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An Analytical Solution for the Motion of a Projectile Accounting for Drag in the Case of Vertical Launch

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We study the problem of projectile motion, subject to atmospheric drag, in the special case of a vertical launch (i.e., straight up). We find that there exists an analytical solution for the time evolution of the projectile position and for the maximum height it will reach, when we assume that the drag coefficient has a constant value. We compare our solution with a full numerical solution in which the drag coefficient is allowed to vary nonlinearly as the projectile changes velocity. We suggest that the analytical solution we present provides a useful pedagogical bridge between full solutions to projectile motion problems, which typically require numerical methods, and the very simplest solutions, which can be inaccurate in many real-world applications.

Quantitative skills

Across many applied sciences, there have been calls to focus on or improve quantitative skill development of students.[1,2] This agenda is aligned with an increase in the quantitative demands of many jobs that graduates of sciences may seek. We propose that physics problems that can be tackled via multiple approaches, of incrementally increasing sophistication, are of particular value in supporting students to improve their quantitative skills. Simple analytical solutions offer an accessible gateway to the problem, at the expense of multiple simplifying assumptions that limit accuracy. As student skill and knowledge increases, more sophisticated approaches become accessible, offering more accurate results. Finally, a full numerical solution can build on these preliminary and intermediate steps, which have helped the student to develop deeper insight into the problem. Here we present an intermediate-level solution to a classic problem: the motion of a ballistic projectile. Exploring such intermediate problems from a mathematical perspective can provide a rich pedagogical toolkit, as well as intriguing and useful solutions for real, applied situations.

The motion of projectiles is a classic physics problem that is used to introduce students to calculus, linear algebra, and general mechanics, all at a wide range of levels of complexity. Such problems are key to understanding the motion of balls in sports,[3] of volcanic ballistic "bombs" thrown out during eruptions,[4,5] and of military projectiles through the ages,[6] among many other applications. Predicting the motion of these objects is straightforward in simple cases where drag forces can be neglected. However, beyond those simple cases, drag forces complicate the solution substantially, and simplifying assumptions must be made to find analytical solutions.[7] Those expedient simplifications can make the solutions less accurate and require calibration against experimental data.[8]

Here, we present an analytical solution to the problem of projectile motion subject to atmospheric drag using minimal approximations. Instead of making an approximation that leads to a simplification, we restrict ourselves to the particular case of a vertical launch trajectory (i.e., launching the projectile straight upward). We find that this special case leads to a simple analytical solution for both the evolution of the projectile height and the maximum height. We explore how this may be deployed in classrooms, using familiar household projectiles as examples, as well as suggesting one real-world applied example where this solution may be of wide utility.

Mathematical model development

The principal complicating factor in the study of projectile motion is the role played by drag. Simple solutions that neglect drag only match experiments at low velocities,[7] and therefore, in many applied examples, drag must be accounted for. There are various intuitive or experimental approaches to understanding the effect of drag on projectile motion,[7,9,10] and the insights that can be gained via mathematics or calculus sometimes take a back seat. This may be because intermediate or advanced calculus skills are often developed later than introductory mechanics skills, such that calculus is rarely used for intuition building. Here, we place the calculus involved in building mechanics models at the center of the problem. The starting point for many mechanics problems is Newton's second law, which states that the sum of the forces that act on an object $\boldsymbol{F}$ is equal to the product of the object's mass m and its acceleration a, so that $\boldsymbol{F} = m\boldsymbol{a}$. In the case of a projectile, classic solutions exist for the case when the only force acting on the object is a gravitational force $m\boldsymbol{g}$, where $\boldsymbol{g}$ is the gravitational acceleration vector. However, when objects are moving in a medium such as air, there is an additional drag force $\boldsymbol{F}_\mathrm{d}$ that arises from friction with the medium. This addition of a second force can lead to a wealth of complexity and cause substantial complications in finding solutions to Newton's second law, necessitating the use of calculus. Here, we start with Newton's second law, with that drag force included, and provide a worked mathematical process to arrive at a specific solution that is possible only when the projectile is launched vertically.

Newton's second law leads to a general equation for the acceleration of a projectile $d\boldsymbol{v}/dt$ as[8,9]

$$\frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}_\mathrm{d}}{m} + \boldsymbol{g} = -\frac{\rho_\mathrm{f} ACv\boldsymbol{v}}{2\rho V} + \boldsymbol{g}, \quad (1)$$

where $\boldsymbol{v}$ is the velocity vector, t is the time since launch, C is the drag coefficient, ρ_f is the fluid density, and ρ, A, V, and m are the projectile density, cross-sectional area, volume, and mass, respectively (note that the projectile mass m is $m = \rho V$). Here v is the magnitude of the vector $\boldsymbol{v}$, and the drag force vector is given by $\boldsymbol{F}_\mathrm{d} = -\rho_\mathrm{f} ACv\boldsymbol{v}/2$. Note that the drag force is negative here because it is a force opposing the

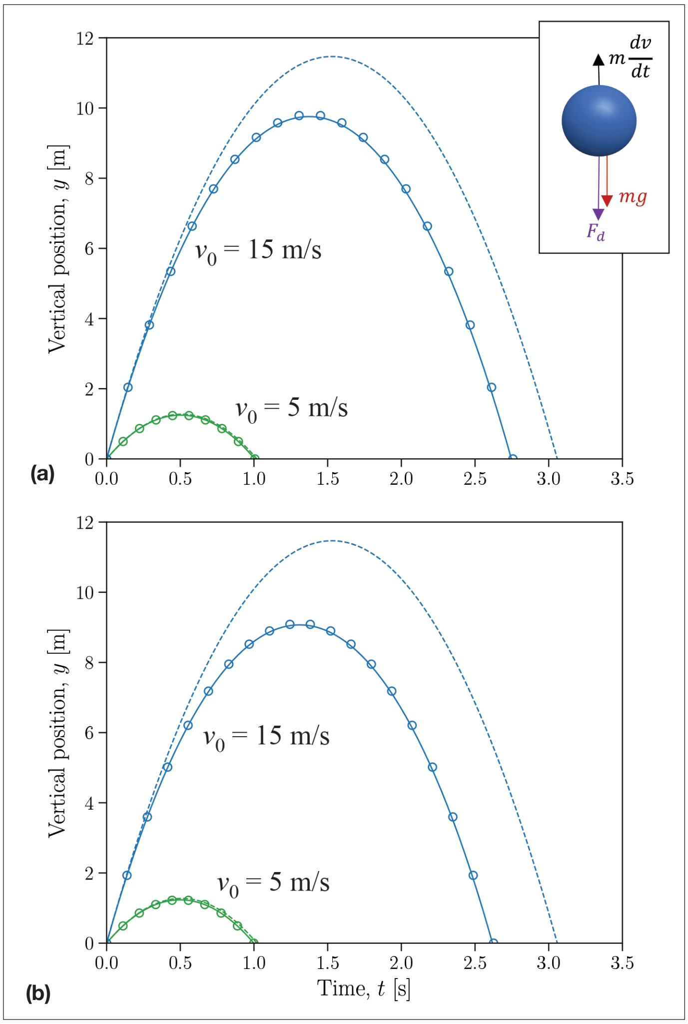

Fig. 1. The evolution of the vertical position y as a function of time t for (a) a tennis ball or (b) a soccer ball, each launched at $v_0 = 5$ m/s (green) or $v_0 = 15$ m/s (blue). The inset shows the forces acting on the ball during upward motion. The model is solved using either Eq. (9) (analytical solution with $C = 0.4$ given by the circle points), a numerical solution to Eq. (1) with variable C (termed the "full model" in the text and given by the solid curve), or the case with no drag (see text; given by the dashed curves). The analytical solution [Eq. (9)] closely matches the full solution in all cases and outperforms the no-drag approximation for $v_0 = 15$ m/s (for $v_0 = 5$ m/s, the no-drag case provides a good approximation). For a tennis ball, we take $m = 0.057$ kg and diameter 0.066 m, and for a soccer ball, we take $m = 0.396$ kg and diameter 0.218 m; these values and a spherical assumption are sufficient to compute A, V, ρ, and v_t. In all cases, $\rho_f = 1.293$ kg/m^3.

motion (see the force diagram in Fig. 1). Equation (1) typically requires a numerical solution to predict the motion of a projectile, or otherwise simplifying approximations must be made to find analytical solutions.[8,11,12] In the case of a purely vertical launch, the product $v\mathbf{v}$ can be simplified to v^2 because there is no horizontal component to consider (i.e., $\mathbf{v} = v$). Casting the problem in one dimension (the vertical direction) and noting that, in this case, the vector $\mathbf{g} = -g$, Eq. (1) becomes

$$\frac{dv}{dt} = -\frac{\rho_f AC}{2\rho V}v^2 - g. \tag{2}$$

If we introduce a terminal velocity $v_t = \sqrt{2\rho Vg/(\rho_f AC)}$, Eq. (2) becomes

$$\frac{dv}{dt} = -g\left(\frac{v^2}{v_t^2}+1\right). \tag{3}$$

Separating variables, we can pose the definite integral for the flight up to time t:

$$\int_{v_0}^{v}\frac{dv}{v^2/v_t^2+1} = -g\int_0^t dt, \tag{4}$$

where v_0 is the initial launch velocity at $t = 0$. Equation (4) can be solved to give

$$v_t\left[\tan^{-1}\left(\frac{v}{v_t}\right) - \tan^{-1}\left(\frac{v_0}{v_t}\right)\right] = -gt. \tag{5}$$

Rearranging for v, we find

$$v = v_t \tan\left[\tan^{-1}\left(\frac{v_0}{v_t}\right) - \frac{g}{v_t}t\right]. \tag{6}$$

The position of the projectile is found from the velocity by noting that $v = dy/dt$, where y is the vertical coordinate from the launch position. Therefore, we can separate variables a second time:

$$\int_0^y dy = v_t\int_0^t \tan\left[\tan^{-1}\left(\frac{v_0}{v_t}\right) - \frac{g}{v_t}t\right]dt. \tag{7}$$

The solution to Eq. (7) is then

$$y = \frac{v_t^2}{g}\left[\ln\left(\cos\left[\tan^{-1}\left(\frac{v_0}{v_t}\right) - \frac{g}{v_t}t\right]\right) - \ln\left(\cos\left[\tan^{-1}\left(\frac{v_0}{v_t}\right)\right]\right)\right], \tag{8}$$

which simplifies to

$$y = \frac{v_t^2}{g}\ln\left[\cos\left(\frac{g}{v_t}t\right) + \frac{v_0}{v_t}\sin\left(\frac{g}{v_t}t\right)\right] \tag{9}$$

and represents an analytical solution for the evolution of the vertical position with time.

If we now look for the maximum height, where $v = 0$ (i.e., at the zenith of the trajectory), then, by inspection of Eq. (6) (and by setting the left side to zero), we can find that this occurs at time t_H:

$$t_H = \frac{v_t}{g}\tan^{-1}\left(\frac{v_0}{v_t}\right), \tag{10}$$

and injecting this time into Eq. (9), we can solve for the maximum vertical position $y = H$ at $t = t_H$:

$$H = \frac{v_t^2}{2g}\ln\left(\frac{v_0^2}{v_t^2}+1\right). \tag{11}$$

Inspecting Eq. (11), it is interesting to note that if we introduce $x = v_0^2/v_t^2$, then we see that Eq. (11) involves the term $\ln(x + 1)$. When x is small (i.e., when v_t is large; or more accurately, when $v_t \gg v_0$), $\ln(x + 1) \rightarrow x$ so that Eq. (11) reduces to $H = v_0^2/(2g)$. This result for H is the well-known result for the maximum height of a projectile in the absence of air resistance (i.e., drag $\mathbf{F}_d = 0$; this result is discussed later).

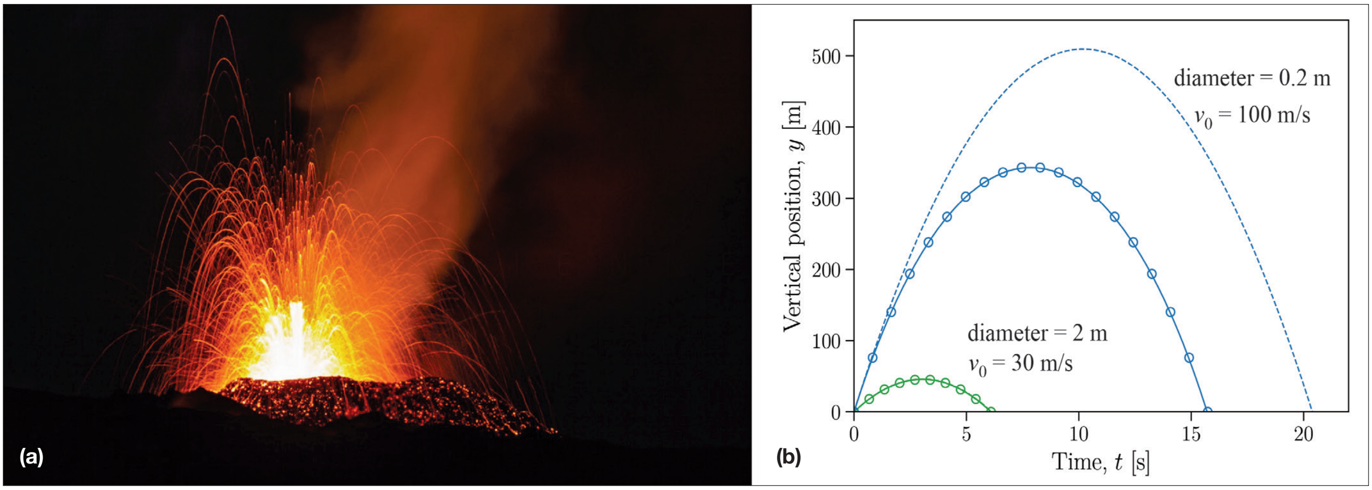

Fig. 2. An example of a volcanic application. (a) An eruption at Piton de la Fournaise (La Réunion) in which volcanic bombs are launched on projectile trajectories, often at near-vertical angles (photograph by Erland De Vienne, Feb. 16, 2017, available via a Creative Commons license[14]). Here a long-exposure shot exploits the glowing pyroclasts to record the projectile trajectories in a still image.[15] (b) The analytical, full, and no-drag solutions (as given in Fig. 1) for two possible volcanic bomb scenarios: in blue is a bomb with diameter 0.2 m launched at v_0 = 100 m/s, and in green is a bomb with diameter 2 m launched at v_0 = 30 m/s. In both cases, we take the density to be ρ = 1800 kg/m^3, and the range of launch velocities is inspired by direct observations.[16] For the latter scenario (i.e., v_0 = 30 m/s), all solutions coincide.

Discussion, application, and concluding remarks

Equation (9) shows us that height of the projectile will evolve with time in a manner that is unique for a given set of projectile attributes in v_t and a given launch velocity v_0. In Fig. 1, we show this evolution in a plot of $y(t)$ for scenarios in which those constants have different values. The maximum height H can be read from each $y(t)$ curve and compared with the predictions of Eqs. (10) and (11).

In grouping the projectile attributes into a constant v_t, we have neglected the fact that C is not strictly constant; in reality, C depends on the velocity of the projectile via a dependence on the Reynolds number Re. In this problem, the Reynolds number is $\mathrm{Re} = \rho_f Dv/\mu_f$, where D is the projectile diameter and μ_f is the fluid viscosity. Using numerical techniques[8] and a functional law relating C to Re (here we use $C = 24/\mathrm{Re} + 4/\sqrt{\mathrm{Re}} + 0.4$ after Clift et al.[13]), we can solve Eq. (1) accounting for both drag and variable C. In Fig. 1, we show these "full solutions" as a comparison with our analytical solution [Eq. (9)], and find excellent agreement. Note that in our analytical solution, we use the constant value C = 0.4, which equates to the constant, high-Re asymptote of the variable C equation used in the full solution (given that at low-to-intermediate Re, C varies substantially, there is no other appropriate value).

In Fig. 1, we also show the solution to Eq. (1) when drag forces are neglected altogether, which is the typical solution developed with students in entry-level quantitative mechanics courses. For vertical launch, this solution is $y = v_0 t - gt^2/2$ with the associated $H = v_0^2/(2g)$. Clearly, this simplest solution is only valid at relatively low launch velocities, and diverges from both the full solution and the analytical solution found here at higher relative velocities. This provides students with at least three solutions, each with different levels of simplification or approximation assumptions to discuss and compare for different projectile scenarios.

The analytical solution that we develop may be especially useful in the case of volcanic eruptions, where Eqs. (9) and (11) can be used to provide information (via fitting to videos of volcanic projectile trajectories) about v_0 and/or the projectile properties, which provide insights into the eruption dynamics that form them. In Fig. 2, we show an example solution for typical volcanic ejecta that are launched upward on projectile trajectories. By comparison with real data acquired via analysis of videos of real eruptions, the analytical solution found herein could be used quickly and easily to minimize for the "source parameters" such as the energy of the volcanic explosion that launches the bombs.

The analytical solution explored here represents a simple tool for analysis of the special case of vertical launch. We suggest that this (i) is a useful pedagogical derivation exercise to be used alongside other approaches to finding analytical solutions to projectile motion problems[11,12] and (ii) has specific application for some applied problems, including the motion of volcanic bombs in flight, which are often launched at only minor deviations from vertical[4,5] (see Fig. 2 for a motivating example image). Key to our motivation here has been to explore a problem that has simple solutions (e.g., projectile motion without drag) and to build up complexity for students, but without necessarily jumping to a full numerical solution. We believe that this approach of building complexity in mathematical methods can support better quantitative skill development for applied science students.[1,2]

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