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Stability of implicit material point methods for geotechnical analysis of large deformation problems

Will Coombs*, R.E. Bird, G. Pretti & C.E. Augarde

*Professor of Computational Mechanics
Department of Engineering, Durham University, UK

w.m.coombs@durham.ac.uk

ALERT 2024


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HARRY POTTER
and the Philosopher's Stone
J.K.ROWLING
9¾
HOGWARTS EXPRESS
"...this is a terrific book." The Sunday Telegraph

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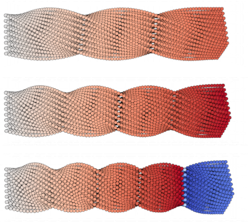
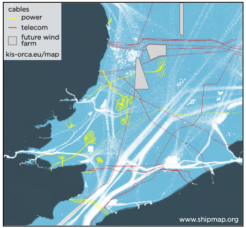
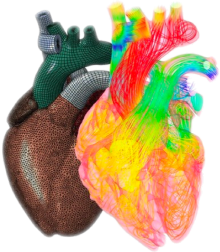


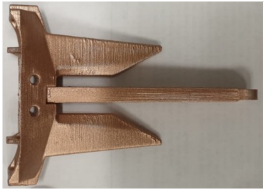




Stability of implicit material point methods for geotechnical analysis of large deformation problems

seabed ploughing
(2014-2018)

screw piles
(2016-2019)



drag anchors
(2022-2025)

offshore decommissioning
(future)

Stability of implicit material point methods for geotechnical analysis of large deformation problems

overview of the MPM

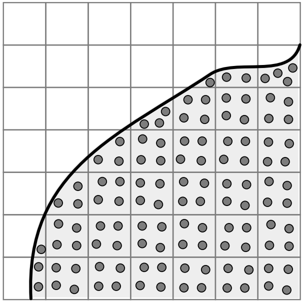
stability issues: cell crossing & small cuts

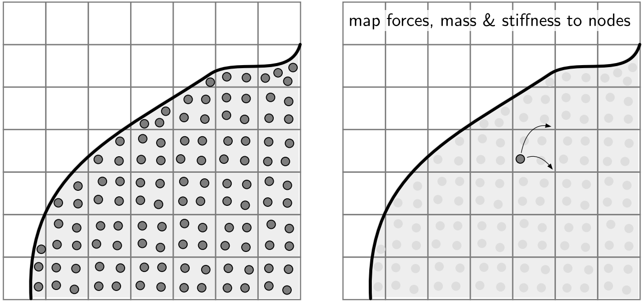
conditioning, implications, avoidance & remedies

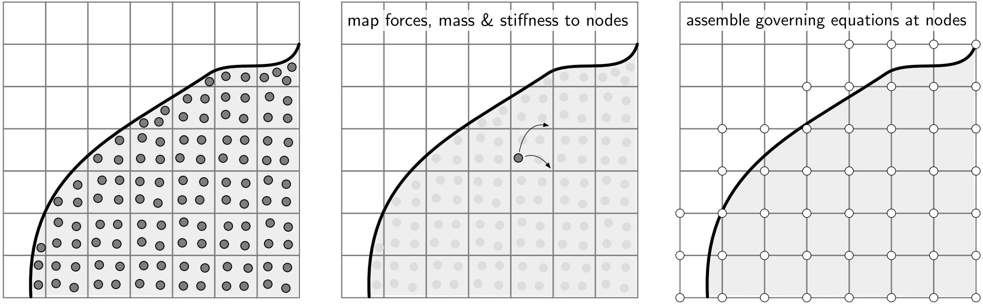
ghost stabilisation

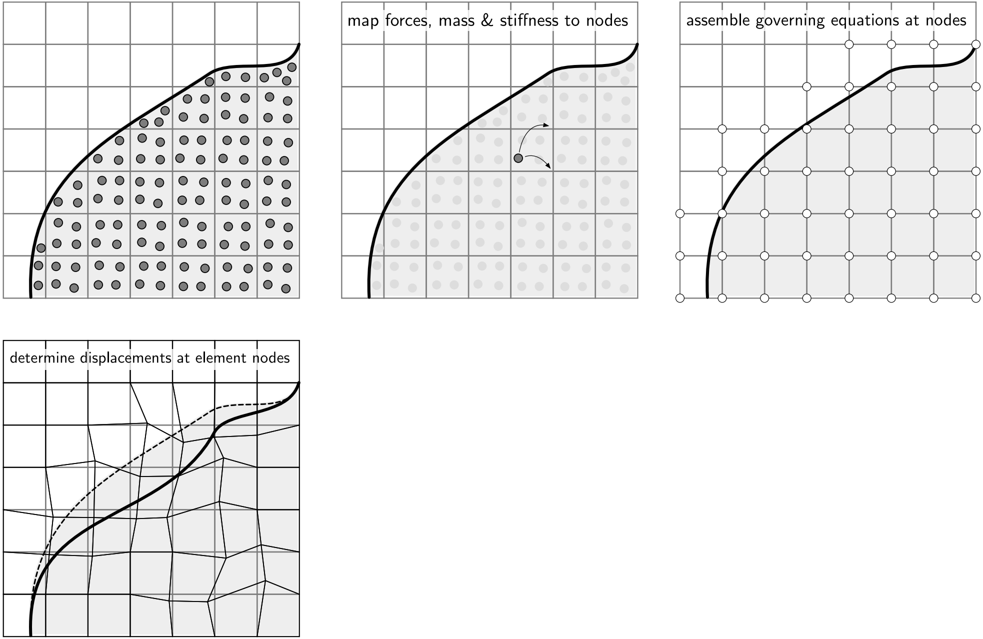
numerical examples

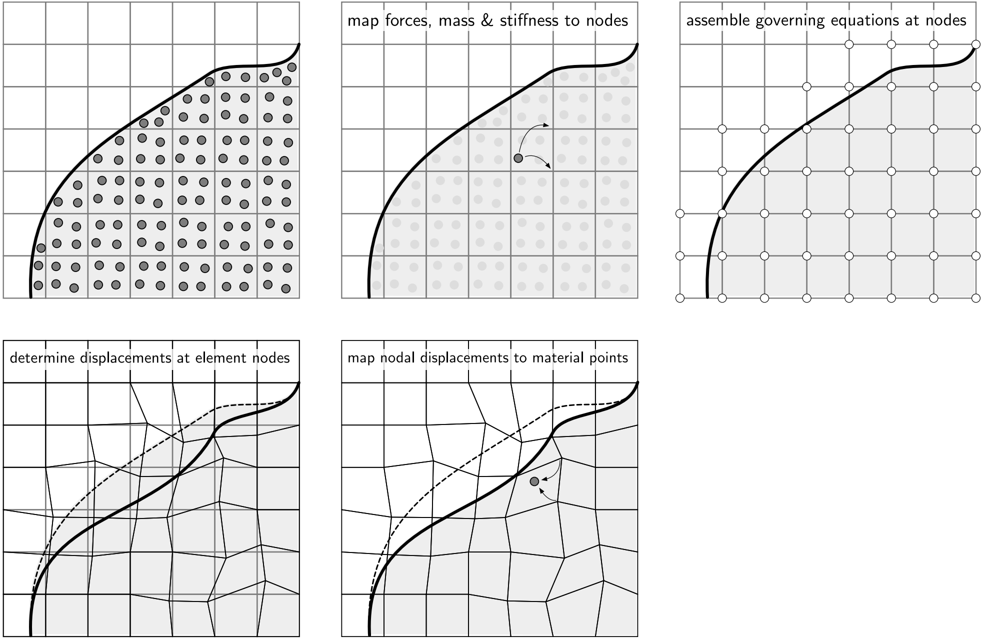
observations

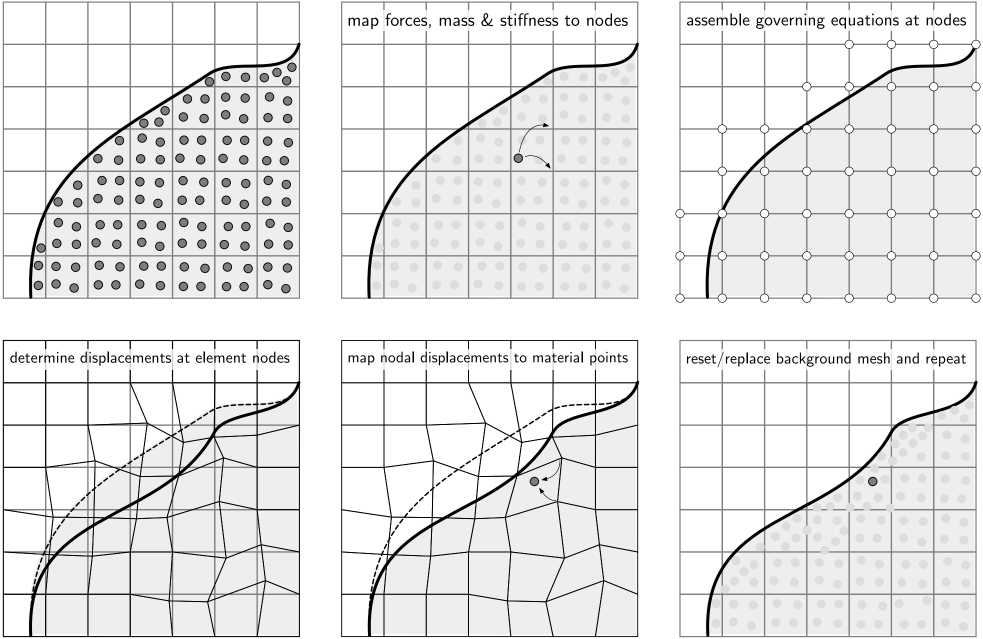



map forces, mass & stiffness to nodes


map forces, mass & stiffness to nodes
assemble governing equations at nodes


map forces, mass & stiffness to nodes
assemble governing equations at nodes
determine displacements at element nodes


map forces, mass & stiffness to nodes
assemble governing equations at nodes
determine displacements at element nodes
map nodal displacements to material points



map forces, mass & stiffness to nodes
assemble governing equations at nodes
determine displacements at element nodes
map nodal displacements to material points
reset/replace background mesh and repeat

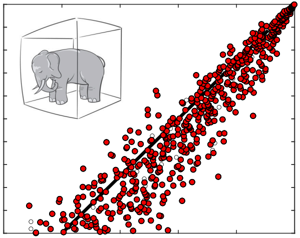
the Material Point Method

the finite element method where Gauss points are allowed to move...

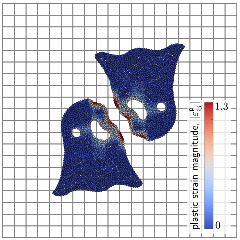
the Material Point Method

the finite element method where
Gauss points are allowed to move...

but there are issues...



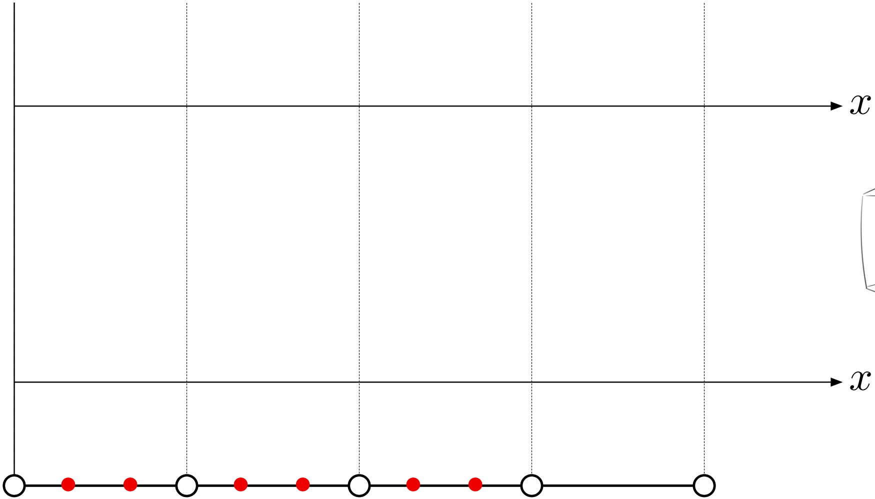
cell crossing

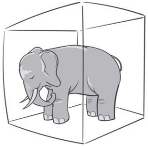


small-cuts

Issues: cell crossing

the *issue* with material point methods





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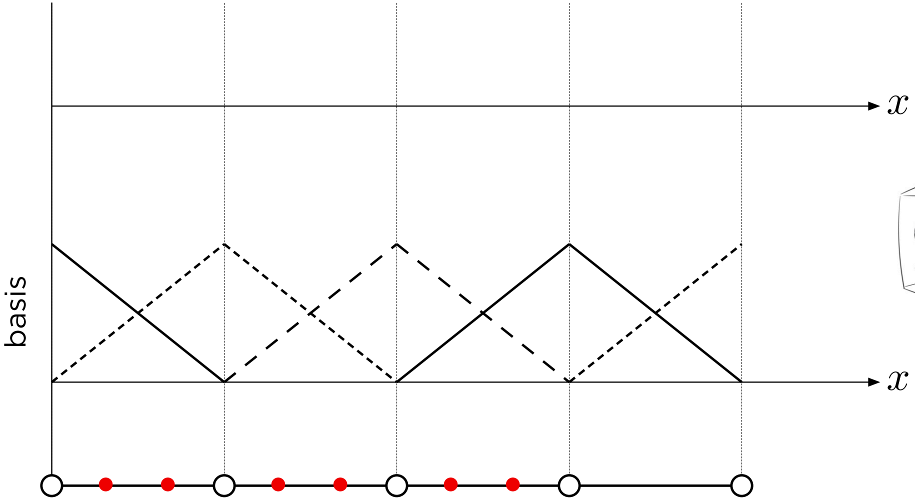

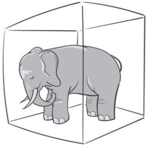

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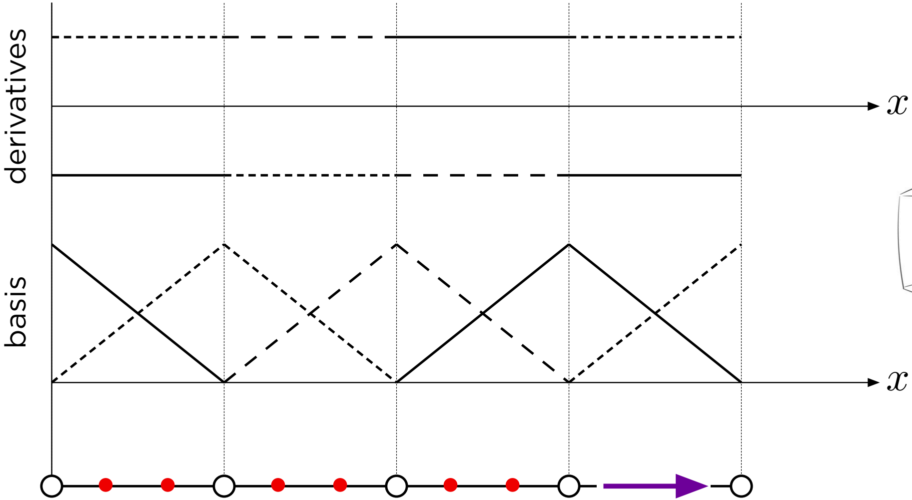

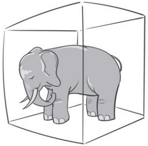

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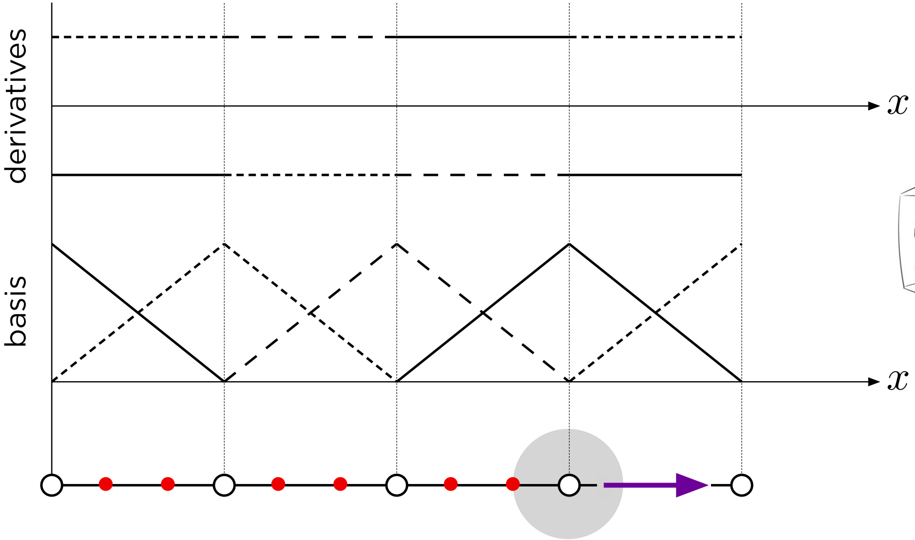

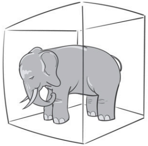

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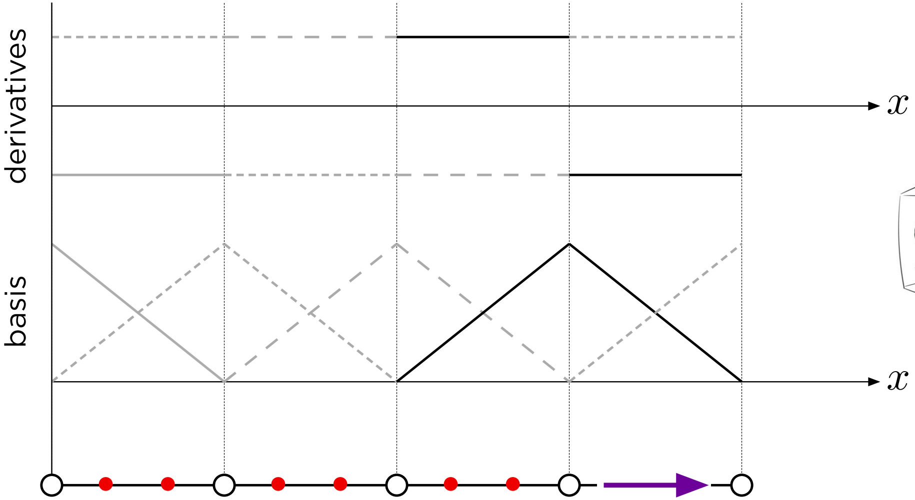

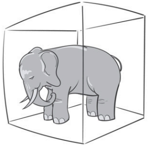

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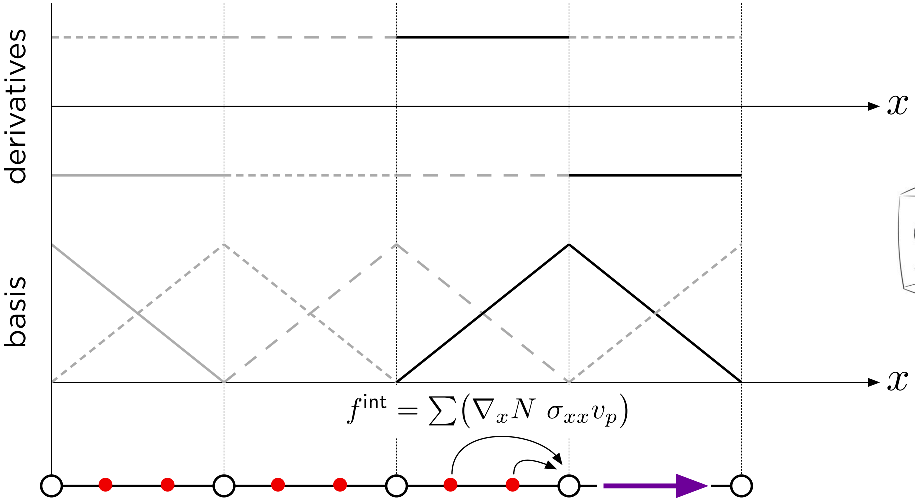

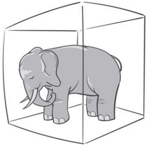

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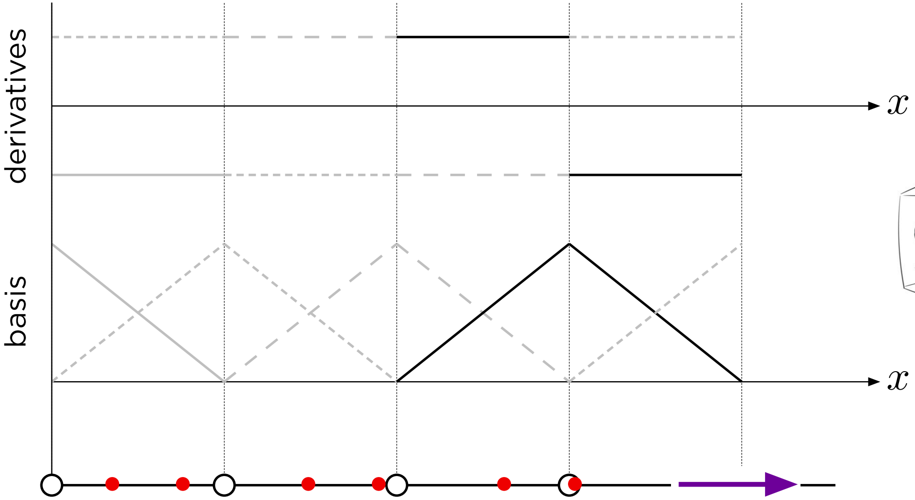

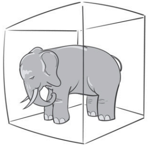

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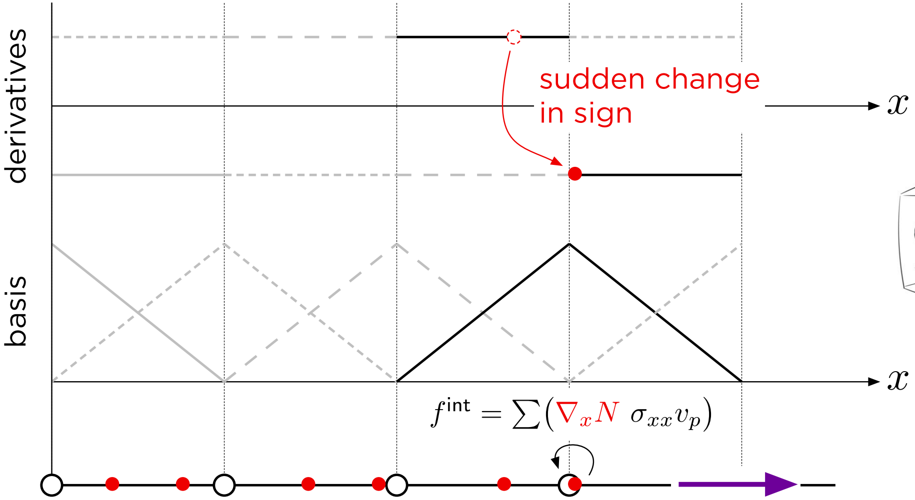
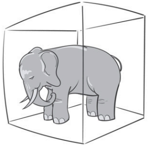
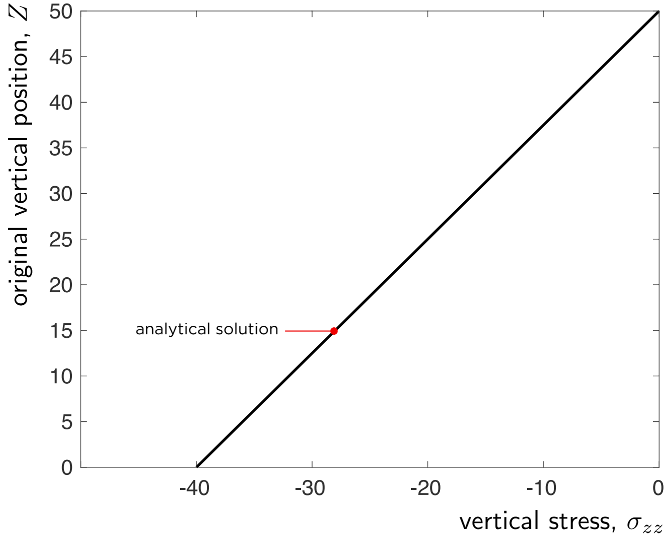
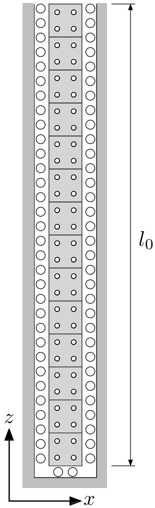




Stability issues: cell crossing

elastic compaction under self weight





initial height, $l_0 = 50$m

$E = 10$kPa
$\nu = 0$

$b = -800\text{N/m}^3$ body force applied over 40 equal load steps

analytical stress solution

$$\sigma^a_{zz} = b(l_0 - Z)$$

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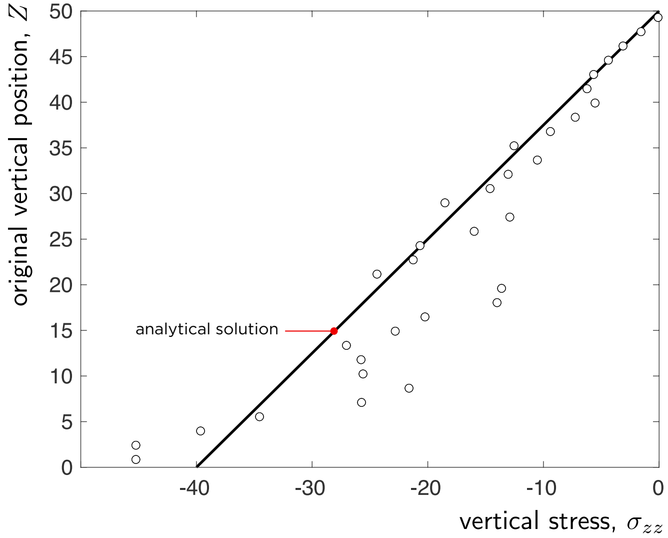

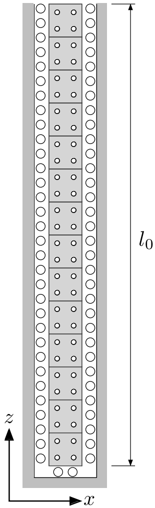

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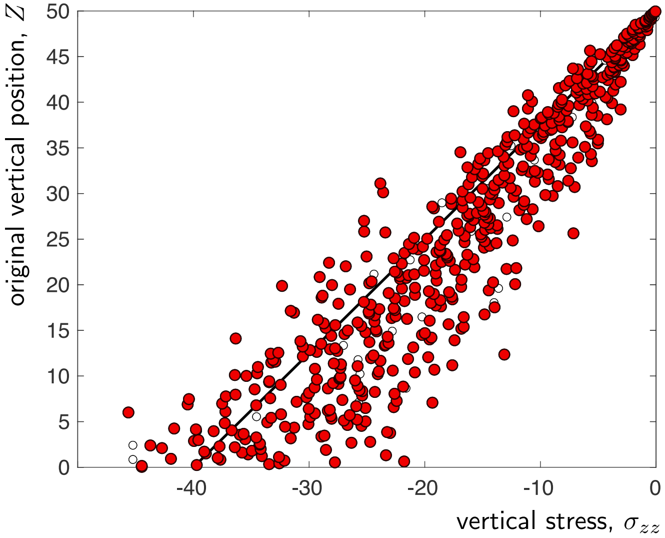

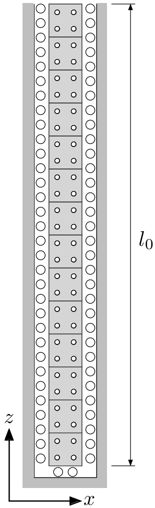

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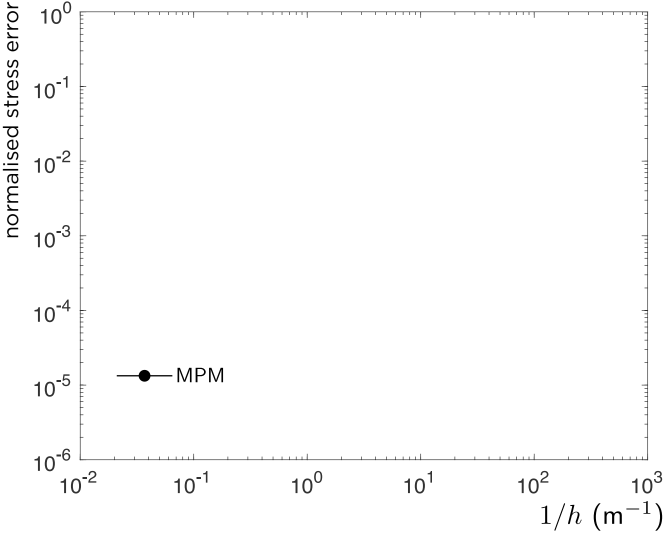

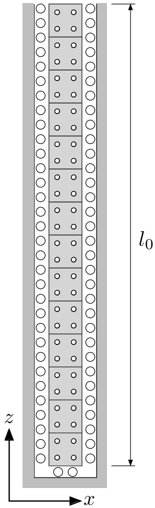

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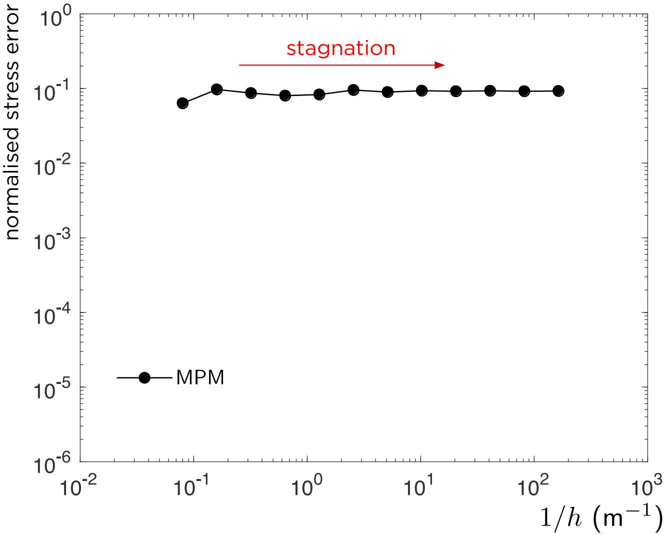

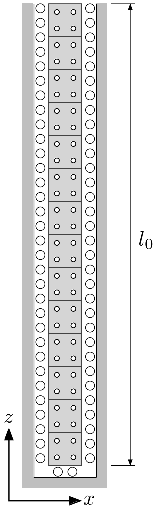

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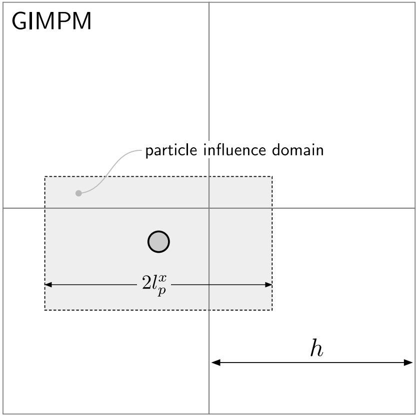
generalised interpolation material point method (GIMPM)



Bardenhagen, S. G., Kober, E. M. (2004). The Generalized Interpolation Material Point Method. *Computer Modeling in Engineering & Sciences*, 5(6), 477–496.

Stability issues: cell crossing

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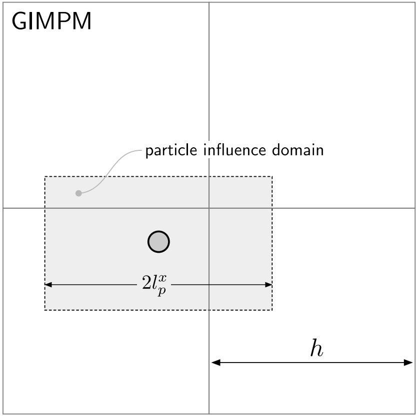

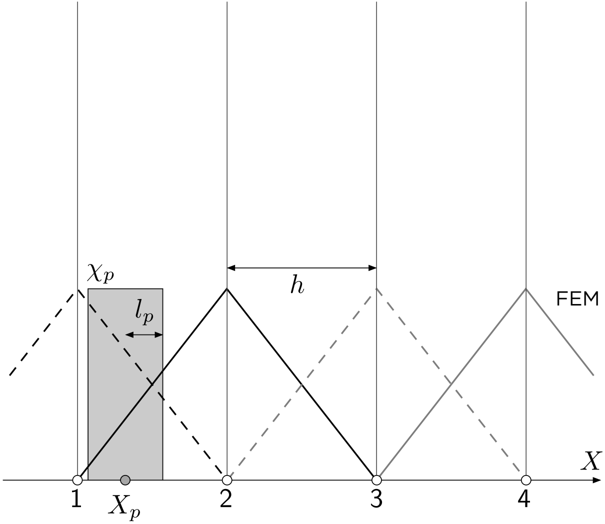

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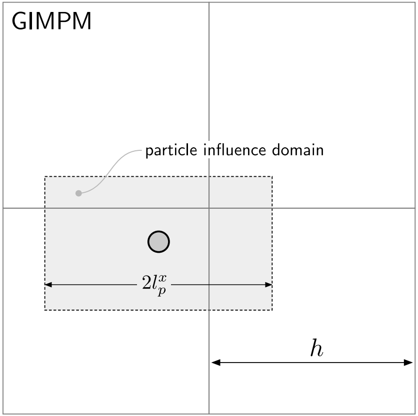

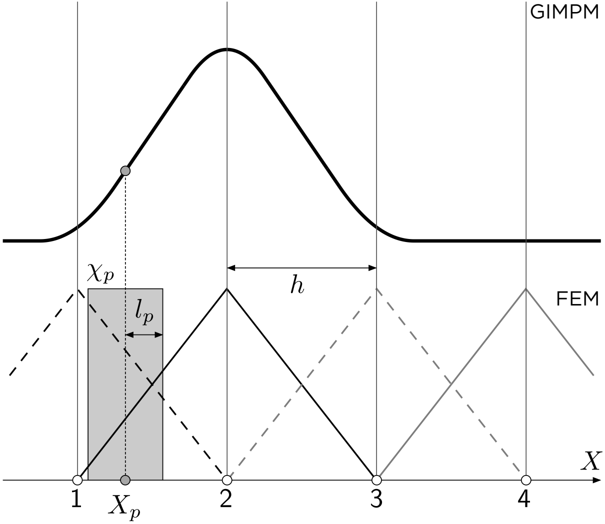

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Stability issues: cell crossing

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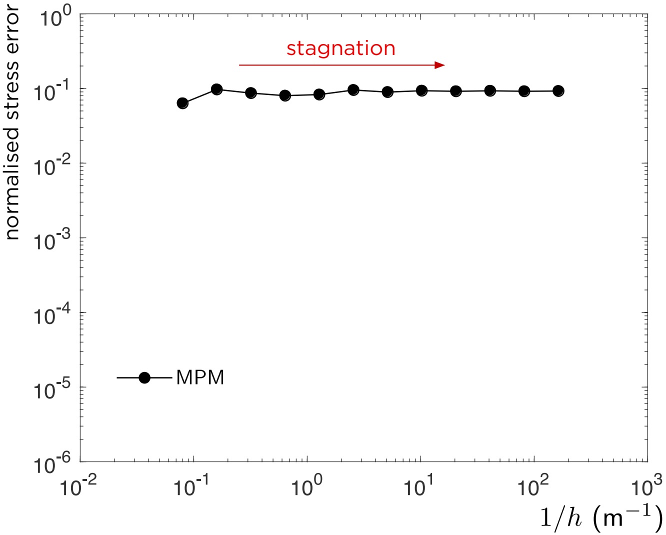

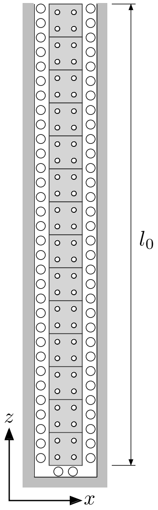

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Stability issues: cell crossing

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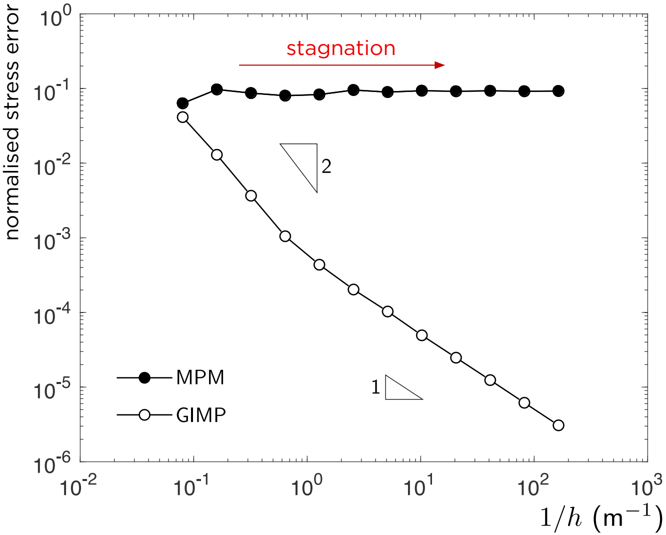

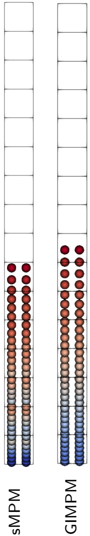

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the Material Point Method

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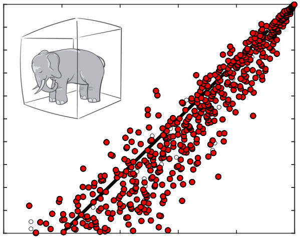
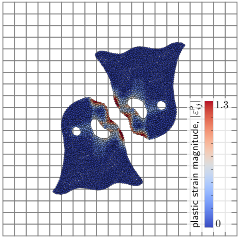


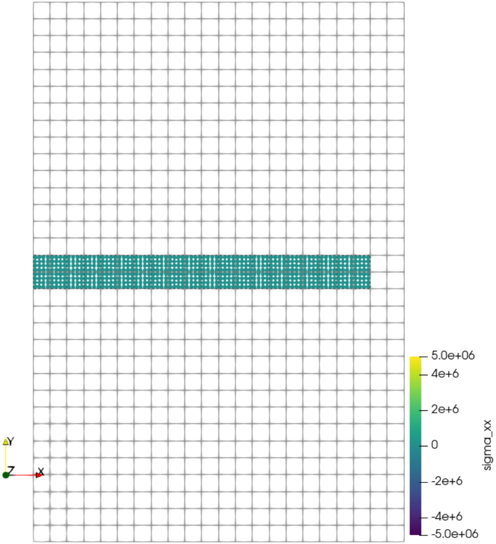
cell crossing

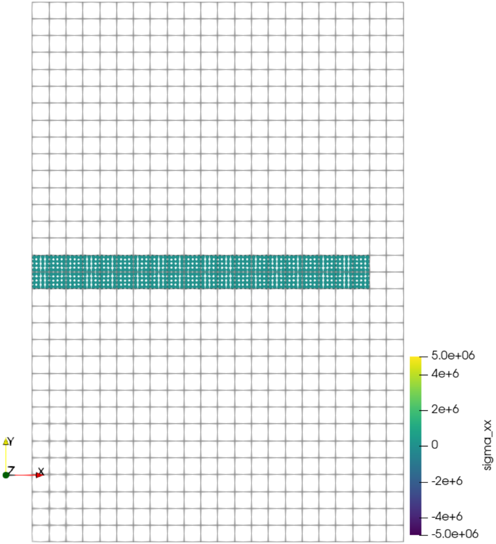


small-cuts

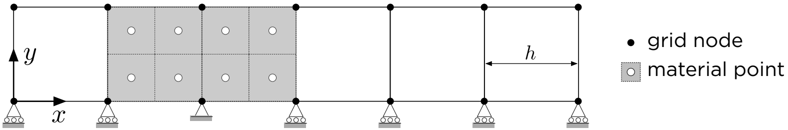
Stability issues

explicit dynamic elastic analysis





Stability issues: conditioning



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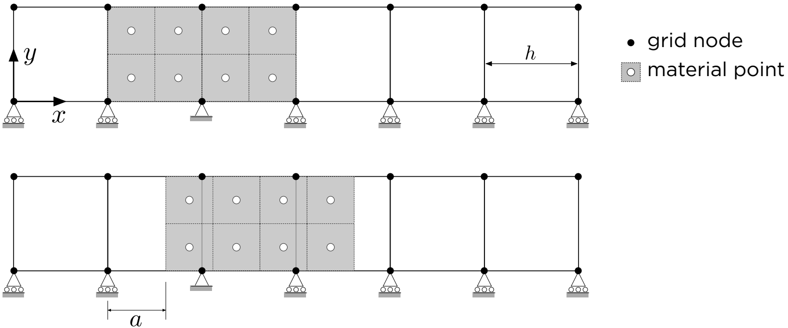

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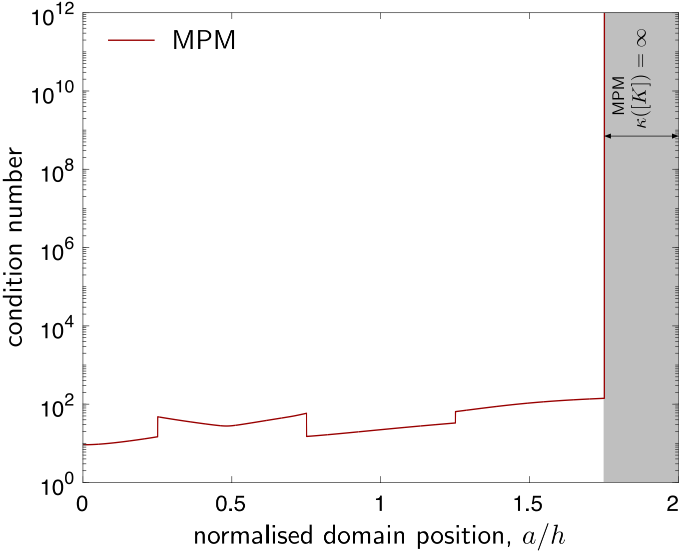
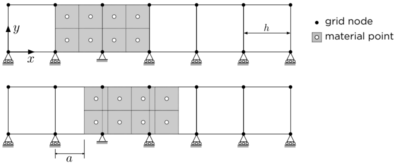




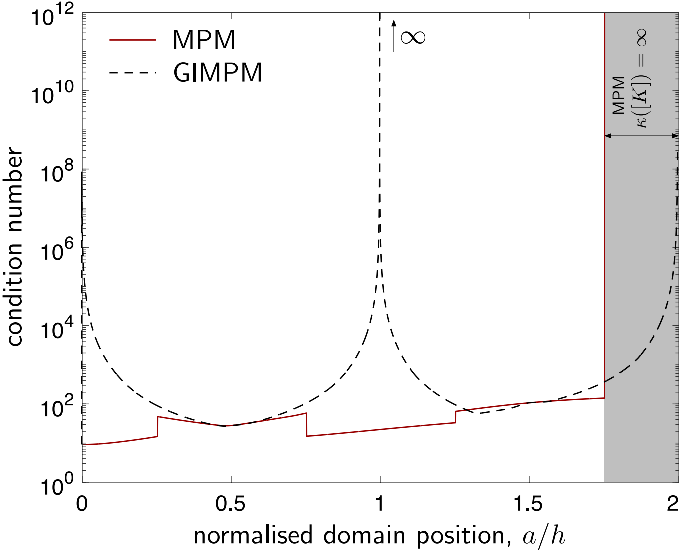
condition number, $\kappa([K])$

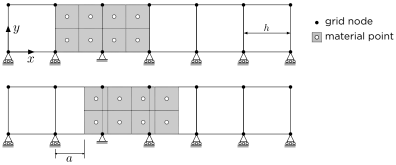
ratio of the largest to smallest eigenvalue of $[K]$

linked to the ease and accuracy of the solution of $\{f\} = [K]\{d\}$

large numbers are *problematic*

Stability issues: conditioning





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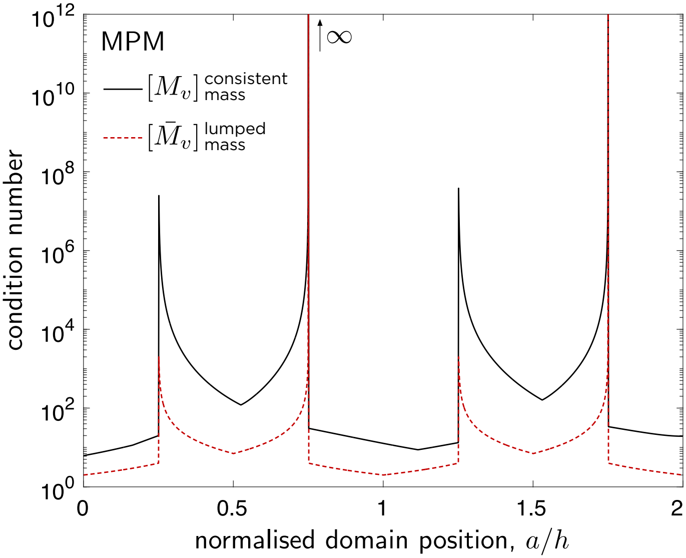
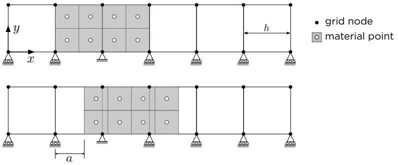




condition number, $\kappa([M])$

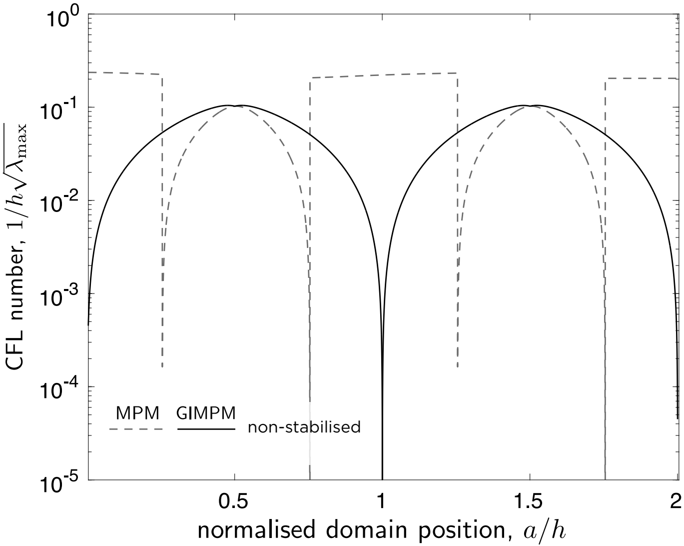
ratio of the largest to smallest eigenvalue of $[M]$

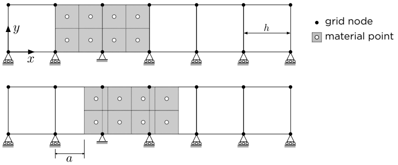
linked to the ease and accuracy of the solution of $\{f\} = [M]\{a\}$ and other mapping issues

large numbers are *problematic*

note that the MPM conserves mass

Stability issues: conditioning





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Stability issues: conditioning

remedies... or avoidance strategies

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"the value of a nodal basis function... may be small. However, the internal force vector... does not approach zero → accelerations... can occasionally be ***unphysical****..."*
Sulsky *et al.* (1995)

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"Such a small mass node not only leads to **tiny time steps**, *but also often results in* **instability** *and failure of numerical simulations."*
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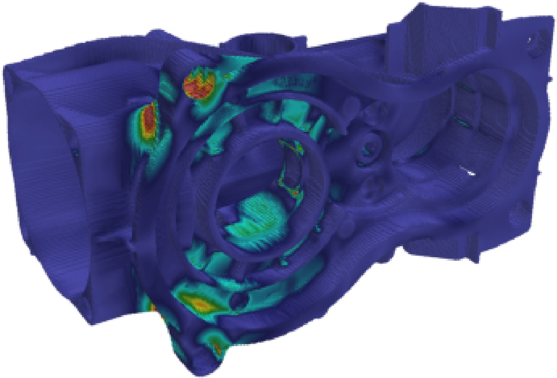
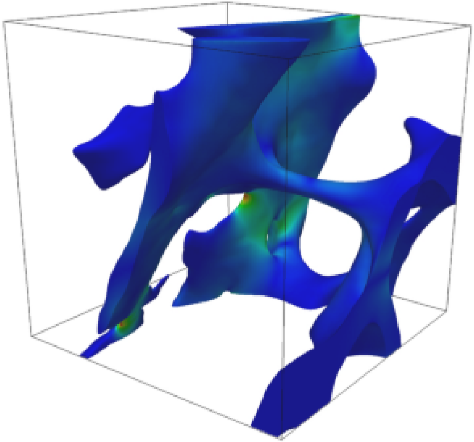
"...occupied by a small physical domain, has a harmful effect on the solution... cause **numerical instability**... **ill-conditioning**..."
Yamaguchi *et al.* (2021)

Unfitted methods & stabilisation

lessons from immersed finite element methods



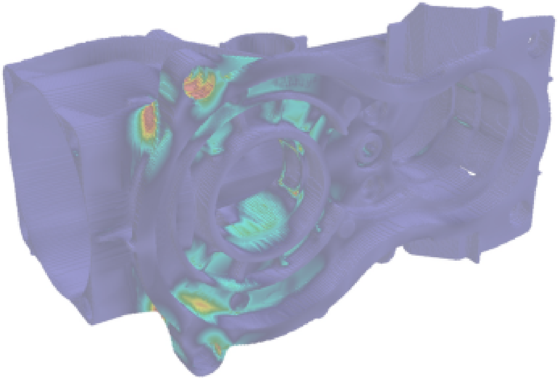


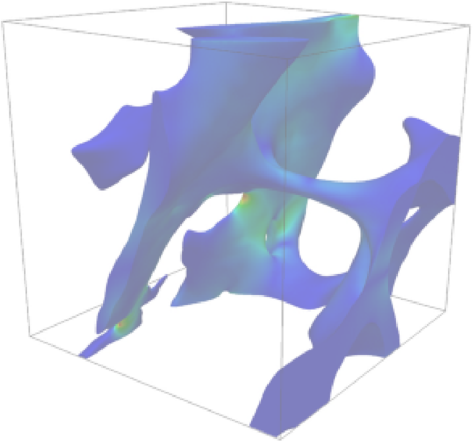
Burman E. Ghost penalty. *Comptes Rendus Mathematique* 2010; 348(21): 1217-1220.

Unfitted methods & stabilisation

lessons from immersed finite element methods

"...the condition number of the finite element matrix **depends on how the domain boundary cuts the mesh**. *If the cut results in elements with* **very small intersections** *with the physical domain, the system matrix may be* **very ill-conditioned**...*"* Burman (2010)





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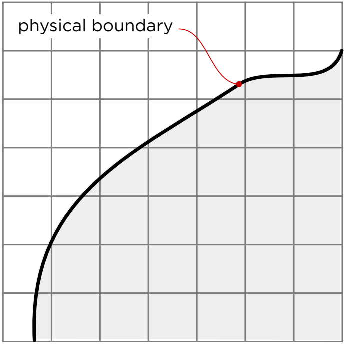
Ghost stabilisation

immersed FEM → MPM



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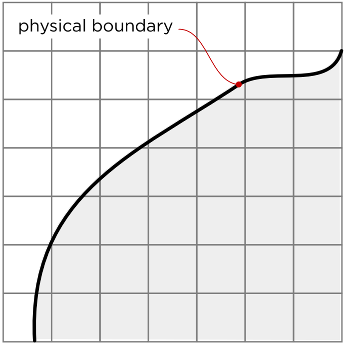

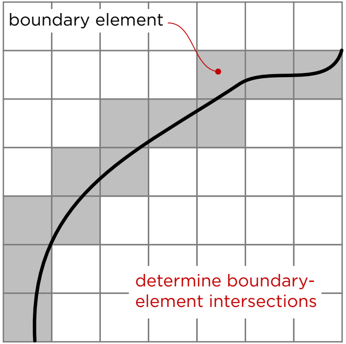

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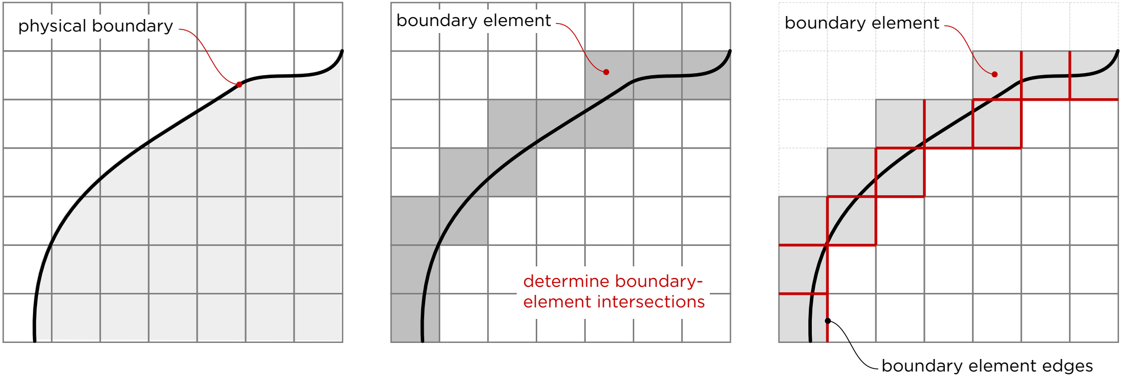

$$j(u_i, w_i) = \sum_{k=1}^{q} \frac{h^{2k+1}}{(2k+1)(k!)^2} \int_\Gamma [[\partial_n^k u_i]]\ [[\partial_n^k w_i]] d\Gamma \qquad \text{where} \qquad [[u_i]] = u_i|_{F^+} - u_i|_{F^-}$$

Ghost stabilisation

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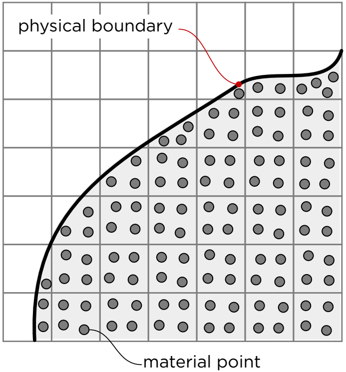

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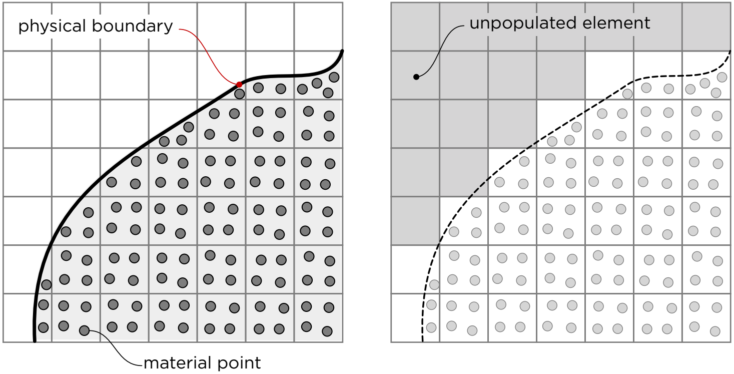

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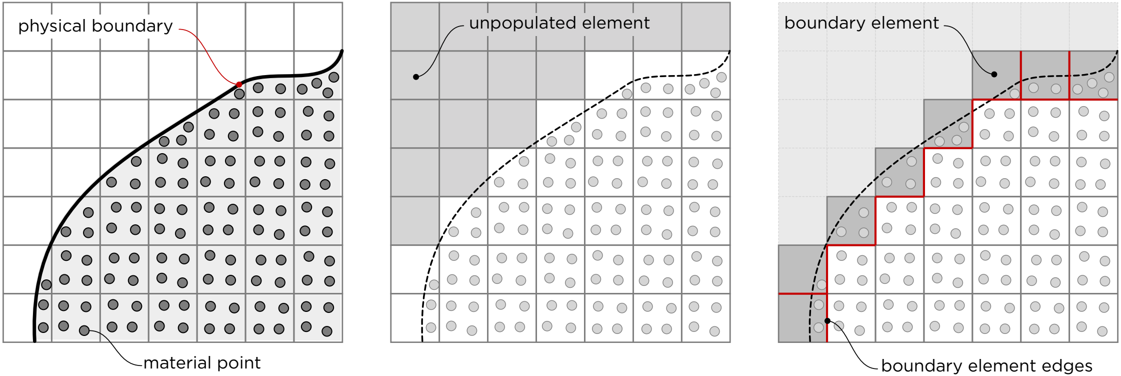
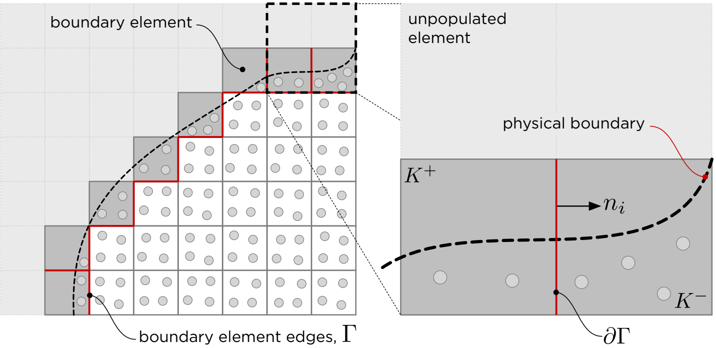


$$j(u_i, w_i) = \sum_{k=1}^{q} \frac{h^{2k+1}}{(2k+1)(k!)^2} \int_\Gamma [[\partial_n^k u_i]]\ [[\partial_n^k w_i]] d\Gamma \quad \rightarrow \quad j(u_i, w_i) = \frac{h^3}{3} \int_\Gamma [[\partial_n u_i]]\ [[\partial_n w_i]] d\Gamma$$

Ghost stabilisation



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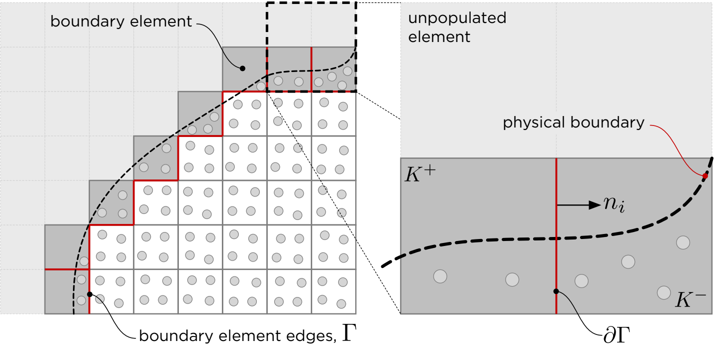
Coombs, W. (2023). Ghost stabilisation of the Material Point Method....
Int. J. Num. Meth. Eng., 124(21), 4841-4875. doi.org/10.1002/nme.7332

Ghost stabilisation



$$[K_G] = \frac{\gamma_k h^3}{3} \int_\Gamma \Big([G]^T [m][G] \Big) d\Gamma$$

penalty parameter, $\gamma_k \propto E$

Coombs, W. (2023). Ghost stabilisation of the Material Point Method....
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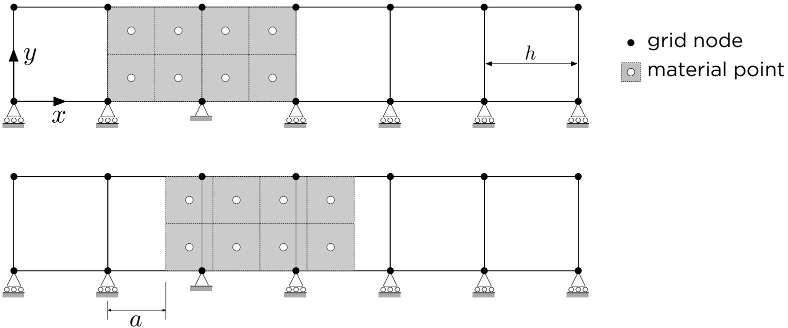
Ghost stabilisation: 1D test



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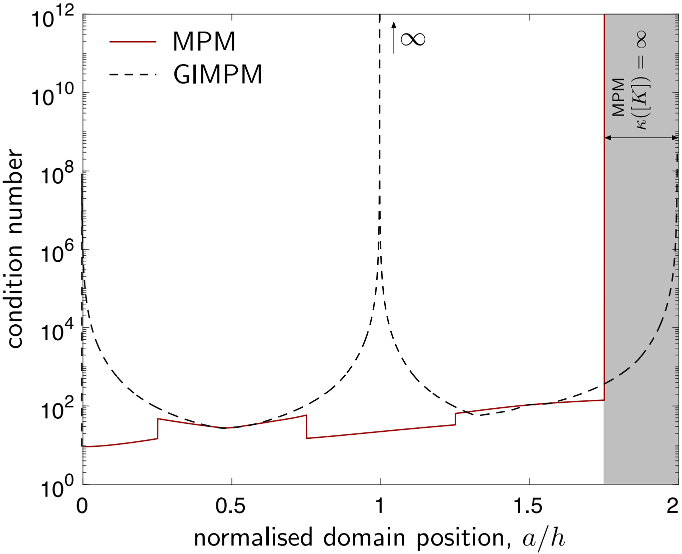
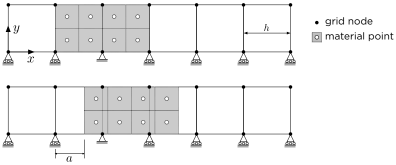




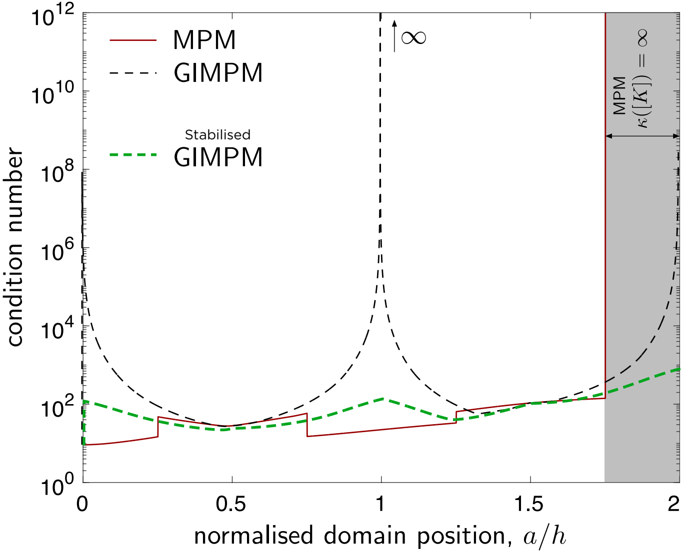
condition number, $\kappa([K])$

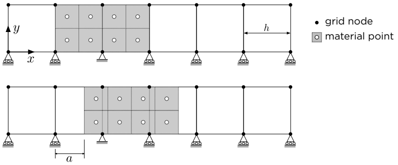
ratio of the largest to smallest eigenvalue of $[K]$

linked to the ease and accuracy of the solution of $\{f\} = [K]\{d\}$

large numbers are *problematic*

Ghost stabilisation: 1D test





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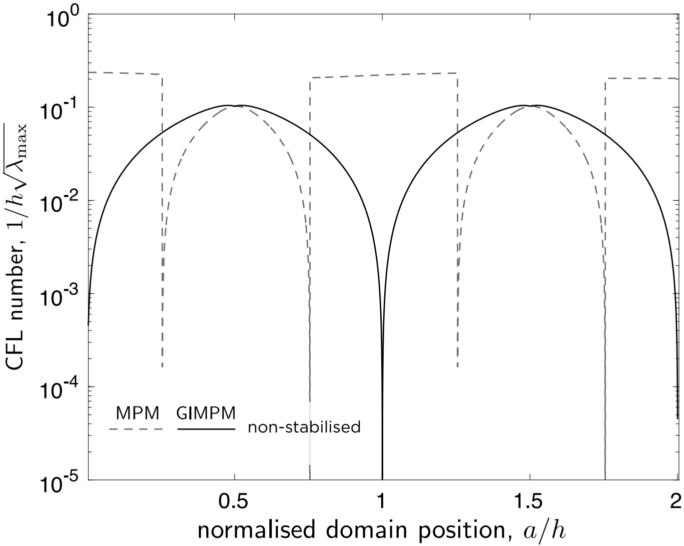

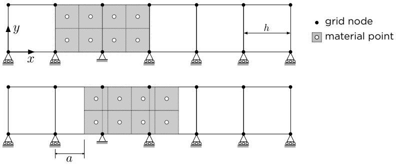

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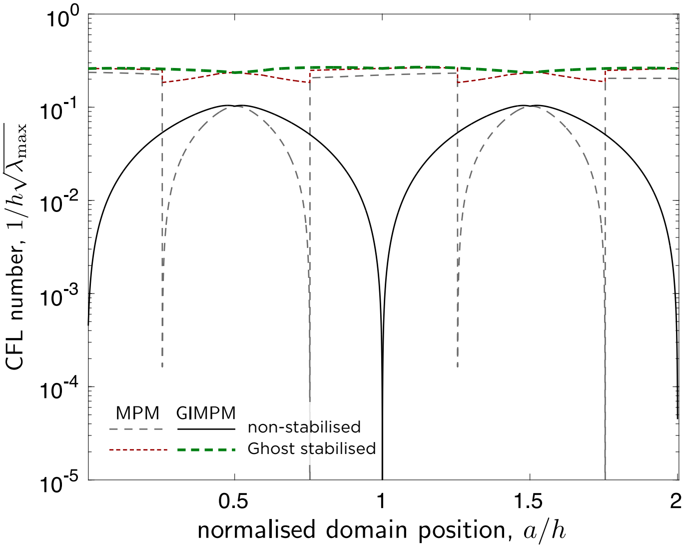

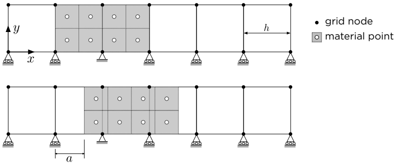

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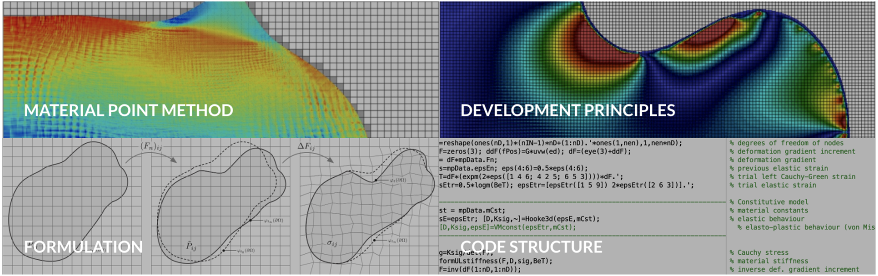
AMPLE: implementation

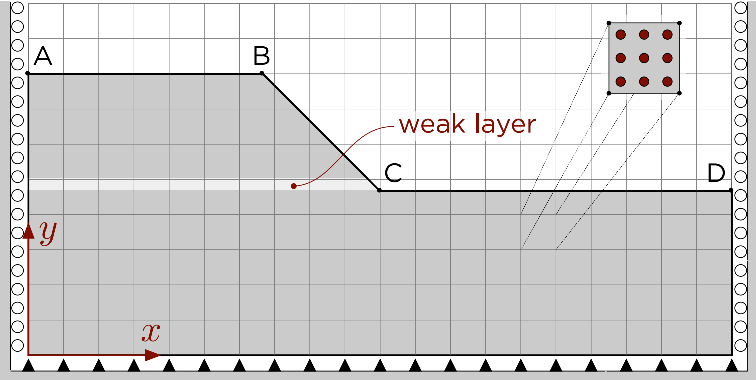
wmcoombs.github.io/

AMPLE: A MATERIAL POINT LEARNING ENVIRONMENT



Coombs, WM & Augarde, CE (2020). AMPLE: A Material Point Learning Environment. Advances in Engineering Software 139: 102748.

Numerical example: slope failure



elastic properties
$E = 1\text{MPa}$, $\nu = 0.3$

von Mises yield strength
$\rho_y = 15\text{kPa}$ (main)
$\rho_y = 7.5\text{kPa}$ (weak)

density, $\rho = 2,400\text{kg/m}^3$
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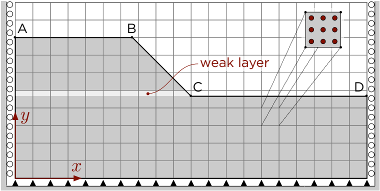

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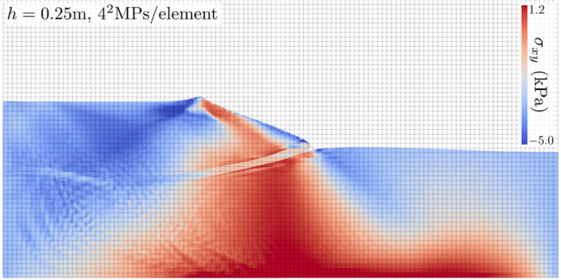
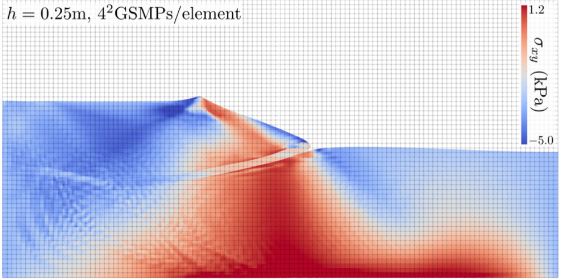
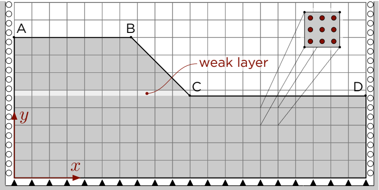
standard GIMPM



Ghost stabilised GIMPM





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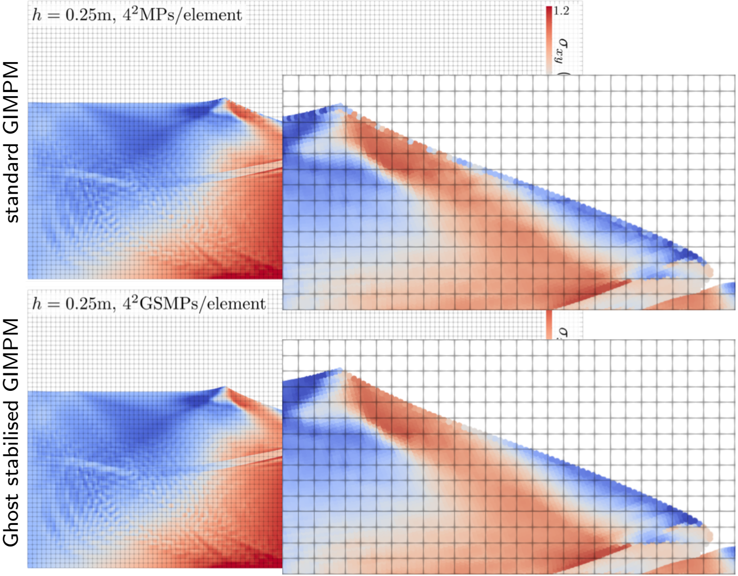

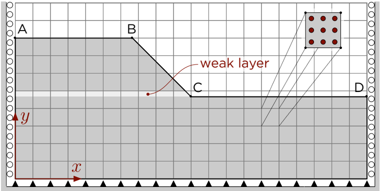

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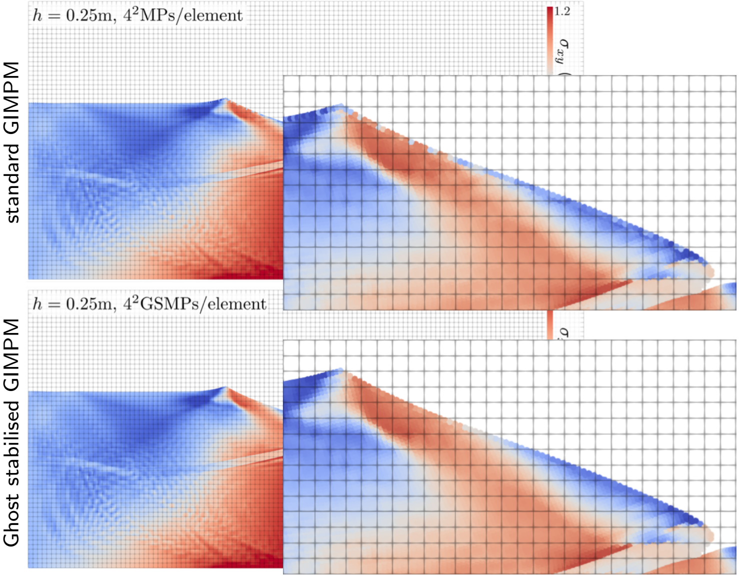
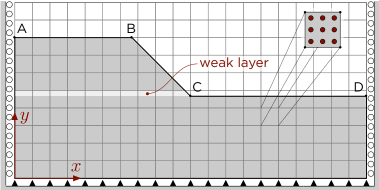
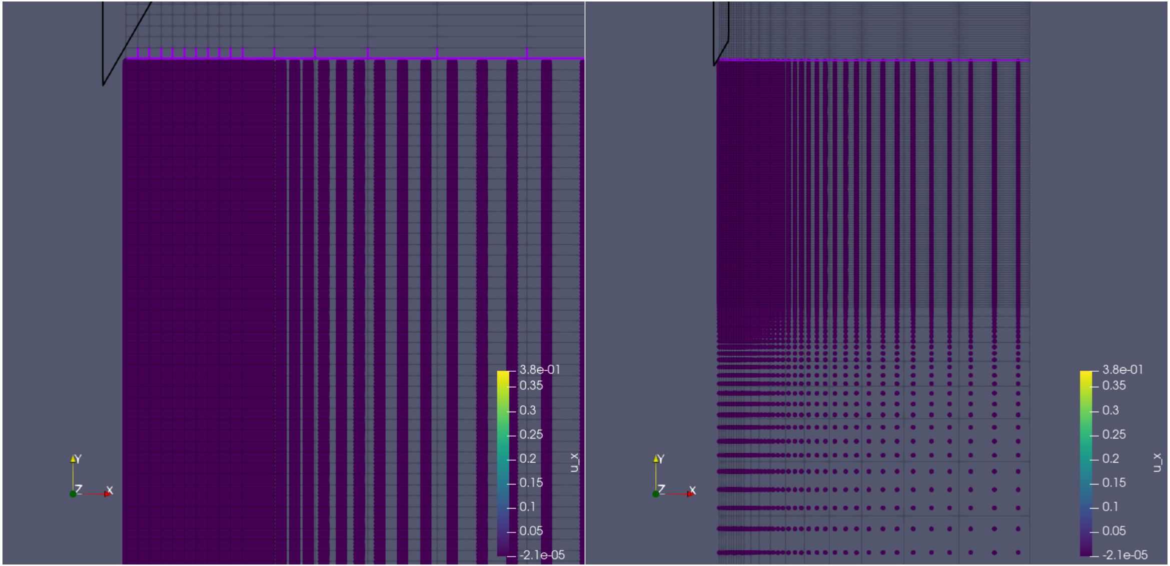


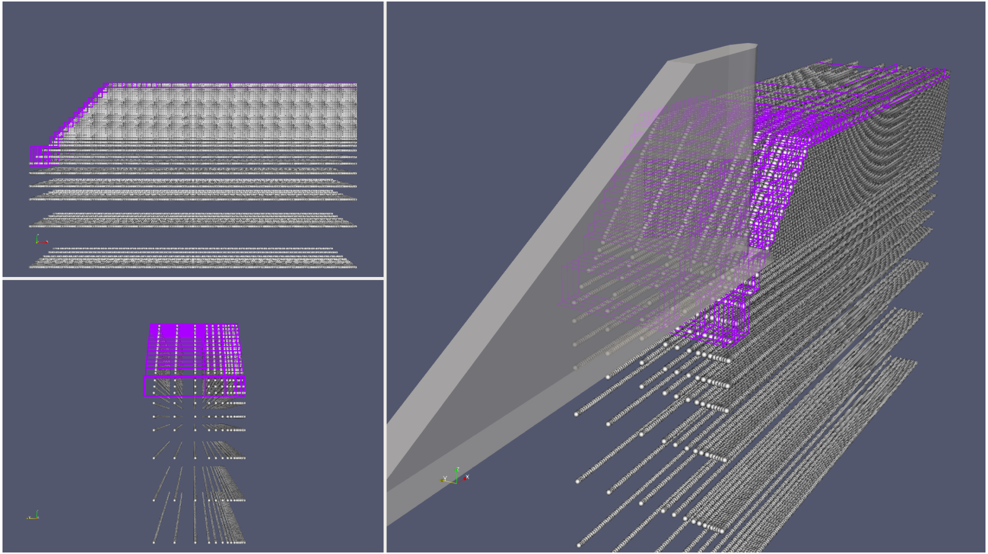


h(m)	1.00	0.50	0.25
2^2 MPs	✓	✗	✓
3^2 MPs	✗	✗	✗
4^2 MPs	✓	✓	✓
2^2 GSMPs	✓	✓	✓
3^2 GSMPs	✓	✓	✓
4^2 GSMPs	✓	✓	✓

Numerical example: Cone Penetration Test (CPT)



Numerical example: seabed ploughing



RD= 44%

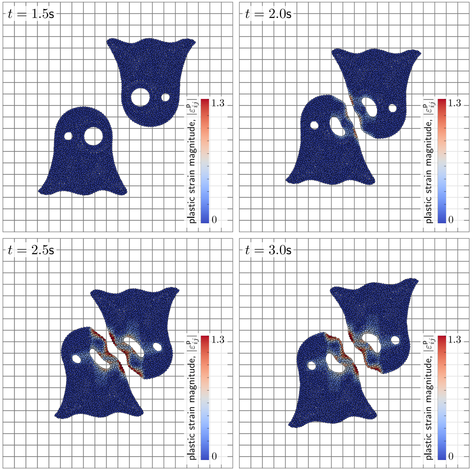
$\rho = 1670\text{kg/m}^3$

$E^* = 26\text{MPa}$
$\nu = 0.3$

$\phi = 33.5^\circ$
$\psi = 3.5^\circ$
$c = 0.3\text{kPa}$

Observations

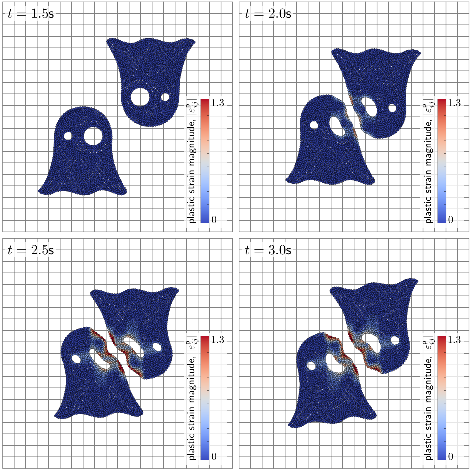
- The MPM suffers from conditioning issues due to the arbitrary interaction between the physical body and the background grid.
- Ghost stabilisation penalises jumps in the spatial gradient of the solution across the *boundary* of the physical body and introduces a bound on the condition number of the linear system - restoring coercivity to some degree (no proof).
- The stabilisation significantly improves the reliability of the MPM and reduces stress oscillations at the physical boundary.



explicit dynamic elasto-plastic analysis

Observations

- 30 years since Sulsky *et al.* (1994)
- MPM is not a meshless method → an unfitted mesh-based method is more appropriate?
- Unless we deal with the instabilities, combining MPM with complex constitutive models is problematic.
- 3D large deformation coupled (soil-water) simulations are still a challenge: inf-sup, stiff system, long run times...
- Boundaries need more work and care.
- Implicit is difficult, but worth the hassle...
  (check with Robert and Ted)



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Sulsky D, Chen Z, Schreyer HL (1994). A particle method for history-dependent materials. Comput Methods Appl Mech Eng. 118(1):179-196.

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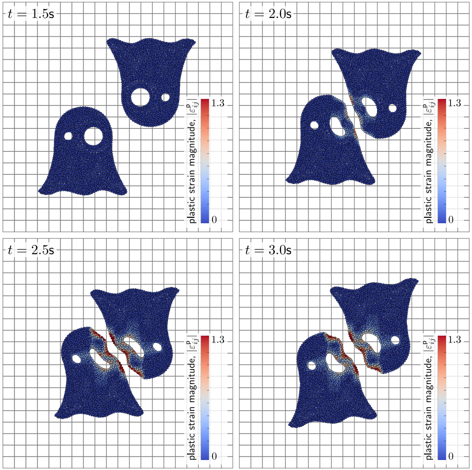

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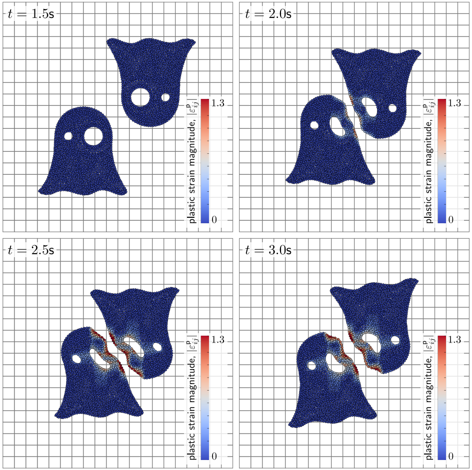
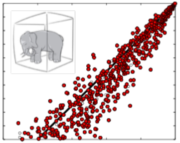


explicit dynamic elasto-plastic analysis

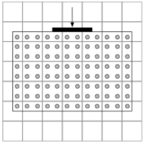
Acknowledgements

Engineering and Physical Sciences Research Council

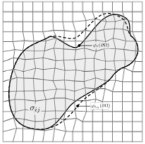
Durham University



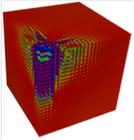
cell crossing



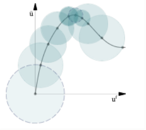
boundary conditions



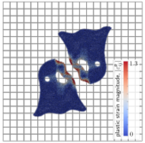
configurations



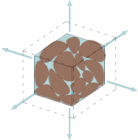
locking



arc length solvers



small-cuts



coupled analysis

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Stability of implicit material point methods for geotechnical analysis of large deformation problems

Will Coombs*, R.E. Bird, G. Pretti & C.E. Augarde

*Professor of Computational Mechanics
Department of Engineering, Durham University, UK

w.m.coombs@durham.ac.uk

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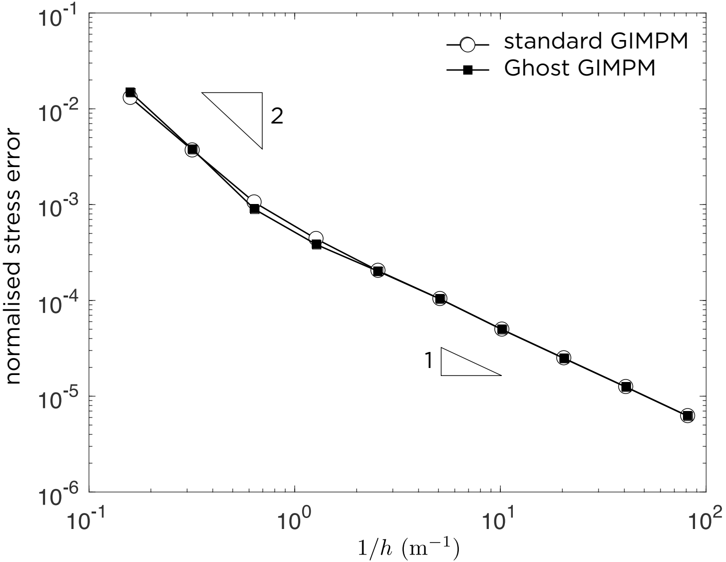
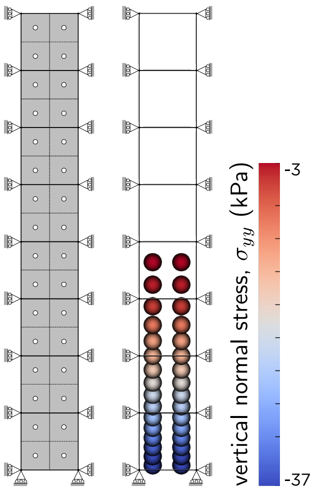
Interested in the details? doi.org/10.1002/nme.7332

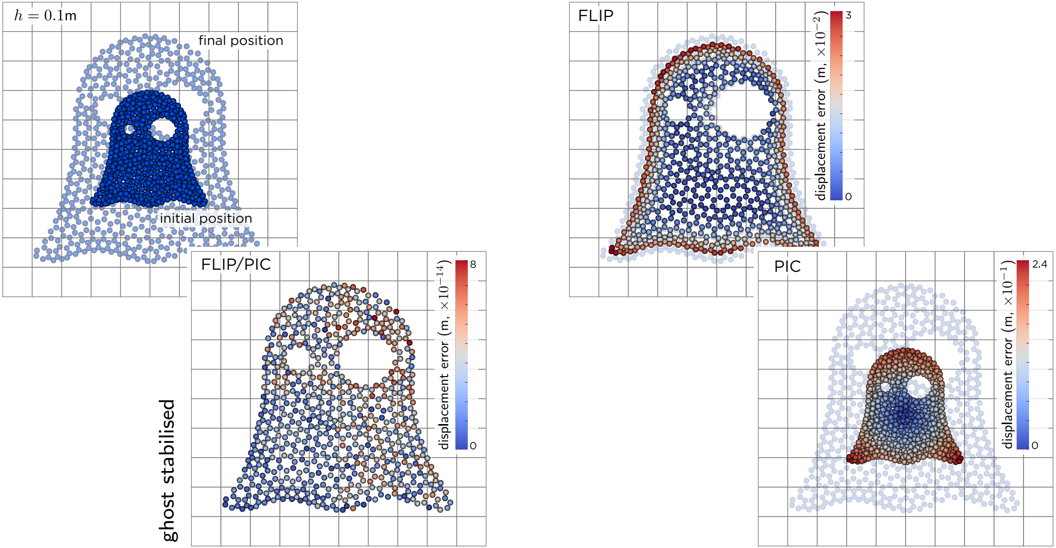
Coupled (soil-water) materials extension: arxiv.org/abs/2405.12814

Validation: column under self weight

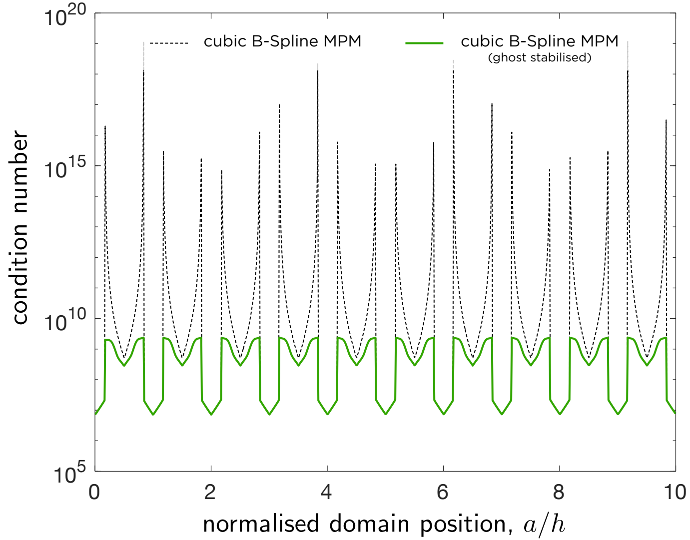


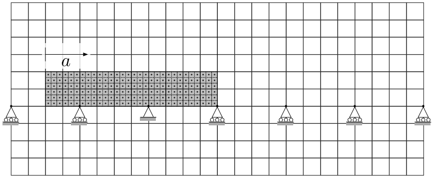


Velocity mapping: expansion



Ghost stabilisation: B-Spline basis functions





$$j(u_i, w_i) = \sum_{k=1}^{q} \frac{h^{2k+1}}{(2k+1)(k!)^2} \int_\Gamma [[\partial_n^k u_i]]\,[[\partial_n^k w_i]] d\Gamma$$

condition number, $\kappa([K])$

ratio of the largest to smallest eigenvalue of $[K]$

linked to the ease and accuracy of the solution of $\{f\} = [K]\{d\}$

large numbers are *problematic*