



Positive Reinforced Generalized Time-Dependent Pólya Urns via Stochastic Approximation

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Abstract

Consider a generalized time-dependent Pólya urn process defined as follows. Let $d \in \mathbb{N}$ be the number of urns/colors. At each time n , we distribute σ_n balls randomly to the d urns, proportionally to f , where f is a valid reinforcement function. We consider a general class of positive reinforcement functions \mathcal{R} assuming some monotonicity and growth condition. The class \mathcal{R} includes convex functions and the classical case $f(x) = x^\alpha, \alpha > 1$. The novelty of the paper lies in extending stochastic approximation techniques to the d -dimensional case and proving that eventually the process will fixate at some random urn and the other urns will not receive any balls anymore.

Keywords Generalized Pólya urn models · Time-dependent Pólya urn models · Positive reinforcement · Stochastic approximation · Dominance · Fixation

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1 Introduction

Pólya urn models

The classical Pólya urn model was introduced by Eggenberger and Pólya in 1923, [13]. Since then, many generalizations and extensions of the classical model have been studied, see e.g. [27] for a survey.

Starting with two different urns containing one ball (or equivalently two different colors in one bin), we add new balls to either urn at each discrete time step depending on some predefined addition rule. Some of the fundamental questions are: understanding

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the proportion of balls in a certain urn as time goes to infinity and how it depends on the addition rule. There are numerous applications in economics, computer science and biology where the model is better known as *balls and bins model*, [1, 4, 11, 12, 28, 30] to mention a few.

A popular generalization is the *nonlinear Pólya urn model* or *balls and bin model with feedback*. The probability of a new ball choosing a bin with x existing balls is proportional to $f(x)$ where f will be referred to as the feedback function, [12]. A common choice of the feedback function is $f(x) = x^\alpha$, $\alpha > 0$.

For the original case $\alpha = 1$, it was proved in [13] that the proportion of balls in each bin converges to a beta-distributed random variable.

Positive and negative feedback functions

In the positive feedback regime $\alpha > 1$, which is also referred to as preferential attachment, the authors in [12] proved *dominance*, i.e., almost surely the proportion of each bin converges to a $\{0, 1\}$ -random variable. A stronger result which we call *fixation* or *monopoly* was proven by [20]. Fixation refers to the event that eventually one bin receives all but a finite number of balls. The onset of the time of fixation (speed of convergence toward the stationary distribution) in the positive feedback regime was studied in [25]. In fact, the author in [25] studies the onset of fixation for more general feedback functions satisfying some growth conditions and being perturbations of the canonical case x^α . In particular for that class of feedback functions, we have that $\sum_{x=1}^{\infty} \frac{1}{f(x)} < \infty$.

The *negative feedback regime* is characterized by $\alpha < 1$, and in this case the proportion of balls in the urns converges toward $U[\{1, \dots, d\}]$, the uniform distribution on $\{1, 2, \dots, d\}$ where d is the number of urns, [12]. A time-dependent version of positively reinforced 2-urn models was studied in [26, 31] where at each time n , σ_n balls (satisfying some growth condition) were randomly added to either one urn [26] or independently to both [31]. In both cases, there will be dominance. Depending on some growth conditions of σ_n , [31] demonstrates there might be no fixation.

General reinforcement functions

Other generalizations include considering d urns or colors, $d > 2$ and more general addition rules like addition generating matrices and convex resp. concave feedback functions, [2, 3, 22, 23]. The replacement matrix $H = ((H_{ij}))_{ij}$ models the placement of H_{ij} balls to urn j when urn i is chosen. In [23], the authors prove that if the feedback function is strictly concave and H bi-stochastic, then the urn composition converges toward the uniform distribution $U[\{1, \dots, d\}]$. The limiting distribution will not be uniform with probability one if f is convex. Finally, they prove in the same paper that for $d = 2$, f convex and H irreducible, the limiting proportion converges toward the equilibrium points of the corresponding mean-field function resulting from the stochastic approximation approach. A concave feedback function, which includes the negative reinforcement regime, tends to equalize the asymptotic distribution of the proportion of different colors, whereas a convex f which includes the positive

reinforcement regime tends to amplify the effect of the generating matrix H . In [19], the author proves CLT type results for the proportion vector of colors around the uniform distribution in the negative reinforcement setting, when H is double stochastic and f Lipschitz.

Infinitely many colors

Urn models with infinitely many colors were treated in [5, 6, 18]. In [6], the authors introduce a class of balanced urn schemes with infinitely many colors indexed by \mathbb{Z}^d where the replacement schemes are given by the transition matrices associated with bounded increment random walks. They show that the urn composition of the n -th selected ball follows a Gaussian distribution. The authors in [24] generalized the possibly infinite space of urns to general Polish spaces and study the asymptotic behavior of these measure-valued Pólya urn processes. The author in [18] generalizes results obtained in [6, 24] and studies measure-valued Pólya urn processes under stochastic replacement matrices H .

Interacting urn models

A simplistic model for the reinforcement of neural connections in the brain using positive reinforced *interacting Pólya urns* was introduced in [16]. The urns/colors represent the edges of a graph. Roughly speaking, one first chooses a random subset of colors (independent of the past) from n colors of balls and then positively reinforces a color from this subset. In [16], the stability of equilibria and examples of different graphs were studied. Interesting follow-up research on percolation questions on different positively reinforced tree-like graphs and its application for neuronal connections were studied in [15, 17].

Generalized time-dependent Pólya urns

In this article, we consider a generalized time-dependent Pólya urn model with d urns, $d \in \mathbb{N}$. More precisely, at each time step, $\sigma_n (\geq 1)$ many balls are added randomly to the d urns, f -proportionally to their weight with instantaneous replacement. We will assume that

$$\sum_{n=1}^{\infty} \left(\frac{\sigma_n}{\sum_{j=1}^n \sigma_j} \right)^2 < \infty.$$

This assumption restricts the growth of σ_n . Polynomial growth is allowed but not exponential growth, since it contradicts this assumption. This condition is necessary to ensure that the process, seen as a stochastic approximation (SA) scheme, is not subjected to a large noise term which will hinder the process to converge. This condition appears naturally when using SA techniques from the dynamical systems viewpoint, see, e.g., [29] or [10, Sect. 1]. The class of reinforcement functions \mathcal{R} (see Sect. 2.1.1

for its definition) is very general and satisfies some natural continuity and monotony conditions. W.l.o.g. we will evaluate f not on the number of balls in an urn but on the proportions. Additionally, we assume that

$$\alpha := \inf_{x \in (0,1)} \frac{xf'(x)}{f(x)} > 1. \quad (1)$$

Note that if f is convex or of the type $f(x) = x^\alpha$, $\alpha > 1$, then (1) is necessarily satisfied, but the converse is not necessarily true. A similar condition can be found in [23, 25]; in [23], the authors assume additionally that f is convex and H irreducible. Our case is not covered in [23], since taking H to be the identity matrix is not irreducible. We will prove in Theorems 1 and 2 that there is dominance and fixation ($\alpha > 3$) for this class of general reinforcement functions \mathcal{R} and generalized urn model. Extending ideas from [31], we will prove that infinitely often the process will move away from non-trivial equilibrium points. Using SA techniques, [7–9] and coupling the process to an appropriate ODE we will prove that the only possible stationary points of the process are the extremal points of the simplex $[0, 1]^d$. In fact, condition (1) ensures that the Jacobian of the ODE at all non-trivial points will have positive eigenvalues. This will imply that the dynamical system is not stable around non-trivial points. The novelty of the paper lies in proving dominance in this general setting where we consider d urns, $d > 2$, f not necessarily convex, and the addition of balls σ_n is variable. We extend SA techniques to this setting. To the best of our knowledge, SA techniques in the positive reinforcement regime were only applied to $d = 2$ case in the literature, e.g. [23].

1.1 Outline of the paper

The structure of the manuscript is as follows. In Sect. 2, we introduce the model and assumptions on the class reinforcement functions f . The results are presented in Sect. 3, whereas Sect. 4 is devoted to their proofs. Finally, in Appendix A we introduce stochastic approximation techniques and relevant results.

2 Model and Definitions

2.1 Model

We assume that all random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the following generalization of Pólya urn scheme where colors are indexed by a non-empty finite set $S := \{1, 2, \dots, d\}$, $d \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote the composition of the urn at time n by $U_n := \{U_{n,j}\}_{j \in S}$, where $U_{n,j}$ is the “weight” of the j -th color at time n .

We start with a non-trivial initial composition U_0 , a given non-negative reinforcement function f and a sequence of positive integers $(\sigma_n)_{n \geq 1}$. At every discrete time point $(n + 1)$, σ_{n+1} balls or colors are added f -proportional to their weight, with

instantaneous replacement. That is, given $U_0, U_1, \dots, U_n, \left(X_i^{(n+1)}\right)_{1 \leq i \leq \sigma_{n+1}}$ are i.i.d. random vectors, such that

$$\mathbb{P}\left(X_i^{(n+1)} = e_j \mid U_0, U_1, \dots, U_n\right) = \frac{f(\theta_{n,j})}{\sum_{k=1}^d f(\theta_{n,k})}, \quad (2)$$

where e_j , $1 \leq j \leq d$ is the unit vector corresponding to the canonical basis in \mathbb{R}^d and $\theta_n = (\theta_{n,k})_{1 \leq k \leq d}$ is defined by $\theta_{n,k} = \frac{U_{n,k}}{\sum_{j=1}^d U_{n,j}}$. In other words, $\theta_{n,k}$ is the proportion of balls of color k at time n . The urn composition is then updated according to the following rule:

$$U_{n+1} = U_n + \sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)}. \quad (3)$$

In words, at time $(n+1)$ if the j -th color is selected at the i -th trial for $1 \leq i \leq \sigma_{n+1}$, then we add a single ball of the same color to the urn.

Let us denote by τ_n the total number of balls at time n . Observe that

$$\tau_n = \|U_n\|_1 = \tau_0 + \sum_{j=1}^n \sigma_j = \sum_{j=1}^d U_{0,j} + \sum_{j=1}^n \sigma_j,$$

where $\|\cdot\|_1$ is the $\ell^1(\mathbb{N}^d)$ -norm. Using this notation, $\theta_n = \frac{U_n}{\tau_n}$ and $\theta_n \in \{y \in [0, 1]^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$.

Definition 2.1 We call the event \mathcal{D} *dominance* if

$$\mathcal{D} = \{\exists i \in \{1, \dots, d\} \text{ s.t. } \lim_{n \rightarrow \infty} (\theta_{n,1}, \dots, \theta_{n,d}) = e_i\} \quad (4)$$

where e_i is the i -th coordinate vector in \mathbb{R}^d .

Dominance means that eventually the proportion of different colors in the urn becomes trivial, except for a single color.

Definition 2.2 We say the urn model $(U_n)_{n \geq 0}$ *fixates*, if almost surely,

$$\mathcal{F} = \{\exists N \geq 1 \text{ and } i \in \{1, \dots, d\} \text{ s.t. for all } n \geq N, U_{n+1,i} = U_{n,i} + \sigma_{n+1}\}. \quad (5)$$

It is important to note here that if the process fixates at some color J out of $\{1, \dots, d\}$ (which is random), then all other colors stop growing after the random time of fixation. It is also clear that if the process fixates, it implies that there is almost surely dominance. The converse is not always true, (see [31] for example).

2.1.1 Class of reinforcement functions \mathcal{R}

For the purpose of this paper, we will assume that $f : [0, 1] \rightarrow \mathbb{R}_+ \cup \{0\}$ satisfies

- (A) f is a strictly increasing and continuous, such that $f(0) = 0$ and $f(1) = 1$.
- (B) $f \in C^1((0, 1))$ and the limits of the derivatives $\lim_{x \rightarrow 0^+} f'(x)$ and $\lim_{x \rightarrow 1^-} f'(x)$ exist and are real numbers.
- (C)

$$\alpha := \inf_{x \in (0, 1)} \frac{xf'(x)}{f(x)} > 1. \quad (6)$$

The assumption (A) that f is strictly increasing and $f(0) = 0$ and $f(1) = 1$ is a natural assumption that ensures that “the higher the proportion of a color, the more likely it is to be chosen.” The assumption (B) is for technical purposes. Let \mathcal{R} denote the class of all functions that satisfy (A), (B) and (C).

Let us consider the following examples:

- (i) $f(x) = x^\alpha$, $\alpha > 1$
- (ii) $f(x) = x^{2+\epsilon}e^{-x+1}$, $\epsilon > 0$

A classical example of such a function is example (i). From assumption (C), it may seem that for all $f \in \mathcal{R}$, f has to be convex. However, this is not the case as we will show in the following counterexample. Let $\epsilon > 0$ and consider example (ii). Then f satisfies trivially assumptions (A) and (B). For the third assumption (C), note that

$$f'(x) = (2 + \epsilon - x)x^{1+\epsilon}e^{-x+1}$$

so that $\inf_{x \in (0, 1)} \frac{xf'(x)}{f(x)} = 1 + \epsilon > 1$. This function is not convex, indeed

$$f''(x) = (x^2 - 2(2 + \epsilon)x + (1 + \epsilon)(\epsilon))x^\epsilon e^{-x+1}$$

and we easily see that for ϵ small, e.g. $\epsilon = 0.1$, the second derivative is changing sign for $x \in (0, 1)$.

Lemma 2.3 *Let $f \in \mathcal{R}$. Then*

- (i) f is Lipschitz on $(0, 1)$.
- (ii) The quantity

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = f'(0) \geq 0,$$

is well defined.

- (iii) The map $x \mapsto \frac{f(x)}{x}$ is strictly increasing for all $x \in (0, 1)$.
- (iv) For all $x \in [0, 1]$, we have that $f(x) \leq x^\alpha$, where α is defined in (6).

Proof (i) follows immediately from assumption (B). For (ii), observe that $f'(0)$ exists finitely is a part of the assumption (B), and $f'(0) \geq 0$ follows from assumption (A).

The statement (iii) follows from the observation that derivative of $\frac{f(x)}{x}$ is given by

$$\frac{xf'(x) - f(x)}{x^2} \geq 0,$$

from assumption (C). Finally, (iv) follows from (C) and

$$\log \left(\frac{f(1)}{f(x)} \right) = \int_x^1 d(\log f(t)) \geq \alpha \log \left(\frac{1}{x} \right)$$

for $x \in (0, 1)$ and α was defined in (6). \square

A major disadvantage of the class of functions \mathcal{R} is that it does not include functions that increase exponentially, for example $f(x) = \frac{1}{a-1}(a^x - 1)$ for $a > 0$, as it fails to satisfy (C) ($\alpha = 1$ instead of $\alpha > 1$).

2.1.2 Connections to Stochastic Approximation Theory

In this section, we will connect the proportion vector $(\theta_n)_{n \geq 0}$ to a standard form of recursive equations of the stochastic approximations (SA) method, see e.g. [8–10]. One of the most general forms of the recursion equation associated with SA is of the form

$$Y_{n+1} = Y_n + \gamma_{n+1} [H(Y_n, Z_n) + r_{n+1}], \quad Y_0 \in \mathbb{R}^d, \quad n \geq 0, \quad (7)$$

where $(\gamma_n)_{n \geq 0}$ is the sequence of step sizes, $(Z_n)_{n \geq 0}$ is a sequence of i.i.d. random vectors and $(r_n)_{n \geq 0}$ is the sequence of “error” or remainder terms and $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable function.

A general practice is to re-write Eq. (7) in the following form

$$Y_{n+1} = Y_n + \gamma_{n+1} [\mathbb{E}[H(Y_n, Z_n) | Y_0, Z_1, \dots, Z_n] + \Delta M_{n+1} + r_{n+1}], \quad (8)$$

where $\Delta M_{n+1} = H(Y_n, Z_n) - \mathbb{E}[H(Y_n, Z_n) | Y_0, Z_1, \dots, Z_n]$.

The advantage of such a representation is that under suitable conditions on H and $(Z_n)_n$, one can relate the asymptotic properties of Y_n to the zeros of the *mean-field function*, which we will define and discuss in details for our model.

Recall our basic recursive equation (3). We can re-write this equation as follows

$$\theta_{n+1} = \theta_n + \frac{1}{\tau_{n+1}} \left[\mathbb{E} \left[\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} \middle| \mathcal{F}_n \right] - \sigma_{n+1} \theta_n + \Delta M_{n+1} \right], \quad (9)$$

where \mathcal{F}_n is the sigma algebra generated by U_0, U_1, \dots, U_n , and

$$\Delta M_{n+1} = \sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} - \mathbb{E} \left[\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} \middle| \mathcal{F}_n \right]. \quad (10)$$

Let us define the function $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as follows $y \mapsto (f(y_1), f(y_2), \dots, f(y_d))$ for any vector $y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$ and $f \in \mathcal{R}$. It is easy to see that

$$\mathbb{E} \left[\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} \mid \mathcal{F}_n \right] = \sigma_{n+1} \frac{\mathbf{f}(\theta_n)}{\|\mathbf{f}(\theta_n)\|_1}.$$

An immediate consequence of the construction is the following corollary.

Corollary 2.4 *Given \mathcal{F}_n , the d -dimensional random vector $\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)}$ follows a Multinomial distribution with parameters $(\sigma_{n+1}, \frac{\mathbf{f}(\theta_n)}{\|\mathbf{f}(\theta_n)\|_1})$, i.e., for $x = (x_1, \dots, x_d) \in \mathbb{N}^d$ such that $\sum_{i=1}^d x_i = \sigma_{n+1}$,*

$$\mathbb{P} \left(\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} = x \right) = \frac{\sigma_{n+1}!}{x_1! \dots x_d!} \frac{f(\theta_{n,1})^{x_1}}{\|\mathbf{f}(\theta_n)\|_1} \dots \frac{f(\theta_{n,d})^{x_d}}{\|\mathbf{f}(\theta_n)\|_1}.$$

Re-writing (9), we have

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{1}{\tau_{n+1}} \left[\sigma_{n+1} \frac{\mathbf{f}(\theta_n)}{\|\mathbf{f}(\theta_n)\|_1} - \sigma_{n+1} \theta_n + \Delta M_{n+1} \right] \\ &= \theta_n + \frac{\sigma_{n+1}}{\tau_{n+1}} \left[h(\theta_n) + \frac{1}{\sigma_{n+1}} \Delta M_{n+1} \right], \end{aligned} \quad (11)$$

where we define $h : \{[0, 1]^d \setminus \{0\} : y_1 + y_2 + \dots + y_d = 1\} \rightarrow \mathbb{R}_+^d$ by $y \mapsto \left(\frac{\mathbf{f}(y)}{\|\mathbf{f}(y)\|_1} - y \right)$. This function h is well defined as f is strictly non-decreasing and $f(0) = 0$. h will be the *mean-field function*, and we later see the relations of h to the limit points of the sequence $(\theta_n)_{n \geq 0}$. Comparing (11) with (8), we observe that it is not exactly in the form of the standard recursive equation as in practice in SA, due to the presence of the coefficient $\frac{1}{\sigma_{n+1}}$ which is multiplied with the martingale difference ΔM_{n+1} . We present all necessary results related to SA required for this paper in Appendix [Appendix A](#).

3 Results

In this section, we will present our main results.

Theorem 1 *Suppose that $(\theta_n)_{n \geq 0}$ is as in (11). Assume the following conditions:*

- (i) *Let $(\sigma_n)_{n \geq 1}$ be such that $\sum_{n \geq 1} \left(\frac{\sigma_n}{\tau_n} \right)^2 < \infty$,*
- (ii) *$f \in \mathcal{R}$.*

Then we have that

$$\mathbb{P}(\mathcal{D}) = 1,$$

where \mathcal{D} is as defined in Definition 2.1.

Theorem 2 Under the assumptions of Theorem 1 and $\alpha > 3$, we have that

$$\mathbb{P}(\mathcal{F}) = 1, \quad (12)$$

where \mathcal{F} is defined in Definition 2.2.

Remark 3.1 It is clear that Theorems 1 and 2 should hold even for exponentially increasing functions, since similar results are shown in [20], where $\sigma_n \equiv 1$. However, we could not apply the general stochastic approximation techniques for exponentially growing functions and for a general sequence $(\sigma_n)_{n \geq 1}$.

4 Proofs

The proof of Theorem 1 will be divided into four parts. The first Lemma 4.1 will identify equilibrium points for the mean-field function h . The *equilibrium set* of a function h is the set

$$\mathcal{E}(h) = \left\{ y \in [0, 1]^d \setminus \{0\} : \sum_{i=1}^d y_i = 1, h(y) = 0 \right\}.$$

We call *trivial equilibrium points* the standard orthonormal basis of \mathbb{R}^d , $\{e_1, \dots, e_d\}$. Lemma 4.2 will ensure that the process $(\theta_n)_{n \geq 0}$ converges and the possible limit points are given by the equilibrium points of the mean-field function h . Proposition 4.3 will show that almost surely the process will infinitely often move away by a small enough distance from the non-trivial equilibrium points. Finally, Lemma 4.4 will conclude that the only possible limit points are the trivial equilibrium points.

Lemma 4.1 Let $\pi = (\pi_1, \pi_2, \dots, \pi_d)$ denote a permutation of $\{1, 2, \dots, d\}$ and for any vector $y := (y_i)_{1 \leq i \leq d}$, let $y \circ \pi := (y_{\pi_i})_{1 \leq i \leq d}$. For $1 \leq k \leq d$, let $\tilde{y}(k) := (y_i)_{1 \leq i \leq d}$, where $y_1 = y_2 = \dots = y_k = \frac{1}{k}$ and $y_i = 0$, elsewhere. Let $I(k) := \cup_{\pi} \tilde{y}(k) \circ \pi$, where the union is taken w.r.t. all permutations π of $\{1, 2, \dots, d\}$. Then if $f \in \mathcal{R}$, we have

$$\mathcal{E}(h) = \cup_{k=1}^d I(k). \quad (13)$$

The above lemma states that the equilibrium set $\mathcal{E}(h)$ contains all vectors $y = (y_i)_{1 \leq i \leq d}$, such that there exists $1 \leq k \leq d$, with $y_i = \frac{1}{k}$, for exactly k co-ordinates and $y_i = 0$, elsewhere.

Proof Observe that to find the equilibrium points of h , we need to find all solutions $y \in \{y \in [0, 1]^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$ for the set of self-consistency equations given by

$$y_1 = \frac{f(y_1)}{\|\mathbf{f}(y)\|_1}, \dots, y_d = \frac{f(y_d)}{\|\mathbf{f}(y)\|_1}. \quad (14)$$

Let $y \in I(k)$. WLOG, we may assume that $y_1 = y_2 = \dots = y_k = \frac{1}{k}$, and $y_j = 0$ for $j \geq k + 1$. Then we obtain that from assumption (A) $\|\mathbf{f}(y)\|_1 = kf(1/k)$. Therefore,

we have $\frac{f(y_i)}{\|\mathbf{f}(y)\|_1} = \frac{1}{k}$ for $1 \leq i \leq k$, and $f(y_i) = 0$ for $i \geq k+1$, which shows that $\bigcup_{k=1}^d I(k) \subset \mathcal{E}(h)$.

Since $y \in \{y \in [0, 1]^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$, it follows immediately that there exists some i , such that $y_i > 0$, and thus, $\|\mathbf{f}(y)\|_1 > 0$. It is clear that the unit coordinate vectors e_i for $1 \leq i \leq d$ are equilibrium points by assumption (A).

So let us assume that $y_1, \dots, y_k \neq 0$ for some $2 \leq k \leq d$. Observe that to prove (13), it is enough to show $y_1 = y_2 = \dots = y_k = \frac{1}{k}$. The previous set of equations (14) can be written as

$$\|\mathbf{f}(y)\|_1 = \frac{f(y_1)}{y_1}, \dots, \|\mathbf{f}(y)\|_1 = \frac{f(y_k)}{y_k}, \quad (15)$$

and $f(y_i) = 0$ for all $(k+1) \leq i \leq d$ by assumption (A). W.l.o.g., we may assume that $y_1 > y_2$. Then by Lemma 2.3 (iii), we know that $\frac{f(y_1)}{y_1} > \frac{f(y_2)}{y_2}$, which is a contradiction to (15). This shows that $y_1 = y_2$, and in particular, $y_1 = y_2 = \dots = y_k$ and also $y_1 = y_2 = \dots = y_k = \frac{1}{k}$, since $\sum_{i=1}^k y_i = 1$. \square

Note that the statement of Lemma 4.1 is similar to Proposition 2.4 from [23]. They prove the statement under the assumption that f is concave or convex, which we do not need here. The following lemma is a consequence of Corollary A.7 from the appendix and Lemma 4.1.

Lemma 4.2 *Let $(\theta_n)_{n \geq 0}$ be a stochastic process defined by the recursion*

$$\theta_{n+1} = \theta_n + \frac{\sigma_{n+1}}{\tau_{n+1}} \left[h(\theta_n) + \frac{1}{\sigma_{n+1}} \Delta M_{n+1} \right].$$

Then

$$\theta_n \xrightarrow{a.s.} \theta^*$$

as $n \rightarrow \infty$, where $\theta^ \in \mathcal{E}(h)$.*

The following proposition is actually a stronger version of Proposition 4.1 of [31]. The case $d = 2$ is covered in Proposition 4.1 of [31], so we extend this to $d \geq 3$.

Proposition 4.3 *Let $(\sigma_n)_n$ and $(\tau_n)_n$ be as in the assumptions of Theorem 1. There exists $\delta_n > 0$, such that*

$$(i) \quad \lim_{n \rightarrow \infty} \delta_n = 0, \quad (16)$$

$$(ii) \quad \sum_{n \geq 1} \frac{\delta_n \sigma_{n+1}}{\tau_{n+1}} < \infty. \quad (17)$$

(iii) For any $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$, where $1 < k \leq d$,

$$\mathbb{P}\left((\theta_{n,1}, \theta_{n,2}, \dots, \theta_{n,d-1}, \theta_{n,d}) \in B_d^c(e, \delta_n) \text{ i.o.}\right) = 1, \quad (18)$$

where for $a \in \mathbb{R}^d$ and $r > 0$, $B_d(a, r) := \{x \in \mathbb{R}^d : \|x - a\|_2 \leq r\}$.

Proof (i)+(ii): Recall that by our choice, we have chosen $(\sigma_n)_{n \geq 1}$, such that, $\sigma_n \geq 1$, for all $n \geq 1$, so we have $\sum_n \frac{\sigma_n}{\tau_n} = \infty$ by the Theorem of Abel and Dini, §39 of [21]. Furthermore, by the assumption $\sum_n \frac{\sigma_n^2}{\tau_n^2} < \infty$, we have $\lim_n \frac{\sigma_n}{\tau_n} = 0$, and therefore, there always exists δ_n that satisfies $\sum_{n \geq 1} \frac{\delta_n \sigma_{n+1}}{\tau_n + \frac{1}{2} \sigma_{n+1}} < \infty$, which implies that $\sum_{n \geq 1} \frac{\delta_n \sigma_{n+1}}{\tau_{n+1}} < \infty$ as $\tau_{n+1} = \tau_n + \sigma_{n+1}$. We can simply choose $0 \leq \delta_n \leq \frac{\sigma_{n+1}}{\tau_n + 2\sigma_{n+1}}$. It is immediately clear that $\lim_{n \rightarrow \infty} \delta_n = 0$.

(iii): For $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$, where $1 < k \leq d$, define

$$\mathcal{H}_m(e) := \{(\theta_{n,1}, \theta_{n,2}, \dots, \theta_{n,d-1}, \theta_{n,d}) \in B_d(e, \delta_n) \text{ for all } n \geq m\}.$$

Since $\mathcal{H}_m(e)$ is non-decreasing in m , therefore it is enough to show that

$$\lim_{m \rightarrow \infty} \mathbb{P}(\mathcal{H}_m(e)) = 0$$

to show (18).

Recall that from (3), we have

$$\theta_{n+1,1} = \frac{\tau_n \theta_{n,1} + B_{n+1,1}}{\tau_{n+1}},$$

where given \mathcal{F}_n , $B_{n+1,1}$ follows $\text{Bin}(\sigma_{n+1}, \Psi_1(\theta_n))$, where $\Psi = (\Psi_1, \dots, \Psi_d)$ and Ψ_i is defined as

$$\Psi_i(x) := \frac{f(x_i)}{\|\mathbf{f}(x)\|_1}, \text{ for } 1 \leq i \leq d, \quad (19)$$

and $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$, $f \in \mathcal{R}$. Re-write the above equation as

$$\theta_{n+1,1} = \frac{\tau_n \theta_{n,1} + \sigma_{n+1} \Psi_1(\theta_n) + \epsilon_n \sqrt{\sigma_{n+1} \Psi_1(\theta_n) (1 - \Psi_1(\theta_n))}}{\tau_{n+1}}, \quad (20)$$

where $\epsilon_n = \frac{B_{n+1,1} - \sigma_{n+1} \Psi_1(\theta_n)}{\sqrt{\sigma_{n+1} \Psi_1(\theta_n) (1 - \Psi_1(\theta_n))}}$.

By Taylor's theorem, we have for any $x \in \{y \in [0, 1]^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$

$$\Psi_1(x) - \tilde{e}_1 = \sum_{i=1}^{d-1} \frac{\partial \Psi_1(\xi_x)}{\partial x_i} (x_i - \tilde{e}_i),$$

where ξ_x lies on the straight line joining x and $e = (\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_d) \in \mathcal{E}(h)$, since $\Psi_1(e) = \tilde{e}_1$.

Re-writing (20), using the Taylor expansion, we get

$$\begin{aligned} \theta_{n+1,1} - \tilde{e}_1 = & \frac{\tau_n}{\tau_{n+1}} (\theta_{n,1} - \tilde{e}_1) + \frac{\sigma_{n+1}}{\tau_{n+1}} \sum_{i=1}^{d-1} \frac{\partial \Psi_1(\xi_{\theta_n})}{\partial x_i} (\theta_{n,i} - \tilde{e}_i) \\ & + \frac{1}{\tau_{n+1}} \epsilon_n \sqrt{\sigma_{n+1} \Psi_1(\theta_n) (1 - \Psi_1(\theta_n))}. \end{aligned} \quad (21)$$

$$\theta_{n+1,1} - \tilde{e}_1 = \kappa_n(\theta_n) (\theta_{n,1} - \tilde{e}_1) + Q_n(\theta_n) + \frac{1}{\tau_{n+1}} \epsilon_n \sqrt{\sigma_{n+1} \Psi_1(\theta_n) (1 - \Psi_1(\theta_n))}, \quad (22)$$

where $\kappa_n(\theta_n) = \left(\frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}} \frac{\partial \Psi_1(\xi_{\theta_n})}{\partial x_1} \right)$, and

$$Q_n(\theta_n) = \frac{\sigma_{n+1}}{\tau_{n+1}} \sum_{i=2}^{d-1} \frac{\partial \Psi_1(\xi_{\theta_n})}{\partial x_i} (\theta_{n,i} - \tilde{e}_i). \quad (23)$$

Re-iterating the above equation, we obtain

$$\begin{aligned} \theta_{n,1} - \tilde{e}_1 = & \prod_{j=m}^{n-1} \kappa_j(\theta_j) \left[\theta_{m,1} - \tilde{e}_1 + \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j \kappa_l(\theta_l) \right)} Q_j(\theta_j) \right] \\ & + \prod_{j=m}^{n-1} \kappa_j(\theta_j) \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j \kappa_l(\theta_l) \right)} \epsilon_j \frac{\sqrt{\sigma_{j+1} \Psi_1(\theta_j) (1 - \Psi_1(\theta_j))}}{\tau_{j+1}}, \end{aligned}$$

By continuity of the partial derivatives of Ψ_1 , for $x \in B_d(e, \delta_n)$ for all n (large enough), we have $\frac{\partial \Psi_1(\xi_x)}{\partial x_i} = \frac{\partial \Psi_1(e)}{\partial x_i} + \mathcal{O}(\delta_n)$.

Step 1 In this step, we analyze the asymptotic behavior of $\prod_{j=1}^n \kappa_j(\theta_j)$. Writing $x = (x_1, x_2, \dots, x_d) \in [0, 1]^d$, such that $\sum_{i=1}^d x_i = 1$, observe that at an equilibrium point $e = (\tilde{e}_1, \dots, \tilde{e}_d)$

$$\frac{\partial \Psi_1(e)}{\partial x_1} = \frac{f'(\tilde{e}_1)}{f(\tilde{e}_1) + \dots + f(\tilde{e}_d)} - \frac{f(\tilde{e}_1) (f'(\tilde{e}_1) - f'(1 - \tilde{e}_1 - \tilde{e}_2 - \dots - \tilde{e}_{d-1}))}{(f(\tilde{e}_1) + \dots + f(\tilde{e}_d))^2}.$$

Therefore,

$$\frac{\partial \Psi_1(e)}{\partial x_1} = \begin{cases} \alpha(\tilde{e}_1), & \text{if } \tilde{e}_1 = \tilde{e}_d, \\ \alpha(\tilde{e}_1) - \frac{f(\tilde{e}_1)(f'(\tilde{e}_1) - f'(0))}{(f(\tilde{e}_1) + \dots + f(\tilde{e}_d))^2}, & \text{if } \tilde{e}_d = 0. \end{cases} \quad (24)$$

From Assumption (C), (15) and the observation that $f'(0) \geq 0$, we have

$$\frac{\partial \Psi_1(e)}{\partial x_1} \begin{cases} = \alpha(\tilde{e}_1) > 1, & \text{if } \tilde{e}_1 = \tilde{e}_d, \\ \geq (1 - \tilde{e}_1) \alpha(\tilde{e}_1), & \text{if } \tilde{e}_d = 0, \end{cases} \quad (25)$$

where $\alpha(x) := \frac{xf'(x)}{f(x)}$ and $f \in \mathcal{R}$. Therefore, on \mathcal{H}_m we obtain for $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$

$$\kappa_n(\theta_n) = \begin{cases} \frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}} \left(\frac{k-1}{k} \alpha(\tilde{e}_1) + \mathcal{O}(\delta_n) \right), & \text{if } 1 < k \leq (d-1), \\ \frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}} (\alpha(\tilde{e}_1) + \mathcal{O}(\delta_n)), & \text{if } k = d. \end{cases}$$

Writing $\pi_{m,n-1}(k) := \prod_{j=m}^{n-1} \left(\frac{\tau_j + \beta_k \sigma_{j+1}}{\tau_{j+1}} \right)$

$$\prod_{j=m}^{n-1} \kappa_j(\theta_j) = \pi_{m,n-1}(k) \prod_{j=m}^{n-1} \left(1 + \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right),$$

where $\beta_k = \frac{k-1}{k} \alpha(\tilde{e}_1)$, if $1 < k \leq d-1$, and $\beta_k = \alpha(\tilde{e}_1)$ for $k = d$.

$$\prod_{j=m}^{n-1} \left(1 + \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right) = \exp \left(\sum_{j=m}^{n-1} \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right).$$

Since $1 \geq (1 - \frac{1}{k}) \geq \frac{1}{2}$ and $\alpha(\tilde{e}_1) > 1$, by our choice of δ_j , in (17), $\sum_j \frac{\delta_j}{\tau_j + \beta_k \sigma_{j+1}} < \infty$. Hence, for all m large enough

$$\prod_{j=m}^{n-1} \left(1 + \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right) = (1 + o(1)).$$

Therefore, for all m large enough and on \mathcal{H}_m , we have

$$\prod_{j=m}^{n-1} \kappa_j(\theta_j) = \pi_{m,n-1}(k)(1 + o(1)). \quad (26)$$

Step 2 In this step, we analyze the sum over $Q_j(\theta_j)$, where Q_j was defined in (23).

From our Assumption (B), for $2 \leq i \leq k$, we have $\frac{\partial \Psi_1(\xi_x)}{\partial x_i} = \frac{\partial \Psi_d(e)}{\partial x_i} + \mathcal{O}(\delta_n)$ is bounded for all $n \geq m$.

On \mathcal{H}_m ,

$$Q_j(\theta_j) = \mathcal{O}(\delta_j) \frac{\sigma_{j+1}}{\tau_{j+1}}. \quad (27)$$

Recall that in *Step 1* we proved that $\prod_{j=m}^l \kappa_j(\theta_j) = \pi_{m,l}(k)(1 + o(1))$, where $\pi_{m,l}(k) = \prod_{j=m}^l \left(\frac{\tau_j + \beta_k \sigma_{j+1}}{\tau_{j+1}} \right)$.

Case 1 When $k = d$,

$$\pi_{m,l}(k) = \exp \left(\sum_{j=m+1}^l \log \left(1 + (\alpha(\tilde{e}_1) - 1) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \right) \rightarrow \infty,$$

as $l \rightarrow \infty$ since $\alpha(\tilde{e}_1) > 1$ and $\sum_j \frac{\sigma_j}{\tau_j} = \infty$. Therefore,

$$\sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} Q_j(\theta_j) = (1 + o(1)) \sum_{j=m}^{n-1} \exp\left(-\sum_{j=m+1}^l (\alpha(\tilde{e}_1) - 1) \frac{\sigma_{j+1}}{\tau_{j+1}}\right) \frac{\sigma_{j+1}}{\tau_{j+1}} \mathcal{O}(\delta_j) < \infty,$$

Case 2 When $1 < k \leq d-1$, $\pi_{m,l}(k) = \exp\left(\sum_{j=m+1}^l \log\left(1 + \left(\frac{k-1}{k}\alpha(\tilde{e}_1) - 1\right) \frac{\sigma_{j+1}}{\tau_{j+1}}\right)\right)$
 $= \exp\left(\sum_{j=m+1}^l \left(\frac{k-1}{k}\alpha(\tilde{e}_1) - 1\right) \frac{\sigma_{j+1}}{\tau_{j+1}}\right) + o(1)$ by the assumptions of Theorem 1.

If $\alpha(\tilde{e}_1) \geq \frac{k}{k-1}$, then it follows similar to Case i that

$$\begin{aligned} & \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} Q_j(\theta_j) \\ &= (1 + o(1)) \sum_{j=m}^{n-1} \exp\left(-\sum_{j=m+1}^l \left(\frac{k-1}{k}\alpha(\tilde{e}_1) - 1\right) \frac{\sigma_{j+1}}{\tau_{j+1}}\right) \frac{\sigma_{j+1}}{\tau_{j+1}} \mathcal{O}(\delta_j) < \infty. \end{aligned}$$

If $\alpha(\tilde{e}_1) < \frac{k}{k-1}$, then because of our choice of δ_j for all m, n (large enough)

$$\begin{aligned} & \left(\prod_{j=m}^{n-1} \kappa_j(\theta_j)\right) \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} Q_j(\theta_j) \\ &= \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l)\right) Q_j(\theta_j) \\ &= (1 + o(1)) \sum_{j=m}^{n-1} \exp\left(\sum_{l=j+1}^{n-1} \left(\frac{k-1}{k}\alpha(\tilde{e}_1) - 1\right) \frac{\sigma_{l+1}}{\tau_{l+1}}\right) \mathcal{O}(\delta_j) \frac{\sigma_{j+1}}{\tau_{j+1}} < \epsilon, \end{aligned} \quad (28)$$

since $\frac{k-1}{k}\alpha(\tilde{e}_1) < 1$, $\sum_j \frac{\sigma_{j+1}}{\tau_{j+1}} = \infty$ and $\sum_j \delta_j \frac{\sigma_{j+1}}{\tau_{j+1}} < \infty$.

Step 3 In this step, we will explore the last part of the summand in

$$\sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j \kappa_l(\theta_l)\right)} \epsilon_j \frac{\sqrt{\sigma_{j+1} \Psi_1(\theta_j)(1 - \Psi_1(\theta_j))}}{\tau_{j+1}}. \text{ The argument is similar to that of}$$

Proposition 4.1 in [31], so we present only those details that are crucial and slightly different from those in [31]. From Step 1, and $\Psi(\theta_{j,1}) = \frac{1}{k} + \mathcal{O}(\delta_j)$, we get for appropriate constants c_k

$$\frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} \epsilon_j \frac{\sqrt{\sigma_{j+1} \Psi_1(\theta_j)(1 - \Psi_1(\theta_j))}}{\tau_{j+1}} = (c_k + o(1)) \epsilon_j \frac{\sqrt{\sigma_{j+1}}}{\pi_{m,j} \tau_{j+1}}.$$

We know that if $X \sim \text{Bin}(n, p)$, then

$\mathbb{E} \left[\exp \left(it \frac{X-np}{\sqrt{np(1-p)}} \right) \right] = \left(1 - \frac{t^2}{2n} + \mathcal{O} \left(\frac{t^3}{n^{\frac{3}{2}}} \right) \right)^n = \exp \left(-\frac{t^2}{2} + \mathcal{O} \left(\frac{t^3}{\sqrt{n}} \right) \right)$, when $|t| < \delta$ for some $\delta > 0$. Therefore,

$$\mathbb{E} [\exp (it\epsilon_j) \mid \mathcal{F}_{j-1}] = \exp \left(-\frac{t^2}{2} + \mathcal{O} \left(\frac{t^3}{\sqrt{\sigma_{j+1}}} \right) \right),$$

since given \mathcal{F}_{j-1} , $\epsilon_j \sim \text{Bin}(\sigma_j, \Psi(\theta_{j-1,1}))$. And hence, we have by tower property

$$\mathbb{E} \left[\exp \left(it \sum_{j=m}^{n-1} \epsilon_j \right) \middle| \mathcal{F}_{m-1} \right] = \exp \left(\sum_{j=m}^{n-1} \frac{-t^2}{2} + \mathcal{O} \left(\frac{t^3}{\sqrt{\sigma_{j+1}}} \right) \right).$$

Hence, we can write

$$\mathbb{E} \left[\exp \left(\frac{it}{\mu_{m,n}} \sum_{j=m}^{n-1} \epsilon_j \right) \middle| \mathcal{F}_{m-1} \right] = \exp \left(-\frac{t^2}{2} + \mathcal{O} \left(\frac{t^3}{\mu_{m,n}^3} \right) \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^3 \tau_{j+1}^3} \right), \quad (29)$$

where $\mu_{m,n} := \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^2 \tau_{j+1}^2}$.

For $e = \frac{1}{d}(e_1 + e_2 + \dots + e_d)$, it follows that $\frac{1}{\mu_{m,n}^3} \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^3 \tau_{j+1}^3} \rightarrow 0$ as $m, n \rightarrow \infty$.

When $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ for $1 < k \leq d-1$, and $\sup_n \sigma_n = \infty$, exactly same argument as in [31] gives us $\frac{1}{\mu_{m,n}^3} \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^3 \tau_{j+1}^3} \rightarrow 0$ as $m, n \rightarrow \infty$.

For $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ for $1 < k \leq d-1$, and $\sup_n \sigma_n < \infty$, we observe that if $\alpha(\tilde{e}_1) > \frac{k}{k-1}$, then similar argument as in [31] works.

Therefore, the only case we need to discuss in details is for $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ for $1 < k \leq d-1$, and $\sup_n \sigma_n < \infty$, and $\alpha \leq \frac{k}{k-1}$.

It is easy to see that

$$\prod_{j=m}^{n-1} \kappa_j(\theta_j) = \exp \left(- \sum_{j=m+1}^{n-1} \left(\frac{k-1}{k} \alpha - 1 + \mathcal{O}(\delta_j) \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \rightarrow 0, \text{ as } m, n \rightarrow \infty, \quad (30)$$

since $\lim_{n \rightarrow \infty} \delta_n = 0$, and $\sum_j \frac{\sigma_j}{\tau_j} = \infty$.

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{it}{\Gamma_{m,n-1}} \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} k_l(\theta_l) \right) \epsilon_j \frac{\sqrt{\sigma_{j+1}}}{\tau_{j+1}} \right) \middle| \mathcal{F}_{m-1} \right] \\ &= \exp \left(-\frac{t^2}{2} + \mathcal{O}(1) \frac{t^3}{\Gamma_{m,n-1}^3} \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^n \kappa_l(\theta_l) \right)^3 \frac{\sigma_{j+1}}{\tau_{j+1}^3} \right), \end{aligned} \quad (31)$$

where $\Gamma_{m,n-1} := \left[\sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^2 \frac{\sigma_{j+1}^2}{\tau_{j+1}^2} \right]^{\frac{1}{2}}$.

Since $\sup_n \sigma_n < \infty$, $j \leq \tau_j \leq Cj$ for suitable constant $C > 0$. From (30), we get

$$\sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^2 \frac{\sigma_{j+1}^2}{\tau_{j+1}^2} = \mathcal{O}(1) \sum_{j=m}^{n-1} \frac{1}{j^2} = \mathcal{O}(1) \left(\frac{1}{m} - \frac{1}{n} \right) \quad (32)$$

By similar arguments,

$$\sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^3 \frac{\sigma_{j+1}^3}{\tau_{j+1}^3} = \mathcal{O}(1) \sum_{j=m}^{n-1} \frac{1}{j^3} = \mathcal{O}(1) \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad (33)$$

Therefore, as $m \rightarrow \infty$,

$$\frac{1}{\Gamma_{m,n-1}^3} \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^3 \frac{\sigma_{j+1}^3}{\tau_{j+1}^3} = \mathcal{O}(1) \frac{1}{\sqrt{m}} \rightarrow 0. \quad (34)$$

For this case, choose the subsequence $n_m = 2m$ and choose $\delta_n = \mathcal{O}(1) \frac{1}{n^{\frac{1}{2}+\delta}}$, for some $\delta > 0$. Finally, from all previous observations,

$$\mathbb{P}(\mathcal{H}_m(e)) \leq \mathbb{P}(\Gamma_{m,n_m} N \in [-\delta_{n_m}, \delta_{n_m}]) \rightarrow 0 \text{ as } m \rightarrow \infty, \quad (35)$$

since $\frac{\delta_{n_m}}{\Gamma_{m,n_m}} = \mathcal{O}(1) \frac{1}{m^\delta} \rightarrow 0$ as $m \rightarrow \infty$. \square

Proposition 4.4 *Let $(\theta_n)_{n \geq 0}$ be the process defined recursively in (11). Then*

$$\theta_n \xrightarrow{a.s.} e^*,$$

as $n \rightarrow \infty$ where $e^* \in \{e_1, \dots, e_d\}$ are the trivial equilibrium points.

The proof of this proposition is similar to parts of Theorem 1.2. from [31] adapted appropriately for the class of reinforcement functions \mathcal{R} ; the main difference is in the final argument leading up to the conclusion. We present all details for the sake of completeness.

Proof For each $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$ we have that

$$\theta_{n+1,i} = \frac{\tau_n \theta_{n,i} + B_{n+1,i}}{\tau_{n+1}} \quad (36)$$

where $B_{n,i} \sim \text{Bin}(\sigma_n, \Psi_i(\theta_n))$ given \mathcal{F}_n and Ψ was defined in (19). We can express the system of equations for $i \in \{1, \dots, d\}$ by

$$\begin{aligned}\theta_{n+1,i} &= \theta_{n,i} + \frac{\sigma_{n+1}}{\tau_{n+1}} \left((\Psi_i(\theta_n) - \theta_{n,i}) + \frac{1}{\sigma_{n+1}} (B_{n+1,i} - \sigma_{n+1} \Psi_i(\theta_n)) \right) \\ &= \theta_{n,i} + \frac{B_{n+1,i} - \sigma_{n+1} \Psi_i(\theta_n)}{\tau_{n+1}} - \frac{\sigma_{n+1}}{\tau_{n+1}} (\theta_{n,i} - \Psi_i(\theta_n)).\end{aligned}\quad (37)$$

For every $\eta \geq 0$, we have by the above recursive relations that

$$\theta_{n+\eta,i} = \theta_{\eta,i} + M_{n,i}(\eta) - R_{n,i}(\eta) \quad (38)$$

where

$$M_{n,i}(\eta) = \sum_{j=\eta+1}^{n+\eta} \frac{B_{j,i} - \sigma_j \Psi_i(\theta_{j-1})}{\tau_j}$$

for $i = 1, \dots, d$ and

$$R_{n,i}(\eta) = \sum_{j=\eta+1}^{\eta+n} \frac{\sigma_j}{\tau_j} (\theta_{j-1,i} - \Psi_i(\theta_{j-1})).$$

Then the vector $M_n(\eta) = (M_{n,1}(\eta), \dots, M_{n,d}(\eta))$ is a martingale w.r.t. $(\mathcal{F}_{n+\eta})_n$, since $\mathbb{E}[B_{n+1,i} | \mathcal{F}_n] = \sigma_{n+1} \Psi_i(\theta_n)$. Note that up to now, we did not use the specific definition of η , which we will define later. For simplicity of notation, we write $M_{n,i}$ instead of $M_{n,i}(\eta)$. Let us show that the martingale is bounded in $L^2(\mathbb{R}^d)$. Note that we can write

$$M_{n,i} = M_{n-1,i} + \frac{B_{n+\eta,i} - \sigma_{n+\eta} \Psi_i(\theta_{n+\eta-1})}{\tau_{n+\eta}}$$

and further estimate

$$\begin{aligned}\mathbb{E}(\langle M_n, M_n \rangle | \mathcal{F}_\eta) &= \sum_{i=1}^d \mathbb{E}((M_{n,i})^2 | \mathcal{F}_\eta) \\ &= \sum_{i=1}^d \mathbb{E}((M_{n-1,i})^2 | \mathcal{F}_\eta) + \mathbb{E} \left(\left(\frac{B_{n+\eta,i} - \sigma_{n+\eta} \Psi_i(\theta_{n+\eta-1})}{\tau_{n+\eta}} \right)^2 \middle| \mathcal{F}_\eta \right) \\ &\quad + 2 \sum_{i=1}^d \mathbb{E} \left(\left[\frac{B_{n+\eta,i} - \sigma_{n+\eta} \Psi_i(\theta_{n+\eta-1})}{\tau_{n+\eta}} \right] M_{n-1,i} \middle| \mathcal{F}_\eta \right) \\ &= \sum_{i=1}^d \mathbb{E}((M_{n-1,i})^2 | \mathcal{F}_\eta) + \frac{1}{\tau_{n+\eta}^2} \mathbb{E} \left((\sigma_{n+\eta} \Psi_i(\theta_{n+\eta-1}) (1 - \Psi_i(\theta_{n+\eta-1}))) \middle| \mathcal{F}_\eta \right).\end{aligned}\quad (39)$$

Since $\sigma_{n+\eta} \Psi_i(\theta_{n+\eta-1})(1 - \Psi_i(\theta_{n+\eta-1})) \leq \sigma_{n+\eta}$ and by recursivity, we can estimate

$$\mathbb{E}(\langle M_n, M_n \rangle | \mathcal{F}_\eta) \leq d \sum_{j=\eta+1}^{\eta+n} \frac{\sigma_j}{\tau_j^2}.$$

$$\sum_{j=\eta+1}^{\eta+n} \frac{\sigma_j}{\tau_j^2} = \sum_{j=\eta+1}^{\eta+n} \int_{\tau_{j-1}}^{\tau_j} \frac{1}{\tau_j^2} dx \leq \sum_{j=\eta+1}^{\infty} \int_{\tau_{j-1}}^{\tau_j} \frac{1}{x^2} dx = \int_{\tau_\eta}^{\infty} \frac{1}{x^2} dx = \frac{1}{\tau_\eta}, \quad (40)$$

so that M_n is bounded in $L^2(\mathbb{R}^d)$:

$$\mathbb{E}(\mathbb{E}(\langle M_n, M_n \rangle | \mathcal{F}_\eta)) = \mathbb{E}(\langle M_n, M_n \rangle) \leq \frac{d}{\tau_\eta} \quad (41)$$

which implies that M_n converges almost surely and in $L^2(\mathbb{R}^d)$ to some $M_\infty \in L^2(\mathbb{R}^d)$.

Observe that there exists a sequence $(\delta_n)_{n \geq 0}$ (see Lemma 5.1 from [31]), such that it satisfies (i) and (ii) from Proposition 4.3, and

$$\lim_{n \rightarrow \infty} \frac{1}{\delta_n^2 \tau_n} = 0.$$

For any $\epsilon > 0$, choose $N_1 > 0$, such that, for all $n \geq N_1$

$$\mathbb{P}\left(M_{n,1}(N_1) \geq \frac{\delta_{N_1}}{2}\right) \leq \frac{4d}{\delta_{N_1}^2 \tau_{N_1}} \leq \epsilon. \quad (42)$$

By Proposition 4.3, for δ_n as in (42) W.l.o.g., we may assume that $\mathbb{P}(N_2 < \infty) > 0$, where $N_2 := \inf \{n : \theta_{n,1} < \frac{1}{k} - \delta_n\}$. Choose $\eta = \max\{N_1, N_2\}$. Henceforth, we will work with $M_{n,1}(\eta)$ and $R_{n,1}(\eta)$ for this choice of η . As before, we write $M_{n,1}$ and $R_{n,1}$ for notational simplicity.

Lemma 4.2 yields that $\theta_n \rightarrow \theta^*$ as $n \rightarrow \infty$; hence, together with $M_n \rightarrow M_\infty$ a.s., we deduce that R_n has to converge as well. It remains to show that $\theta^* \neq e$ where $e = \frac{1}{k}(e_1 + \dots + e_k)$ and $2 \leq k \leq d$. Assume that $\theta^* = e$. Recall that $\Psi_i(\theta^*) = \frac{1}{k}$ for $1 \leq i \leq k$, and $\Psi_i(\theta^*) = 0$ for all other $i > k$. Since $\Psi_i(\cdot)$ is continuous for all i , and $\mathcal{E}(h)$ consists of isolated points, either $\Psi_i(x) < x_i$ for all $x_i \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$ and $x_i < \Psi_i(x)$ for all $x_i \in \left(\frac{1}{k}, \frac{1}{k-1}\right)$ or $\Psi_i(x) > x_i$ for all $x_i \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$ and $x_i > \Psi_i(x)$ for all $x_i \in \left(\frac{1}{k}, \frac{1}{k-1}\right)$ and i fixed. W.l.o.g., we may assume that $\Psi_i(x) < x_i$ for all $x_i \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$.

On the event $E_\eta := \{\theta_{\eta,1} < \frac{1}{k} - \delta_\eta\} \cap \left\{\sup_{n \geq \eta} M_{n,1}(\eta) \leq \frac{\delta_\eta}{2}\right\}$, we have $\theta_{\eta,1} - \Psi_1(\theta_\eta) > 0$ and from (42) $\mathbb{P}(E_\eta) > 0$. Now let show by induction that on E_η , $\theta_{n,1} < \frac{1}{k} - \frac{\delta_\eta}{2}$ for $n \geq \eta$. The base step of the induction for $n = \eta$ is obvious. Let us

assume that $\theta_{n-1,1} < \frac{1}{k} - \frac{\delta_\eta}{2}$, which implies that $R_{n,1} > 0$ and $\theta_{j,1} - \Psi_1(\theta_j) > 0$ for $\eta \leq j \leq n-1$. Therefore,

$$\theta_{n,1} < \theta_{\eta,1} + M_{n,1} \leq \frac{1}{k} - \delta_\eta + \frac{\delta_\eta}{2} = \frac{1}{k} - \frac{\delta_\eta}{2}.$$

Therefore, on E_η , we have $\theta_{n,1} - \Psi_1(\theta_n) > 0$ for all $n \geq \eta$ and $0 < \lim_{n \rightarrow \infty} R_{n,1} < \infty$. But this is a contradiction, since on E_η

$$\lim_{n \rightarrow \infty} R_{n,1} = \lim_{n \rightarrow \infty} (\theta_{\eta,1} + M_{n,1} - \theta_{n,1}) \leq -\frac{\delta_\eta}{2}. \quad (43)$$

□

Remark 4.5 Ideally, Theorem 1 should be stated as a corollary of Theorem 2. However, we state the theorems in reverse order. This is because we will prove Theorem 2 using Theorem 1.

Our proof for Theorem 2 is simpler than the proof of Proposition 7.1 from [31]. This is due to the assumption $\sum_n \left(\frac{\sigma_n}{\tau_n}\right)^2 < \infty$.

Proof of Theorem 2 From Theorem 1, we know that

$$\mathbb{P}(\mathcal{D}) = 1.$$

By symmetry, W.l.o.g. we may assume $\theta_{n,1} \rightarrow 0$ as $n \rightarrow \infty$. To prove (12), it is enough to show that, conditioned on the event $\theta_{n,1} \rightarrow 0$ as $n \rightarrow \infty$, if $\sup_{n \geq 1} U_{n,1} = \infty$, then we get a contradiction.

Let us assume that $\sup_{n \geq 1} U_{n,1} = \infty$. Since $U_{n,1}$ is non-decreasing, it follows $U_{n,1} \rightarrow \infty$, as $n \rightarrow \infty$.

W.l.o.g. we may assume that for some m, n (large enough), $U_{i+1,1} - U_{i,1} > 0$ for $n \leq i \leq m$. For some $C > 0$ by assumption (C),

$$\sum_{i=n}^{m-1} \frac{U_{i+1,1} - U_{i,1}}{U_{i,1}^\alpha} \geq \int_{U_{n,1}}^{U_{m,1}} \frac{1}{x^\alpha} dx \sim \frac{C}{U_{n,1}^{\alpha-1}}. \quad (44)$$

Recall that we know from (3)

$$U_{n+1,1} = U_{n,1} + B_{n+1,1},$$

where $B_{n+1,1}$ follows $\text{Bin}(\sigma_{n+1}, \Psi_1(\theta_n))$.

Let us re-write the above equation as

$$U_{n+1,1} = U_{n,1} + \sigma_{n+1} \Psi_1(\theta_n) + \epsilon_n \sqrt{\sigma_{n+1} \Psi_1(\theta_n) (1 - \Psi_1(\theta_n))}, \quad (45)$$

where $\epsilon_n = \frac{B_{n+1,1} - \sigma_{n+1} \Psi_1(\theta_n)}{\sqrt{\sigma_{n+1} \Psi_1(\theta_n) (1 - \Psi_1(\theta_n))}}$.

Recall the definition of $\Psi_i(x)$ as in (19). Since $x \in \{[0, 1]^d \setminus \{0\}\}$ and $x_1 + x_2 + \dots + x_d = 1$, and f is strictly increasing, we have $\sum_{i=1}^d f(x_i) \geq f(\frac{1}{d}) =: C_d^{-1}$, which implies that

$$\Psi_i(x) \leq C_d f(x_i). \quad (46)$$

By the estimate from Lemma 2.3 (iv) applied to (45), we get

$$\begin{aligned} U_{n+1,1} &\leq U_{n,1} + C_d \sigma_{n+1} \theta_{n,1}^\alpha + \epsilon_n \sqrt{\sigma_{n+1} \Psi_1(\theta_n)(1 - \Psi_1(\theta_n))} \\ \frac{U_{n+1,1} - U_{n,1}}{U_{n,1}^\alpha} &\leq C_d \frac{\sigma_{n+1}}{\tau_n^\alpha} + \frac{1}{U_{n,1}^\alpha} \epsilon_n \sqrt{\sigma_{n+1} \Psi_1(\theta_n)(1 - \Psi_1(\theta_n))}. \end{aligned} \quad (47)$$

Thus,

$$\sum_{i=n}^{\infty} \frac{U_{i+1,1} - U_{i,1}}{U_{i,1}^\alpha} \leq C_d \sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^\alpha} + \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi_1(\theta_i)(1 - \Psi_1(\theta_i))}. \quad (48)$$

Combining this with (44), we get

$$\frac{C}{U_{n,1}^{\alpha-1}} \leq C_d \sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^\alpha} + \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi_1(\theta_i)(1 - \Psi_1(\theta_i))}. \quad (49)$$

Furthermore, write

$$\frac{\sigma_{n+1}}{\tau_n} = \frac{\frac{\sigma_{n+1}}{\tau_{n+1}}}{1 - \frac{\sigma_{n+1}}{\tau_{n+1}}}. \quad (50)$$

Observe that by our assumption $\sum_{n \geq 1} \frac{\sigma_n^2}{\tau_n^2} < \infty$, we have $\lim_{n \rightarrow \infty} \frac{\sigma_n}{\tau_n} = 0$, and hence, from (54) we have $\frac{\sigma_{n+1}}{\tau_n} \rightarrow 0$, as $n \rightarrow \infty$.

Therefore, using arguments similar to (40), for some suitable constant $C' > 0$, we obtain

$$\sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^\alpha} = \sum_{i \geq n} \frac{(1 + \frac{\sigma_{i+1}}{\tau_i})^\alpha \sigma_{i+1}}{\tau_{i+1}^\alpha} \leq C' \int_{\tau_n}^{\infty} \frac{1}{x^\alpha} dx = C' \frac{1}{\tau_n^{\alpha-1}}.$$

Using the above bound in (49), we obtain where $C, C' > 0$ are suitable constants that may change accordingly

$$\begin{aligned} \frac{C}{U_{n,1}^{\alpha-1}} &\leq C' \frac{1}{\tau_n^{\alpha-1}} + \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi_1(\theta_i)(1 - \Psi_1(\theta_i))} \\ C &\leq C' \theta_{n,1}^{\alpha-1} + U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi_1(\theta_i)(1 - \Psi_1(\theta_i))}. \end{aligned} \quad (51)$$

Let us denote by

$$T_n := U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi_1(\theta_i)(1 - \Psi_1(\theta_i))}, \quad (52)$$

then

$$|T_n| \leq U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} |\epsilon_i| \sqrt{\sigma_{i+1} \Psi_1(\theta_i)(1 - \Psi_1(\theta_i))}.$$

Using (46) and $f(x_i) \leq x_i^\alpha$, we get for some $C' > 0$

$$|T_n| \leq C' U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^{\frac{\alpha}{2}}} |\epsilon_i| \sqrt{\sigma_{i+1} \frac{1}{\tau_i^\alpha} (1 - \Psi_1(\theta_i))}.$$

Since $U_{n,1}$ is non-decreasing in n , we have $\frac{U_{n,1}}{U_{i,1}} \leq 1$ for $i \geq n$. Therefore, from the above equation, we get

$$\begin{aligned} |T_n| &\leq C' \frac{1}{U_{n,1}} \sum_{i \geq n} |\epsilon_i| \sqrt{\sigma_{i+1} \frac{U_{n,1}^\alpha}{\tau_i^\alpha} (1 - \Psi_1(\theta_i))} \\ &= C' \frac{1}{\sqrt{U_{n,1}}} \sum_{i \geq n} |\epsilon_i| \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{U_{n,1}}{\tau_i} \right)^{\alpha-1} (1 - \Psi_1(\theta_i))} \\ &\leq C' \frac{1}{\sqrt{U_{n,1}}} \sum_{i \geq n} |\epsilon_i| \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i} \right)^{\alpha-1}}, \end{aligned} \quad (53)$$

where we have used $(1 - \Psi_1(\theta_i)) \leq 1$, and $U_{n,1} \leq \tau_n$. Now, consider the following quantity

$$N_n := \sum_{i=1}^n |\epsilon_i| \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i} \right)^{\alpha-1}}.$$

It is easy to see that $(N_n)_n$ is a non-negative submartingale w.r.t. $(\mathcal{F}_n)_n$. Furthermore, write

$$\frac{\sigma_{n+1}}{\tau_n} = \frac{\frac{\sigma_{n+1}}{\tau_{n+1}}}{1 - \frac{\sigma_{n+1}}{\tau_{n+1}}}. \quad (54)$$

Observe that by our assumption $\sum_{n \geq 1} \frac{\sigma_n^2}{\tau_n^2} < \infty$, we have $\lim_{n \rightarrow \infty} \frac{\sigma_n}{\tau_n} = 0$, and hence, from (54) we have $\frac{\sigma_{n+1}}{\tau_n} \rightarrow 0$, as $n \rightarrow \infty$.

Recall that $\frac{\sigma_{n+1}}{\tau_n} \rightarrow 0$, as $n \rightarrow \infty$ follows from (54). Therefore, for all n (large enough), we obtain

$$\begin{aligned} \frac{\tau_i}{\tau_n} &= \prod_{j=n}^{i-1} \frac{\tau_{j+1}}{\tau_j} \\ &= \prod_{j=n}^{i-1} \left(1 + \frac{\sigma_{j+1}}{\tau_j} \right) \\ &= (1 + o(1))^{i-n}, \text{ for all } i \geq n. \end{aligned} \quad (55)$$

Therefore, using the above inequality, $\alpha > 3$, and $\tau_i \geq i$ we obtain

$$\sum_{i \geq n} \left(\frac{\tau_n}{\tau_i} \right)^{(\alpha-1)/2} = \sum_{i=1}^{\infty} \left(\frac{1}{(1 + o(1))^i} \right)^{(\alpha-1)/2} < \infty. \quad (56)$$

We have for some $C > 0$ that

$$\sup_{n \geq 1} \mathbb{E}(N_n) \leq C \sup_{n \geq 1} \sum_{i=1}^{\infty} \left(\frac{\tau_n}{\tau_i} \right)^{(\alpha-1)/2} < \infty, \quad (57)$$

where we have used (55) and $\mathbb{E}(|\epsilon_i|) \leq (\mathbb{E}(\epsilon_i^2))^{1/2} \leq 1$, which follows from the Cauchy-Schwartz inequality. By the martingale convergence theorem, Theorem 2.5 [14], we have that almost surely $N_n \rightarrow N_{\infty}$ and $\mathbb{P}(N_{\infty} < \infty) = 1$.

Therefore, since we assumed that $U_{n,1} \rightarrow \infty$ as $n \rightarrow \infty$, we get for all n large enough

$$|T_n| \leq \frac{1}{\sqrt{U_{n,1}}} \sum_{i \geq n} |\epsilon_i| \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i} \right)^{\alpha-1}} < \epsilon \quad (58)$$

for any $\epsilon > 0$. From (51), we get on the event $\{\theta_{n,1} \rightarrow 0\}$,

$$C \leq C' \theta_{n,1}^{\alpha-1} + T_n \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which is a contradiction since $C > 0$. \square

Remark 4.6 We believe that $\mathbb{P}(\mathcal{F}) = 1$ when $\alpha > 1$, but our submartingale works only when $\alpha > 3$. The other alternative method which we have not done is to try and use an argument similar to the ones available in [31] to show that

$$\mathbb{P} \left(\liminf_{n \rightarrow \infty} \frac{1}{U_{n,1}^{\frac{\alpha}{2}} \sqrt{A_n}} \sum_{i \geq n} \frac{1}{U_{i,1}^{\alpha}} \epsilon_i \sqrt{\sigma_{i+1} \Psi_1(\theta_{i,1}) (1 - \Psi_1(\theta_{i,1}))} < \infty \right) = 1,$$

where $A_n := \sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^{\alpha}}$.

Appendix A Stochastic Approximation

In this section, we will introduce relevant quantities and existing results regarding stochastic approximation. We will use notation from [7, 9]. Define the following recursive scheme on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ with values in a compact set $\Gamma = [0, 1]^d \subset \mathbb{R}^d$:

$$\theta_{n+1} = \theta_n + \frac{\sigma_{n+1}}{\tau_{n+1}} \left[h(\theta_n) + \frac{1}{\sigma_{n+1}} \Delta M_{n+1} \right], \quad (59)$$

where $(\frac{\sigma_n}{\tau_n})_{n \geq 1}$ is a positive sequence of numbers, $h : \Gamma \rightarrow \mathbb{R}^d$ is a continuous function, $(\Delta M_n)_{n \geq 0}$ a $(\mathcal{F}_n)_{n \geq 0}$ -martingale increment, defined in (10). The mean-field function h was defined as

$$h(y) = \frac{f_d(y)}{\|\mathbf{f}(y)\|_1} - y \quad (60)$$

for $y \in \Gamma$.

The main idea of stochastic approximation techniques is to show that the recursive system defined in (59) behaves like an ODE perturbed by a small noise term. Then the recursive system is shown to converge toward the equilibrium points of the deterministic flow induced by the driving term h .

The *limit set* $L(\{w_n\}_{n \geq 1})$ of the sequence $\{w_n\}_{n \geq 1}$ is the set of points $x \in \Gamma$ such that there exists a subsequence $(n_k)_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} n_k = \infty$ for which $\lim_{k \rightarrow \infty} w_{n_k} = x$.

Consider furthermore the ODE

$$\dot{y} = h(y) \quad (61)$$

and associated with h look at $(\Phi(t, x))_{t \geq 0, x \in \Gamma}$ the Γ -valued *flow* of the system. The family $\{\Phi_t\}_{t \in \mathbb{R}_+}$, where $\Phi_t(x) = \Phi(t, x)$, satisfies the group property, is defined as $\Phi_0 = \text{Id}$ and for all $(t, s) \in \mathbb{R}_+^2$

$$\Phi_{t+s} = \Phi_t \circ \Phi_s.$$

For every $x \in \Gamma$, $(\Phi(t, x))_{t \geq 0}$ is the unique solution to the ODE (61)

$$\frac{d}{dt} \Phi_t(x) = h(\Phi_t(x)), \quad (62)$$

where $\Phi_0(x) = x$. It exists since h is locally Lipschitz which follows from continuity of the function f . Denote by $\mathcal{E}(\Phi) = \{p \in \Gamma : \Phi_t(p) = p \text{ for all } t > 0\}$ the *equilibrium set* for the flow Φ . Note that by definition, this set is invariant.

Definition A.1 Let $\delta > 0$, $T > 0$ and Γ a metric space with metric d . A point x is said to be (δ, T) -recurrent if there exist $k \in \mathbb{N}$ points $y_0, \dots, y_{k-1} \in \Gamma$ and t_1, \dots, t_{k-1} such that for all $i = 0, \dots, k-1$ and $y_k = x$

$$t_i \geq T; \quad d(y_0, x) < \delta; \quad d(\Phi_{t_i}(y_i), y_{i+1}) < \delta.$$

A point is called *chain recurrent* if it is (δ, T) -recurrent for all $\delta > 0, T > 0$. The set of all chain recurrent points of the flow Φ will be denoted by $\text{CR}(\Phi)$.

Definition A.2 A compact set Γ is called internally chain recurrent if each $x \in \Gamma$ is chain recurrent for the flow Φ .

The omega limit set of Φ is defined by $\mathcal{L}(\Phi) = \bigcup_{x \in \Gamma} \omega(x)$, where

$$\omega(x) = \left\{ p \in \Gamma : \exists (t_k)_k \text{ such that } t_k \rightarrow \infty \text{ and } p = \lim_{k \rightarrow \infty} \Phi_{t_k}(x) \right\}.$$

One has that

$$\mathcal{E}(\Phi) \subset \mathcal{L}(\Phi) \subset \text{CR}(\Phi), \quad (63)$$

see page 21 of [9].

We will need the notion of a strict Lyapunov function for the flow Φ , see also Sect. 3 of [7].

Definition A.3 A Lyapunov function for the flow Φ (or h) on Λ , $\Lambda \subset \Gamma$ invariant set, is a continuous map $V : \Gamma \rightarrow \mathbb{R}$ such that $t \mapsto V(\Phi_t(x))$ is constant for $x \in \Lambda$ and strictly decreasing for all $x \in \Gamma \setminus \Lambda$ for all $t > 0$. When $\Lambda = \mathcal{E}(\Phi)$, we call the function strictly Lyapunov. If such a V exists, we call h a gradient-like vector field.

Lemma A.4 Let the flow Φ be defined in (62) and h in (60) and $F : \Gamma \rightarrow \mathbb{R}$ by

$$F(y) = \sum_{i=1}^d \int_0^{y_i} \frac{f(z_i)}{z_i} dz_i.$$

Then $V = -F$ is a strict Lyapunov function for Φ on $\mathcal{E}(\Phi)$.

Proof For $i = 1, \dots, d$, write the ODE (61) as

$$\begin{aligned} \dot{y}_i(t) &= \frac{y_i(t)}{\|\mathbf{f}(y(t))\|_1} \left(\frac{f(y_i(t))}{y_i(t)} - \sum_{j=1}^d y_j(t) \frac{f(y_j(t))}{y_j(t)} \right) \\ &= \frac{y_i(t)}{\|\mathbf{f}(y(t))\|_1} \left(\frac{\partial}{\partial y_i} F(y(t)) - \sum_{j=1}^d y_j(t) \frac{\partial}{\partial y_j} F(y(t)) \right) \end{aligned} \quad (64)$$

where $F : \mathbb{R}^d \rightarrow \mathbb{R}$, $F(y) = (F_1(y_1), \dots, F_d(y_d))$, and $F(y) = \sum_{i=1}^d F_i(y_i)$. We have that $F(0) = 0$ and

$$\frac{\partial}{\partial y_i} F(y) = \frac{f(y_i)}{y_i}.$$

For each coordinate $i = 1, \dots, d$ we can write

$$F_i(y_i) = \int_0^{y_i} \frac{\partial}{\partial z} F_i(z) dz = \int_0^{y_i} \frac{f(z)}{z} dz$$

where $z \in \mathbb{R}$. The function $-F$ is strictly Lyapunov for Φ since

$$\begin{aligned} \frac{d}{dt} F(y(t)) &= \sum_{i=1}^d \dot{y}_i(t) \frac{\partial}{\partial y_i} F(y(t)) \\ &= \frac{1}{\|\mathbf{f}(y)\|_1} \left\{ \sum_{i=1}^d y_i(t) \left(\frac{\partial}{\partial y_i} F(y(t)) \right)^2 - \left(\sum_{i=1}^d y_i(t) \frac{\partial}{\partial y_i} F(y(t)) \right)^2 \right\} > 0 \end{aligned}$$

where the last step follows from Jensen's inequality. Finally, note that it is constant on the equilibrium set $\mathcal{E}(h) = \mathcal{E}(\Phi)$:

$$\begin{aligned} &\sum_{i=1}^d y_i(t) \left(\frac{\partial}{\partial y_i} F(y(t)) \right)^2 - \left(\sum_{i=1}^d y_i(t) \frac{\partial}{\partial y_i} F(y(t)) \right)^2 \\ &= \sum_{i=1}^d \frac{f^2(y_i(t))}{y_i(t)} - \left(\sum_{i=1}^d f(y_i(t)) \right)^2 \\ &= \sum_{i=1}^d \|\mathbf{f}(y(t))\|_1 f(y_i(t)) - \|\mathbf{f}(y(t))\|_1^2 = 0, \end{aligned}$$

where in the before last equality we used (15). \square

Proposition A.5 *Let Φ be the flow on Γ , where Γ is compact. Furthermore, let $\Lambda \subset \Gamma$ be compact invariant set and $V : \Gamma \rightarrow \mathbb{R}$ a Lyapunov function for Φ on Λ with finite $V(\Lambda)$, then*

$$CR(\Phi) \subset \Lambda.$$

The last proposition is same as Proposition 3.2 from [7]. Choosing $V = -F$ and as invariant set $\Lambda = \mathcal{E}(\Phi)$ in the last proposition together with (63) implies that

$$\mathcal{E}(\Phi) = \mathcal{L}(\Phi) = CR(\Phi).$$

The following theorem is equivalent to Theorem 6 from [19], based on Theorem 5.7 from [9].

Theorem A.6 *Let $\{\theta_n\}_{n \geq 1}$ be a solution to (59) and assume that*

- (i) $\sum_{n=1}^{\infty} \left(\frac{\sigma_n}{\tau_n} \right)^2 < \infty$.
- (ii) h is a local Lipschitz function.
- (iii) $\sup_{n \geq 1} \mathbb{E}(\|\Delta M_n\|_2^2 | \mathcal{F}_n) < \infty$ a.s.

Then almost surely, $L(\{\theta_n\}_{n \geq 1})$ is a connected set, internally chain recurrent for the flow Φ induced by h , $L(\{\theta_n\}_{n \geq 1}) \subset CR(\Phi)$.

Proof The first assumption is trivially satisfied, the second follows from the fact that f is locally Lipschitz since it is $C^1((0, 1))$, and the third follows trivially from writing the martingale difference as in (10). \square

The next corollary is analogous to Corollary 3.3. from [7]; the proof is adapted.

Corollary A.7 *Let $\{\theta_n\}_{n \geq 1}$ be defined in (59). Then $\theta_n \rightarrow \theta^*$ a.s. as $n \rightarrow \infty$ where $\theta^* \in \mathcal{E}(h)$.*

Proof Let Φ be the flow induced by h defined in (60) on $\Gamma = [0, 1]^d$ and let $L(\{\theta_n\}_{n \geq 1})$ be the set of all almost sure limiting points of $\{\theta_n\}_{n \geq 1}$. By Theorem A.6, $L(\{\theta_n\}_{n \geq 1})$ is a connected set, internally chain recurrent for the flow Φ induced by h . By Lemma A.4, we know that there exists a strict Lyapunov function for the flow on Γ and by Proposition A.5 the set $L(\{\theta_n\}_{n \geq 1})$ consists of equilibria, $\mathcal{E}(\Phi) = \mathcal{E}(h)$. Since they are isolated (Lemma 4.1), $L(\{\theta_n\}_{n \geq 1})$ coincides with the set of equilibrium points. \square

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Data availability We do not analyze or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

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