^-, 3^-) = 0. \quad (7.26)$$

The effective pentagon coefficient $\hat{e}_{1 \times 2 \times 3 \times 4}(3^-, 1^-, 2^-)$ is obtained by simply permuting the arguments in the defining eq. (7.21), noting also that tr_5 does not flip sign under this even permutation. The reduction coefficients are obtained similarly.

The remaining box coefficients are given by,

$$d_{1 \times 4 \times 2}(1^-, 2^-, 3^-) = \frac{1}{[12][23][31]} \left[\bar{\mathcal{C}}_1^{3 \times 2 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(3^-, 2^-, 1^-) + \bar{\mathcal{C}}_1^{3 \times 1 \times 4 \times 2} \hat{e}_{1 \times 2 \times 4 \times 3}(3^-, 1^-, 2^-) \right] + \hat{d}_{1 \times 4 \times 2}(1^-, 2^-, 3^-), \quad (7.27)$$

$$\hat{d}_{1 \times 4 \times 2}(1^-, 2^-, 3^-) = \frac{(4m^2 - M_H^2)(s_{14}s_{24} - M_H^2 s_{35})}{4[12][23][31]}, \quad (7.28)$$

and,

$$d_{4 \times 1 \times 23}(1^-, 2^-, 3^-) = \frac{1}{[12][23][31]} \left[\mathcal{C}_2^{3 \times 2 \times 1 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^-, 2^-, 1^-) + \mathcal{C}_2^{2 \times 3 \times 1 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(2^-, 3^-, 1^-) \right] + \hat{d}_{4 \times 1 \times 23}(1^-, 2^-, 3^-) \quad (7.29)$$

$$\hat{d}_{4 \times 1 \times 23}(1^-, 2^-, 3^-) = \frac{\langle 23 \rangle \langle 1|5|4|1 \rangle}{[23]} \frac{m^2}{2\text{tr}_5} (s_{45} + 2M_H^2 - 8m^2), \quad (7.30)$$

and,

$$d_{1 \times 4 \times 23}(1^-, 2^-, 3^-) = \frac{1}{[12][23][31]} \left[\bar{\mathcal{C}}_2^{3 \times 2 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(3^-, 2^-, 1^-) + \bar{\mathcal{C}}_2^{2 \times 3 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(2^-, 3^-, 1^-) \right] + \hat{d}_{1 \times 4 \times 23}(1^-, 2^-, 3^-). \quad (7.31)$$

Note that these last two boxes have the same remainder contribution.

$$\hat{d}_{1 \times 4 \times 23}(1^-, 2^-, 3^-) = \hat{d}_{4 \times 1 \times 23}(1^-, 2^-, 3^-). \quad (7.32)$$

This fully specifies the five integrals that enter the basis set indicated in table 2. The remainder are related by,

$$\begin{aligned} d_{4 \times 1 \times 2}(1^-, 2^-, 3^-) &= -d_{1 \times 2 \times 4}(2^-, 1^-, 3^-) \\ d_{34 \times 1 \times 2}(1^-, 2^-, 3^-) &= d_{1 \times 2 \times 4}(1^-, 2^-, 3^-) \{4 \leftrightarrow 5\} \\ d_{34 \times 2 \times 1}(1^-, 2^-, 3^-) &= -d_{1 \times 2 \times 4}(2^-, 1^-, 3^-) \{4 \leftrightarrow 5\} \\ d_{2 \times 34 \times 1}(1^-, 2^-, 3^-) &= d_{1 \times 4 \times 2}(1^-, 2^-, 3^-) \{4 \leftrightarrow 5\} \\ d_{1 \times 23 \times 4}(1^-, 2^-, 3^-) &= d_{4 \times 1 \times 23}(1^-, 2^-, 3^-) \{4 \leftrightarrow 5\}, \end{aligned} \quad (7.33)$$

with the full set obtained by performing cyclic permutations of $(1, 2, 3)$.

7.2.3 Triangles

The following triangle coefficients are all zero,

$$c_{4 \times 12}(1^-, 2^-, 3^-) = 0, \quad c_{4 \times 123}(1^-, 2^-, 3^-) = 0, \quad c_{1 \times 2}(1^-, 2^-, 3^-) = 0, \quad (7.34)$$

whereas the following two triangle coefficients only have contributions at order m^2

$$c_{3 \times 12}^{(0)}(1^-, 2^-, 3^-) = 0, \quad c_{3 \times 12}^{(2)}(1^-, 2^-, 3^-) = \frac{(s_{13} + s_{23})}{[1\,2][2\,3][3\,1]}, \quad (7.35)$$

$$c_{1 \times 4}^{(0)}(1^-, 2^-, 3^-) = 0, \quad c_{1 \times 4}^{(2)}(1^-, 2^-, 3^-) = -2 \frac{\langle 1|4|1 \rangle}{[1\,2][2\,3][3\,1]}. \quad (7.36)$$

7.2.4 Bubbles and rational terms

As discussed in section 6.4.1, for each helicity configuration we need only give results for a single bubble coefficient. In this case it vanishes,

$$b_{12}(1^-, 2^-, 3^-) = 0. \quad (7.37)$$

The rational term is

$$R(1^-, 2^-, 3^-) = \frac{s_{12} + s_{23} + s_{31}}{[1\,2][2\,3][3\,1]}. \quad (7.38)$$

7.3 $g^-g^-g^+HH$

7.3.1 Effective pentagons

Turning now to the $1^-2^-3^+$ helicity, the first effective pentagon coefficient is,

$$\begin{aligned} \hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^-, 3^+) = & -\frac{m^2}{4} \langle 13 \rangle \langle 23 \rangle [1\,2] \left\{ \frac{\langle 1\,2 \rangle [2\,3] [1|5|4|3]}{\text{tr}_5} (s_{45} - 2s_{12} - 2M_H^2 + 8m^2) \right. \\ & \left. + [3|5|4|3] \right\}. \end{aligned} \quad (7.39)$$

The second effective pentagon coefficient is,

$$\hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^-, 3^+) = -\frac{m^2}{4\text{tr}_5} \langle 13 \rangle \langle 23 \rangle [1\,2]^2 \langle 1|5|3 \rangle \langle 2|4|3 \rangle (s_{45} - 2s_{12} - 2M_H^2 + 8m^2). \quad (7.40)$$

The basis set specified in table 2 requires us to also define the effective pentagon coefficients for the $(1^-, 2^+, 3^-)$ configuration. The first of these, $\hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^+, 3^-)$, is symmetric under $(1 \leftrightarrow 3, 4 \leftrightarrow 5)$. This effective pentagon coefficient reads,

$$\begin{aligned} \hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^+, 3^-) = & \left[\frac{s_{12}^2 s_{23} \langle 1|5|4|3 \rangle (\langle 3|4|5|(1+2)|3 \rangle + 4m^2 \langle 3|(1+2)|3 \rangle)}{4 \langle 1\,3 \rangle \text{tr}_5} \right. \\ & \left. + \frac{1}{4} s_{12} m^2 (s_{23}(s_{15} - s_{23}) - \langle 1\,2 \rangle [2\,3] \langle 3|5|1 \rangle) \right] + \left[\right]_{1 \leftrightarrow 3, 4 \leftrightarrow 5} \\ & + m^2 s_{12} s_{23} \left([1\,3] \langle 1|5|4|3 \rangle \frac{3s_{123} - 2s_{13} + 2M_H^2 - 4m^2}{4\text{tr}_5} - m^2 \frac{\langle 1\,3 \rangle [1|5|4|3]}{\text{tr}_5} - 3m^2 \right) \end{aligned} \quad (7.41)$$

The last effective pentagon is $\hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^+, 3^-)$,

$$\begin{aligned}
 \hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^+, 3^-) = & \frac{m^2}{4} \langle 12 \rangle \left(\begin{aligned} & [23] \langle 3|5|1 \rangle (s_{12} - s_{24} - M_H^2 + 8m^2) \\ & - [12] \langle 3|5|3 \rangle (\langle 2|(3+4)|2 \rangle - 8M_H^2) + [13] \langle 3|4|2 \rangle \langle 2|5|2 \rangle \end{aligned} \right) \\
 & + \frac{m^2}{4} s_{12} \langle 1|5|3 \rangle \frac{\left(\begin{aligned} & \langle 23 \rangle [13] \langle 3|4|2 \rangle (s_{123} - 2s_{13} - 2M_H^2 + 8m^2) \\ & - 8 \langle 3|5|3 \rangle ([12] \langle 23 \rangle M_H^2 + \langle 3|5|1 \rangle s_{12}) \end{aligned} \right)}{\text{tr}_5} \\
 & + \frac{1}{4} s_{12} \langle 12 \rangle \langle 3|5|1 \rangle \langle 3|4|2 \rangle \langle 1|5|3 \rangle \left[\frac{1}{\langle 13 \rangle} + \frac{[13](s_{24} + s_{35} - 8m^2) + [12] \langle 2|5|3 \rangle}{\text{tr}_5} \right] \\
 & + \frac{1}{4} s_{12} \langle 12 \rangle \langle 3|5|1 \rangle \langle 3|5|3 \rangle \frac{\langle 1|5|3 \rangle \langle 3|4|2 \rangle (s_{123} - 2M_H^2) - 8m^2 \langle 1|5|2 \rangle \langle 3|5|3 \rangle}{\langle 13 \rangle \text{tr}_5}. \quad (7.42)
 \end{aligned}$$

7.3.2 Boxes

The first box coefficient can be written in terms of effective pentagons as,

$$\begin{aligned}
 d_{1 \times 2 \times 3}(1^-, 2^-, 3^+) = & \left\{ \frac{\langle 12 \rangle}{[12]^2 \langle 23 \rangle \langle 13 \rangle} \mathcal{C}_5^{1 \times 2 \times 3 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^-, 3^+) \right\} + \left\{ 4 \leftrightarrow 5 \right\} \\
 & + \hat{d}_{1 \times 2 \times 3}(1^-, 2^-, 3^+), \quad (7.43)
 \end{aligned}$$

$$\hat{d}_{1 \times 2 \times 3}(1^-, 2^-, 3^+) = \frac{m^2}{2} \frac{\langle 12 \rangle^2 [23]}{[12] \langle 13 \rangle}. \quad (7.44)$$

The next box coefficient again has no remainder,

$$\begin{aligned}
 d_{1 \times 2 \times 4}(1^-, 2^-, 3^+) = & \frac{\langle 12 \rangle}{[12]^2 \langle 23 \rangle \langle 13 \rangle} \\
 & \times \left[\mathcal{C}_1^{3 \times 1 \times 2 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^+, 1^-, 2^-) + \bar{\mathcal{C}}_5^{1 \times 2 \times 4 \times 3} \hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^-, 3^+) \right]. \quad (7.45)
 \end{aligned}$$

The next box coefficient can be decomposed as,

$$\begin{aligned}
 d_{1 \times 4 \times 2}(1^-, 2^-, 3^+) = & \frac{\langle 12 \rangle}{[12]^2 \langle 23 \rangle \langle 13 \rangle} \left[\bar{\mathcal{C}}_1^{3 \times 2 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(3^+, 2^-, 1^-) + \right. \\
 & \left. \bar{\mathcal{C}}_1^{3 \times 1 \times 4 \times 2} \hat{e}_{1 \times 2 \times 4 \times 3}(3^+, 1^-, 2^-) \right] \\
 & + \hat{d}_{1 \times 4 \times 2}(1^-, 2^-, 3^+). \quad (7.46)
 \end{aligned}$$

The remainder is anti-symmetric under the exchange $1 \leftrightarrow 2$,

$$\begin{aligned}
 \hat{d}_{1 \times 4 \times 2}(1^-, 2^-, 3^+) = & \frac{\langle 12 \rangle \langle 1|4|2 \rangle \langle 2|4|1 \rangle}{4[12]} \\
 & \times \left\{ \frac{s_{13} + s_{23} - 2M_H^2 + 8m^2}{\langle 13 \rangle \langle 23 \rangle} \left[\frac{\langle 1|5|3 \rangle \langle 23 \rangle + \langle 2|5|3 \rangle \langle 13 \rangle}{\text{tr}_5} + \frac{1}{2[12]} \right] + \frac{[3|4|5|3]}{\text{tr}_5} \right\} \quad (7.47)
 \end{aligned}$$

The next box coefficient is,

$$\begin{aligned}
 d_{4 \times 1 \times 23}(1^-, 2^-, 3^+) &= \frac{\langle 12 \rangle}{[12]^2 \langle 23 \rangle \langle 13 \rangle} \\
 &\times \left[\mathcal{C}_2^{3 \times 2 \times 1 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^+, 2^-, 1^-) + \mathcal{C}_2^{2 \times 3 \times 1 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(2^-, 3^+, 1^-) \right] \\
 &+ \hat{d}_{4 \times 1 \times 23}(1^-, 2^-, 3^+), \tag{7.48}
 \end{aligned}$$

$$\begin{aligned}
 \hat{d}_{4 \times 1 \times 23}(1^-, 2^-, 3^+) &= \\
 &- \frac{[13] \langle 145 | (2+3) | 1 \rangle}{4[12][145|1]} \\
 &\times \left[\frac{[354|3]}{[23]} + [13](s_{123} - 2M_H^2 + 8m^2) \left(\frac{1}{2[12]} - \frac{[145|3]}{[23][145|1]} \right) \right] \\
 &+ \frac{\langle 12 \rangle}{4\text{tr}_5} \left[M_H^2 (\langle 153 | \langle 243 | + \langle 143 | \langle 253 |) - \langle 143 | \langle 243 | (s_{123} - 2M_H^2) \right] \\
 &+ \frac{s_{13}}{4[12]\text{tr}_5} \left[\langle 12 \rangle [13] \langle 153 | (s_{123} + M_H^2) - \langle 12 \rangle [23] \langle 253 | (M_H^2 - 8m^2) - 2 \langle 253 | \langle 153 | s_{123} \right] \\
 &+ \frac{\langle 12 \rangle [13]}{4[12]\text{tr}_5} \left[s_{123} (\langle 152 | \langle 253 | + \langle 14 | (2-3) | 43 |) + \langle 12 \rangle [23] (s_{13} s_{123} - 2 \langle 343 | M_H^2) \right] \\
 &+ [13][23] \frac{s_{123} - 2M_H^2 + 8m^2}{4[12]^2} \left[s_{13} \frac{\langle 145 | 2 \rangle + \langle 12 \rangle s_{35}}{\text{tr}_5} - \langle 12 \rangle \right] \\
 &- m^2 \frac{\langle 23453 | - \langle 25423 |}{2[12] \langle 23 \rangle} \left[(s_{123} - 2M_H^2 + 8m^2) \left(\frac{\langle 12 \rangle}{\text{tr}_5} + \frac{[13]}{[23][145|1]} \right) - \frac{2 \langle 12 \rangle s_{12}}{\text{tr}_5} \right] \\
 &+ \frac{s_{123}}{8[12]^2} \left[[23] \langle 243 | + [13] \langle 153 | + \frac{\langle 152 | \langle 253 | - \langle 142 | \langle 243 |}{\langle 13 \rangle} - \frac{[13] \langle 145 | 2 \rangle}{\langle 23 \rangle} \right] \\
 &+ \langle 12 \rangle \frac{\langle 143 | (s_{123} - M_H^2) - M_H^2 \langle 153 |}{8[12] \langle 13 \rangle}. \tag{7.49}
 \end{aligned}$$

The last box coefficient is the most complicated,

$$\begin{aligned}
 d_{1 \times 4 \times 23}(1^-, 2^-, 3^+) &= \frac{\langle 12 \rangle}{[12]^2 \langle 23 \rangle \langle 13 \rangle} \left[(\bar{\mathcal{C}}_2^{3 \times 2 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(3^+, 2^-, 1^-)) + (4 \leftrightarrow 5) \right] \\
 &+ (\hat{d}_{1 \times 4 \times 23}^{\text{unsym.}}(1^-, 2^-, 3^+)) + (4 \leftrightarrow 5) \tag{7.50}
 \end{aligned}$$

$$\begin{aligned}
 \hat{d}_{1 \times 4 \times 23}^{\text{unsym.}}(1^-, 2^-, 3^+) &= \\
 &- \frac{[13] \langle 15 | (2+3) | 41 |}{4[12] \langle 23 \rangle [145|1]} \\
 &\times \left(\langle 241 | \frac{\langle 241 | M_H^2 + 4 \langle 251 | m^2}{[145|1]} + \langle 251 | \frac{s_{123} - 2M_H^2 + 8m^2}{2[12]} \right) \\
 &+ \frac{[13]m^2}{2[12][145|1]} \left(\langle 12 \rangle \langle 251 | \frac{M_H^2 - 4m^2}{\langle 23 \rangle} - \langle 153 | \frac{[145|3] + 4[13]m^2}{[23]} \right)
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{\langle 12 \rangle \langle 1|4|1 \rangle}{8[12]^2 \langle 13 \rangle \langle 23 \rangle} (s_{123} - s_{12} - 2M_H^2)(s_{123} - s_{23} - 2M_H^2 + 8m^2 + \langle 1|4|1 \rangle) \\
 & + \frac{\langle 12 \rangle m^2}{2[12]^2 \langle 13 \rangle \langle 23 \rangle} [s_{12}(2s_{35} - 3s_{25}) + 4[12] \langle 13 \rangle \langle 2|4|3 \rangle - 2 \langle 1|4|1 \rangle (s_{25} + s_{35} - 2M_H^2)] \\
 & + \frac{\langle 12 \rangle m^2}{2[12] \text{tr}_5} \left[(\langle 2|(2+3)|4|5|3 \rangle - \langle 2|5|4|(2+3)|3 \rangle) \frac{s_{12} - 4m^2}{\langle 23 \rangle} - \langle 1|4|1 \rangle \langle 1|5|3 \rangle \frac{s_{123} - 4s_{23} - 2M_H^2}{\langle 13 \rangle} \right] \\
 & + \frac{[23] \langle 1|4|1 \rangle}{4[12] \text{tr}_5} \left\{ \langle 12 \rangle \langle 2|5|3 \rangle (s_{123} - s_{12} - 2M_H^2 + s_{35} + 2s_{25}) + \langle 13 \rangle \langle 2|5|3 \rangle^2 \right. \\
 & + \langle 12 \rangle \langle 2|4|3 \rangle M_H^2 - \langle 12 \rangle^2 \frac{\langle 3|4|2 \rangle \langle 2|5|3 \rangle + s_{123}(s_{123} - M_H^2) + 2 \langle 3|5|3 \rangle M_H^2 - \langle 2|5|2 \rangle^2}{\langle 13 \rangle} \\
 & \left. + \langle 1|4|1 \rangle \langle 1|5|4|2 \rangle \frac{s_{123} - 2M_H^2 + 8m^2}{[12] \langle 13 \rangle} \right\} \quad (7.51)
 \end{aligned}$$

Since the above relationships involve permuting the arguments 1^- , 2^- and 3^+ they result in contributions from effective pentagon coefficients with other helicity orderings. These can be simply related to our basis set by reading off the momenta in the opposite direction around the loop:

$$\begin{aligned}
 \hat{e}_{1 \times 2 \times 3 \times 4}(3^+, 1^-, 2^-) &= \hat{e}_{1 \times 2 \times 3 \times 4}(2^-, 1^-, 3^+)(4 \leftrightarrow 5) \\
 \hat{e}_{1 \times 2 \times 4 \times 3}(3^+, 2^-, 1^-) &= \hat{e}_{1 \times 2 \times 4 \times 3}(2^-, 3^+, 1^-)(4 \leftrightarrow 5) \\
 \hat{e}_{1 \times 2 \times 4 \times 3}(3^+, 1^-, 2^-) &= \hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 3^+, 2^-)(4 \leftrightarrow 5) \\
 \hat{e}_{1 \times 2 \times 3 \times 4}(3^+, 2^-, 1^-) &= \hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^-, 3^+)(4 \leftrightarrow 5). \quad (7.52)
 \end{aligned}$$

This fully specifies the five integrals that enter the basis set indicated in table 2. The remainder are related by,

$$\begin{aligned}
 d_{4 \times 1 \times 2}(1^-, 2^-, 3^+) &= -d_{1 \times 2 \times 4}(2^-, 1^-, 3^+) \\
 d_{34 \times 1 \times 2}(1^-, 2^-, 3^+) &= d_{1 \times 2 \times 4}(1^-, 2^-, 3^+)\{4 \leftrightarrow 5\} \\
 d_{34 \times 2 \times 1}(1^-, 2^-, 3^+) &= -d_{1 \times 2 \times 4}(2^-, 1^-, 3^+)\{4 \leftrightarrow 5\} \\
 d_{2 \times 34 \times 1}(1^-, 2^-, 3^+) &= d_{1 \times 4 \times 2}(1^-, 2^-, 3^+)\{4 \leftrightarrow 5\} \\
 d_{1 \times 23 \times 4}(1^-, 2^-, 3^+) &= d_{4 \times 1 \times 23}(1^-, 2^-, 3^+)\{4 \leftrightarrow 5\}, \quad (7.53)
 \end{aligned}$$

with the full set obtained by performing cyclic permutations of $(1, 2, 3)$.

7.3.3 Triangles

The following triangle coefficients are zero,

$$c_{4 \times 12}(1^-, 2^-, 3^+) = 0, \quad c_{1 \times 2}(1^-, 2^-, 3^+) = 0, \quad (7.54)$$

whereas the coefficients $c_{3 \times 12}(1^-, 2^-, 3^+)$ and $c_{1 \times 4}^{(2)}(1^-, 2^-, 3^+)$ only have a contribution at order m^2

$$c_{3 \times 12}^{(0)}(1^-, 2^-, 3^+) = 0, \quad c_{3 \times 12}^{(2)}(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle}{[12]^2} \frac{(s_{13} + s_{23})}{\langle 13 \rangle \langle 23 \rangle} \quad (7.55)$$

In addition for $c_{1 \times 4}(1^-, 2^-, 3^+)$ we have,

$$\begin{aligned} c_{1 \times 4}^{(0)}(1^-, 2^-, 3^+) &= 0, \\ c_{1 \times 4}^{(2)}(1^-, 2^-, 3^+) &= -2 \frac{\langle 1|4|1 \rangle}{[12] \langle 23 \rangle [23]} \left[\frac{[13]^2 \langle 2|4|1 \rangle (\langle 1|4|1 \rangle - \langle 2|3|2 \rangle)}{[1|4|5|1]^2} \right. \\ &\quad \left. - \frac{(\langle 3|4|1 \rangle \langle 2|5|3 \rangle [13] + \langle 2|4|1 \rangle \langle 2|5|1 \rangle [23])}{[1|4|5|1] \langle 3|4|1 \rangle} \right]. \end{aligned} \quad (7.56)$$

$c_{4 \times 123}(1^-, 2^-, 3^+)$ has contributions at both order m^0 and m^2 ,

$$\begin{aligned} c_{4 \times 123}^{(0)}(1^-, 2^-, 3^+) &= \frac{(s_{45} - 2M_H^2)}{8} c_{4 \times 123}^{(2)}(1^-, 2^-, 3^+), \\ c_{4 \times 123}^{(2)}(1^-, 2^-, 3^+) &= \left\{ 2 \frac{[13]^3 s_{45} (s_{45} - 4M_H^2)}{[12][23][1|4|5|1]^2} \right. \\ &\quad \left. - 4 \frac{[13](2[23]\langle 2|4|3 \rangle + \langle 1|4|3 \rangle [13])}{[12][23][1|4|5|1]} - 2 \frac{(s_{34} + M_H^2 - s_{14} - s_{24})[13]^2}{[12]^2 [1|4|5|1]} \right\} \\ &\quad - \{1 \leftrightarrow 2\}. \end{aligned} \quad (7.58)$$

It is interesting to note that $s_{45}(s_{45} - 4M_H^2)$ is the factorized form of the Källén function $\Delta_{123|4|5}$ under the constraint $m_4 = m_5$. Without the constraint it is unfactorizable. Furthermore, the second power of the pole $[12]$ is spurious. In appendix C we show how to eliminate it and the consequences this has on the other poles in light of primary decompositions in the covariant spinor ring.

7.3.4 Bubbles and rational terms

All the bubble coefficients vanish,

$$b_{12}(1^-, 2^-, 3^+) = 0. \quad (7.59)$$

The rational term is

$$R(1^-, 2^-, 3^+) = -\frac{\langle 12 \rangle^2}{[12]} \left[\frac{1}{\langle 13 \rangle \langle 23 \rangle} + \frac{[23]}{\langle 13 \rangle (s_{12} + s_{23})} + \frac{[13]}{\langle 23 \rangle (s_{12} + s_{13})} \right]. \quad (7.60)$$

7.4 $g^- g^+ g^- HH$

7.4.1 Effective pentagons

Turning now to the $1^- 2^+ 3^-$ helicity combination all the effective pentagon coefficients necessary for this amplitude have already been introduced in section 7.3.1.

7.4.2 Boxes

In terms of effective pentagons the first box is,

$$d_{1 \times 2 \times 3}(1^-, 2^+, 3^-) = \frac{\langle 13 \rangle}{[13]^2 \langle 23 \rangle \langle 12 \rangle} \left[(\mathcal{C}_5^{1 \times 2 \times 3 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(1^-, 2^+, 3^-)) + (4 \leftrightarrow 5) \right] + \hat{d}_{1 \times 2 \times 3}(1^-, 2^+, 3^-), \quad (7.61)$$

$$\hat{d}_{1 \times 2 \times 3}(1^-, 2^+, 3^-) = \left[\frac{\langle 13 \rangle [23] \langle 3|5|2 \rangle s_{12} s_{123}}{4[13] \text{tr}_5} \right] + \left[4 \leftrightarrow 5 \right] - \left[1 \leftrightarrow 3 \right] - \left[1 \leftrightarrow 3, 4 \leftrightarrow 5 \right] + \frac{[12][23] s_{12} s_{23}}{2[13]^3} + \frac{[12] \langle 13 \rangle [23] (s_{123} - 6m^2)}{4[13]^2} - \frac{[12] \langle 13 \rangle^2 [23]}{4[13]} \quad (7.62)$$

The second box is,

$$d_{1 \times 2 \times 4}(1^-, 2^+, 3^-) = \frac{\langle 13 \rangle}{[13]^2 \langle 23 \rangle \langle 12 \rangle} \left[\mathcal{C}_1^{3 \times 1 \times 2 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^-, 1^-, 2^+) + \bar{\mathcal{C}}_5^{1 \times 2 \times 4 \times 3} \hat{e}_{1 \times 2 \times 4 \times 3}(1^-, 2^+, 3^-) \right] + \hat{d}_{1 \times 2 \times 4}(1^-, 2^+, 3^-), \quad (7.63)$$

$$\hat{d}_{1 \times 2 \times 4}(1^-, 2^+, 3^-) = \frac{s_{12}}{8 \langle 23 \rangle [13]^2} \left[\langle 3|4|2 \rangle (M_H^2 - 2s_{35}) - \langle 3|5|2 \rangle s_{24} - \langle 3|5|1 \rangle \frac{s_{24}(s_{24} - 8m^2) + M_H^4}{\langle 2|4|1 \rangle} \right] + \frac{\langle 3|5|1 \rangle \langle 1|4|2 \rangle s_{12} [\langle 3|4|2 \rangle \langle 2|5|3 \rangle + \langle 3|4|3 \rangle M_H^2 - \langle 3|(1+5)|3 \rangle (s_{24} - 8m^2)]}{4 \langle 23 \rangle [13]^2 \text{tr}_5} \quad (7.64)$$

The last box we need to define is,

$$d_{1 \times 4 \times 2}(1^-, 2^+, 3^-) = \frac{\langle 13 \rangle}{[13]^2 \langle 23 \rangle \langle 12 \rangle} \left[\bar{\mathcal{C}}_1^{3 \times 2 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(3^-, 2^+, 1^-) + \bar{\mathcal{C}}_1^{3 \times 1 \times 4 \times 2} \hat{e}_{1 \times 2 \times 4 \times 3}(3^-, 1^-, 2^+) \right] + \hat{d}_{1 \times 4 \times 2}(1^-, 2^+, 3^-), \quad (7.65)$$

$$\hat{d}_{1 \times 4 \times 2}(1^-, 2^+, 3^-) = \frac{\langle 1|4|2 \rangle}{8[13]^2} \left(\frac{1}{\langle 12 \rangle} - 2 \frac{[23] \langle 3|5|1 \rangle}{\text{tr}_5} \right) \times \left[\langle 2|4|1 \rangle \langle 1|5|2 \rangle + \langle 1|4|1 \rangle M_H^2 - (\langle 2|4|2 \rangle - \langle 3|5|3 \rangle)(s_{24} - 8m^2) \right] \quad (7.66)$$

Again we can relate some of these pentagon coefficients to already-specified ones, after permutation:

$$\hat{e}_{1 \times 2 \times 3 \times 4}(2^+, 3^-, 1^-) = e_{1 \times 2 \times 3 \times 4}(1^-, 3^-, 2^+)(4 \leftrightarrow 5) \quad (7.67)$$

$$\hat{e}_{1 \times 2 \times 4 \times 3}(2^+, 3^-, 1^-) = e_{1 \times 2 \times 4 \times 3}(3^-, 2^+, 1^-)(4 \leftrightarrow 5) \quad (7.68)$$

This fully specifies the three integrals that enter the basis set indicated in table 2. The remainder are related by,

$$\begin{aligned}
 d_{4 \times 1 \times 2}(1^-, 2^+, 3^-) &= -d_{1 \times 2 \times 4}(2^+, 1^-, 3^-) \\
 d_{34 \times 1 \times 2}(1^-, 2^+, 3^-) &= d_{1 \times 2 \times 4}(1^-, 2^+, 3^-)\{4 \leftrightarrow 5\} \\
 d_{34 \times 2 \times 1}(1^-, 2^+, 3^-) &= -d_{1 \times 2 \times 4}(2^+, 1^-, 3^-)\{4 \leftrightarrow 5\} \\
 d_{2 \times 34 \times 1}(1^-, 2^+, 3^-) &= d_{1 \times 4 \times 2}(1^-, 2^+, 3^-)\{4 \leftrightarrow 5\} \\
 d_{1 \times 23 \times 4}(1^-, 2^+, 3^-) &= -d_{4 \times 1 \times 23}(1^-, 3^-, 2^+)\{4 \leftrightarrow 5\} \\
 d_{4 \times 1 \times 23}(1^-, 2^+, 3^-) &= -d_{4 \times 1 \times 23}(1^-, 3^-, 2^+) \\
 d_{1 \times 4 \times 23}(1^-, 2^+, 3^-) &= -d_{1 \times 4 \times 23}(1^-, 3^-, 2^+), \tag{7.69}
 \end{aligned}$$

with the full set obtained by performing cyclic permutations of $(1, 2, 3)$. Note that two of these relations involve the box coefficients for the $(+, -, -)$ configuration that will be given in the following section.

7.4.3 Triangles

The coefficient $c_{1 \times 2}(1^-, 2^+, 3^-)$ is very simple,

$$c_{1 \times 2}(1^-, 2^+, 3^-) = -\frac{s_{12} [1\,2] [2\,3]}{2 [1\,3]^3} \tag{7.70}$$

The following two triangle coefficients only have contributions at order m^2 ,

$$c_{3 \times 12}^{(0)}(1^-, 2^+, 3^-) = 0 \tag{7.71}$$

$$\begin{aligned}
 c_{3 \times 12}^{(2)}(1^-, 2^+, 3^-) &= \left[\frac{2(s_{45} - 2M_H^2) [2\,3]^3 (s_{13} + s_{23})}{[1\,2] [1\,3] [3|4|5|3]^2} - \frac{2 [2\,3] \langle 1\,3 \rangle^2}{[1\,3] \langle 1\,2 \rangle (s_{13} + s_{23})} \right. \\
 &\quad \left. + \frac{[2\,3] \langle 1\,3 \rangle}{[1\,3]^2 \langle 1\,2 \rangle} - \frac{\langle 1\,3 \rangle^2}{[1\,3] \langle 1\,2 \rangle \langle 2\,3 \rangle} \right] \tag{7.72}
 \end{aligned}$$

and,

$$c_{1 \times 4}^{(0)}(1^-, 2^+, 3^-) = -c_{1 \times 4}^{(0)}(1^-, 3^-, 2^+) = 0 \tag{7.73}$$

$$c_{1 \times 4}^{(2)}(1^-, 2^+, 3^-) = -c_{1 \times 4}^{(2)}(1^-, 3^-, 2^+) \tag{7.74}$$

$$\begin{aligned}
 &= 2 \frac{\langle 1|4|1 \rangle}{[1\,3] \langle 3\,2 \rangle [3\,2]} \left[\frac{[1\,2]^2 \langle 3|4|1 \rangle (\langle 1|4|1 \rangle - \langle 3|2|3 \rangle)}{\langle 1|5|4|1 \rangle^2} \right. \\
 &\quad \left. + \frac{(\langle 2|4|1 \rangle \langle 3|5|2 \rangle [1\,2] + \langle 3|4|1 \rangle \langle 3|5|1 \rangle [3\,2])}{\langle 1|5|4|1 \rangle [1|4|2]} \right] \tag{7.75}
 \end{aligned}$$

The last triangle coefficient is simply related to one previously defined,

$$c_{4 \times 12}(1^-, 2^+, 3^-) = -c_{4 \times 12}(2^+, 1^-, 3^-), \tag{7.76}$$

$$c_{4 \times 123}(1^-, 2^+, 3^-) = c_{4 \times 123}(3^-, 1^-, 2^+). \tag{7.77}$$

7.4.4 Bubbles and rational terms

The bubble coefficient $b_{12}(1^-, 2^+, 3^-)$ is given by,

$$b_{12}(1^-, 2^+, 3^-) = -\frac{\langle 13 \rangle [12]}{\langle 23 \rangle [13]^2} - \frac{\langle 13 \rangle^2 [12]}{\langle 23 \rangle [13] (s_{13} + s_{23})} - \frac{\langle 13 \rangle^2 [12] [23]}{[13] (s_{13} + s_{23})^2} \quad (7.78)$$

The rational term is

$$R(1^-, 2^+, 3^-) = -\frac{\langle 13 \rangle^2}{\langle 12 \rangle \langle 23 \rangle [13]} \left[1 - \frac{s_{12}}{(s_{12} + s_{13})} - \frac{s_{23}}{(s_{13} + s_{23})} \right] \quad (7.79)$$

7.5 $g^+ g^- g^- HH$

7.5.1 Effective pentagons

Turning now to the $1^+ 2^- 3^-$ helicity combination, all the effective pentagon coefficients necessary for this amplitude have already been introduced in section 7.3.1.

7.5.2 Boxes

There is one independent box with two lightlike external lines,

$$\begin{aligned} d_{1 \times 2 \times 4}(1^+, 2^-, 3^-) &= \frac{\langle 23 \rangle}{[23]^2 \langle 13 \rangle \langle 12 \rangle} \\ &\times \left[\mathcal{C}_1^{3 \times 1 \times 2 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^-, 1^+, 2^-) + \bar{\mathcal{C}}_5^{1 \times 2 \times 4 \times 3} \hat{e}_{1 \times 2 \times 4 \times 3}(1^+, 2^-, 3^+) \right] \\ &+ \hat{d}_{1 \times 2 \times 4}(1^+, 2^-, 3^-), \end{aligned} \quad (7.80)$$

where the remainder $\hat{d}_{1 \times 2 \times 4}(1^+, 2^-, 3^-)$ reads,

$$\begin{aligned} \hat{d}_{1 \times 2 \times 4}(1^+, 2^-, 3^-) &= \\ &- \frac{[12] s_{24}}{8 \langle 13 \rangle [23]^2} \left\{ \frac{\langle 13 \rangle [s_{12}(s_{14} + M_H^2) + 2M_H^2(s_{24} - M_H^2)] + 8 \langle 12 \rangle \langle 3|5|2] m^2}{\langle 1|4|2]} \right. \\ &+ \langle 23 \rangle (s_{13} + 2s_{12} - 2M_H^2) \left. \right\} + \frac{[12] \langle 2|4|1]}{4[23]^2 \text{tr}_5} \left\{ - \langle 12 \rangle [13] [\langle 3|4|2] (s_{15} - s_{35} + 2M_H^2) \right. \\ &+ 8 \langle 3|5|2] m^2 \left. \right\} + \left[(s_{12}(\langle 3|5|3] (s_{123} - 2M_H^2) + (s_{35} - s_{34})M_H^2) + (2 \leftrightarrow 3, 4 \leftrightarrow 5)) \right] \\ &- \langle 12 \rangle \langle 3|4|2] \frac{\langle 3|5|3] (s_{12} - s_{13} - 2M_H^2 + 8m^2) + [12] \langle 13 \rangle \langle 2|5|3]}{\langle 13 \rangle} \left. \right\} \end{aligned} \quad (7.81)$$

The first box with only one lightlike external line is,

$$\begin{aligned} d_{4 \times 1 \times 23}(1^+, 2^-, 3^-) &= \frac{\langle 23 \rangle}{[23]^2 \langle 13 \rangle \langle 12 \rangle} \\ &\times \left[\mathcal{C}_2^{3 \times 2 \times 1 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(3^-, 2^-, 1^+) + \mathcal{C}_2^{2 \times 3 \times 1 \times 4} \hat{e}_{1 \times 2 \times 3 \times 4}(2^-, 3^-, 1^+) \right] \\ &+ \hat{d}_{4 \times 1 \times 23}(1^+, 2^-, 3^-), \end{aligned} \quad (7.82)$$

where the effective box coefficient $\hat{d}_{4 \times 1 \times 23}(1^+, 2^-, 3^-)$ is given by

$$\hat{d}_{4 \times 1 \times 23}(1^+, 2^-, 3^-) = \frac{1}{2} m^2 (s_{123} - 2s_{23} - 2M_H^2 + 8m^2) \frac{\langle 23 \rangle}{[23]} \frac{[1|4|5|1]}{\text{tr}_5}. \quad (7.83)$$

It is manifestly $(4 \leftrightarrow 5)$ symmetric.

The other box with one lightlike external line has the same effective box contribution,

$$\begin{aligned} d_{1 \times 4 \times 23}(1^+, 2^-, 3^-) &= \frac{\langle 23 \rangle}{[23]^2 \langle 13 \rangle \langle 12 \rangle} \\ &\times \left[\bar{\mathcal{C}}_2^{3 \times 2 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(3^-, 2^-, 1^+) + + \bar{\mathcal{C}}_2^{2 \times 3 \times 4 \times 1} \hat{e}_{1 \times 2 \times 4 \times 3}(2^-, 3^-, 1^+) \right] \\ &+ \hat{d}_{1 \times 4 \times 23}(1^+, 2^-, 3^-), \end{aligned} \quad (7.84)$$

with

$$\hat{d}_{1 \times 4 \times 23}(1^+, 2^-, 3^-) = \hat{d}_{4 \times 1 \times 23}(1^+, 2^-, 3^-). \quad (7.85)$$

This fully specifies the three integrals that enter the basis set indicated in table 2. The remainder are related by,

$$\begin{aligned} d_{1 \times 2 \times 3}(1^+, 2^-, 3^-) &= -d_{1 \times 2 \times 3}(3^-, 2^-, 1^+) \\ d_{4 \times 1 \times 2}(1^+, 2^-, 3^-) &= -d_{1 \times 2 \times 4}(2^-, 1^+, 3^-) \\ d_{34 \times 1 \times 2}(1^+, 2^-, 3^-) &= d_{1 \times 2 \times 4}(1^+, 2^-, 3^-) \{4 \leftrightarrow 5\} \\ d_{34 \times 2 \times 1}(1^+, 2^-, 3^-) &= -d_{1 \times 2 \times 4}(2^-, 1^+, 3^-) \{4 \leftrightarrow 5\} \\ d_{1 \times 4 \times 2}(1^+, 2^-, 3^-) &= -d_{1 \times 4 \times 2}(2^-, 1^+, 3^-) \\ d_{2 \times 34 \times 1}(1^+, 2^-, 3^-) &= -d_{1 \times 4 \times 2}(2^-, 1^+, 3^-) \{4 \leftrightarrow 5\} \\ d_{1 \times 23 \times 4}(1^+, 2^-, 3^-) &= d_{4 \times 1 \times 23}(1^+, 2^-, 3^-) \{4 \leftrightarrow 5\}, \end{aligned} \quad (7.86)$$

with the full set obtained by performing cyclic permutations of $(1, 2, 3)$.

7.5.3 Triangles

The simplest triangle coefficient is,

$$c_{1 \times 2}(1^+, 2^-, 3^-) = -\frac{s_{12}}{2} \frac{[12]}{[23]^3} \frac{[13]}{[23]}. \quad (7.87)$$

The order m^2 coefficients $c_{3 \times 12}^{(2)}(1^+, 2^-, 3^-)$ and $c_{1 \times 4}^{(2)}(1^+, 2^-, 3^-)$ are given by,

$$c_{3 \times 12}^{(2)}(1^+, 2^-, 3^-) = -c_{3 \times 12}^{(2)}(2^-, 1^+, 3^-) \quad (7.88)$$

$$\begin{aligned} &= -\left[\frac{-2(s_{45} - 2M_H^2)[13]^3(s_{13} + s_{23})}{[21][23][3|4|5|3]^2} - \frac{2[13]\langle 23 \rangle^2}{[23]\langle 21 \rangle(s_{13} + s_{23})} \right. \\ &\quad \left. + \frac{[13]\langle 23 \rangle}{[23]^2\langle 21 \rangle} - \frac{\langle 23 \rangle^2}{[23]\langle 21 \rangle\langle 13 \rangle} \right], \end{aligned} \quad (7.89)$$

and,

$$c_{1 \times 4}^{(2)}(1^+, 2^-, 3^-) = 2 \frac{\langle 1|4|1 \rangle}{[23]^2} \left[\frac{\langle 3|5|2 \rangle}{\langle 13 \rangle \langle 1|4|2 \rangle} - \frac{\langle 2|5|3 \rangle}{\langle 12 \rangle \langle 1|4|3 \rangle} + \frac{\langle 32 \rangle}{\langle 12 \rangle \langle 13 \rangle} \right], \quad (7.90)$$

with the order m^0 pieces determined by the infrared relations given in eqs. (6.13) and (6.14).

The remaining triangle is specified by,

$$\begin{aligned}
 c_{4 \times 12}^{(0)}(1^+, 2^-, 3^-) = & \frac{\langle 2|4|1 \rangle (s_{24} - s_{14})}{4s_{12} [3|4|5|3]} \left[[13] \langle 23 \rangle + \langle 13 \rangle \frac{[12] \langle 2|5|3 \rangle + [13] (s_{35} - s_{24})}{\langle 1|4|2 \rangle} \right] \\
 & - \frac{\langle 23 \rangle \Delta_{12|4|35} ([12] \langle 2|4|3 \rangle - [13] (s_{14} - M_H^2))}{4s_{12} \langle 1|4|2 \rangle [3|4|5|3]} \\
 & + \frac{\langle 2|4|1 \rangle (s_{24} - s_{14}) (s_{35} - s_{24}) (s_{123} - 2M_H^2)}{4s_{12} \langle 1|4|2 \rangle [3|4|5|3]} - \frac{\langle 2|4|1 \rangle^2 (s_{12} - 2M_H^2)}{2s_{12} [3|4|5|3]} \\
 & + \frac{\langle 2|5|3 \rangle \Delta_{12|4|35} (s_{123} - 2M_H^2) ([12] \langle 2|5|3 \rangle + [13] (s_{35} - s_{24}))}{4s_{12} \langle 1|4|2 \rangle [3|4|5|3]^2}, \quad (7.91)
 \end{aligned}$$

$$\begin{aligned}
 c_{4 \times 12}^{(2)}(1^+, 2^-, 3^-) = & 2 \langle 2|4|1 \rangle \frac{(s_{24} - s_{14}) (s_{35} - s_{24}) - 2 \langle 1|4|2 \rangle \langle 2|4|1 \rangle}{\langle 1|4|2 \rangle [3|4|5|3] s_{12}} \\
 & + 2 \frac{\langle 2|5|3 \rangle \Delta_{12|4|35} ([12] \langle 2|5|3 \rangle + [13] (s_{35} - s_{24}))}{\langle 1|4|2 \rangle [3|4|5|3]^2 s_{12}}, \quad (7.92)
 \end{aligned}$$

where the Källén function $\Delta_{12|4|35}$ was defined in eq. (4.30) and

$$c_{4 \times 123}(1^+, 2^-, 3^-) = c_{4 \times 123}(2^-, 3^-, 1^+). \quad (7.93)$$

It is evident that eq. (7.92) could be reabsorbed into eq. (7.91) by adding $8m^2$ to the parentheses involving M_H^2 in the last three fractions.

7.5.4 Bubbles and rational terms

The bubble coefficient is given by,

$$b_{12}(1^+, 2^-, 3^-) = -\frac{\langle 23 \rangle [12] [13]}{[23]} \left[\frac{\langle 23 \rangle}{(s_{13} + s_{23})^2} - \frac{1}{[23] (s_{13} + s_{23})} \right] \quad (7.94)$$

The rational term is

$$R(1^+, 2^-, 3^-) = -\frac{\langle 23 \rangle^2}{[23]} \left[\frac{1}{\langle 12 \rangle \langle 13 \rangle} + \frac{[13]}{\langle 12 \rangle (s_{13} + s_{23})} + \frac{[12]}{\langle 13 \rangle (s_{12} + s_{23})} \right] \quad (7.95)$$

8 Implementation of NLO $pp \rightarrow HH$ calculation

The one-loop matrix elements that we have computed here have been cross-checked against OpenLoops [63], finding full agreement. Our analytic calculation of the $0 \rightarrow gggHH$ process is approximately 90 times faster to evaluate than OpenLoops, while the simpler $0 \rightarrow \bar{q}qgHH$ amplitude is only 35 times quicker. The implementation of the corresponding dipole subtraction terms in MCFM [27–29], to complete the real radiation computation, is straightforward.

The 2-loop virtual matrix element contribution is implemented using hhgrid [21], a package that uses a grid to interpolate the two-loop result (available from <https://github.com/mppmu/hhgrid>). The package provides a Fortran interface to the interpolating Python code, which we have linked to MCFM.²

²We thank Stephen Jones for help producing grid files that can be loaded efficiently, so that it is straightforward to run our calculations on multiple cores simultaneously.

The `hhgrid` code provides the value of $\mathcal{V}_{\text{fin}}$, which is defined in terms of the virtual contribution $\mathcal{V}_b$ — the interference of the 2-loop and 1-loop amplitudes including all overall coupling and averaging factors [21] — and takes the general form,

$$\mathcal{V}_b = \mathcal{N} \frac{\alpha_s}{2\pi} \left[\frac{1}{\epsilon^2} a \mathcal{B} + \frac{1}{\epsilon} \sum_{i \neq j} c_{ij} \mathcal{B}_{ij} + \mathcal{V}_{\text{fin}} \right], \quad \mathcal{N} = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{Q^2} \right)^\epsilon, \quad (8.1)$$

where $\mathcal{B}$ is the Born contribution. From colour conservation, for this process we have $\mathcal{B}_{12} = \mathcal{B}_{21} = C_A \mathcal{B}$. The coefficients of the pole terms are determined by $a = -2C_A$ and $c_{12} = c_{21} = -\beta_0/C_A - \log(\mu^2/\hat{s})$, where $\beta_0 = (11C_A - 2n_f)/6$.³ Finally, the value of $\mathcal{V}_{\text{fin}}$ is provided for the scale $\mu_0 = \sqrt{\hat{s}}/2$, from which the result at an arbitrary scale μ can be found using,

$$\mathcal{V}_{\text{fin}}(\mu) = \mathcal{V}_{\text{fin}}(\mu_0) \cdot \frac{\alpha_s^2(\mu)}{\alpha_s^2(\mu_0)} + C_A \mathcal{B}(\mu) \left[\log^2 \left(\frac{\mu_0^2}{\hat{s}} \right) - \log^2 \left(\frac{\mu^2}{\hat{s}} \right) \right] \quad (8.2)$$

The virtual contribution is implemented in `MCFM` by using eqs. (8.1) and (8.2), setting $\mathcal{N} = 1$ since such an overall factor is implicit in the rest of the code.

8.1 Validation

We first compare with the 14 and 100 TeV total cross section results presented in ref. [14]. These are obtained with $m = 173$ GeV, $M_H = 125$ GeV and the PDF set `PDF4LHC15_nlo_100_pdfas` (for both LO and NLO calculations). The top quark width is set to zero.

In order to address issues of numerical stability, we have implemented a rescue system in the calculation of the real radiation corrections. This compares the calculation of the matrix elements at two phase-space points related by a rotation in order to provide an estimate of the numerical accuracy. If this suggests that the result is not accurate to at least eight digits we switch on the fly from double to quadruple precision and repeat the calculation of the real emission matrix elements. Although this rescue system clearly requires two evaluations of the matrix elements, since they are already very fast to compute this is not a great additional burden. With this in place we obtain the same integrated cross-sections as when using `OpenLoops`, but in a factor 50 less time. To eliminate unnecessary further numerical instability, we have also imposed a technical cut $p_T(HH)/\sqrt{\hat{s}} > 10^{-2}$. We have checked that variation of this cut in the range 10^{-6} to 10^{-2} makes a difference of less than one per mille in the total NLO result.

At the level of total cross sections, the perfect agreement between the NLO results from the two codes is demonstrated in table 3. The `hhgrid` package also provides 14 TeV validation data for the distributions of the Higgs boson pair invariant mass and rapidity, as well as the transverse momentum of a random Higgs boson. We compare these with the `MCFM` results in figures 1, 2 and 3, which again demonstrate excellent agreement.

³This corrects the definition of c_{12} and c_{21} found in the published version of ref. [21]; the arXiv version has been updated.

$\sqrt{s}[\text{TeV}]$	Calculation	LO [fb]	NLO [fb]
14	MCFM	$19.85^{+27.6\%}_{-20.5\%}$	$32.91^{+13.6\%}_{-12.6\%}$
	Ref. [14]	$19.85^{+27.6\%}_{-20.5\%}$	$32.91^{+13.6\%}_{-12.6\%}$
100	MCFM	$730.9^{+20.9\%}_{-15.9\%}$	$1149^{+10.8\%}_{-10.0\%}$
	Ref. [14]	$731.3^{+20.9\%}_{-15.9\%}$	$1149^{+10.8\%}_{-10.0\%}$

Table 3. Validation of 14 and 100 TeV cross sections against the results of ref. [14]. The numerical uncertainty in the MCFM results is beyond the last digit. The percentage deviations correspond to estimates of uncertainty from 7-point variation of the scale according to the procedure described in ref. [14].

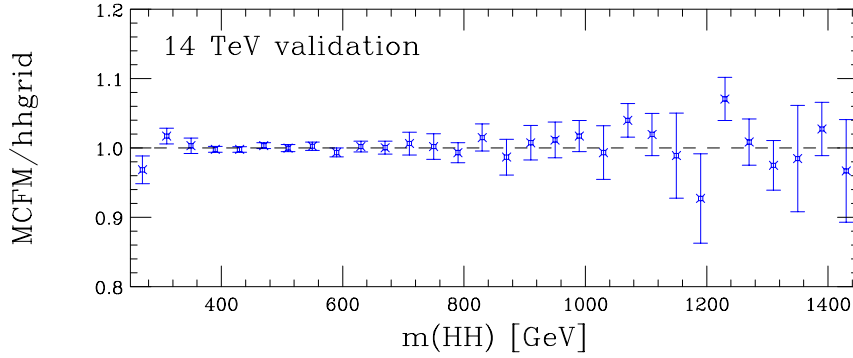


Figure 1. Validation plot for the $m(HH)$ distribution, comparing against the `hhgrid` result of ref. [21].

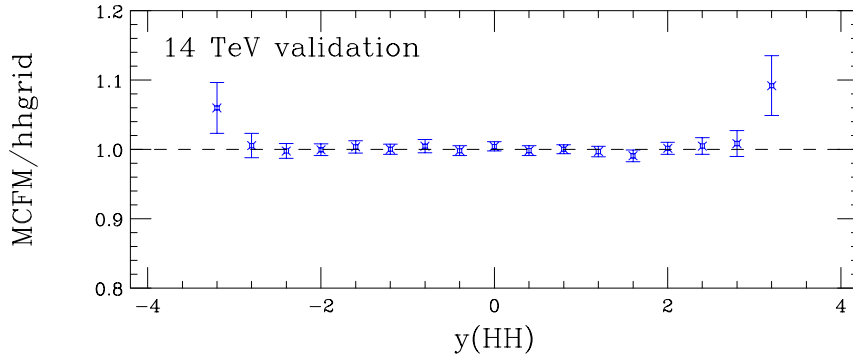


Figure 2. Validation plot for the $y(HH)$ distribution, comparing against the `hhgrid` result of ref. [21].

8.2 Phenomenology

With the fixed order Higgs pair production process implemented in MCFM it is straightforward to extend the existing framework to provide a resummed prediction for the transverse momentum of the Higgs boson pair. This is implemented using the `CuTe-MCFM` framework [64, 65]; we refer the reader to the original papers for more details. For our purposes it is important to note that for our matched resummed prediction, which combines the NNLL result at small q_T with the fixed order one at high q_T , we use a transition function with parameter $x = q_T^2(HH)/m_{HH}^2$ and $x_{\max} = 0.1$.

As an example, in figure 4 we show the matched, resummed q_T spectrum of the Higgs boson pair at a 100 TeV pp collider, together with the results obtained from pure fixed

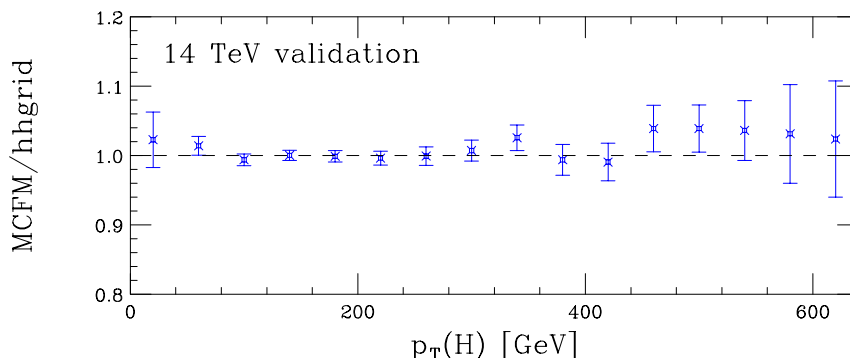


Figure 3. Validation plot for the $p_T(H)$ (random Higgs boson) distribution, comparing against the hhgrid result of ref. [21].

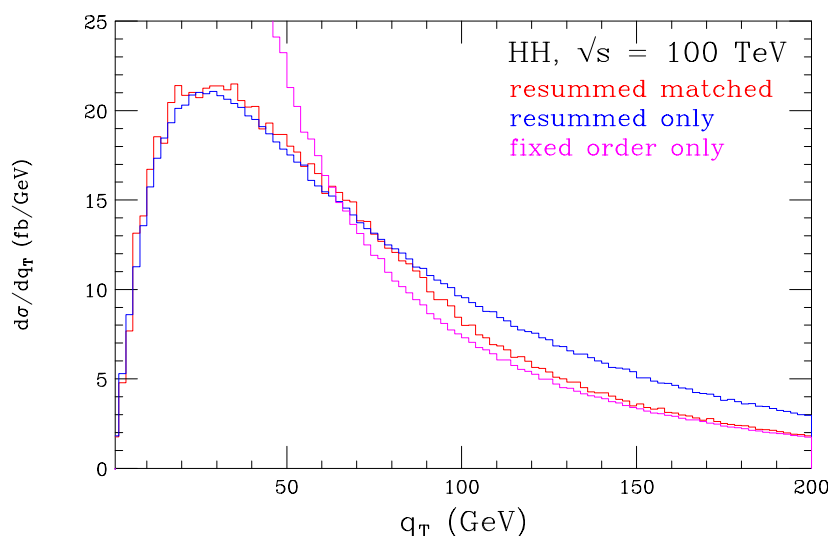


Figure 4. The q_T spectrum of the Higgs boson pair at a 100 TeV pp collider. The plot shows the matched resummed prediction (red), compared with the pure resummed result (blue) and the fixed order calculation (magenta).

order and resummed calculations. The fixed order result clearly diverges at small q_T while the resummed result ameliorates this behavior. The matched resummed result smoothly interpolates between the two calculations, transitioning to the fixed-order result for q_T around $q_T^{\max} = \sqrt{x_{\max}} m_{HH} \sim 130 \text{ GeV}$ (using the fact that the peak of the m_{HH} distribution is around 400 GeV). The matched result begins to differ substantially from the resummed result a little before that, around $q_T \sim 90 \text{ GeV}$.

9 Conclusions

This paper has addressed the calculation of the amplitude for a pair of Higgs bosons in association with three partons at one-loop level. The calculation proceeded in two steps. First the coefficients of the needed scalar integrals were calculated using both methods based on the work of Passarino and Veltman, as well as more modern techniques based on generalized unitarity. The initial results for the box coefficients obtained using unitarity

were subsequently simplified using the technique of analytic reconstruction. Compared to previous uses of this technique, new strategies were introduced in order to handle particular features of this processes, in particular the presence of two massive external particles and a massive particle circulating in the loop. This latter step, which yielded the simpler results for the box coefficients and some of the triangle coefficients given in this paper, also improved the speed of the numerical evaluation. The resulting code, in combination with previous work [21] on the two loop corrections to the Higgs boson pair + 2 parton process, allows the fast evaluation of the next-to-leading order corrections to $pp \rightarrow HH$. This calculation will be included in an upcoming release of the MCFM code, also providing machine-readable versions of the analytic amplitude results presented in this paper.

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A Spinor algebra

All results are presented using the standard notation for the kinematic invariants of the process,

$$s_{ij} = (p_i + p_j)^2, s_{ijk} = (p_i + p_j + p_k)^2, s_{ijkl} = (p_i + p_j + p_k + p_l)^2, \quad (\text{A.1})$$

and the Gram determinant,

$$\Delta_3(i, j, k, l) = (s_{ijkl} - s_{ij} - s_{kl})^2 - 4s_{ij}s_{kl}. \quad (\text{A.2})$$

In the case where momentum j is not lightlike, we put the corresponding subscript in boldface, e.g. s_{ij} . We express the amplitudes in terms of spinor products defined as,

$$\langle i j \rangle = \bar{u}_-(p_i)u_+(p_j), \quad [i j] = \bar{u}_+(p_i)u_-(p_j), \quad \langle i j \rangle [j i] = 2p_i \cdot p_j, \quad (\text{A.3})$$

and we further define the spinor sandwiches for massless momenta j and k ,

$$\begin{aligned} \langle i|(j+k)|l \rangle &= \langle i j \rangle [j l] + \langle i k \rangle [k l] \\ [i|(j+k)|l \rangle &= [i j] \langle j l \rangle + [i k] \langle k l \rangle \end{aligned} \quad (\text{A.4})$$

The spinor sandwich with momentum k not lightlike is distinguished by putting the momentum k in boldface, e.g. $\langle i|\mathbf{k}|l \rangle$.

In the Weyl representation for the Dirac gamma matrices, following the conventions of ref. [66], we have

$$\not{p} = \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 = \begin{pmatrix} 0 & 0 & p^- & -p^1 + ip^2 \\ 0 & 0 & -p^1 - ip^2 & p^+ \\ p^+ & p^1 - ip^2 & 0 & 0 \\ p^1 + ip^2 & p^- & 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

The massless spinors solutions of Dirac equation are

$$u_-(p) = \begin{bmatrix} (-p^1 + ip^2)/\sqrt{p^+} \\ \sqrt{p^+} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{p^-} e^{-i\varphi_p} \\ \sqrt{p^+} \\ 0 \\ 0 \end{bmatrix}, \quad (\text{A.6})$$

and

$$u_+(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ (p^1 + ip^2)/\sqrt{p^+} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \end{bmatrix}, \quad (\text{A.7})$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \quad p^\pm = p^0 \pm p^3. \quad (\text{A.8})$$

In this representation the Dirac conjugate spinors are

$$\bar{u}_-(p) \equiv u_-^\dagger(p) \gamma^0 = [0, 0, -\sqrt{p^-} e^{i\varphi_p}, \sqrt{p^+}] \quad (\text{A.9})$$

$$\bar{u}_+(p) \equiv u_+^\dagger(p) \gamma^0 = [\sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}, 0, 0] \quad (\text{A.10})$$

More complicated spinor products, follow in an obvious way,

$$\langle i|\mathbf{k}|j\rangle = \bar{u}_-(p_i) \not{\mathbf{k}} u_-(p_j), \quad \langle i|\mathbf{k}|\mathbf{l}|j\rangle = \bar{u}_-(p_i) \not{\mathbf{k}} \not{\mathbf{l}} u_+(p_j), \quad (\text{A.11})$$

where $\mathbf{k}$ and $\mathbf{l}$ are the momenta of non lightlike particles.

B Loop integral definitions

We work in the Bjorken-Drell metric so that $l^2 = l_0^2 - l_1^2 - l_2^2 - l_3^2$. The affine momenta q_i are given by sums of the external momenta, p_i , where $q_n \equiv \sum_{i=1}^n p_i$ and $q_0 = 0$. The propagator denominators are defined as $d_i = (l + q_i)^2 - m^2 + i\varepsilon$. The definition of the relevant scalar integrals is as follows,

$$\begin{aligned} B_0(p_1; m) &= \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma} \int \frac{d^D l}{d_0 d_1}, \\ C_0(p_1, p_2; m) &= \frac{1}{i\pi^2} \int \frac{d^4 l}{d_0 d_1 d_2}, \\ D_0(p_1, p_2, p_3; m) &= \frac{1}{i\pi^2} \int \frac{d^4 l}{d_0 d_1 d_2 d_3}, \\ E_0(p_1, p_2, p_3, p_4; m) &= \frac{1}{i\pi^2} \int \frac{d^4 l}{d_0 d_1 d_2 d_3 d_4}. \end{aligned} \quad (\text{B.1})$$

For the purposes of this paper we take the masses in the propagators to be real. Near four dimensions we use $D = 4 - 2\epsilon$. (For clarity the small imaginary part which fixes the analytic continuations is specified by $+i\epsilon$). μ is a scale introduced so that the integrals preserve their natural dimensions, despite excursions away from $D = 4$. We have removed the overall constant which occurs in D -dimensional integrals,

$$r_\Gamma \equiv \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\Gamma(1-\epsilon)} + \mathcal{O}(\epsilon^3) = 1 - \epsilon\gamma + \epsilon^2 \left[\frac{\gamma^2}{2} - \frac{\pi^2}{12} \right] + \mathcal{O}(\epsilon^3). \quad (\text{B.2})$$

C Spinor decompositions

We supplement the primary decompositions presented in ref. [34, section 3.3.1] with the following new decomposition

$$\begin{aligned} & \left\langle [1|(4+5)|(6+7)|1], [2|(4+5)|(6+7)|2] \right\rangle = \\ & \left\langle [1\,2], [1\,3], [2\,3] \right\rangle \cap \left\langle [1\,2], \langle 3|(4+5)|2 \rangle, \langle 3|(4+5)|1 \rangle \right\rangle \cap \\ & \left\langle [1|(4+5)|(6+7)|1], [2|(4+5)|(6+7)|2], \right. \\ & \quad \left. |(1+3)|(4+5)|2][1+|2][1|(4+5)|(2+3)| - |2]\langle 3|(4+5)|3 \rangle [1] \right\rangle, \end{aligned} \quad (\text{C.1})$$

where it appears that the last ideal on the right-hand side may require further decompositions, even if this form suffices for the current discussion. The generator with two open indices was obtained by fitting a covariant ansatz to numerical evaluations obtained from a component expression obtained from **Singular**, after quotienting the left-hand side ideal by the first two prime ideals on the right-hand side. The equality holds in R_7 without imposing the additional constraint $s_{45} = s_{67}$. Changing the ring, e.g. to $\mathcal{R}_5$, causes non-trivial modification to the decomposition.

The decomposition of eq. (C.1) can be understood in light of the following identity

$$\begin{aligned} & [1] [2|(4+5)|(6+7)|2] [1 - |2] [1|(4+5)|(6+7)|1] [2] = \\ & - [1\,2] \left(|(1+3)|(4+5)|2][1+|2][1|(4+5)|(2+3)| - |2][3|(4+5)|3][1] \right) \end{aligned} \quad (\text{C.2})$$

where on the left-hand side we clearly have a member of the maximal codimension ideal being decomposed in eq. (C.1), while on the right-hand side we have a polynomial with two factors. This manifestly shows that a non-trivial primary decomposition is needed, and identifies the two factors as generators of ideals in the decomposition.

We can see this decomposition reflected in the structure of the coefficient $c_{4 \times 123}^{(2)}(1^-, 2^-, 3^+)$ from eq. (7.58). In order to fully separate the poles $[1\,4\,5|1]$ and $[2\,4\,5|2]$ into separate fractions, it is necessary to introduce a spurious second power of the pole $[1\,2]$. Without it, the common numerator does not vanish on all branches of (C.1), thus preventing the partial fraction decomposition by Hilbert's Nullstellensatz.

Alternatively, it is possible to avoid introducing the spurious double pole, if $[1|4|5|1]$ and $[2|4|5|2]$ are kept in the same denominator

$$c_{4 \times 123}^{(2)}(1^-, 2^-, 3^+) = \left\{ 2 \frac{[13]^3 s_{45}(s_{45} - 4M_H^2)}{[12][23][1|4|5|1]^2} + 4 \frac{[13][3|4|5|3]}{[12][23][1|4|5|1]} - 2 \frac{[13]\langle 2|4-5|3 \rangle}{[12][1|4|5|1]} \right\} - \left\{ 1 \leftrightarrow 2 \right\} \\ - \frac{1}{2} \frac{\text{tr}(3-1-2|4-5)([13][23]\text{tr}(3-1-2|4-5) - [12][3|(4-5)|(1-2)|3])}{[12][1|4|5|1][2|4|5|2]}. \quad (\text{C.3})$$

This form also manifests the symmetry under $4 \leftrightarrow 5$ term by term. The numerator of the last fraction is now a contraction of the covariant generator in (C.1),

$$\frac{1}{2} [12] ([13][23]\text{tr}(3-1-2|4-5) - [12][3|(4-5)|(1-2)|3]) = [13]^2 [2|4|5|2] - [23]^2 [1|4|5|1], \quad (\text{C.4})$$

where for convenience we have written this as $[3] \times (\text{C.2}) \times [3]$. As before the trace is understood as being of rank-two spinors: $(3-1-2)_{\alpha\dot{\alpha}}(4-5)^{\dot{\alpha}\alpha}$.

D Box manipulations and spurious singularities at infinity

In this appendix we analyse the singularity structure of one of the box coefficients, $d_{1 \times 2 \times 3}(1^-, 2^-, 3^+)$, which was presented eq. (7.43). That form with the effective pentagon involves a spurious single pole in tr_5 , as well a degree two (instead of one) pole in $[12]$, a degree one (instead of two) zero in $\langle 12 \rangle$, etc. A form with more manifest analytic properties is,

$$d_{1 \times 2 \times 3}(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^2 \langle 23 \rangle [23] m^2 \left([3|4|5|3] \langle 13 \rangle [3|4|5|1] - [13] \langle 1|5|4|3 \rangle \right) + \langle 12 \rangle [23] (s_{123} - 2s_{12} - 2M_H^2 + 8m^2) [3|4|5|1]}{8(-\langle 12 \rangle [12] \langle 23 \rangle [23] \langle 1|5|4|3 \rangle [3|4|5|1] + m^2 \text{tr}_5^2)} + (4 \leftrightarrow 5) \\ + \frac{\langle 12 \rangle^3 [13] \langle 23 \rangle [23]^2 m^4 \text{tr}_5 (\langle 1|4|5|3 \rangle [3|5|4|1] - (4 \leftrightarrow 5)) (s_{123} - 2s_{12} - 2M_H^2 + 8m^2)}{4(-\langle 12 \rangle [12] \langle 23 \rangle [23] \langle 1|5|4|3 \rangle [3|4|5|1] + m^2 \text{tr}_5^2) \times (4 \leftrightarrow 5)} \\ + m^2 \frac{\langle 12 \rangle^2 [23]}{2[12] \langle 13 \rangle}, \quad (\text{D.1})$$

where the first line is a doublet and the latter two lines are singlets under the exchange of the Higgs bosons, $4 \leftrightarrow 5$. In eq. (D.1) this is shown explicitly. Except for the contribution in the last line of eq. (D.1), the only singularities are $|S^{1 \times 2 \times 3 \times 4}|$ and $|S^{1 \times 2 \times 3 \times 4}|(4 \leftrightarrow 5)$, as introduced in eq. (7.11).

Let us now analyse the large m limit. As discussed in section 3.3, we consider a projective space in m , rather than an affine space. This box coefficient diverges linearly in m^2 . In the limit, it reads,

$$\lim_{m \rightarrow \infty} d_{1 \times 2 \times 3}(1^-, 2^-, 3^+) = +m^2 \frac{\langle 12 \rangle^2 [23]}{2[12] \langle 13 \rangle} - m^2 \frac{\langle 12 \rangle^3 [13] \langle 23 \rangle [23]^2 (s_{123} - 2M_H^2)}{\text{tr}_5^2} \\ + 2m^2 \frac{\langle 12 \rangle^3 [13] \langle 23 \rangle [23]^2 (\langle 1|4|5|3 \rangle [3|5|4|1] - (4 \leftrightarrow 5))}{\text{tr}_5^3}. \quad (\text{D.2})$$

This form can be read out from eq. (D.1), using the following identity,

$$[1|4|5|3] + [1|5|4|3] = [13](s_{123} - 2M_H^2). \quad (\text{D.3})$$

We can now understand why the two $|S|$ denominators in the general m expression of eq. (D.1) cannot be separated without introducing a spurious pole in $\text{tr}_5 \rightarrow 0$ or $m \rightarrow \infty$. An ansatz where the $|S|$ denominators are separated would read, schematically,

$$d_{1 \times 2 \times 3} \sim \sum_{\text{terms}} \frac{m^{\alpha \text{tr}_5^\beta}}{|S|} \quad (\text{D.4})$$

On the irreducible codimension-two projective variety $V(\langle \text{tr}_5, m^{-1} \rangle)$ we must have a pole of degree five, as manifest in the second line in eq. (D.2), thus we have the constraint,

$$\alpha - \beta = 5. \quad (\text{D.5})$$

We impose the equality because at least one of the terms must saturate the limit. On the irreducible codimension-one projective variety $V(\langle m^{-1} \rangle)$ we have a double pole, i.e.

$$\alpha \leq 4, \quad (\text{D.6})$$

since $|S|$ goes quadratically. We allow the inequality since not all terms need to saturate this limit. On the irreducible codimension-one variety $V(\langle \text{tr}_5 \rangle)$ the coefficient is regular,

$$\beta \geq 0. \quad (\text{D.7})$$

In the $\alpha - \beta$ plane the line $\alpha - \beta = 5$ does not intersect the semi-infinite region defined by $\alpha \leq 4, \beta \geq 0$. Therefore, no simultaneous solution exists to these three constraints.

Consider one last form for this coefficient,

$$\begin{aligned} d_{1 \times 2 \times 3}(1^-, 2^-, 3^+) = & \left\{ \frac{\langle 1\,2 \rangle^2 [23] m^2}{8(-\langle 1\,2 \rangle [12] \langle 2\,3 \rangle [23] \langle 1|5|4|3 \rangle [3|4|5|1] + m^2 \text{tr}_5^2)} \times \right. \\ & \left[(s_{123} - 2s_{12} - 2M_H^2 + 8m^2)(-2m^2 \text{tr}_5 \frac{[13]}{[12]} + \langle 1\,2 \rangle \langle 2\,3 \rangle [23] [3|4|5|1]) \right. \\ & \left. \left. + \langle 2\,3 \rangle [3|4|5|3] (\langle 1\,3 \rangle [3|4|5|1] - [13] \langle 1|5|4|3 \rangle) \right] \right\} + \left\{ 4 \leftrightarrow 5 \right\} \\ & + m^2 \frac{\langle 1\,2 \rangle^2 [23]}{2[12] \langle 1\,3 \rangle} \end{aligned} \quad (\text{D.8})$$

The above curly bracket goes like m^4 in the large m limit (the numerator goes like m^6 , the denominator as m^2), while we have already shown that this box coefficient only scales as m^2 . In terms of α and β , we have $\alpha = 6$ and $\beta = 1$. This form has a spurious pole at infinity, meaning it appears in the numerator rather than the denominator.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has associated code in a code repository.

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