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Global aspects of 3-form gauge theory: implications for axion-Yang-Mills systems

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Abstract: We investigate the proposition that axion-Yang-Mills systems are characterized by a 3-form gauge theory in the deep infrared regime. This hypothesis is rigorously examined by initially developing a systematic framework for analyzing 3-form gauge theory coupled to an axion, specifically focusing on its global properties. The theory consists of a BF term deformed by marginal and irrelevant operators and describes a network of vacua separated by domain walls converging at the junction of an axion string. It encompasses 0- and 3-form spontaneously broken global symmetries. Utilizing this framework, in conjunction with effective field theory techniques and 't Hooft anomaly-matching conditions, we argue that the 3-form gauge theory faithfully captures the infrared physics of the axion-Yang-Mills system. The ultraviolet theory is an SU(N) Yang-Mills theory endowed with a massless Dirac fermion coupled to a complex scalar and is characterized by chiral and genuine $\mathbb{Z}_m^{(1)}$ 1-form center symmetries, with a mixed anomaly between them. It features two scales: the vev of the complex scalar, v, and the strong-coupling scale, Λ, with $\Lambda \ll v$. Below v, the fermion decouples and a $\mathrm{U}(1)^{(2)}$ 2-form winding symmetry emerge, while the 1-form symmetry is enhanced to $\mathbb{Z}_N^{(1)}$. As we flow below Λ, matching the mixed anomaly necessitates introducing a dynamical 3-form gauge field of $\mathrm{U}(1)^{(2)}$, which appears as the incarnation of a long-range tail of the color field. The infrared theory possesses spontaneously broken chiral and emergent 3-form global symmetries. It passes several checks, among which: it displays the expected restructuring in the hadronic sector upon transition between the vacua, and it is consistent under the gauging of the genuine $\mathbb{Z}_m^{(1)} \subset \mathbb{Z}_N^{(1)}$ symmetry.

Keywords: Anomalies in Field and String Theories, Effective Field Theories, Global Symmetries

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Contents

1 Introduction

Three-form gauge theory is a fascinating topic that attracted attention since Lüscher's seminal work [1]. There, it was argued that the non-trivial topology of the 4-D Yang-Mills theory shows up in the infrared as an abelian long-range 3-form gauge field c_3. It does not correspond to a physical massless particle, i.e., a propagating degree of freedom. Nevertheless, it contributes to the theory's vacuum energy, i.e., cosmological constant. In this formulation, the CP-violating θ term can be written as

$$\theta \int_{\mathcal{M}_4} dc_3 \,, \tag{1.1}$$

where c_3 is given in terms of the nonabelian Chern-Simons current density: $c_3 = \mathrm{tr}\left[\frac{1}{3}\left(a_1^c\right)^3 + a_1^c \wedge da_1^c\right]/(8\pi^2)$, a_1^c is the color field, and $\mathcal{M}_4$ is the 4-D manifold. The correlator of the derivatives of two Chern-Simons current densities is the topological susceptibility χ of Yang-Mills theory, which develops a pole as the momentum vanishes, the Kogut-Susskind pole [2], corresponding to a pole of the c_3 correlator. This is also known as the Veneziano ghost [3] because the pole appears with the opposite sign compared to those of conventional particles, and it provides an alternative means (to Yang-Mills instantons) to solve the axial U(1) problem. One can write down an effective action that reproduces these findings [4]:

$$S_{\mathrm{IR}} = \frac{1}{2\chi} \int_{\mathcal{M}_4} |dc_3|^2 + \theta \int_{\mathcal{M}_4} dc_3 \,, \tag{1.2}$$

where a kinetic-energy term of c_3 is added to give the correct long-range interaction form of the c_3 correlator. The effective action (1.2) yields the vacuum energy [5]:

$$E_0(\theta) = \frac{\chi}{2} \mathrm{min}_k \left(\theta + 2\pi k\right)^2 \,, \quad k \in \mathbb{Z} \,. \tag{1.3}$$

This multi-branch function is periodic in θ and develops cusps at $\theta = \pi, 3\pi, \ldots$, a result derived by Witten in the large-N limit [6]. While this renowned result lends support to the validity of (1.2), it is important to emphasize that the effective description (1.2) is strictly derived in the large-N limit. See [5] for a review and [7–12] for further works on 3-form gauge theory, including supersymmetric versions.

When massless quarks are introduced, the θ term can be rotated away, and the theory restores its CP invariance, thus solving the strong CP problem. A similar effect can be achieved by introducing an axion a through the Peccei-Quinn mechanism [13]. This approach employs an anomalous global U(1) symmetry along with a complex scalar field whose phase corresponds to the axion. The complex scalar undergoes spontaneous symmetry breaking at a scale v, which is much higher than the strong-coupling scale Λ. Below Λ, Yang-Mills instantons generate an effective potential for $a + \theta$, which is minimized at a point in the field space restoring the CP symmetry. The same conclusion can be reached using the 3-form gauge theory, as demonstrated by Dvali in [14, 15]. In this framework, c_3 undergoes Higgsing when it absorbs the axion, resulting in c_3 becoming a short-range field. This mechanism can be elegantly seen in the Kalb-Ramond frame, where the axion is dualized to a 2-form gauge theory. This process effectively eliminates the second term in (1.2) and restores the CP invariance.

Since c_3 does not carry a physical degree of freedom, it is reasonable to question whether the 3-form gauge theory is essential for formulating Yang-Mills theory in the deep IR (without or with axions) or if it is merely a redundant description lacking true physical significance. This work offers a new perspective on the validity of the 3-form gauge theory beyond the large-N limit. Under certain conditions, we argue that the 3-form gauge theory is a faithful IR effective description in axion-Yang-Mills systems. This is achieved by examining the global symmetries of such systems as well as certain types of 't Hooft anomaly-matching conditions.

Our understanding of symmetries has undergone a conceptual paradigm shift over the past decade [16]; see [17] for a review and [18–22] for examples of works that discussed the 3-form symmetries from a modern perspective. In the contemporary paradigm, a p-form global symmetry in 4-D acts on a p-dimensional object and is generated by operators (symmetry defects) living on $(3 - p)$-dimensional topological manifolds. Gauging a global symmetry is performed by introducing a background field of the symmetry and including an arbitrary sum over inequivalent classes of this background in the path integral. Moreover, it has been realized recently that the concept of symmetry can be extended to include operations that lack inversions; see, e.g., [23–28] and the reviews [29, 30]. Our primary goal is to conduct a systematic study of the 3-form gauge theory and apply it to Yang-Mills theory coupled to matter fields and axions, with a focus on its global aspects. By utilizing newly developed mathematical tools, we can thoroughly examine the 3-form effective field theory of axion-Yang-Mills systems. This effective description successfully passes several consistency checks.

This work is divided into two main parts. The first part discusses the symmetry aspects of a general low-energy 3-form gauge theory coupled to axion, including the multi-field case. Starting from a low-energy Lagrangian exhibiting a $\mathrm{U}(1)^{(2)}$ 2-form along with $\mathrm{U}(1)^{(0)}$ 0-form global symmetries, we construct a 3-form gauge theory by gauging the former symmetry. The resulting theory is a topological quantum field theory (TQFT) of the BF type, modified by marginal and irrelevant operators, and describes a set of q domain walls (q is a free parameter) separating q distinct vacua and forming a junction at the locus of an axion string. Generally, the gauge theory exhibits $\mathbb{Z}_q^{(0)}$ 0-form and $\mathbb{Z}_q^{(3)}$ 3-form global symmetries, with a mixed anomaly between them valued in $\mathbb{Z}_q$. This anomaly is matched by breaking the two participating symmetries, leading to q distinct vacua separated by domain walls. Furthermore, we point out that the theory encompasses a gauged $\mathrm{U}(1)^{(-1)}$ (-1)-form symmetry, which undergoes spontaneous breaking (Higgsing), signifying that the vacuum energy has no contribution coming from c_3. We also construct the symmetry defects associated with these symmetries in two dual frames: the axion and the Kalb-Ramond frames. We demonstrate the action of such symmetries within an example of a domain-wall system. When the discrete 3-form global symmetry or a subgroup thereof, $\mathbb{Z}_p^{(3)} \subseteq \mathbb{Z}_q^{(3)}$, is gauged, we are left with only q/p distinct vacua.

In the second part of the paper, we apply this formalism to $\mathrm{SU}(N)$ Yang-Mills theory endowed with a single massless Dirac fermion in a representation $\mathcal{R}$ coupled to a neutral complex scalar field. In the UV, this theory exhibits both a $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ 0-form chiral and $\mathbb{Z}_m^{(1)}$ 1-form center symmetries, with a possible mixed anomaly between the two symmetries. Here, $T_\mathcal{R}$ is the Dynkin index of $\mathcal{R}$ and $m = \gcd(N, n)$, where n is the N-ality of $\mathcal{R}$. When matched below the strong-coupling scale, this non-vanishing mixed anomaly necessitates introducing a dynamical c_3. We summarize the idea here, with details provided in the main body of the paper.

Let v and Λ be the complex scalar vev and the Yang-Mills strong-coupling scales, respectively, and we take $\Lambda \ll v$. To see the $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)} - \mathbb{Z}_m^{(1)}$ mixed anomaly, we turn on a 2-form background field of $\mathbb{Z}_m^{(1)}$. This is achieved, as in [31–34], by first introducing a pair of 1-form and 2-form U(1) gauge fields (B_1^c, B_2^c) along with the constraint $mB_2^c = dB_1^c$. The field strength of the 1-form field satisfies the quantization condition $\int_{\mathcal{M}_2} dB_1^c \in 2\pi\mathbb{Z}$. Then the constraint implies that B_2^c has a vanishing field strength $dB_2^c = 0$, while its holonomy is fractional $\int_{\mathcal{M}_2} B_2^c \in \frac{2\pi\mathbb{Z}}{m}$. To couple the background field to fermions, in the second step, we enlarge $\mathrm{SU}(N)$ to $\mathrm{U}(N)$ and embed the $\mathbb{Z}_m^{(1)}$ background field into the $\mathrm{U}(1)^{(1)}$ 1-form symmetry of $\mathrm{U}(N)$ gauge theory. However, we must ensure that the enlargement from $\mathrm{SU}(N)$ to $\mathrm{U}(N)$ does not introduce new degrees of freedom, which can be done by postulating that the theory is invariant under an auxiliary 1-form gauge symmetry that acts simultaneously on the $\mathrm{U}(N)$ and (B_1^c, B_2^c) fields. As we flow to energy scales $\Lambda \ll E \ll v$, the magnitude of the complex scalar field freezes at v, while the winding of its phase (the axion) leads to the emergence of a $\mathrm{U}(1)^{(2)}$ 2-form symmetry that couples to the axion strings. Below Λ, the strong dynamics set in, leading to the confinement of the color field. Here, one is faced with a puzzle: the confinement of the color field means that one should no longer incorporate the nonabelian field in the calculations. If true, the low-energy theory is no longer invariant under the postulated auxiliary 1-form gauge symmetry, indicating that something is missing. The way out is to introduce the dynamical 3-form gauge field c_3 of $\mathrm{U}(1)^{(2)}$. The latter transforms non-trivially under the auxiliary gauge symmetry, ensuring the full low-energy effective field theory (in the background of the $\mathbb{Z}_m^{(1)}$ flux) is invariant under this auxiliary symmetry. Moreover, the IR theory reproduces the mixed anomaly, which is an important check on the analysis since anomalies are all-scale phenomena. In this regard, c_3 can be thought of as the long-range tail of the nonabelian dynamics. Even though it does not carry a physical degree of freedom, its presence is essential for the consistency of the theory deep in the IR.

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Below the scale, v, the fermions become massive, with a mass of order v, and decouple, leading to the enhancement of $\mathbb{Z}_m^{(1)}$ to $\mathbb{Z}_N^{(1)}$ 1-form symmetry. The groups $\mathrm{U}(1)^{(2)}$ and $\mathbb{Z}_N^{(1)}$ constitute a higher-group structure, where the former is the parent and the latter is the daughter symmetries. One may gauge the parent without gauging the daughter, but not conversely. As we flow below Λ, we may freely gauge $\mathrm{U}(1)^{(2)}$ and introduce the dynamical c_3 without worrying about $\mathbb{Z}_N^{(1)}$. The latter stays an enhanced symmetry below Λ.

One of our main results is the IR effective field theory at energy scale $E \ll \Lambda$ given by eq. (5.12), which we display here for convenience:

$$\mathcal{L}_{E\ll\Lambda} = \frac{v^2}{2} da \wedge \star da + \frac{T_{\mathcal{R}} a}{2\pi}\left(dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c\right) + \Lambda^4 \mathcal{K}\left(\frac{dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c}{\Lambda^4}\right), \tag{1.4}$$

where $\mathcal{K}$ is the kinetic energy term of c_3, and the background of $\mathbb{Z}_m^{(1)}$ is activated. This theory exhibits $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)} \times \mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ global symmetries. Dynamically, the IR theory forms axion domain walls, separating $T_{\mathcal{R}}$ distinct minima and breaking $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)}$ and $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ maximally. The enhanced $\mathbb{Z}_N^{(1)}$ symmetry is explicitly broken by higher-order operators down to the genuine $\mathbb{Z}_m^{(1)}$ symmetry, which remains intact.

The field strength of the 3-form gauge field satisfies the quantization condition $\int_{\mathcal{M}_4} dc_3 = 2\pi m$, where m is an integer equivalent to the topological charge of the Yang-Mills instantons. The full partition function of the IR theory (at energy scale $E \ll \Lambda$) includes a sum over all integers m. We may integrate out c_3, and using the Poisson resummation formula, we obtain the Euclidean partition function:

$$Z[a] \sim \sum_{k\in\mathbb{Z}} \exp\left[-i\frac{kN}{4\pi}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c\right] \exp\left[-\int_{\mathcal{M}_4} \frac{v^2}{2} da \wedge \star da + \frac{\Lambda^4}{8\pi^2}\left(T_{\mathcal{R}} a + 2\pi k\right)^2\right]. \tag{1.5}$$

This partition function reproduces the chiral-center anomaly upon shifting $a \to a + \frac{2\pi}{T_{\mathcal{R}}}$. It also displays an infinite number of vacua, with the true vacuum energy given by

$$V(a) \sim \Lambda^4 \mathrm{min}_k \left(T_{\mathcal{R}} a + 2\pi k\right)^2 . \tag{1.6}$$

The potential $V(a)$ has $T_{\mathcal{R}}$ minima at $2\pi\ell/T_{\mathcal{R}}$, $\ell = 0, 1, \ldots, T_{\mathcal{R}} - 1$, as well as cusps at $a = \pi(2\ell+1)/T_{\mathcal{R}}$, reflecting two facts. First, the cusps indicate that additional degrees of freedom, not accounted for by $V(a)$, are sandwiched between the true minima of the theory. These are the hadronic walls, which are very thin compared to the thickness of the axion domain walls [35, 36]. Second, a restructuring in the hadronic sector occurs as one goes between one minimum and the other. These results are consistent with the large-N limit (1.3).

There exists a higher group structure between $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ and the enhanced $\mathbb{Z}_N^{(1)}$ symmetries. However, this structure trivialises for the genuine $\mathbb{Z}_m^{(1)} \subset \mathbb{Z}_N^{(1)}$ symmetry. This means we can gauge $\mathbb{Z}_m^{(1)}$ without worrying about $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$. Gauging the former gives $\mathrm{SU}(N)/\mathbb{Z}_m$ theory. This theory still exhibits a spontaneously broken IR $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ symmetry due to the formation of domain walls. However, the chiral symmetry $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)}$ becomes noninvertible. This results in dressing the domain walls with an IR TQFT. This intricate structure works as a consistency check on the adequacy of using the 3-form gauge theory to describe the axion-Yang-Mills systems' IR physics.

This paper is organized as follows. In section 2, we set the stage by considering the field theory of a compact scalar, which possesses two global symmetries: shift and winding symmetries. The theory encounters a mixed 't Hooft anomaly, and thus, the shift symmetry breaks into a discrete group upon gauging the winding symmetry. Next, we couple the gauge field of the winding symmetry, the 3-form gauge field, to the compact scalar and analyze the resulting theory in great detail: we identify the global symmetries, their mixed anomalies, and the noninvertible symmetries within this theory. Section 3 is devoted to studying the compact scalar in the dual frame, the Kalb-Ramond gauge theory, while section 4 generalizes these findings to two or more 3-form gauge fields. In section 5, we use the machinery built in the previous sections to examine our proposal that the deep IR regime of the axion-Yang-Mills systems is described by a 3-form gauge theory and apply various checks to this proposal. Finally, we conclude in section 6 with a brief discussion.

In this paper, dynamical fields (summed over in the path integral) are denoted by lowercase letters, while background fields use uppercase. We primarily use differential forms with subscripts indicating the form's degree.

2 The Axion theory

We consider the 4-D theory of a 2π-periodic scalar field a, the axion, i.e., we identify $a(\mathcal{P}) \equiv a(\mathcal{P}) + 2\pi\mathbb{Z}$ at the spacetime point $\mathcal{P}$. The basic Lagrangian is

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da \,, \tag{2.1}$$

where v is a constant of mass dimension 1. When coupling the axion to a Yang-Mills theory via the Peccei-Quinn mechanism, v is the axion's symmetry-breaking scale. The Lagrangian (2.1) has a global $\mathrm{U}(1)^{(0)}$ 0-form shift symmetry acting on the axion as $a \to a + \alpha$, where α is a constant. The corresponding Noether's 1-form current is

$$j_1 = v^2 da \,, \tag{2.2}$$

which is conserved thanks to the equation of motion:

$$d \star j_1^{(0)} = v^2 d \star da = 0 \,. \tag{2.3}$$

The topological symmetry generator (symmetry defect) enacting this transformation is defined on a closed co-dimension-1 manifold $\mathcal{M}_3$:

$$U_\alpha^{(0)}(\mathcal{M}_3) = e^{i\alpha \int_{\mathcal{M}_3} v^2 \star da} \,. \tag{2.4}$$

The superscript emphasizes that the operator implements the action of a 0-form symmetry. We can take $U_\alpha^{(0)}(\mathcal{M}_3)$ to surround the local operator

$$V(\mathcal{M}_0 = \mathcal{P}) = e^{ia(\mathcal{P})} \,, \tag{2.5}$$

and then topologically deforming $U_\alpha^{(0)}(\mathcal{M}_3)$ past V to find

$$U_\alpha^{(0)}(\mathcal{M}_3) V(\mathcal{M}_0) = e^{i\alpha} V(\mathcal{M}_0) \,. \tag{2.6}$$

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The axion theory is also endowed with a $\mathrm{U}(1)^{(2)}$ 2-form global symmetry with a corresponding 3-form current

$$j_3 = \star da\,, \tag{2.7}$$

which is conserved because of the Bianchi identity:

$$d\star j_3 = d^2 a = 0\,. \tag{2.8}$$

We can also define the symmetry defect of the $\mathrm{U}(1)^{(2)}$ 2-form symmetry by integrating the Hodge-dual of j_3 on a co-dimension-3 manifold $\mathcal{M}_1$ as:

$$U_\beta^{(2)}(\mathcal{M}_1) = e^{i\beta\int_{\mathcal{M}_1} da}\,. \tag{2.9}$$

This symmetry defect acts on the 2-dimensional axion-string worldsheet $\mathcal{M}_2$ [19]. Let $V(\mathcal{M}_2)$ be the axion-string Wilson surface, which has no local description[1] in terms of the axion field a. Deforming the symmetry defect $U_\beta^{(2)}(\mathcal{M}_1)$ past $V(\mathcal{M}_2)$ transforms the latter by a phase:

$$U_\beta^{(2)}(\mathcal{M}_1)V(\mathcal{M}_2) = e^{i\beta \mathrm{Link}(\mathcal{M}_1,\mathcal{M}_2)}V(\mathcal{M}_2)\,, \tag{2.10}$$

where $\mathrm{Link}(\mathcal{M}_1,\mathcal{M}_2)$ is the linking number between the two manifolds.

There is a mixed 't Hooft anomaly between $\mathrm{U}(1)^{(0)}$ and $\mathrm{U}(1)^{(2)}$ symmetries. To see it, we examine the commutation relation between the symmetry defects $U_\alpha^{(0)}(\mathcal{M}_3)$ and $U_\beta^{(2)}(\mathcal{M}_1)$. One way to perform the calculations is by foliating $\mathcal{M}_4$ into constant time slices and orienting both $\mathcal{M}_3$ and $\mathcal{M}_1$ to be time-like surfaces:

$$U_\alpha^{(0)}(\mathcal{M}_3(t)) = e^{i\alpha v^2\int_{\mathcal{M}_3(t)} d^3x\partial_0 a(x,t)}\,,\quad U_\beta^{(2)}(\mathcal{M}_1(t)) = e^{i\beta\int_{\mathcal{M}_1(t)}\partial_i a dx^i}\,, \tag{2.11}$$

where $i\in\{1,2,3\}$. Then, using the equal-time commutation relation $[a(\boldsymbol{x},t),\Pi_a(\boldsymbol{y},t)] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, where $\Pi_a = v^2\partial_0 a$, and employing the Baker-Campbell-Hausdorff formula, we find

$$U_\alpha^{(0)}(\mathcal{M}_3(t))U_\beta^{(2)}(\mathcal{M}_1(t)) = e^{-i\alpha\beta}U_\beta^{(2)}(\mathcal{M}_1(t))U_\alpha^{(0)}(\mathcal{M}_3(t))\,. \tag{2.12}$$

The phase manifests the mixed anomaly: one cannot move the symmetry defects freely without encountering non-trivial phases. The anomaly implies that gauging one symmetry breaks the other into, at most, a discrete subgroup.

2.1 Gauging the $\mathrm{U}(1)^{(2)}$ 2-form symmetry: 3-form gauge theory and domain walls

Now, we gauge the $\mathrm{U}(1)^{(2)}$ symmetry, meaning that we introduce the 3-form gauge field c_3 of the 2-form symmetry and perform the path integral over c_3. We couple c_3 to its current j_3 by adding the BF term

$$\frac{q}{2\pi}\star j_3\wedge c_3 = \frac{q}{2\pi}da\wedge c_3 \tag{2.13}$$

[1]It is, however, possible to define an operator $e^{i\int_{\Sigma_3} v^2\star da}$ on an open surface Σ_3, with the string positioned at $\Sigma_2 = \partial\Sigma_3$, the boundary of Σ_3. This approach mirrors the implicit definition of 't Hooft lines in earlier formulations. A direct definition of the Wilson surface operator as an integral over a closed 2-dimensional surface will be given in section 3 using the dual Kalb-Ramnod field.

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to the Lagrangian (2.1). We introduced the positive integer $q \in \mathbb{N}$ as a free parameter of the theory, and its physical significance will be apparent below. The gauge field c_3 transforms as $c_3 \to c_3 + d\lambda_2$ under the $\mathrm{U}(1)^{(2)}$ gauge transformation, and via integration by parts, we see that the new term (2.13) is invariant under this transformation. The field strength of c_3 is $f_4 = dc_3$, and it satisfies the quantization condition:

$$\int_{\mathcal{M}_4} f_4 \in 2\pi\mathbb{Z} \tag{2.14}$$

on a closed $\mathcal{M}_4$. The consistency of the theory under $\mathrm{U}(1)^{(2)}$ large gauge transformations implies that $d\lambda_2$ satisfies the condition

$$\int_{\mathcal{M}_3} d\lambda_2 \in 2\pi\mathbb{Z}\,. \tag{2.15}$$

In addition, since c_3 is a dynamical field,[2] we can include a kinetic energy term $\mathcal{K}$ for c_3. The total Lagrangian is:[3]

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{q}{2\pi} da \wedge c_3 + \Lambda^4 \mathcal{K}\left(\frac{dc_3}{\Lambda^4}\right)\,, \tag{2.16}$$

and we introduced the new scale Λ. More on $\mathcal{K}$ will be discussed momentarily. The presence of the c_3 field reduces the $\mathrm{U}(1)^{(0)}$ symmetry to a $\mathbb{Z}_q^{(0)} \subset \mathrm{U}(1)^{(0)}$ symmetry; the Lagrangian (2.16) is only invariant under the shift $a \to a + 2\pi/q$. This demonstrates the earlier assertion that the mixed anomaly between the $\mathrm{U}(1)^{(2)}$ and $\mathrm{U}(1)^{(0)}$ symmetries leads to the latter being broken into a discrete subgroup when the former is gauged.[4] The current conservation law (2.3) of the 0-form symmetry is modified to:

$$v^2 d \star da - \frac{q}{2\pi} dc_3 = 0\,. \tag{2.17}$$

The corresponding $\mathbb{Z}_q^{(0)}$ symmetry defect is topological only when we include this combination of fields - (2.4) is modified to

$$U_\alpha^{(0)}(\mathcal{M}_3) = e^{i\alpha \int_{\mathcal{M}_3}\left(v^2 \star da - \frac{q}{2\pi} c_3\right)}\,. \tag{2.18}$$

Using the quantization condition on $d\lambda_2$, i.e., $\int_{\mathcal{M}_3} d\lambda_2 \in 2\pi\mathbb{Z}$, we readily see that $U_\alpha^{(0)}(\mathcal{M}_3)$ is gauge-invariant under a $\mathrm{U}(1)^{(2)}$ gauge transformation if and only if $\alpha = 2\pi\ell/q, \ell \in \mathbb{Z}$. This seconds the above assertion that introducing the term (2.13) reduces the $\mathrm{U}(1)^{(0)}$ symmetry down to a $\mathbb{Z}_q^{(0)}$ subgroup.

Next, we focus on $\mathcal{K}$, the kinetic energy term of the 3-form gauge field. In the following, it will be helpful to write and analyze the Lagrangian (2.16) in index notation:[5]

$$\mathcal{L} = \frac{v^2}{2} \partial_\mu a\, \partial^\mu a - \frac{q}{2\pi}\frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu a)\, c_{\nu\rho\sigma} + \Lambda^4 \mathcal{K}\,, \tag{2.19}$$

[2]The 3-form gauge field has mass dimension 3.

[3]In a manifold with boundary, we also need to consider a boundary term so that the variation of the kinetic term at the boundary vanishes; see, e.g., [11] and references therein. We do not run into this subtlety in this work.

[4]As we shall see, this mixed anomaly is the IR incarnation of the axial-color ABJ anomaly in an axion-Yang-Mills UV complete theory.

[5]Translating from the d-forms to the index notation, it helps to remember that we are working in Minkowski space, with metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, such that $\star\star = -1$ for even forms and $\star\star = +1$ for odd forms.

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where $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and the Greek indices run over $0, 1, 2, 3$. The canonical (quadratic) form of the kinetic energy term is:

$$\mathcal{K}_{\text{can}} = -\frac{1}{2 \cdot 4! \Lambda^8} f^{\mu\nu\alpha\beta} f_{\mu\nu\alpha\beta} \,, \tag{2.20}$$

where $f_{\mu\nu\alpha\beta} = \partial_\mu c_{\nu\alpha\beta} - \partial_\nu c_{\mu\alpha\beta} + \partial_\alpha c_{\mu\nu\beta} - \partial_\beta c_{\mu\nu\alpha}$. Since $f_{\mu\nu\alpha\beta}$ is totally anti-symmetric in the 4 indices, we can always write it as

$$f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} f(x) \,, \tag{2.21}$$

for some scalar function $f(x)$. Therfore, $\mathcal{K}_{\text{can}}$ takes the simple form[6]

$$\mathcal{K}_{\text{can}} = \frac{f^2(x)}{2\Lambda^8} \,. \tag{2.22}$$

The mathematical statement (2.21) is equivalent to saying that the free 3-form field c_3 does not carry propagating degrees of freedom. To see that, use the canonical kinetic term of c_3, ignore the axion in (2.19), and vary the Lagrangian with respect to $c_{\mu\nu\alpha}$ to find

$$\partial_\mu f^{\mu\nu\alpha\beta} = 0 \,, \tag{2.23}$$

which admits the general solution $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} f$, where f, in this case, is a constant. The constant field strength of a 3-form gauge field carries no propagating degrees of freedom, much like free electrodynamics in 2-D. In the absence of a, the 4-form field f_4 is a cosmological constant, which is easily seen by substituting $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} f$ into the Lagrangian.[7]

From the perspective of effective field theory, the kinetic energy term can take a more generalized form, with $\mathcal{K}$ represented as a polynomial in f:

$$\mathcal{K}\left(\frac{f}{\Lambda^4}\right) = \theta \frac{f}{\Lambda^4} + \frac{f^2}{2\Lambda^8} + c' \frac{f^4}{\Lambda^{16}} + \dots \,, \tag{2.24}$$

where θ and c' are some real parameters.The first term is topological, and since $\int_{\mathcal{M}^4} f_4 = \int_{\mathcal{M}_4} d^4x f \in 2\pi\mathbb{Z}$, the theory is invariant under the shift $\theta \to \theta + 2\pi$, and thus, θ is 2π-periodic. This term breaks the CP invariance unless $\theta = \{0, \pi\}$. However, θ can be rotated away by combining $\theta \frac{f}{\Lambda^4}$ with the second term in (2.19), after integrating by parts and shifting $qa \to qa - \theta$. Other possible higher-order kinetic energy terms may involve higher derivatives of f_4. Yet, these terms may be plagued with ghosts [12], and thus, we ignore such terms in our construction.

By rescaling the fields a and f as $a \to av$ and $f \to \Lambda f$, we observe that the Lagrangian (2.19) manifests as a BF theory deformed by marginal terms (the canonical kinetic terms of a and f) and by irrelevant operators (the higher-order terms of f). The full quantum theory guarantees that the system has q degenerate ground states.[8] To see that, we

[6] We used $\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\alpha\beta} = -4!$. We can also express f in terms of $f_{\mu\nu\alpha\beta}$ as $f = \frac{1}{4!}\epsilon^{\mu\nu\alpha\beta} f_{\mu\nu\alpha\beta}$.

[7] However, a subtle issue arises because the cosmological constant obtained in this manner does not align with the correct value derived by varying (2.19) with respect to the metric tensor [12]. This discrepancy can be resolved by including boundary terms in (2.19). Nonetheless, to obtain the correct form of the energy-momentum tensor and, consequently, the cosmological constant, we will rely on the variation of the action with respect to the metric tensor, disregarding boundary terms.

[8] We thank T. Sulejmanpasic and T. Tanizaki for discussions about this point.

may want to find an effective axion potential $V_{\text{eff}}(a)$ by integrating out f_4 and imposing the constraint (2.14). To streamline the analysis, we proceed by disregarding the axion kinetic energy term in (2.16), which is a good approximation assuming $\Lambda \ll v$. Then, the Euclidean partition function reads:

$$Z[a] = \int [Dc_3] \sum_{m\in\mathbb{Z}} \delta\left(2\pi m - \int_{\mathcal{M}_4} f_4\right) e^{-S_E}\,, \tag{2.25}$$

and

$$S_E = -\int_{\mathcal{M}_4} \Lambda^4 \mathcal{K}\left(i f_4/\Lambda^4\right) + i\frac{q}{2\pi} a \wedge f_4\,. \tag{2.26}$$

We can further simplify our analysis by recalling that f_4 does not carry propagating degrees of freedom and can be expressed by eq. (2.21). We take $\mathcal{M}_4$ to be a closed manifold, and therefore, we have $\int_{\mathcal{M}_4} f_4 = \int dV_{\mathcal{M}_4} f(x) = f_0 V_{\mathcal{M}_4}$, where f_0 is the zero mode of $f(x)$ and $V_{\mathcal{M}_4}$ is the 4-volume of $\mathcal{M}_4$, and we assume $\Lambda^4 V_{\mathcal{M}_4} \gg 1$. Focusing only on the zero modes of f and a, we find

$$Z[a] \sim \left[\sum_{m\geq 0} e^{\Lambda^4 V_{\mathcal{M}_4}\mathcal{K}(2\pi i m)} \cos(m q a_0)\right] \times \text{higher modes of}\, a\,, \tag{2.27}$$

where a_0 is the zero mode of a. The higher modes of a are suppressed by inverse powers of $\Lambda^4 V_{\mathcal{M}_4}$ and can be neglected deep in the IR. The effective potential is defined via $V_{\text{eff}}(a) = -V_{\mathcal{M}_4}^{-1} \log Z[a]$, and thus, $V_{\text{eff}}(a_0) \sim \Lambda^4 \mathcal{F}(q a_0)$, where $\mathcal{F}$ is a periodic function with period $2\pi/q$. We conclude that integrating out f_4 yields a periodic potential for the axion that respects the $\mathbb{Z}_q^{(0)}$ shift symmetry, as it should be. Minimizing $V_{\text{eff}}(a_0)$ yields q-degenerate ground states connected via domian walls.

An alternative way to perform the path integral in (2.25) is to use the Poisson resummation formula $\sum_{m\in\mathbb{Z}} \delta\left(2\pi m - \int_{\mathcal{M}_4} f_4\right) = \sum_{k\in\mathbb{Z}} e^{-ik\int_{\mathcal{M}_4} f_4}$. Taking $\mathcal{K}$ in the canonical form $\mathcal{K}_{\text{can}} = f_4^2/(2\Lambda^2)$, focusing on the zero modes, and performing the Gaussian integral, we obtain

$$Z[a] \sim \sum_{k\in\mathbb{Z}} e^{-\frac{\Lambda^4 V_{\mathcal{M}_4}}{8\pi^2}(qa+2\pi k)^2}\,, \tag{2.28}$$

which, again, is a periodic function that respects the $\mathbb{Z}_q^{(0)}$ shift symmetry. The result (2.28) is remarkable. This partition function displays an infinite number of vacua, most of which are false. The true vacuum energy is given by

$$V(a) \sim \Lambda^4 \text{min}_k \left(qa + 2\pi k\right)^2\,. \tag{2.29}$$

This potential has q distinct minima, with cusps at $a = \pi/q$, $3\pi/q$, etc. These cusps appear only upon including the infinite sum over m in (2.27) and performing the Poisson resummation formula. In other words, the cusps are a feature of the full quantum theory. This observation will have far-reaching consequences in axion-Yang-Mills systems.

Let us return to the Lagrangian (2.19) and study its classical aspects. Varying it with respect to $c_{\nu\rho\sigma}$, assuming the general form of $\mathcal{K}$, yields the equation of motion of the 3-form field:

$$\frac{q}{2\pi}\partial_\mu a = -\Lambda^4 \partial_\mu \mathcal{K}'\left(\frac{f}{\Lambda^4}\right), \tag{2.30}$$

where the prime denotes the derivative with respect to the argument of $\mathcal{K}$. Equation (2.30) is pivotal to our subsequent analysis. Integrating once, we obtain

$$\frac{q}{2\pi}(a-\tilde{a}_0) = -\Lambda^4 \mathcal{K}'\left(\frac{f}{\Lambda^4}\right), \tag{2.31}$$

for some integration constant $\tilde{a}_0$. Assuming $\mathcal{K}'$ is invertible, we can rearrange the equation of f:

$$f = \Lambda^4 (\mathcal{K}')^{-1}\left(\frac{q}{2\pi\Lambda^4}(\tilde{a}_0 - a)\right). \tag{2.32}$$

From (2.19), the equation of motion for the axion a is

$$v^2 \partial_\mu \partial^\mu a - \frac{q}{2\pi} f = 0\,. \tag{2.33}$$

Here, f acts as the derivative of a classical effective potential for the axion field: in the presence of a classical effective potential for the axion, the equation of motion is $v^2\partial_\mu\partial^\mu a + \frac{\partial V_{\text{cl-eff}}(a)}{\partial a} = 0$. This means we can set $\frac{q}{2\pi} f = -\frac{\partial V_{\text{cl-eff}}(a)}{\partial a}$, and using (2.32), we conclude

$$\frac{\partial V_{\text{cl-eff}}(a)}{\partial a} = -\frac{q}{2\pi} f = -\frac{q}{2\pi}\Lambda^4 (\mathcal{K}')^{-1}\left(\frac{q}{2\pi\Lambda^4}(\tilde{a}_0 - a)\right). \tag{2.34}$$

Then, one can integrate (2.34) to obtain an expression of $V_{\text{cl-eff}}(a)$. Unlike the effective potential obtained from the full partition function, the classical effective potential does not need to yield q degenerate ground states. The form of the classical effective potential depends on the kinetic energy term for c_3. To elucidate this point, we consider two examples of the resulting $V_{\text{cl-eff}}(a)$: the quadratic and the cosine potentials. The corresponding kinetic energy functions are [14]:

$$V^{\text{quadratic}}_{\text{cl-eff}}(a) = \frac{\Lambda^4}{2}(a-\tilde{a}_0)^2 \iff \mathcal{K}_{\text{can}}\left(\frac{f}{\Lambda^4}\right) = \frac{q^2 f^2}{2\Lambda^8}, \tag{2.35}$$

and

$$\begin{aligned} V^{\cos}_{\text{cl-eff}}(a) &= \Lambda^4(1-\cos(n(a-\tilde{a}_0))) \iff \\ \mathcal{K}_{\cos}\left(\frac{f}{\Lambda^4}\right) &= -1 + \sqrt{1-\left(\frac{qf}{2\pi n\Lambda^4}\right)^2} + \frac{qf}{2\pi n \Lambda^4}\arcsin\left(\frac{qf}{2\pi n\Lambda^4}\right). \end{aligned} \tag{2.36}$$

The integration constant in $\mathcal{K}$ is chosen such that $\mathcal{K}\left(\frac{f}{\Lambda^4}=0\right)$=0.

The kinetic energy term (2.36) is designed to produce $V^{\cos}_{\text{cl-eff}}(a)$, and when $n \in q\mathbb{N}$, it exhibits multiple-of-q minima. These minima are located at values of a satisfying $\frac{\partial V^{\cos}_{\text{cl-eff}}(a)}{\partial a} = 0$, and from eq. (2.34) we see that f vanishes there; the 3-form gauge field c_3 is gapped at these minima. Expanding $\mathcal{K}_{\cos}\left(\frac{f}{\Lambda^4}\right)$ to the leading order in f about one of the minima results, up

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to a proportionality constant, in the canonical kinetic energy term (2.35). This, however, does not imply that the canonical kinetic energy term fails to produce q-degenerate ground states. As previously discussed, regardless of the form of $\mathcal{K}$, the full partition function of the Lagrangian (2.16), incorporating the quantization condition (2.14), will always lead to q-fold degeneracy in the deep IR regime. Nonetheless, $\mathcal{K}_{\cos}\left(\frac{f}{\Lambda^4}\right)$ proves invaluable as it facilitates connections with textbook examples of axion domain walls. One can think of it as a UV completion of the canonical kinetic energy.

Domin walls. In the following, we proceed to discuss the classical domain wall solutions in the $\mathrm{U}(1)^{(2)}$ gauge theory. We shall use the effective cosine potential in (2.36) to carry out our analysis. However, this section's conclusions also hold for arbitrary potential, i.e., for arbitrary forms of $\mathcal{K}$.

The cosine potential yields n vacua $a_\ell = \frac{2\pi\ell}{n}$, where $\ell = 0, 1, \ldots, n-1$. There are n domain walls separating the adjacent vacua with a kink-like profile given by (here, we may set $\tilde{a}_0 = 0$):

$$a(z) = \frac{2\pi\ell}{n} + \frac{4}{n}\arctan\left(e^{nm_a z}\right), \quad -\infty < z < \infty, \tag{2.37}$$

and $m_a = \frac{\Lambda^2}{v}$ is related to the axion mass (the actual mass of the axion is obtained after using the canonical kinetic term, which yields the mass $\frac{n\Lambda^2}{v}$). We assumed the walls are space-filling in the x and y directions with a profile along the z-direction, taking $\mathcal{M}_4 = \mathbb{R}^4$ for simplicity. We also assumed that the walls are separated by distances much larger than their width $\sim m_a^{-1}$. In the following, the statement $z \to \pm\infty$ means that $|z| \gg m_a^{-1}$ but still away from the adjacent walls. We observe that the reality of the kinetic energy term $\mathcal{K}$, see (2.36), implies the inequality $|f| \leq 2\pi n\Lambda^4/q$. The value of $\mathcal{K}$ attains its minimum value, $\mathcal{K} = 1$, at $f = 0$, while it is maximized at $|f| = 2\pi n\Lambda^4/q$. In addition, the first equality in (2.34) yields:

$$f(z) = -\frac{2\pi n}{q}\Lambda^4 \sin\left(na(z)\right) = -\frac{2\pi n}{q}\Lambda^4 \sin\left[4\arctan\left(e^{nm_a z}\right)\right]. \tag{2.38}$$

The theory has a $\mathbb{Z}_q^{(0)}$ 0-form symmetry that acts as $a \to a + \frac{2\pi}{q}$. The invariance of the cosine potential under the 0-form symmetry demands that

$$\frac{n}{q} \in \mathbb{N}, \tag{2.39}$$

and thus $n \geq q$. Later, we shall discuss that a stack of n domain walls intersects at the locus of an axion string carrying a charge q under c_3. It is more energetically favorable for the charge-q string to support only $n = q$ domain walls. Nevertheless, we maintain the generality of n and q in our subsequent discussion[9].

In the following, the derivatives of $\mathcal{K}$ will prove useful for our study:

$$\mathcal{K}' = \frac{q}{2\pi n\Lambda^4}\arcsin\left(\frac{qf}{2\pi n\Lambda^4}\right), \quad \mathcal{K}'' = \frac{q^2}{(2\pi n\Lambda^4)^2\sqrt{1-\left(\frac{qf}{2\pi n\Lambda^4}\right)^2}}. \tag{2.40}$$

[9] For example, we could have a kinetic energy term $\mathcal{K}$ that corresponds to a more general form of the effective potential $V_{\text{cl-eff}}(a) = \sum_{m\geq 1}\Lambda_m^4\left(1-\cos(mqa)\right)$.

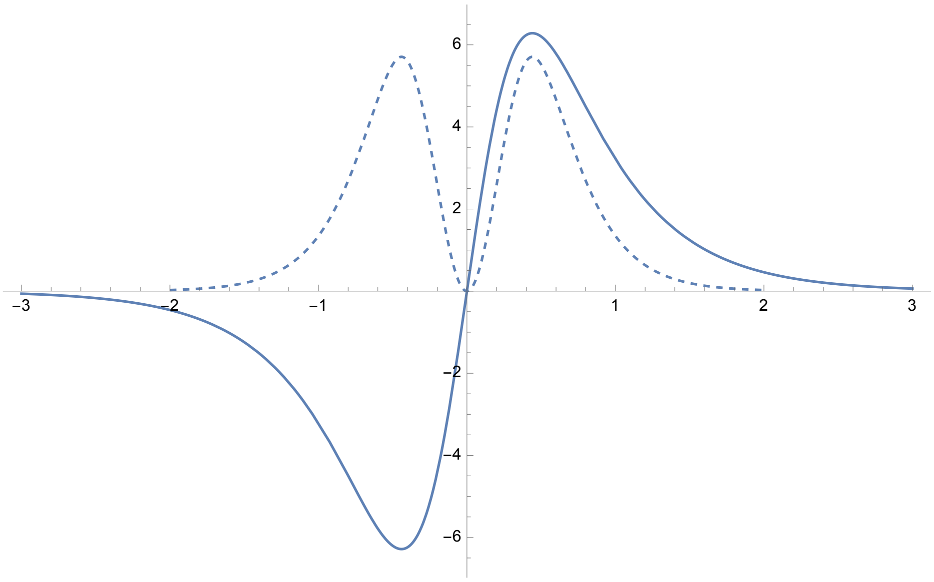

Figure 1. The profiles of f (solid line) and $\mathcal{K}_{\cos}$ (dashed line) as functions of the coordinate z (not to scale). We take $q = n = 2$ and set $\Lambda = 1$.

Also, two important limiting behaviours of $f(z)$ are worth noticing:

$$\begin{aligned} f(z \to 0) &= \frac{4\pi n^2}{q} z \Lambda^4 \,, \\ f(z \to \pm\infty) &= \frac{8\pi n}{q} \Lambda^4 e^{-n m_a |z|} \text{sign}(z) \,. \end{aligned} \tag{2.41}$$

At the wall core, $z = 0$, we find that f attains its mimimum value $f(0) = 0$, where $\mathcal{K}$ is also minimized, while in the vacum $z \to \pm\infty$, we similarly have $f(\pm\infty) = 0$, where again $\mathcal{K}$ is minimized. However, there exists a distance inside the wall, $\pm|z_m|$, at which $|f(z_m)| = \frac{2\pi n}{q}\Lambda^4$, i.e., it is maximized:

$$m_a |z_m| = \frac{1}{n} \log \tan\left(\frac{\pi}{8}\right) \,. \tag{2.42}$$

At $|z_m|$, the kinetic energy $\mathcal{K}$ is also maximized. Thus, $|f(z)|$ monotonically increases from the core of the domain wall until it reaches z_m, after which it starts decreasing exponentially; see figure 1. The exponential decay observed in $f(z)$ is a defining trait of a gapped system. In this scenario, the 4-form field f_4 eats the axion, resulting in its acquisition of mass. This is also evident from

$$\int_{\mathbb{R}^4} f_4 = 0 \,, \tag{2.43}$$

a result consistent with the quantization condition (2.14). In contrast, in a gapless system, $|f(z)|$ would remain at the constant value of $\frac{2\pi n}{q}\Lambda^4$ from $\pm z_m$ to infinity. Below, we shall show that $\frac{q}{2\pi}\left(a(\infty) - a(-\infty)\right) = q/n$ is the domain wall charge under a $\mathbb{Z}_q^{(3)}$ 3-form global symmetry. Thus, the stack of n domain walls carries a total charge of q under the 3-form symmetry.

2.2 The 3- and (−1)-form symmetries, and their anomalies

In this section, we show that the pure $\mathrm{U}(1)^{(2)}$ gauge theory possesses a $\mathrm{U}(1)^{(3)}$ 3-form global symmetry, which undergoes a breakdown into a $\mathbb{Z}_q^{(3)}$ symmetry in the presence of charge-q matter. In addition, the theory is endowed with a gauged (−1)-form symmetry that can be used to diagnose the existence of a cosmological constant. We further demonstrate that there is a mixed 't Hooft anomaly between $\mathbb{Z}_q^{(3)}$ and $\mathbb{Z}_q^{(0)}$ symmetries. Gauging the former gives rise to a $\mathrm{U}(1)^{(2)}/\mathbb{Z}_q$ gauge theory.

3-form global symmetry. To begin, we rewrite eq. (2.30), the equation of motion of c_3, in vacuum, i.e., setting the left-hand-side to 0, as

$$\mathcal{K}''\left(\frac{f}{\Lambda^2}\right)\partial_\mu f^{\mu\nu\alpha\beta}=0\,, \tag{2.44}$$

which implies either $\mathcal{K}''=0$ or $\partial_\mu f^{\mu\nu\alpha\beta}=0$. The first equation holds only when $\mathcal{K}$ is extremized, and thus, we concentrate solely on the latter equation that can be rewritten in the d-form language as

$$d\star dc_3=0\,. \tag{2.45}$$

This takes the form of the conservation law of the Hodge dual of a 4-form current:

$$d\star j_4=0\,,\quad \star j_4=\frac{\star dc_3}{\Lambda^4}=\frac{\star f_4}{\Lambda^4}\,, \tag{2.46}$$

implying that gauging the $\mathrm{U}(1)^{(2)}$ 2-form symmetry in vacuum gives rise to an emergent $\mathrm{U}(1)^{(3)}$ 3-form global symmetry. The 3-form symmetry couples to 3-surfaces $\mathcal{M}_3$, such that the Wilson surface operator is given by

$$V(\mathcal{M}_3)=e^{ip\int_{\mathcal{M}_3}c_3}\,,\quad p\in\mathbb{Z}\,. \tag{2.47}$$

The value $p=1$ gives the fundamental Wilson surface, while values of $p>1$ are higher representations. The conserved charge of this symmetry $Q^{(3)}$ is given by integrating $\star j_4$ over a 0-dimensional manifold, or in other words, it is the local operator $\star j_4(\mathcal{P})=\star f_4(\mathcal{P})/\Lambda^4$ at the spacetime point $\mathcal{P}$. Using $\star f_4=\frac{1}{4!}\epsilon^{\mu\nu\alpha\beta}f_{\mu\nu\alpha\beta}$ along with $f_{\mu\nu\alpha\beta}=-\epsilon_{\mu\nu\alpha\beta}f$ and $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta}=-4!$, we find

$$Q^{(3)}=\star j_4(\mathcal{P})=\frac{f(\mathcal{P})}{\Lambda^4}\,, \tag{2.48}$$

and, thus, the generator of the symmetry (symmetry defect) is

$$U_\gamma^{(3)}(\mathcal{M}_0=\mathcal{P})=e^{i\gamma\star j_4}=e^{i\gamma f/\Lambda^4}\,, \tag{2.49}$$

where $\gamma\in[0,2\pi)$. The symmetry defect measures the amount of flux carried by a Wilson surface. Upon pushing $U_\gamma^{(3)}(\mathcal{M}_0)$ past $V(\mathcal{M}_3)$, we obtain the algebra:

$$U_\gamma^{(3)}(\mathcal{M}_0)V(\mathcal{M}_3)=e^{ip\gamma\,\mathrm{Link}(\mathcal{M}_0,\mathcal{M}_3)}V(\mathcal{M}_3)\,. \tag{2.50}$$

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Let us repeat the analysis in the presence of matter. We shall use two approaches. First, we will investigate the 3-form symmetry in the vicinity of the wall but far enough from its core. Next, we shall redo the analysis, this time without making any approximations or assumptions about the nature of the wall. We start with the approximate method, analyzing the situation near the domain walls of the cosine potential we studied above. We assume that we are far from the domain wall core, i.e., we are considering distances $|z| > z_m$, where z_m is given by (2.42), which is the distance at which f and $\mathcal{K}$ are maximized. At z_m, $\mathcal{K}''$ is ill-defined, invalidating our analysis; this is why we need to perform the calculations far from the core. Keeping this constraint in mind, we start by rewriting eq. (2.30) in the d-form language as:

$$\Lambda^4\mathcal{K}''(f)d\star dc_3 = -\frac{q}{2\pi}da\,. \tag{2.51}$$

Far from the core, we can safely set $f \cong 0$, as $f(z)$ decays exponentially fast at distances $z > z_m$. We have[10] $\mathcal{K}''(f \cong 0) = q^2/(4\pi^2 n^2 \Lambda^8)$. Using this information, we can rearrange (2.51) as a conservation law:

$$\begin{aligned} d\star j_4 = 0\,, \quad Q^{(3)} = \star j_4(\mathcal{P}) &= \frac{q^2}{(2\pi n)^2\Lambda^2}\star f_4(\mathcal{P}) + \frac{q}{2\pi}a(\mathcal{P}) \\ &= \frac{q^2}{(2\pi n)^2\Lambda^2}f(\mathcal{P}) + \frac{q}{2\pi}a(\mathcal{P})\,. \end{aligned} \tag{2.52}$$

The symmetry defect is

$$U_\gamma^{(3)}(\mathcal{M}_0 = \mathcal{P}) = e^{i\gamma\star j_4} = e^{i\gamma\left(\frac{q^2 f(\mathcal{P})}{(2\pi n)^2\Lambda^4} + \frac{qa(\mathcal{P})}{2\pi}\right)}\,. \tag{2.53}$$

Since $a(\mathcal{P})$ and $a(\mathcal{P})+2\pi$ are identified, $U_\gamma^{(3)}(\mathcal{M}_0 = \mathcal{P})$ is meaningful only when $\gamma \in 2\pi\mathbb{Z}/q\mathbb{Z} \equiv 2\pi\mathbb{Z}_q$. The charged object under the global symmetry is still the 3-dimensional Wilson surface $V(\mathcal{M}_3)$ given by (2.47). Now, the algebra of $U_\ell^{(3)}(\mathcal{M}_0)$, $\ell \in \mathbb{Z}_q$, and $V(\mathcal{M}_3)$ is given by:

$$U_\ell^{(3)}(\mathcal{M}_0)V(\mathcal{M}_3) = e^{i\frac{2\pi p\ell}{q}\mathrm{Link}(\mathcal{M}_0,\mathcal{M}_3)}V(\mathcal{M}_3)\,. \tag{2.54}$$

As we emphasized above, expression (2.53) is valid only far from the domain wall core, i.e., (2.53) is consistent with setting $f \cong 0$. However, inspired by the preceding analysis, we can repeat the treatment without making any approximations or assumptions about the nature of the walls. Our central equation, as usual, is (2.30), which we will write as a conservation law:[11]

$$\partial_\mu Q^{(3)} = 0\,, \quad Q^{(3)}(\mathcal{P}) = \frac{q}{2\pi}a(\mathcal{P}) + \Lambda^4\mathcal{K}'\left(\frac{f(\mathcal{P})}{\Lambda^4}\right)\,, \tag{2.55}$$

with corresponding symmetry defect

$$U_\ell^{(3)}(\mathcal{M}_0) = e^{i\frac{2\pi\ell}{q}\left(\frac{q}{2\pi}a(\mathcal{P}) + \Lambda^4\mathcal{K}'\left(\frac{f(\mathcal{P})}{\Lambda^4}\right)\right)}\,, \quad \ell = 1, 2, \ldots, q\,. \tag{2.56}$$

[10]Consider if we had employed the quadratic potential, as represented by eq. (2.35). In such a scenario, we would obtain $\mathcal{K}'' = q^2/\Lambda^8$. The disparity of $(2\pi n)^2$ between these two cases becomes evident when we recognize that the function f reaches its zero precisely at the local minimum of $\mathcal{K}$ in the case of the cosine potential, where we may approximate $\mathcal{K}$ by a quadratic function $\mathcal{K} \cong 1 + \frac{q^2 f^2}{2(2\pi n)^2\Lambda^8}$.

[11]In the cosine potential example, the derivative of $Q^{(3)}$ is ill-defined at the core. Nevertheless, $Q^{(3)}$ is well-defined everywhere.

This is the generator of a $\mathbb{Z}_q^{(3)}$ symmetry for a generic form of the kinetic energy $\mathcal{K}$. It is easy to check that (2.55) reproduces the approximate expression (2.53) of the cosine potential near $f \cong 0$. The charge $Q^{(3)}(\mathcal{P})$ is a constant of motion unless one encounters a domain wall: crossing an elementary wall changes $Q^{(3)}$ by

$$\Delta Q^{(3)} = \frac{q}{2\pi}\left(a(\infty) - a(-\infty)\right) = \frac{q}{n} \tag{2.57}$$

units. In other words, $\Delta Q^{(3)}$ is the domain wall charge under the $\mathbb{Z}_q^{(3)}$ global symmetry. Interestingly, when $n = q$, the most natural scenario, $\Delta Q^{(3)}$ coincides with the concept of topological charge in the theory of solitons. As we shall discuss in section 3, the domain walls intersect in a line; this is the locus of a string. From the flux conservation, this string carries a charge q, evenly distributed among the n intersecting domain walls. We conclude that there are n dynamical walls attached to a string, carrying a total charge $Q^{(3)} = q$ under $\mathbb{Z}_q^{(3)}$.

The 3-form symmetry $\mathbb{Z}_q^{(3)}$ acts on q distinct Wilson surfaces $V(\mathcal{M}_3) = e^{ip\int_{\mathcal{M}_3} c_3}$, $p = 1, 2, \ldots, q$. When $p \neq 0$ Mod q, the flux carried by these Wilson surfaces cannot be absorbed by the dynamical domain walls since the latter always comes in a stack of a total charge q.

It remains to discuss the fate of the 3-form global symmetry. We start with the $\mathrm{U}(1)^{(3)}$ symmetry in the absence of matter, i.e., taking $v \to \infty$. In this case, as we discussed before, the equation of motion of the 3-form gauge field c_3 yields a constant solution: $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} f$, where f is a constant. Two walls experience a constant force, meaning that the $\mathrm{U}(1)^{(3)}$ is unbroken and the Wilson surface $V(\mathcal{M}_3)$ exhibits the "area" law $\langle V(\mathcal{M}_3)\rangle = 0$. Introducing the axion field and the coupling $-\frac{q}{2\pi} da \wedge c_3$ breaks $\mathrm{U}(1)^{(3)}$ down to $\mathbb{Z}_q^{(3)}$ and endows the theory with a $\mathbb{Z}_q^{(0)}$ 0-form global symmetry. Now, the 4-form field f_4 is gapped, and the Wilson surfaces exhibit the "perimeter" law $\langle V(\mathcal{M}_3)\rangle \neq 0$, meaning that $\mathbb{Z}_q^{(3)}$ is spontaneously broken. Moreover, if the theory forms domain walls, $\mathbb{Z}_q^{(0)}$ also breaks spontaneously.

The (-1)-form symmetry, its gauging, and the cosmological constant. In addition, the theory possess a $\mathrm{U}(1)^{(-1)}$ (-1)-form symmetry. The Bianchi identity $d^2 c_3 = 0$ can be written as the conservation law of the Hodge-dual of a "magnetic" current $\star j_4^m = dc_3 = f_4$. The corresponding symmetry defect is

$$U_\gamma^{(-1)}(\mathcal{M}_4) = e^{i\gamma\int_{\mathcal{M}_4} f_4}, \quad \text{noting that} \quad \int_{\mathcal{M}_4} f_4 \in 2\pi\mathbb{Z}\,. \tag{2.58}$$

The operator $U_\gamma^{(-1)}(\mathcal{M}_4)$ does not act on any physical objects directly. However, the two-point correlator $\langle c_3(x) c_3(0)\rangle$ can be used to determine whether the (-1)-form symmetry is preserved or spontaneously broken. A massless pole in the correlator signifies symmetry breaking; if absent, the symmetry remains unbroken [37].

In the absence of axions, the (-1)-form symmetry functions as a global symmetry. The gauge field c_3 is massless, leading to a pole in its two-point correlator, which indicates that the (-1)-form symmetry is spontaneously broken. In this context, c_3 can be viewed as the Nambu-Goldstone field associated with this breaking.

When we couple c_3 to the axion through the term $\frac{q}{2\pi} a \wedge f_4$, the axion can be interpreted as the background gauge field for the (-1)-form symmetry. By introducing a kinetic term for a, we effectively sum over this background gauge field in the path integral, meaning we are gauging the (-1)-form symmetry.[12] As previously discussed, this leads to the axion acquiring mass, which can be understood as a absorbing the would-be Goldstone field c_3 and becoming massive. Consequently, the gauged (-1)-form symmetry is spontaneously broken, and the correlator $\langle c_3(x)c_3(0)\rangle$ no longer exhibits a massless pole.

From our discussion, we find that the (-1)-form symmetry is intricately linked to the presence/absence of a cosmological constant sourced by c_3. The energy-momentum tensor of c_3 can be derived directly from (2.16) by varying with respect to the metric tensor:[13]

$$T_{\mu\nu} = \eta_{\mu\nu}\Lambda^4\left[\mathcal{K}\left(\frac{f}{\Lambda^4}\right) - \frac{f}{\Lambda^4}\mathcal{K}'\left(\frac{f}{\Lambda^4}\right)\right], \tag{2.59}$$

which takes the form of a cosmological constant. Without axions, the global (-1)-form symmetry is spontaneously broken and $f = \text{constant}$, meaning that the vacuum energy gets a contribution from the long-range field c_3. Coupling to axions, the (-1)-form symmetry is gauged and Higgsed. Now, the system is gapped, i.e., $f = 0$ (this is true far from the domain wall core); thus, the vacuum energy does not receive contribution from c_3, and we have $T_{\mu\nu} = 0$.

The mixed 't Hooft anomaly between $\mathbb{Z}_q^{(3)}$ and $\mathbb{Z}_q^{(0)}$ symmetries. An important question is whether there is a mixed anomaly between the two global symmetries $\mathbb{Z}_q^{(3)}$ and $\mathbb{Z}_q^{(0)}$. To answer this question, we examine the commutation relation between $U_\ell^{(3)}(\mathcal{M}_0)$ and $U_{\ell'}^{(0)}(\mathcal{M}_3)$ given by (2.56) and (2.18), respectively. The calculations can be performed, as before, by foliating $\mathcal{M}_4$ into constant time slices and orienting $\mathcal{M}_3$ to be a time-like surface:

$$U_{\ell'}^{(0)}(\mathcal{M}_3(t)) = e^{i\frac{2\pi\ell'}{q}\int_{\mathcal{M}_3(t)}\left(v^2\partial_0 a - \frac{q}{2\pi}\frac{c_{ijk}\epsilon^{ijk}}{3!}\right)}. \tag{2.60}$$

Then, using the equal-time commutation relation $[a(\boldsymbol{x},t),\Pi_a(\boldsymbol{y},t)] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, where $\Pi_a = v^2\partial_0 a - \frac{q}{2\pi}\frac{c_{ijk}\epsilon^{ijk}}{3!}$, we obtain

$$U_{\ell'}^{(0)}(\mathcal{M}_3(t))U_\ell^{(3)}(\mathcal{M}_0(t)) = e^{i\frac{2\pi\ell\ell'}{q}}U_\ell^{(3)}(\mathcal{M}_0(t))U_{\ell'}^{(0)}(\mathcal{M}_3(t)). \tag{2.61}$$

Thus, a $\mathbb{Z}_q$-valued mixed anomaly exists between the two symmetries. The anomaly implies one or both symmetries are broken. Our previous discussion reveals that both symmetries are broken in a theory that forms domain walls in the IR.

We further explore the consequences of the mixed anomaly (2.61), now working in a Hamiltonian formalism. Let H be the Hamiltonian of the $\mathrm{U}(1)^{(2)}$ gauge theory under study. Since the theory has a $\mathbb{Z}_q^{(3)}$ 3-form global symmetry, its generators commute with the Hamiltonian: $\left[H, U_\ell^{(3)}\right] = 0$, and we can take the physical states in Hilbert space to be simultaneous eigenstates of these two operators. Thus, we have

$$|\psi\rangle_{\text{phy}} = |E(e), e\rangle, \quad e = 0, 1, \ldots, q-1. \tag{2.62}$$

[12]We thank an anonymous referee for pointing this out.

[13]Remember that we use the boundary condition $\mathcal{K}(f=0) = 0$.

Here, $E(e)$ labels the energy and e labels the "flux" of the state (under the 3-form symmetry) such that

$$H|E(e),e\rangle = E(e)|E(e),e\rangle\,, \quad U_\ell^{(3)}|E(e),e\rangle = e^{i\frac{2\pi\ell e}{q}}|E(e),e\rangle\,. \tag{2.63}$$

Notice that the state's energy $E(e)$ can also depend on the value of the flux carried by the state.

To show that e labels the flux carried by the state $|E(e),e\rangle$, let us insert a Wilson surface $V(\mathcal{M}_3) = e^{ip\int_{\mathcal{M}_3} c_3}$ in the state $|E(e),e\rangle$ and then measure the new flux; the measurement is performed by acting with $U_{\ell=1}^{(3)}$ on the new state. Since this Wison surface carries a flux p, we expect inserting it increases the state flux by p unit. To confirm this, we perform the operation

$$U_{\ell=1}^{(3)}V(\mathcal{M}_3)|E(e),e\rangle = e^{i\frac{2\pi(e+p)}{q}}V(\mathcal{M}_3)|E(e),e\rangle\,, \tag{2.64}$$

where we used (2.54) and (2.63). The relation (2.64) shows that the state $V(\mathcal{M}_3)|E(e),e\rangle$ carries a flux $e+p$, and thus, indeed, e in (2.62) labels the flux carried by a state as anticipated.

Next, we act with both sides of the anomaly (2.61) on the state $|E(e),e\rangle$ to find that $U_{\ell'=1}^{(0)}|E(e),e\rangle$ is an eigenstate of $U_{\ell=1}^{(3)}$ with eigenvalue $e-1$. Since $\mathbb{Z}_q^{(0)}$ is a symmetry of the theory, we have $[H,U_{\ell'}^{(0)}]=0$, and thus, the state $U_{\ell'=1}^{(0)}|E(e),e\rangle$ has the same energy as $|E(e),e\rangle$. Repeating the statement q times, we conclude that there are q degenerate eigenstates of the same energy, labeled by the q different values of e, as in (2.62). This q-fold degeneracy is a direct consequence of the mixed anomaly (2.61), which is true for both the ground and excited states of the system. In the thermodynamic limit, i.e., as we take the manifold $\mathcal{M}_4$ to be very large, we become interested mainly in the ground states, which stay q-fold degenerate.

Gauging $\mathbb{Z}_q^{(3)}$. Next, we consider the $\mathrm{U}(1)^{(2)}/\mathbb{Z}_p$ gauge theory resulting from gauging a subgroup $\mathbb{Z}_p^{(3)} \subseteq \mathbb{Z}_q^{(3)}$. Gauging the discrete symmetry $\mathbb{Z}_p^{(3)}$ means introducing a background gauge field of the symmetry and including a sum over arbitrary insertions of this background in the path integral. We can introduce the $\mathbb{Z}_p^{(3)}$ background F_4 into the path integral by replacing every f_4 in (2.16) by $f_4 \to f_4 + F_4$, where F_4 can be expressed as $pF_4 = dF_3$ and dF_3 satisfies the quantization condition $\int_{\mathcal{M}_4} dF_3 \in 2\pi\mathbb{Z}$. In turn, this implies the quantization of F_4 in units of $1/p$, i.e., $\int_{\mathcal{M}_4} F_4 \in \frac{2\pi}{p}\mathbb{Z}$, such that $dF_4=0$.

Now consider the second term in (2.16) in the presence of the $\mathbb{Z}_q^{(3)}$ background. Its Euclidean form is $i\frac{q}{2\pi}a\wedge(f_4+F_4)$. Since, $\int_{\mathcal{M}_4} F_4 \in \frac{2\pi}{p}\mathbb{Z}$, only the shift $a \to a + \frac{2\pi p}{q}$, i.e., $\mathbb{Z}_{q/p}^{(0)}$, survives as a genuine discrete symmetry. Actually, gauging $\mathbb{Z}_p^{(3)}$ renders the symmetry operator of $\mathbb{Z}_q^{(0)}$ a projective operator. To see this, consider the commutation relation between the $\mathbb{Z}_q^{(0)}$ symmetry defect $U_\ell^{(0)}(\mathcal{M}_3)$ and the generator of the $\mathbb{Z}_p^{(3)}$ symmetry $\left[U_1^{(3)}(\mathcal{M}_0)\right]^{q/p}$ in the original $\mathrm{U}(1)^{(2)}$ gauge theory. From (2.61) we have

$$\left[U_1^{(3)}(\mathcal{M}_0)\right]^{-q/p} U_\ell^{(0)}(\mathcal{M}_3)\left[U_1^{(3)}(\mathcal{M}_0)\right]^{q/p} = e^{i\frac{2\pi\ell}{p}}U_\ell^{(0)}(\mathcal{M}_3)\,. \tag{2.65}$$

This relation shows that $U_\ell^{(0)}(\mathcal{M}_3)$ fails to be a gauge-invariant operator in the $\mathrm{U}(1)^{(2)}/\mathbb{Z}_p$ gauge theory. To remedy the problem, we sum over an arbitrary number of gauge transformations under the operator $\left[U_1^{(3)}(\mathcal{M}_0)\right]^{q/p}$ by defining the new $\mathrm{U}(1)^{(2)}/\mathbb{Z}_p$ gauge-invariant

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symmetry defect:

$$\begin{aligned}\mathcal{U}^{(0)}_\ell(\mathcal{M}_3) &\equiv \sum_{r\in\mathbb{Z}} \left[U^{(3)}_1(\mathcal{M}_0)\right]^{-rq/p} U^{(0)}_\ell(\mathcal{M}_3) \left[U^{(3)}_1(\mathcal{M}_0)\right]^{rq/p} \\ &= U^{(0)}_\ell(\mathcal{M}_3)\sum_{r\in\mathbb{Z}} e^{i\frac{2\pi\ell r}{p}} \\ &= U^{(0)}_\ell(\mathcal{M}_3)\delta_{\ell\in p\mathbb{Z}}\,. \end{aligned} \tag{2.66}$$

Thus, $\mathcal{U}^{(0)}_\ell(\mathcal{M}_3)$ is a projective operator. Recalling $\ell = 0, 1, \ldots, q$, we conclude that the subgroup $\mathbb{Z}^{(0)}_{q/p}$ survives as a genuine symmetry.[14]

The $\mathrm{U}(1)^{(2)}/\mathbb{Z}_p$ gauge theory has a remaining $\mathbb{Z}^{(3)}_{q/p}$ global symmetry with symmetry defect given by

$$\left[U^{(3)}_\ell(\mathcal{M}_0)\right]^p = e^{i\frac{2\pi\ell p}{q}\left(\frac{q}{2\pi}a(\mathcal{P})+\Lambda^4\mathcal{K}'\left(\frac{f(\mathcal{P})}{\Lambda^4}\right)\right)}\,, \quad \ell = 1, 2, \ldots, q/p. \tag{2.67}$$

The $\mathbb{Z}^{(3)}_{q/p}$ symmetry acts on the $\mathrm{U}(1)^{(2)}/\mathbb{Z}_p$ gauge-invariant Wilson surfaces $V(\mathcal{M}_3) = e^{imp\int_{\mathcal{M}_3} c_3}$, $m \in \mathbb{Z}_{q/p}$. The action of $\mathbb{Z}^{(0)}_{q/p}$ takes us between the q/p vacua of the theory.

Let us discuss the consequences of gauging $\mathbb{Z}^{(3)}_p$ in the Hamiltonian formalism. When $\mathcal{U}^{(0)}_\ell(\mathcal{M}_3)$ acts on the state $|E(e), e\rangle$, with $e = 0, 1, \ldots, q-1$, it annihilates it unless ℓ is a multiple of p:

$$\mathcal{U}^{(0)}_\ell(\mathcal{M}_3)|E(e), e\rangle = \delta_{\ell\in p\mathbb{Z}} U^{(0)}_\ell(\mathcal{M}_3)|E(e), e\rangle\,, \tag{2.68}$$

i.e., only q/p states are not annihilated by $\mathcal{U}^{(0)}_\ell(\mathcal{M}_3)$. When we fully gauge $\mathbb{Z}^{(3)}_q$, i.e., in $\mathrm{U}(1)^{(2)}/\mathbb{Z}_q$ gauge theory, we have neither genuine 0-form nor 3-form global symmetries.

What has just happened, especially concerning a generic kinetic energy term $\mathcal{K}$, that leads to the formation of dynamical domain walls? We analyze the situation by considering the cosine potential (2.36), setting $n = q$ for simplicity. We shall also gauge the full $\mathbb{Z}^{(3)}_q$ symmetry. In this case, what we are operationally doing is that we are declaring the equivalence between all the q vacua: $a \equiv a + \frac{2\pi\ell}{q}$, $\ell = 1, 2, \ldots, q$. Therefore, it is more meaningful to define $\varphi \equiv qa$ and replace $V_{\text{eff}}(a) = \Lambda^4(1-\cos(qa))$ with $V_{\text{eff}}(\varphi) = \Lambda^4(1-\cos\varphi)$. The latter potential supports a single domain wall interpolating between $\varphi = 0$ and $\varphi = 2\pi$. However, such a wall is unstable quantum mechanically as it decays by instanton effects. We end up with a theory with a unique vacuum, supporting only strings but no domain walls.

3 The dual description: the Kalb-Ramond frame

While the axion framework provided valuable insights into the global symmetries within our system, the notion of strings remained implicit. To address this, we transition to the Kalb-Ramond frame [38], where the presence and properties of strings become more evident and accessible.

[14] We thank E. Poppitz for discussion about this point.

To dualize the axion Lagrangian (2.1) to a theory of a 2-form Kalb-Ramond field, we add to the Lagrangian (2.1) an extra term [14]:

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{1}{2\pi} b_2 \wedge d^2 a \,. \tag{3.1}$$

Here, b_2 is a Lagrange multiplier used to impose the Bianchi identity[15]- integrating out b_2 gives $d^2 a = 0$. We can also integrate out a via its equation of motion:

$$2\pi v^2 \star da = db_2 \,. \tag{3.2}$$

Substituting back into (3.1) gives the dual Kalb-Ramond theory:

$$\mathcal{L}_{\text{dual}} = \frac{1}{2(2\pi)^2 v^2} db_2 \wedge \star db_2 \,. \tag{3.3}$$

The Kalb-Ramond field b_2 couples to the 2-dimensional axion-string worldsheet $\mathcal{M}_2$. Thus, the Wilson-like operator of an axion-string is:

$$V(\mathcal{M}_2) = e^{i \int_{\mathcal{M}_2} b_2} \,. \tag{3.4}$$

Unlike $V(\mathcal{M}_2)$, the operator $V(\mathcal{M}_0)$, which in the original theory was given by e^{ia}, has no local description in terms of b_2. Therefore, in the Kalb-Ramond frame, $V(\mathcal{M}_2)$ and $V(\mathcal{M}_0)$ behave respectively like Wilson and 't Hooft operators in electrodynamics. This picture is reversed in the axion frame.

In the dual description, the 0-form and 2-form global symmetry currents are:

$$j_1 = \frac{1}{2\pi} \star db_2, \qquad j_3 = 2\pi v^2 db_2 \,. \tag{3.5}$$

These currents satisfy the conservation laws:

$$d \star j_1 = 0 \,, \quad d \star j_3 = 0 \,, \tag{3.6}$$

which are the results of the Bianchi identity $d^2 b_2 = 0$ and the equation of motion $d \star db_2 = 0$, respectively. The corresponding $\mathrm{U}(1)^{(0)}$ and $\mathrm{U}(1)^{(2)}$ symmetry defects are given by

$$U_\alpha^{(0)}(\mathcal{M}_3) = e^{i\alpha \int_{\mathcal{M}_3} \frac{1}{2\pi} db_2} \,, \quad U_\beta^{(2)}(\mathcal{M}_1) = e^{i\beta \int_{\mathcal{M}_1} 2\pi v^2 \star db_2} \,. \tag{3.7}$$

With this dual formulation, we can see the action of $\mathrm{U}(1)^{(2)}$ global symmetry- it shifts b_2 by a constant 2-form Λ_2:

$$b_2 \to b_2 + \Lambda_2 \,. \tag{3.8}$$

This transforms the axion string by a U(1) phase:

$$U_\beta^{(2)}(\mathcal{M}_2) V(\mathcal{M}_2) = e^{i\beta \mathrm{Link}(\mathcal{M}_1, \mathcal{M}_2)} V(\mathcal{M}_2) \,. \tag{3.9}$$

[15] The form of the extra term implies that b_2 has mass dimension 2.

3.1 Gauging the $\mathrm{U}(1)^{(2)}$ 2-form symmetry

Here, we derive the Lagrangian of the dual Kalb-Ramond gauge theory, which results by gauging the $\mathrm{U}(1)^{(2)}$ symmetry. Our starting point is the Lagrangian (2.16) after adding the term $-b_2 \wedge d^2a/(2\pi)$ to enforce the Bianchi identity $d^2a = 0$:

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{q}{2\pi} da \wedge c_3 - \frac{1}{2\pi} b_2 \wedge d^2 a + \Lambda^4 \mathcal{K}\left(\frac{dc_3}{\Lambda^4}\right) . \tag{3.10}$$

The equation of motion of a is:

$$d \star da - \frac{1}{2\pi} d^2 b_2 - \frac{q}{2\pi} dc_3 = 0 \,, \tag{3.11}$$

and integrating once we find

$$\star da = \frac{1}{2\pi v^2} \left(db_2 + qc_3\right) . \tag{3.12}$$

Substituting (3.12) into (3.10), we obtain the dual Lagrangian

$$\mathcal{L}_{\text{dual}} = \frac{1}{2(2\pi)^2 v^2} (db_2 + qc_3) \wedge \star (db_2 + qc_3) + \Lambda^4 \mathcal{K}\left(\frac{dc_3}{\Lambda^4}\right) . \tag{3.13}$$

In this formulation, the $\mathrm{U}(1)^{(2)}$ symmetry is gauged by introducing the 3-form gauge field c_3, which couples minimally to the Kalb-Ramond field b_2. The minimal coupling $db_2 + qc_3$ means that b_2 carries a charge q under c_3. This is also manifest in the fact that the dual Lagrangian is invariant under the $\mathrm{U}(1)^{(2)}$ local gauge transformation $c_3 \to c_3 + d\lambda_2$, $b_2 \to b_2 - q\lambda_2$. We may think of b_2 as the Stuckelberg field of c_3; as c_3 eats up the b_2 field, it aquires a mass $\sim \frac{\Lambda^2}{v}$. In the limit $v \to \infty$, the Kalb-Ramond field decouples, leaving us with a pure $\mathrm{U}(1)^{(2)}$ gauge theory.

The effect of gauging the $\mathrm{U}(1)^{(2)}$ symmetry is that the spectrum of extended operator changes. The Wlison surface operator (3.4) is no longer gauge-invariant under the 2-form gauge symmetry. In the Kalb-Ramond frame, a gauge-invariant operator is

$$e^{i \int_{\mathcal{M}_3} db_2 + qc_3} = e^{i \int_{\mathcal{M}_2 = \partial \mathcal{M}_3} b_2} e^{iq \int_{\mathcal{M}_3} c_3} \,, \tag{3.14}$$

which can be interpreted as a string attached to a stack of domain walls with a cumulative charge of q (the charge under the c_3 field). An elementary axion that winds around this configuration cannot detect a nontrivial phase. This can be envisaged by computing the commutator

$$\left[e^{ia}, e^{i \int_{\mathcal{M}_3} db_2 + qc_3}\right] = 0 \,, \tag{3.15}$$

where we used (3.12) along with $[a(\boldsymbol{x}, t), \Pi_a(\boldsymbol{y}, t)] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$ and $\Pi_a = v^2 \partial_0 a - \frac{q}{2\pi} \frac{c_{ijk} \epsilon^{ijk}}{3!}$.

We end this section by discussing the global symmetries in the Kalb-Ramond frame. First, the generator of the $\mathbb{Z}_q^{(0)}$ 0-form symmetry is given by $U_\alpha^{(0)}(\mathcal{M}_3)$ in (3.7). This generator, however, must be invariant under a $\mathrm{U}(1)^{(2)}$ gauge transformation $b_2 \to b_2 - q\lambda_2$. Using $\int_{\mathcal{M}_3} d\lambda_2 \in 2\pi\mathbb{Z}$, we find $\alpha = \frac{2\pi\ell}{q}$, $\ell = 1, 2, \ldots, q$, as expected for the $\mathbb{Z}_q^{(0)}$ symmetry. Second, we also have a $\mathbb{Z}_q^{(3)}$ 3-form global symmetry. However, the generator of this symmetry has no local description in the Kalb-Ramond frame. It is crucial to highlight that transitioning between the axion and Kalb-Ramond frames does not eliminate global symmetries. Rather, certain symmetries may not be manifest in a local description.

4 Multi 3-form gauge theory

Consider an axion a coupled to 2 distinct 3-form gauge fields c_3 and $\tilde{c}_3$. The Lagrangian reads

$$\mathcal{L} = \frac{v^2}{2}|da|^2 - \frac{q_1}{2\pi} da \wedge c_3 - \frac{q_2}{2\pi} da \wedge \tilde{c}_3 + \Lambda^4 \mathcal{K}\left(\frac{f_4}{\Lambda^4}\right) + \Lambda^4 \tilde{\mathcal{K}}\left(\frac{\tilde{f}_4}{\Lambda^4}\right), \tag{4.1}$$

where $f_4 = dc_3$ and $\tilde{f}_4 = d\tilde{c}_3$ are the field strengths of c_3 and $\tilde{c}_3$, respectively, and we assumed that the scale Λ is the same for all the 3-form fields. Notice that we did not include a kinetic-mixing term to avoid complications. The 3-form gauge fields c_3 and $\tilde{c}_3$ are invariant under $\mathrm{U}(1)^{(2)} \times \mathrm{U}(1)^{(2)}$ gauge transformations $c_3 \to c_3 + d\lambda_2$ and $\tilde{c}_3 \to \tilde{c}_3 + d\tilde{\lambda}_2$, where $\int_{\mathcal{M}_3} d\lambda_2, \int_{\mathcal{M}_3} d\tilde{\lambda}_2 \in 2\pi\mathbb{Z}$, while the field strengths satisfy the quantization conditions

$$\int_{\mathcal{M}_4} f_4\,, \int_{\mathcal{M}_4} \tilde{f}_4 \in 2\pi\mathbb{Z}\,. \tag{4.2}$$

The equations of motion of a, c_3, and $\tilde{c}_3$ read

$$v^2 d\star da - \left(\frac{q_1}{2\pi}dc_3 + \frac{q_2}{2\pi}d\tilde{c}_3\right) = 0\,, \qquad \begin{aligned} \frac{q_1}{2\pi}\partial_\mu a &= -\Lambda^4 \partial_\mu \mathcal{K}'\left(\frac{f}{\Lambda^4}\right), \\ \frac{q_2}{2\pi}\partial_\mu a &= -\Lambda^4 \partial_\mu \tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda^4}\right). \end{aligned} \tag{4.3}$$

Trading f and $\tilde{f}$ for a classical axion effective potential yields

$$\frac{\partial V_{\text{cl-eff}}(a)}{\partial a} = -\frac{1}{2\pi}\left(q_1 f + q_2 \tilde{f}\right). \tag{4.4}$$

This relationship asserts that the combination of fields $q_1 f + q_2 \tilde{f}$ vanishes at the extrema of $V_{\text{cl-eff}}(a)$. This implies that this particular combination of the 3-form fields is gapped at the theory's vacua. Conversely, the independent combination $q_2 f - q_1 \tilde{f}$ remains ungapped [14].

The system (4.1) enjoys a multitude of global symmetries. First, the theory is invariant under a $\mathbb{Z}_q^{(0)}$ symmetry that acts on a as $a \to a + \frac{2\pi}{q}$ and $q = \gcd(q_1, q_2)$. The generator of $\mathbb{Z}_q^{(0)}$ is

$$U_\ell^{(0)}(\mathcal{M}_3) = e^{i\frac{2\pi\ell}{q}\int_{\mathcal{M}_3}\left(v^2\star da - \frac{q_1 c_3 + q_2\tilde{c}_3}{2\pi}\right)}, \quad \ell = 1, 2, \ldots, q\,, \tag{4.5}$$

which acts on the local operator $e^{ia(\mathcal{P})}$.

In addition, the system exhibits two independent 3-form global symmetries. To find them, we use two methods. We start with the first method, which was not discussed previously but works as an alternative view on the global 3-form symmetry. In this method, we shift c_3 and $\tilde{c}_3$ by two independent closed but not exact 3-forms Λ_3 and $\tilde{\Lambda}_3$:

$$c_3 \to c_3 + \Lambda_3\,, \quad \tilde{c}_3 \to \tilde{c}_3 + \tilde{\Lambda}_3\,, \tag{4.6}$$

under which the action gets shifted by

$$S \to S + \frac{q_1}{2\pi}\int_{\mathcal{M}_4} da \wedge \Lambda_3 + \frac{q_2}{2\pi}\int_{\mathcal{M}_4} da \wedge \tilde{\Lambda}_3 = S + q_1 k\alpha + q_2 k\beta\,, \tag{4.7}$$

where we defined $\int_{\mathcal{M}_3} \Lambda_3 = \alpha$, $\int_{\mathcal{M}_3} \tilde{\Lambda}_3 = \beta$, and we recalled that $\int_{\mathcal{M}_1} da = 2\pi k, k \in \mathbb{Z}$ since a is a compact scalar. Under the Λ_3 and $\tilde{\Lambda}_3$ shifts, the path integral picks up a phase:

$$Z \to e^{iq_1 k\alpha + iq_2 k\beta} Z \,. \tag{4.8}$$

It is easily seen that there are two combinations of α and β that lead to two independent 3-form global symmetries that leave the action invariant: a $\mathrm{U}(1)^{(3)}$ symmetry is obtained by setting $q_1\alpha = -q_2\beta$ and a $\mathbb{Z}_q^{(3)}$ symmetry is obtained upon taking $\alpha, \beta \in \frac{2\pi}{q}\mathbb{Z}$, where $q = \gcd(q_1, q_2)$. These are linearly independent transformations, so there are no redundancies, and the faithful 3-form symmetry group is

$$\mathbb{Z}_q^{(3)} \times \mathrm{U}(1)^{(3)} \,. \tag{4.9}$$

Another way to obtain the same result is by combining the equations of motion of c_3 and $\tilde{c}_3$ in the form of two independent conservation laws. Using (4.3) we find

$$\begin{aligned} \partial_\mu \left(\frac{q_1 a + q_2 a}{2\pi} + \Lambda^4 \mathcal{K}'\left(\frac{f}{\Lambda}\right) + \Lambda^4 \tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right) \right) &= 0 \,, \\ \partial_\mu \left(-q_2 \Lambda^4 \mathcal{K}'\left(\frac{f}{\Lambda}\right) + q_1 \Lambda^4 \tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right) \right) &= 0 \,, \end{aligned} \tag{4.10}$$

from which we define the two symmetry defects:

$$U^{(3)}_{\alpha_1}(\mathcal{M}_0) = e^{i\alpha_1\left(\frac{q_1 a + q_2 a}{2\pi} + \Lambda^4 \mathcal{K}'\left(\frac{f}{\Lambda}\right) + \Lambda^4 \tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right)\right)} \,, \quad U^{(3)}_{\alpha_2}(\mathcal{M}_0) = e^{i\alpha_2\left(-q_2 \Lambda^4 \mathcal{K}'\left(\frac{f}{\Lambda}\right) + q_1 \Lambda^4 \tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right)\right)} \,. \tag{4.11}$$

While the phase α_2 is an arbitrary U(1) phase, implying that $U^{(3)}_{\alpha_2}$ is the symmetry defec of a $\mathrm{U}(1)^{(3)}$ 3-form global symmetry, the single-valuedness of $U^{(3)}_{\alpha_1}$ as $a \sim a + 2\pi$ implies that $\alpha_1 = \frac{2\pi\mathbb{Z}}{q}$, reducing the second symmetry group from $\mathrm{U}(1)^{(3)}$ down to $\mathbb{Z}_q^{(3)}$, in accordance with our earlier finding.

The theory also possesses two distinct $\mathrm{U}(1)^{(-1)}$ (-1)-form symmetries associated with the Bianchi's identities: $d^2 c_3 = d^2 \tilde{c}_3 = 0$. As we mentioned above, only the field combination $q_1 c_3 + q_2 \tilde{c}_3$ is gapped while the other combination $-q_2 c_3 + q_1 \tilde{c}_3$ remains gappless. This implies that only one of the two (-1)-form symmetries is gauged and spontaneously broken (Higgsed), resulting in the axion acquiring a mass. The other (-1)-form symmetry is a global symmetry, which also exhibits spontaneous breaking, resulting in a massless 3-form gauge field that sources a cosmological constant in the deep IR.

This treatment is easily generalized to any K distinct 3-form fields to find that the full faithful symmetry group is

$$\mathbb{Z}_q^{(0)} \times \mathbb{Z}_q^{(3)} \times \prod_{i=1}^{K-1} \mathrm{U}(1)^{(i)(3)} \tag{4.12}$$

where $q = \gcd(q_1, \ldots, q_K)$.

5 UV completion: axion-Yang-Mills theory

In this section, we argue that the 3-form gauge theory in either the axion or the Kalb-Ramond frame emerges in the IR from a UV-complete axion-Yang-Mills system.[16] This conclusion is reached by using effective field theory methods empowered by new 't Hooft anomaly matching conditions. We put the IR effective field theory under scrutiny by testing its adequacy under various checks.

To this end, consider an SU(N) gauge theory endowed with a massless Dirac fermion in a representation $\mathcal{R}$ under SU(N). In addition, consider a complex scalar Φ that is inert under SU(N) but otherwise couples to the Dirac fermion. The Lagrangian of the system reads [34]:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}\left(f_2^c \wedge \star f_2^c\right) + \bar{\psi}\bar{\sigma}^\mu D_\mu \psi + \bar{\tilde{\psi}}\bar{\sigma}^\mu D_\mu \tilde{\psi} + |d\Phi|^2 - V(\Phi) + y\Phi\tilde{\psi}\psi + \mathrm{h.c.}\,. \qquad (5.1)$$

$f_2^c = da_1^c - i a_1^c \wedge a_1^c$ is the field strength of the SU(N) color field a_1^c. ψ and $\tilde{\psi}$ are two left-handed Weyl fermions in representations $\mathcal{R}$ and $\overline{\mathcal{R}}$ under SU(N), respectively, constituting together a single Dirac fermion. The covariant derivative is $D_\mu = \partial_\mu - i a_\mu^c$ and y is the Yukawa coupling. The potential of the complex scalar field is $V(\Phi) = \lambda\left(|\Phi|^2 - v^2/2\right)$, where λ is $\mathcal{O}(1)$ parameter. We take $v \gg \Lambda$, where Λ is the strong scale of the gauge sector. The Lagrangian (5.1) is invariant under two classical 0-form symmetries $\mathrm{U}(1)_B^{(0)} \times \mathrm{U}(1)_\chi^{(0)}$, the baryon-number and axial symmetries. The ABJ anomaly in the color background breaks $\mathrm{U}(1)_\chi^{(0)}$ down to $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$, and we find that the full good global symmetry of the SU(N) axion-YM theory is [39, 40]

$$G^{\mathrm{global}} = \frac{\mathrm{U}(1)_B^{(0)} \times \mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}}{\mathbb{Z}_{N/m} \times \mathbb{Z}_2^F} \times \mathbb{Z}_m^{(1)}\,. \qquad (5.2)$$

The 1-form global symmetry $\mathbb{Z}_m^{(1)}$ acts on Wilson's lines of a_1^c, where $m = \gcd(N, n)$ and n is the N-ality of the representation $\mathcal{R}$, i.e., the boxes in the Young tableaux modulo N. The baryon-number $\mathrm{U}(1)_B^{(0)}$ and the chiral $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ symmetries act on the local fields as

$$\begin{aligned} \mathrm{U}(1)_B^{(0)}: &\quad \psi \to e^{i\alpha}\psi\,, \quad \tilde{\psi} \to e^{-i\alpha}\tilde{\psi}\,, \quad \Phi \to \Phi\,, \\ \mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}: &\quad \psi \to e^{i\frac{2\pi\ell}{2T_\mathcal{R}}}\psi\,, \quad \tilde{\psi} \to e^{i\frac{2\pi\ell}{2T_\mathcal{R}}}\tilde{\psi}\,, \quad \Phi \to e^{-i\frac{4\pi\ell}{2T_\mathcal{R}}}\Phi\,, \end{aligned} \qquad (5.3)$$

and $\ell = 1, 2, \ldots, 2T_\mathcal{R}$ and $T_\mathcal{R}$ is the Dynkin index of $\mathcal{R}$ (in our normalization, $T_\square = 1$, where $\square$ is the fundamental representation). The modding by $\mathbb{Z}_{N/m} \times \mathbb{Z}_2^F$ in (5.2) is important to remove redundancies. Here, $\mathbb{Z}_2^F$ is the $(-1)^F$ fermion number subgroup of the Lorentz group.[17] The complex scalar field can be written as $\Phi = |\Phi| e^{ia}$, where a is the axion. At energy scales $\ll v$, we can set $|\Phi| = v/\sqrt{2}$, and thus, one may only work with the axion as the lightest degree of freedom.

[16]The reader might object referring to the system we study in this section as UV complete since we use a scalar field that exhibits a Landau pole. Here, by a UV-complete, we just mean a model that couples the axion to Yang-Mills theory and gives the desired symmetries and natural hierarchy of scales.

[17]Notice that the gauge group that faithfully acts on the fermions is $\mathrm{SU}(N)/\mathbb{Z}_m$. Thus, the fermions are charged under $\mathbb{Z}_{N/m}$ subgroup of the center of SU(N) gauge group. When N/m is even, the fermion number is a subgroup of $\mathbb{Z}_{N/m}$, i.e., the fermion number is gauged. In this case, all gauge-invariant operators are bosons.

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5.1 The mixed 't Hooft anomaly and IR Lagrangian

Energy scale $E \gg v$.

Among the anomalies of the axion-YM theory, the mixed anomaly between the $\mathbb{Z}_m^{(1)}$ 1-form center and $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ chiral symmetries is essential in connection with the 3-form gauge theory. To see the link, we first review this anomaly from the UV point of view [31, 32, 34, 41]. We shall be general and examine the anomaly between a subgroup of the full center $\mathbb{Z}_m^{(1)}$ and chiral symmetries. We shall also work in the Euclidean space.

To this end, we turn on a background of $\mathbb{Z}_p^{(1)} \subseteq \mathbb{Z}_m^{(1)}$. This can be implemented by introducing the pair of U(1) fields (B_1^c, B_2^c) and the constraint $pB_2^c = dB_1^c$. Demanding the quantization condition $\int_{\mathcal{M}_2} dB_1^c \in 2\pi\mathbb{Z}$ implies the fractional quantization of B_2^c flux: $\int_{\mathcal{M}_2} B_2^c \in \frac{2\pi\mathbb{Z}}{p}$. We couple B_2^c to fermions as follows. First, we enlarge the gauge group from SU(N) to U(N); we introduce the $\hat{a}_1^c$ gauge field of U(N) such that $\hat{a}_1^c \equiv a_1^c + \frac{B_1^c}{p}$ with field strength $\hat{f}_2^c = d\hat{a}_1^c + \hat{a}_1^c \wedge \hat{a}_1^c$. This, in turn, implies the relation $\text{tr}(\hat{f}_2^c) = NB_2^c$. Enlarging the group form SU(N) to U(N) introduces an extra degree of freedom, which can be eliminated by postulating the invariance of the theory under the action of an additional U$(1)^{(1)}$ 1-form gauge symmetry: $\hat{a}_1^c \to \hat{a}_1^c - \lambda_1^c$. This implies that $\hat{f}_2^c$, B_1^c, and B_2^c transform as $\hat{f}_2^c \to \hat{f}_2^c - d\lambda_1^c$, $B_1^c \to B_1^c - p\lambda_1^c$, and $B_2^c \to B_2^c - d\lambda_1^c$, such that the condition $dB_1^c = pB_2^c$ remains invariant.

The mixed anomaly between the $\mathbb{Z}_p^{(1)}$ center and $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ chiral symmetries is envisaged by examining the partition function in the background of the U(N) and $\mathbb{Z}_p^{(1)}$ fluxes. In such backgrounds, the topological charge is determined by replacing f_2^c with the combination $\hat{f}_2^c - B_2^c$ in the expression for the topological charge. Importantly, this expression remains invariant under gauge transformations by λ_1^c. Thus, the topological charge is

$$\begin{aligned} Q^c &= \frac{1}{8\pi^2}\int_{\mathcal{M}_4} \text{tr}_\Box \left[\left(\hat{f}_2^c - B_2^c\right) \wedge \left(\hat{f}_2^c - B_2^c\right)\right] \\ &= \frac{1}{8\pi^2}\int_{\mathcal{M}_4} \text{tr}_\Box \left[\hat{f}_2^c \wedge \hat{f}_2^c\right] - \frac{N}{8\pi^2}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c\,, \end{aligned} \tag{5.4}$$

and is fractional. Recalling that $\int_{\mathcal{M}_4} \text{tr}_\Box \left[\hat{f}_2^c \wedge \hat{f}_2^c\right] \in 8\pi^2\mathbb{Z}$, the partition function transforms by the phase $-\frac{N}{8\pi^2}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c = -\frac{N}{8\pi^2 p^2}\int_{\mathcal{M}_4} dB_1^c \wedge dB_1^c \in \frac{N\mathbb{Z}}{p^2}$. This is the mixed anomaly between the $\mathbb{Z}_p^{(1)}$ 1-form center and $\mathbb{Z}_{2T_\mathcal{R}}^{(0)}$ discrete chiral symmetries. The anomaly is nontrivial, provided that p^2 is not a divisor of N. It is important to highlight the group-theoretical result

$$\mathbb{Z}_{m/\gcd(m,m')} \subseteq \mathbb{Z}_{T_\mathcal{R}}\,, \tag{5.5}$$

where we have expressed $N = mm'$. This result can be verified numerically; we shall use it in our analysis below.

Energy scale $\Lambda \ll E \ll v$.

At energy scale $\Lambda \ll E \ll v$, the magnitude of Φ freezes and we may set $\Phi \cong \frac{v}{\sqrt{2}}e^{ia}$. Also, the fermions acquire a mass $\sim yv$ and decouple. Then, the effective Lagrangian is:

$$\mathcal{L}_{\Lambda \ll E \ll v} = -\frac{1}{2g^2}\text{tr}\left(f_2^c \wedge \star f_2^c\right) + \frac{v^2}{2}da \wedge \star da + T_\mathcal{R} a q^c\,, \tag{5.6}$$

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where q^c the topological charge density: $Q^c = \int_{\mathcal{M}_4} q^c$, and Q^c is given by the expression (5.4). Thus, we have

$$q^c = \frac{1}{8\pi^2}\left[\mathrm{tr}_\Box\left(\hat{f}_2^c \wedge \hat{f}_2^c\right) - N B_2^c \wedge B_2^c\right]. \tag{5.7}$$

In particular, one can easily see that the Euclidean version of (5.6) reproduces the anomaly $e^{-i\frac{2\pi N}{p^2}}$ under the transformation $a \to a + \frac{2\pi}{T_\mathcal{R}}$.

In the absence of the center background, the Lagrangian (5.6) is invariant under the global symmetry group[18]

$$G^{\mathrm{global}} = \mathbb{Z}_{T_\mathcal{R}}^{(0)} \times \left(\mathbb{Z}_N^{(1)} \tilde{\times} \mathrm{U}(1)^{(2)}\right). \tag{5.8}$$

The 2-form symmetry $\mathrm{U}(1)^{(2)}$ is an emergent winding-number symmetry that acts on axion strings, while $\mathbb{Z}_N^{(1)}$ is an enhanced 1-form symmetry (remember that the UV genuine 1-form symmetry is $\mathbb{Z}_m^{(1)}$) resulting from the decoupling of fermions. Notice that there can be a higher group structure between $\mathbb{Z}_N^{(1)}$ and $\mathrm{U}(1)^{(2)}$ symmetries, and we used the symbol $\tilde{\times}$ to denote this structure. To see it, we activate a background for $\mathbb{Z}_N^{(1)}$ by introducing the pair $(B_1^{(N)}, B_2^{(N)})$ such that $NB_2^{(N)} = dB_1^{(N)}$ and demanding $\int_{M_2} dB_1^{(N)} \in 2\pi\mathbb{Z}$. This, in turn, implies the flux of $B_2^{(N)}$ is fractional: $\int_{M_2} B_2^{(N)} \in \frac{2\pi\mathbb{Z}}{N}$. The pair of fields $(B_1^{(N)}, B_2^{(N)})$ transforms under a $\mathrm{U}(1)^{(1)}$ gauge transformation as $B_1^{(N)} \to B_1^{(N)} + N\lambda_1^{(N)}$ and $B_2^{(N)} \to B_2^{(N)} + d\lambda_1^{(N)}$, which leaves the relation $NB_2^{(N)} = dB_1^{(N)}$ invariant. We also introduce C_3, the background gauge field of the global $\mathrm{U}(1)^{(2)}$ symmetry.

Inspection of (5.6), (5.7) reveals that the backgrounds of $\mathbb{Z}_N^{(1)}$ and $\mathrm{U}(1)^{(2)}$ couple to the axion via the term [42]

$$\mathcal{L} \supset \frac{1}{2\pi} a G_4\,, \tag{5.9}$$

where G_4 is the field strength of the combined backgrounds. It is given by

$$G_4 = dC_3 - \frac{T_\mathcal{R} N}{4\pi} B_2^{(N)} \wedge B_2^{(N)}. \tag{5.10}$$

G_4 is invariant under a gauge transformation by $\lambda_1^{(N)}$ provided that C_3 transforms as

$$C_3 \to C_3 + d\lambda_2 + \frac{T_\mathcal{R} N}{2\pi} \lambda_1^{(N)} \wedge B_2^{(N)} + \frac{T_\mathcal{R} N}{4\pi} \lambda_1^{(N)} \wedge d\lambda_1^{(N)}. \tag{5.11}$$

The interplay among C_3, $B_2^{(N)}$, and $\lambda_1^{(N)}$ indicates a higher-group structure, where $\mathbb{Z}_N^{(1)}$ represents the daughter symmetry and $\mathrm{U}(1)^{(2)}$ the parent symmetry. Notably, the former cannot exist independently of the latter [43], imposing constraints on the emergent (enhanced) symmetry scales: $E_{\mathbb{Z}_N^{(1)}} \lesssim E_{\mathrm{U}(1)^{(2)}}$. This condition aligns well with effective field theory expectations: $E_{\mathbb{Z}_N^{(1)}} \cong \sqrt{\lambda} v$, $E_{\mathbb{Z}_N^{(1)}} \cong yv$, and $\lambda \ll y^2 \ll 1$; see [19] for details.

In a higher-group structure, one cannot gauge the daughter symmetry without gauging the parent. But the reverse is possible. This observation shall play an important role

[18] In fact, $\mathrm{U}(1)^{(2)}$ is only approximate global symmetry. See our discussion after eq. (5.15).

below. Notice that the higher-group symmetry becomes split (trivialized) if one can write G_4 as a total derivative [42]. For example, there is no higher-group structure between the genuine $\mathbb{Z}_m^{(1)}$ symmetry of the UV theory and $\mathrm{U}(1)^{(2)}$. To see that, we replace $B_2^{(N)} \wedge B_2^{(N)}$ in eq. (5.10) by $B_2^c \wedge B_2^c$, where we use the pair (B_1^c, B_2^c) (which satisfies the constraint $mB_2^c = dB_1^c$) to activate the background of $\mathbb{Z}_m^{(1)}$. Thus, $G_4 = dC_3 - \frac{T_\mathcal{R} N}{4\pi m^2} dB_1^c \wedge dB_1^c$. Since $\mathbb{Z}_{m/\gcd(m,m')} \subseteq \mathbb{Z}_{T_\mathcal{R}}$ (remember that $N = mm'$), we can write $\frac{T_\mathcal{R} N}{m^2} = m''$, $m'' \in \mathbb{N}$, and hence, $G_4 = dC_3' \equiv d\left(C_3 - \frac{m''}{4\pi} B_1^c \wedge dB_1^c\right)$, trivializing the higher-group. This observation is important for our subsequent analysis.

Energy scale $E \ll \Lambda$.

Next, we flow to the deep IR at energy scale $\ll \Lambda$, where we assume the theory confines, and hence, the color gauge field is gapped. We must write down an effective Lagrangian that captures the UV center-chiral anomaly. This can be achieved by (i) gauging the $\mathrm{U}(1)^{(2)}$ symmetry and introducing the dynamical 3-form gauge field c_3 and (ii) replacing $\mathrm{tr}_\Box\left(\hat{f}_2^c \wedge \hat{f}_2^c\right)/4\pi$ in eqs. (5.6), (5.7) by dc_3:

$$\mathcal{L}_{E\ll\Lambda} = \frac{v^2}{2} da \wedge \star da + \frac{T_\mathcal{R} a}{2\pi}\left(dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c\right) + \Lambda^4 \mathcal{K}\left(\frac{dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c}{\Lambda^4}\right), \tag{5.12}$$

and we added a kinetic energy term for c_3. The field strength of c_3 satisfies the quantization condition $\int_{\mathcal{M}_4} dc_3 \in 2\pi\mathbb{Z}$, which is simply the infrared manifestation of the quantization of topological charges in Yang-Mills theory. The reader will notice that the coupling between a and dc_3 has an extra factor of $T_\mathcal{R}$ compared to the coupling in eq. (5.9). This is because c_3 in eq. (5.12) is a dynamical rather than a background field, and as the dynamical field absorbs the axion, it should describe the formation of $T_\mathcal{R}$ domain walls. As we shall discuss later, at an energy scale below Λ, the theory has enhanced $\mathbb{Z}_N^{(1)}$ 1-form symmetry. As noted above, we are allowed to gauge $\mathrm{U}(1)^{(2)}$ without gauging the daughter symmetry $\mathbb{Z}_N^{(1)}$. This is important; otherwise, we would have changed the theory's global structure and run into trouble since $\mathbb{Z}_N^{(1)}$ is not a genuine symmetry of the theory.

The Lagrangian (5.12) must pass several checks. First, it must be invariant under the $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ chiral symmetry in the absence of the center background, which is evident from the transformation $a \to a + \frac{2\pi}{T_\mathcal{R}}$ along with the condition $\int_{\mathcal{M}_4} dc_3 \in 2\pi\mathbb{Z}$. Second, the Lagrangian must be invariant under the same auxiliary $\mathrm{U}(1)^{(1)}$ gauge transformation, by λ_1^c, of the UV theory. This is the case provided that c_3 transforms as

$$c_3 \to c_3 + d\lambda_2 + \frac{p'}{2\pi} B_1^c \wedge d\lambda_1^c + \frac{pp'}{4\pi} \lambda_1^c \wedge d\lambda_1^c\,, \tag{5.13}$$

and we wrote $N = pp'$. Also, the Lagrangian (5.12) must reproduce the mixed $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$-$\mathbb{Z}_p^{(1)}$ anomaly of the UV theory. This can be easily verified by observing that the partition function acquires the phase $e^{-i\frac{2\pi N}{p^2}}$ when a is shifted by $a \to a + \frac{2\pi}{T_\mathcal{R}}$ in the presence of the center background. In the absence of a center background, the Lagrangian (5.12) exactly matches (2.16) in section 2.1, and everything we said there applies here.

Another check[19] on the validity of (5.12) is to integrate out c_3 along the lines of our discussion that led from eq. (2.25) to eq. (2.28). Thus, we sum over arbitrary values of the

[19]We thank T. Tanizaki for pointing this out.

integers $\int_{\mathcal{M}_4} f_4 \in 2\pi\mathbb{Z}$ and use the Poisson resummation formula $\sum_{m\in\mathbb{Z}} \delta\left(2\pi m - \int_{\mathcal{M}_4} f_4\right) = \sum_{k\in\mathbb{Z}} e^{-ik\int_{\mathcal{M}_4} f_4}$. We also use the change of variables $\hat{f}_4 = f_4 - \frac{N}{4\pi} B_2^c \wedge B_2^c$. Specifying to the canonical kinetic term $\mathcal{K}_{\text{can}}$, performing the Gaussian integral over $\hat{f}_4$, and focusing only on the zero modes, we obtain the Euclidean partition function

$$Z[a] \sim \sum_{k\in\mathbb{Z}} e^{-i\frac{kN}{4\pi}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c} e^{-\frac{\Lambda^4 V_{\mathcal{M}_4}}{8\pi^2}(T_{\mathcal{R}} a + 2\pi k)^2} . \tag{5.14}$$

This effective partition function picks up the anomaly $e^{i\frac{N}{4\pi}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c} = e^{-i\frac{2\pi N}{p^2}}$ upon shifting $a \to a + \frac{2\pi}{T_{\mathcal{R}}}$. It also displays the expected structure of the Yang-Mills theory: it has an infinite number of vacua, with the true vacuum energy density given by

$$V(a) = \frac{\Lambda^4}{8\pi^2} \text{min}_k \left(T_{\mathcal{R}} a + 2\pi k\right)^2 . \tag{5.15}$$

The potential $V(a)$ has $T_{\mathcal{R}}$ minima with cusps at $a = \pi/T_{\mathcal{R}}, 3\pi/T_{\mathcal{R}}$, etc. . . The cusps indicate that the potential $V(a)$ is missing degrees of freedom at these locations. These are the hadronic walls sandwiched between the axion domain walls. An axion wall has width $\sim v/\Lambda^2$, while a hadronic wall is much thinner with width $\sim \Lambda^{-1}$. Including the infinite sum over all the integers $\int_{\mathcal{M}_4} f_4 \in 2\pi m$, $m \in \mathbb{Z}$ was crucial to see these cusps. As emphasized above, the integer m is the IR manifestation of the Yang-Mills instantons' topological charge. Below Λ, the theory is strongly coupled, and the vacuum receives contributions from all the topological charge sectors.

Let us examine the theory's behavior at energy scales $\Lambda \ll E \ll v$, such as at a corresponding temperature. In this case, it suffices to include the contribution from minimal charges $\int_{\mathcal{M}_4} f_4 = m = \pm 1$ in the partition function. Translating this into the language of Yang-Mills instantons, the dilute instanton-gas approximation is reliable at this temperature because it serves as an infrared cut-off on the instanton's scale modulus [44]. Thus, summing over the smallest instantons, which possess topological charges of ± 1, is adequate. Limiting the sum over m to the lowest charge sector means that one can no longer perform the Poisson resummation that leads to (5.14), and thus, one can no longer make sense of (5.15) or the cusps. This is consistent with the expectation that the hadronic walls melt away at a temperature $\sim \Lambda$. Nevertheless, at temperatures in the range $\Lambda \ll T \ll v$, the classical theory (5.12) still possesses $T_{\mathcal{R}}$ vacua with axion domain walls interpolating between them. Notice that in this energy range, c_3 does not strongly fluctuate (since$\int_{\mathcal{M}_4} f_4 = \pm 1$), and we can consider the $\mathrm{U}(1)^{(2)}$ 3-form gauge field c_3 as a background rather than a dynamical field. Thus, one may still regard $\mathrm{U}(1)^{(2)}$ as an approximate global symmetry. Eventually, the axion domain walls melt at temperature $T \gtrsim v$.

As noted in [1, 5, 14] and discussed in the Introduction, it was recognized that the IR behavior of both pure YM theory in the large-N limit and the axion-YM theory can be effectively described using the 3-form gauge field c_3. Here, incorporating c_3 in our discussion has been essential for aligning with the infrared constraints of the 't Hooft anomaly. Our method provides a systematic approach to argue for a consistent infrared effective field theory of the axion-YM system.

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At energy scales below Λ, the theory acquires the global symmetry

$$G^{\text{global}} = \mathbb{Z}_{T_{\mathcal{R}}}^{(0)} \times \left(\mathbb{Z}_N^{(1)} \tilde{\times} \mathbb{Z}_{T_{\mathcal{R}}}^{(3)}\right) . \tag{5.16}$$

In general, a higher-group structure may exist between $\mathbb{Z}_N^{(1)}$ and $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$, which becomes apparent when both symmetries' backgrounds are activated. The background of $\mathbb{Z}_N^{(1)}$ was discussed earlier, while that of $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ can be activated by introducing the pair (F_3, F_4), satisfying the constraint $dF_3 = T_{\mathcal{R}} F_4$, along with the quantization condition $\int_{\mathcal{M}4} dF_3 \in 2\pi\mathbb{Z}$ [45]. The axion coupled to these backgrounds is represented by

$$\mathcal{L} \supset \frac{a}{2\pi}\left(T_{\mathcal{R}} dc_3 + dF_3 - \frac{T_{\mathcal{R}} N}{4\pi} B_2^{(N)} \wedge B_2^{(N)}\right) . \tag{5.17}$$

Maintaining invariance under a gauge transformation by $\lambda_1^{(N)}$ requires F_3 to transform as

$$F_3 \to F_3 + d\lambda_2 + \frac{T_{\mathcal{R}} N}{2\pi} d\lambda_1^{(N)} \wedge B_2^{(N)} + \frac{T_{\mathcal{R}} N}{4\pi} d\lambda_1^{(N)} \wedge d\lambda_1^{(N)} . \tag{5.18}$$

The interplay among F_3, $B_2^{(N)}$, and $\lambda_1^{(N)}$ indicates a higher-group symmetry $\mathbb{Z}_N^{(1)} \tilde{\times} \mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$. However, this higher-group structure becomes trivial if $dF_3 - \frac{N T_{\mathcal{R}}}{4\pi N^2} dB_1^{(N)} \wedge dB_1^{(N)}$ can be expressed as a total derivative. This holds particularly true for the $\mathbb{Z}_m^{(1)}$ symmetry, as demonstrated above in a similar case involving $\mathrm{U}(1)^{(2)}$ and $\mathbb{Z}_m^{(1)}$. Understanding this aspect is pivotal when gauging the genuine center, as this operation should be executed without gauging $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$.

In the IR, the symmetries $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)}$ and $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ undergo spontaneous breaking. The enhanced symmetry $\mathbb{Z}_N^{(1)}$ remains unbroken until length scales $\sim yv/\Lambda^2$, at which point it also undergoes explicit breaking due to the heavy fermions that pop up from vacuum as we take the Wilson lines to be larger than $\sim yv/\Lambda^2$. This leaves $\mathbb{Z}_m^{(1)}$ as the sole surviving unbroken symmetry.

5.2 SU(N)/$\mathbb{Z}_p$ and noninvertible chiral symmetry

Let us investigate whether our construction yields the desired results when we gauge the genuine center or any of its subgroups, aligning with the well-established findings in the literature [23, 46, 47]. We shall see that the answer is affirmative, lending support to the picture that the deep IR regime of the system is genuinely described by the 3-form gauge theory.

We consider the same axion-YM theory with matter, but now let us gauge a subgroup of the center $\mathbb{Z}_p^{(1)} \subseteq \mathbb{Z}_m^{(1)}$, i.e., we consider SU($N$)/$\mathbb{Z}_p$ axion-YM theory with matter.[20] This theory is constructed by promoting (B_1^c, B_2^c) to dynamical fields (b_1^c, b_2^c) and performing the sum over the fractional instantons in the path integral. Let us define the new 3-form gauge field $\hat{c}_3$:

$$\hat{c}_3 \equiv c_3 - \frac{N}{4\pi p^2} b_1^c \wedge db_1^c , \tag{5.19}$$

[20]In principle, there are p distinct theories: $(\mathrm{SU}(N)/\mathbb{Z}_p)_k$, $k = 0, 1, \ldots, p-1$ differing by the admissible genuine (electric, magnetic, or dyonic) line operators [48]. In this paper, we limit our treatment to $(\mathrm{SU}(N)/\mathbb{Z}_p)_{k=0}$. The Hilbert space and the noninvertible chiral symmetry in $(\mathrm{SU}(N)/\mathbb{Z}_p)_{k=0}$ theory were considered in [47].

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keeping in mind the quantization condition $\int_{\mathcal{M}_4} dc_3 \in 2\pi\mathbb{Z}$. The Lagrangian of this theory at energy scale $E \ll \Lambda$ reads

$$\mathcal{L}_{E\ll\Lambda}[(b_1^c, b_2^c)] = \frac{v^2}{2} da \wedge \star da + \frac{T_{\mathcal{R}}}{2\pi} a \wedge d\hat{c}_3 + \Lambda^4 \mathcal{K}\left(\frac{d\hat{c}_3}{\Lambda^4}\right) , \tag{5.20}$$

and the partition function is

$$Z = \sum_{(b_1^c, b_2^c)} \int [Dc_3][Da] e^{i\int_{\mathcal{M}_4} \mathcal{L}_{E\ll\Lambda}[(b_1^c, b_2^c)]} . \tag{5.21}$$

In the Kalb-Ramond frame, we replace $q \to T_{\mathcal{R}}$ and $c_3 \to \hat{c}_3$ in (3.13). It is important to repeat what we stated above: we can gauge the genuine $\mathbb{Z}_m^{(1)}$ center symmetry or a subgroup thereof without spoiling $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ since the pair does not constitute a higher-group. On the contrary, gauging the enhanced $\mathbb{Z}_N^{(1)}$ is disastrous: it entails that we also gauge $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$, which destroys the domain walls.

The chiral symmetry defect is given by (2.18) after replacing c_3 with $\hat{c}_3$ and summing over (b_1^c, b_2^c):

$$\tilde{U}_\ell^{(0)}(\mathcal{M}_3) \sim \sum_{(b_1^c, b_2^c)} e^{i\frac{2\pi\ell}{T_{\mathcal{R}}}\int_{\mathcal{M}_3} v^2 \star da - i\ell\int_{\mathcal{M}_3}\left(c_3 - \frac{N}{4\pi p^2} b_1^c \wedge db_1^c\right)} , \quad \ell = 1, 2, \ldots, T_{\mathcal{R}} . \tag{5.22}$$

The new defect $\tilde{U}_\ell^{(0)}(\mathcal{M}_3)$ defines a noninvertible chiral symmetry $\tilde{\mathbb{Z}}_{T_{\mathcal{R}}}^{(0)}$. To simplify the form of $\tilde{U}_\ell^{(0)}(\mathcal{M}_3)$, we write N as $N = pp'$ and assume that $p' = 1$ Mod p. Then, $\frac{N}{4\pi p^2} b_1^c \wedge db_1^c \sim \frac{1}{4\pi p} b_1^c \wedge db_1^c$, i.e., this is an improperly quantized quantum Hall term. We may rewrite it in terms of an auxiliary 1-form gauge field φ_1 that lives on $\mathcal{M}_3$:

$$\tilde{U}_\ell^{(0)}(\mathcal{M}_3) \sim \sum_{(b_1^c, b_2^c)} e^{i\frac{2\pi\ell}{T_{\mathcal{R}}}\int_{\mathcal{M}_3} v^2 \star da - i\ell\int_{\mathcal{M}_3} c_3} \int [D\varphi_1] e^{-i\ell\int_{\mathcal{M}_3}\left(\frac{p}{4\pi}\varphi_1 \wedge d\varphi_1 + \frac{1}{2\pi}\varphi_1 \wedge db_1^c\right)} . \tag{5.23}$$

The last term is the minimal abelian TQFT $\mathcal{A}^{p,1}$ discussed in [49]. When ℓ is a multiple of p, the TQFT is trivialized, giving us an invertible symmetry.

6 Discussion

In this letter, we critically evaluated the hypothesis that the 3-form gauge theory offers more than an alternative framework for the deep IR regime of axion-Yang-Mills systems. We commenced by rigorously investigating the 3-form gauge theory coupled to axions, as encapsulated by the Lagrangian (2.16). Our analysis revealed that this theory represents a network of domain walls terminating on an axion string, with particular emphasis placed on its global structure. A dual formulation, the Kalb-Ramond Lagrangian (3.13), was also considered, which describes the same physical phenomenon. Notably, while some symmetry defects are explicit in one formulation, others are evident in the alternative frame. Crucially, in the absence of gravitational effects,[21] both formulations are equivalent, possessing identical global symmetries.

[21] Inclusion of gravity may differentiate between the axion and Kalb-Ramond frames, see [8].

Subsequently, we examined the SU(N) Yang-Mills theory with a Dirac fermion coupled to an SU(N)-neutral complex scalar, highlighting the necessity of an emergent 3-form gauge field for IR matching of the theory's mixed center-chiral anomaly. Consider varying the complex scalar vev such that we go to the limit $v \ll \Lambda$. In this case, the strong dynamics set in well before the axions are amenable to the weak-coupling treatment. In this opposite limit, the theory still forms domain walls leading to $T_{\mathcal{R}}$ distinct vacua, and thus, we do not expect a bulk phase transition to take place as we vary v below or above Λ. We may not rigorously justify the introduction of the 3-form gauge field in this scenario. However, by continuity, we expect that the 3-form gauge theory remains a valid description in the deep IR. This reasoning, combined with the large-N limit analysis discussed in the introduction, supports the notion that the vacuum of Yang-Mills theory should likely be described by a 3-form gauge theory.

Incorporating gravity offers further insights into the significance of the 3-form description, as the cosmological constant in this context can be interpreted as arising from a gauge principle. Theoretically, one can distinguish between a pure cosmological constant and a 3-form gauge theory, as the latter yields a non-vanishing contribution to the trace anomaly proportional to the Gauss-Bonnet invariant [8, 50]. Irrespective of this subtle effect, Brown and Teitelboim realized that the action (1.2) taken as a starting point with no reference to its UV completion leads to the quantum creation of closed membranes localized on the boundary of $\mathcal{M}_4$ [51, 52]. As the membranes are produced, the vacuum energy density associated with c_3 decreases, reducing the effective value of the cosmological constant. This idea, when refined, may lead to a solution to the cosmological constant problem [53, 54]. Also, connections between the QCD vacuum and the cosmological constant problem were discussed in [55, 56].

As discussed in the paper, introducing the 3-form gauge field c_3 was necessary to match the chiral-center anomaly in an axion-Yang-Mills system. Eventually, c_3 eats the axion, becoming a short-range field, and the cosmological constant vanishes. Yet, one can think of an alternative scenario with an axion, two distinct Yang-Mills fields, and a chiral-center anomaly. In this case, two 3-form gauge fields are anticipated. One combination of these fields eats the axion, while an orthogonal combination remains gapless. The latter can source a cosmological constant. Intriguingly, in this scenario, the infrared cosmological constant can be considered a by-product of the 't Hooft anomaly-matching condition. However, in such a scenario, global symmetries should only be considered approximate since exact global symmetries are forbidden in quantum gravity, see, e.g., [57–59].

The method presented in this paper can be extended in various ways. One immediate application is to address the problem of multi-flavor quarks by incorporating the 3-form gauge theory description in the chiral Lagrangian. Another venue is applying our formalism to the Standard Model (SM) and its variants, potentially through coupling with axions. It is well-established that the SM exhibits a $\mathbb{Z}_6^{(1)}$ 1-form symmetry, and the true gauge group might be modded by $\mathbb{Z}_6$ or a subgroup thereof (see, e.g., [60–66]). Exploring whether and how a 3-form gauge theory may emerge deep in the IR of the SM and its extensions and whether the modded discrete group could play a significant role in this formalism will be an exciting avenue of research. Additionally, linking the emergent 3-form to the observed cosmological constant presents another intriguing possibility. These investigations are worthy of future exploration.

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References

[1] M. Luscher, *The Secret Long Range Force in Quantum Field Theories With Instantons*, *Phys. Lett. B* **78** (1978) 465 [INSPIRE].

[2] J.B. Kogut and L. Susskind, *How to Solve the $\eta \to 3\pi$ Problem by Seizing the Vacuum*, *Phys. Rev. D* **11** (1975) 3594 [INSPIRE].

[3] G. Veneziano, *U(1) Without Instantons*, *Nucl. Phys. B* **159** (1979) 213 [INSPIRE].

[4] P. Di Vecchia and G. Veneziano, *Chiral Dynamics in the Large n Limit*, *Nucl. Phys. B* **171** (1980) 253 [INSPIRE].

[5] G. Gabadadze and M. Shifman, *QCD vacuum and axions: What's happening?*, *Int. J. Mod. Phys. A* **17** (2002) 3689 [hep-ph/0206123] [INSPIRE].

[6] E. Witten, *Large N Chiral Dynamics*, *Annals Phys.* **128** (1980) 363 [INSPIRE].

[7] A. Aurilia, H. Nicolai and P.K. Townsend, *Hidden Constants: The Theta Parameter of QCD and the Cosmological Constant of N=8 Supergravity*, *Nucl. Phys. B* **176** (1980) 509 [INSPIRE].

[8] M.J. Duff and P. van Nieuwenhuizen, *Quantum Inequivalence of Different Field Representations*, *Phys. Lett. B* **94** (1980) 179 [INSPIRE].

[9] P.G.O. Freund and M.A. Rubin, *Dynamics of Dimensional Reduction*, *Phys. Lett. B* **97** (1980) 233 [INSPIRE].

[10] H. Hata, T. Kugo and N. Ohta, *Skew Symmetric Tensor Gauge Field Theory Dynamically Realized in QCD U(1) Channel*, *Nucl. Phys. B* **178** (1981) 527 [INSPIRE].

[11] M. Nitta and R. Yokokura, *Higher derivative three-form gauge theories and their supersymmetric extension*, *JHEP* **10** (2018) 146 [arXiv:1809.03957] [INSPIRE].

[12] M. Nitta and R. Yokokura, *Topological couplings in higher derivative extensions of supersymmetric three-form gauge theories*, *JHEP* **05** (2019) 102 [arXiv:1810.12678] [INSPIRE].

[13] R.D. Peccei and H.R. Quinn, *Constraints Imposed by CP Conservation in the Presence of Instantons*, *Phys. Rev. D* **16** (1977) 1791 [INSPIRE].

[14] G. Dvali, *Three-form gauging of axion symmetries and gravity*, hep-th/0507215 [INSPIRE].

[15] G. Dvali, *Strong-CP with and without gravity*, arXiv:2209.14219 [INSPIRE].

[16] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].

[17] T.D. Brennan and S. Hong, *Introduction to Generalized Global Symmetries in QFT and Particle Physics*, arXiv:2306.00912 [INSPIRE].

[18] Y. Hidaka, M. Nitta and R. Yokokura, *Emergent discrete 3-form symmetry and domain walls*, *Phys. Lett. B* **803** (2020) 135290 [arXiv:1912.02782] [INSPIRE].

[19] T.D. Brennan and C. Cordova, *Axions, higher-groups, and emergent symmetry*, *JHEP* **02** (2022) 145 [arXiv:2011.09600] [INSPIRE].

[20] R. Yokokura, *Non-invertible symmetries in axion electrodynamics*, arXiv:2212.05001 [INSPIRE].

[21] N. Yamamoto and R. Yokokura, *Topological mass generation in gapless systems*, *Phys. Rev. D* **104** (2021) 025010 [arXiv:2009.07621] [INSPIRE].

[22] Y. Hidaka, M. Nitta and R. Yokokura, *Global 4-group symmetry and 't Hooft anomalies in topological axion electrodynamics*, *PTEP* **2022** (2022) 04A109 [arXiv:2108.12564] [INSPIRE].

[23] J. Kaidi, K. Ohmori and Y. Zheng, *Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories*, *Phys. Rev. Lett.* **128** (2022) 111601 [arXiv:2111.01141] [INSPIRE].

[24] C. Cordova and K. Ohmori, *Noninvertible Chiral Symmetry and Exponential Hierarchies*, *Phys. Rev. X* **13** (2023) 011034 [arXiv:2205.06243] [INSPIRE].

[25] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Global Symmetries in the Standard Model*, *Phys. Rev. Lett.* **129** (2022) 161601 [arXiv:2205.05086] [INSPIRE].

[26] L. Bhardwaj, S. Schafer-Nameki and J. Wu, *Universal Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200143 [arXiv:2208.05973] [INSPIRE].

[27] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible symmetries and higher representation theory I*, *SciPost Phys.* **17** (2024) 015 [arXiv:2208.05993] [INSPIRE].

[28] M.M. Anber and E. Poppitz, *Noninvertible anomalies in SU(N) × U(1) gauge theories*, *JHEP* **08** (2023) 149 [arXiv:2305.14425] [INSPIRE].

[29] S.-H. Shao, *What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries*, arXiv:2308.00747 [INSPIRE].

[30] S. Schafer-Nameki, *ICTP lectures on (non-)invertible generalized symmetries*, *Phys. Rept.* **1063** (2024) 1 [arXiv:2305.18296] [INSPIRE].

[31] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and Duality*, *JHEP* **04** (2014) 001 [arXiv:1401.0740] [INSPIRE].

[32] D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, *Theta, Time Reversal, and Temperature*, *JHEP* **05** (2017) 091 [arXiv:1703.00501] [INSPIRE].

[33] N. Seiberg, Y. Tachikawa and K. Yonekura, *Anomalies of Duality Groups and Extended Conformal Manifolds*, *PTEP* **2018** (2018) 073B04 [arXiv:1803.07366] [INSPIRE].

[34] M.M. Anber and E. Poppitz, *Deconfinement on axion domain walls*, *JHEP* **03** (2020) 124 [arXiv:2001.03631] [INSPIRE].

[35] T. Fugleberg, I.E. Halperin and A. Zhitnitsky, *Domain walls and theta dependence in QCD with an effective Lagrangian approach*, *Phys. Rev. D* **59** (1999) 074023 [hep-ph/9808469] [INSPIRE].

[36] G. Gabadadze and M.A. Shifman, *D walls and junctions in supersymmetric gluodynamics in the large N limit suggest the existence of heavy hadrons*, *Phys. Rev. D* **61** (2000) 075014 [hep-th/9910050] [INSPIRE].

JHEP10(2024)113

[37] D. Aloni, E. García-Valdecasas, M. Reece and M. Suzuki, *Spontaneously broken (-1)-form U(1) symmetries*, *SciPost Phys.* **17** (2024) 031 [arXiv:2402.00117] [INSPIRE].

[38] M. Kalb and P. Ramond, *Classical direct interstring action*, *Phys. Rev. D* **9** (1974) 2273 [INSPIRE].

[39] M.M. Anber and E. Poppitz, *On the baryon-color-flavor (BCF) anomaly in vector-like theories*, *JHEP* **11** (2019) 063 [arXiv:1909.09027] [INSPIRE].

[40] M.M. Anber and E. Poppitz, *Generalized 't Hooft anomalies on non-spin manifolds*, *JHEP* **04** (2020) 097 [arXiv:2002.02037] [INSPIRE].

[41] M.M. Anber, *Condensates and anomaly cascade in vector-like theories*, *JHEP* **03** (2021) 191 [arXiv:2101.04132] [INSPIRE].

[42] Y. Choi, M. Forslund, H.T. Lam and S.-H. Shao, *Quantization of Axion-Gauge Couplings and Noninvertible Higher Symmetries*, *Phys. Rev. Lett.* **132** (2024) 121601 [arXiv:2309.03937] [INSPIRE].

[43] C. Córdova, T.T. Dumitrescu and K. Intriligator, *Exploring 2-Group Global Symmetries*, *JHEP* **02** (2019) 184 [arXiv:1802.04790] [INSPIRE].

[44] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and Instantons at Finite Temperature*, *Rev. Mod. Phys.* **53** (1981) 43 [INSPIRE].

[45] Y. Tanizaki and M. Ünsal, *Modified instanton sum in QCD and higher-groups*, *JHEP* **03** (2020) 123 [arXiv:1912.01033] [INSPIRE].

[46] R. Argurio and R. Vandepopeliere, *When $\mathbb{Z}_2$ one-form symmetry leads to non-invertible axial symmetries*, *JHEP* **08** (2023) 205 [arXiv:2306.01414] [INSPIRE].

[47] M.M. Anber and S.Y.L. Chan, *Noninvertible symmetries and anomalies from gauging 1-form electric centers*, *JHEP* **03** (2024) 169 [arXiv:2311.07662] [INSPIRE].

[48] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115 [arXiv:1305.0318] [INSPIRE].

[49] P.-S. Hsin, H.T. Lam and N. Seiberg, *Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d*, *SciPost Phys.* **6** (2019) 039 [arXiv:1812.04716] [INSPIRE].

[50] S.M. Christensen and M.J. Duff, *New Gravitational Index Theorems and Supertheorems*, *Nucl. Phys. B* **154** (1979) 301 [INSPIRE].

[51] J.D. Brown and C. Teitelboim, *Dynamical Neutralization of the Cosmological Constant*, *Phys. Lett. B* **195** (1987) 177 [INSPIRE].

[52] J.D. Brown and C. Teitelboim, *Neutralization of the Cosmological Constant by Membrane Creation*, *Nucl. Phys. B* **297** (1988) 787 [INSPIRE].

[53] R. Bousso and J. Polchinski, *Quantization of four form fluxes and dynamical neutralization of the cosmological constant*, *JHEP* **06** (2000) 006 [hep-th/0004134] [INSPIRE].

[54] R. Bousso, *TASI Lectures on the Cosmological Constant*, *Gen. Rel. Grav.* **40** (2008) 607 [arXiv:0708.4231] [INSPIRE].

[55] N. Kaloper and J. Terning, *Landscaping the Strong CP Problem*, *JHEP* **03** (2019) 032 [arXiv:1710.01740] [INSPIRE].

[56] F.R. Urban and A.R. Zhitnitsky, *The cosmological constant from the QCD Veneziano ghost*, *Phys. Lett. B* **688** (2010) 9 [arXiv:0906.2162] [INSPIRE].

JHEP10(2024)113

[57] C.W. Misner and J.A. Wheeler, *Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space*, *Annals Phys.* **2** (1957) 525 [INSPIRE].

[58] T. Banks and N. Seiberg, *Symmetries and Strings in Field Theory and Gravity*, *Phys. Rev. D* **83** (2011) 084019 [arXiv:1011.5120] [INSPIRE].

[59] D. Harlow and H. Ooguri, *Symmetries in quantum field theory and quantum gravity*, *Commun. Math. Phys.* **383** (2021) 1669 [arXiv:1810.05338] [INSPIRE].

[60] D. Tong, *Line Operators in the Standard Model*, *JHEP* **07** (2017) 104 [arXiv:1705.01853] [INSPIRE].

[61] M.M. Anber and E. Poppitz, *Nonperturbative effects in the Standard Model with gauged 1-form symmetry*, *JHEP* **12** (2021) 055 [arXiv:2110.02981] [INSPIRE].

[62] J. Wang and X.-G. Wen, *Nonperturbative definition of the standard models*, *Phys. Rev. Res.* **2** (2020) 023356 [arXiv:1809.11171] [INSPIRE].

[63] Z. Wan and J. Wang, *Beyond Standard Models and Grand Unifications: Anomalies, Topological Terms, and Dynamical Constraints via Cobordisms*, *JHEP* **07** (2020) 062 [arXiv:1910.14668] [INSPIRE].

[64] H.-L. Li and L.-X. Xu, *Understanding the SM gauge group from SMEFT*, *JHEP* **07** (2024) 199 [arXiv:2404.04229] [INSPIRE].

[65] R. Alonso, D. Dimakou and M. West, *Fractional-charge hadrons and leptons to tell the Standard Model group apart*, arXiv:2404.03438 [INSPIRE].

[66] S. Koren and A. Martin, *Fractionally Charged Particles at the Energy Frontier: The SM Gauge Group and One-Form Global Symmetry*, arXiv:2406.17850 [INSPIRE].

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