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Weyl-Lewis-Papapetrou coordinates, self-dual Yang-Mills equations and the single copy

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Abstract: We consider the dimensional reduction to two dimensions of certain gravitational theories in $D \geq 4$ dimensions at the two-derivative level. It is known that the resulting field equations describe an integrable system in two dimensions which can also be obtained by a dimensional reduction of the self-dual Yang-Mills equations in four dimensions. We use this relation to construct a single copy prescription for classes of gravitational solutions in Weyl-Lewis-Papapetrou coordinates. In contrast with previous proposals, we find that the gauge group of the Yang-Mills single copy carries non-trivial information about the gravitational solution. We illustrate our single copy prescription with various examples that include the extremal Reissner-Nordstrom solution, the Kaluza-Klein rotating attractor solution, the Einstein-Rosen wave solution and the self-dual Kleinian Taub-NUT solution.

Keywords: Black Holes, Classical Theories of Gravity, Integrable Field Theories, Solitons Monopoles and Instantons

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Contents

1 Introduction

In this paper, we consider certain gravitational theories in $D \geq 4$ dimensions at the two-derivative level, in the absence of a cosmological constant. We focus on solutions of the associated field equations that have sufficiently many commuting isometries so that they can also be regarded as solutions of the field equations that result from the dimensional reduction of these theories down to two dimensions using a two step procedure [1–4]. The

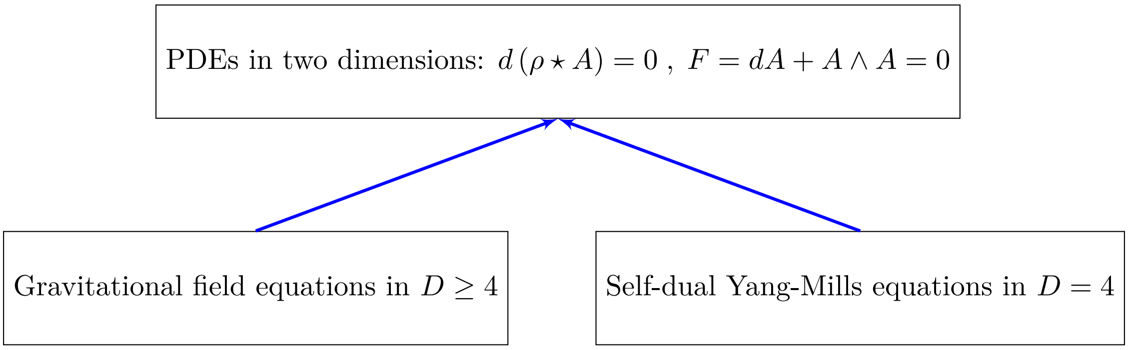

Figure 1. Relating gravitational solutions in $D \geq 4$ to solutions of the self-dual Yang-Mills equations in $D = 4$ through PDEs in two dimensions.

resulting field equations in two dimensions are partial differential equations (PDEs) in the so-called Weyl coordinates (ρ, v), with $\rho > 0$ and $v \in \mathbb{R}$. As we will review in section 2, they are written in terms of a 1-form A and take the form

$$d\,(\rho \star A) = 0\,, \quad F = dA + A \wedge A = 0\,. \tag{1.1}$$

These field equations are known to describe an integrable system in two dimensions [4, 5], namely, they are the compatibility condition for an auxiliary linear system of differential equations (a Lax pair), called the Breitenlohner-Maison (BM) linear system.

It is also known that the dimensional reduction of the self-duality equations for Yang-Mills fields in four dimensions can give rise to PDEs in lower dimensions that describe integrable systems [6, 7], and that a particular reduction to two dimensions gives the field equations (1.1) [6, 8, 9].

Thus, solutions to the PDEs (1.1) give rise, on the one hand, to solutions of the gravitational field equations in $D \geq 4$, and on the other hand to solutions of the self-dual Yang-Mills equations in four dimensions. In this way, (non self-dual) solutions of the gravitational field equations in $D \geq 4$ can be related to self-dual solutions of the Yang-Mills equations in four dimensions. This is depicted in figure 1.

Here, we will focus on two classes of solutions to the two-dimensional PDEs (1.1), which we call class I and class II. These will be introduced in section 2. Given a solution belonging to one of these classes, we give a procedure for constructing further solutions belonging to this class. Let us illustrate this by considering a class I solution. These are of the form

$$A = df\, C\,, \tag{1.2}$$

where f denotes a function of the Weyl coordinates (ρ, v) and C denotes a constant matrix. By integrating f over v, we obtain a new function $G(\rho, v)$, and therefore a 1-form

$$A = dG\, C \tag{1.3}$$

that also solves (1.1), cf. (2.31) and (2.33). This procedure can, in principle, be repeated: integrating G over v will yield a new function from which one then constructs a 1-form

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	A		single copy $\mathcal{A}$
Class I	$df\, C$	$G(\rho,v) = \int_a^v f(\rho,\tilde{v})\, d\tilde{v} + g(\rho)$	$\mathcal{A}_U = \mathcal{A}_w = 0,$ $\mathcal{A}_\tau = A_V = \frac{1}{4} f\, C\,,\ \mathcal{A}_{\bar{w}} = \frac{1}{2}\, e^{i\phi}\, \partial_\rho G\, C$

Table 1. Single copy description in a four-dimensional Minkowski space-time with line element $ds^2 = -dUdV + dwd\bar{w}$, where $U = \tau - v$, $V = \tau + v$, $w = \rho\, e^{i\phi}$, $\bar{w} = \rho\, e^{-i\phi}$.

A, as just described, that again solves (1.1). Each of these solutions, obtained from f by integration over v, will correspond to a solution of the gravitational theory in space-time dimensions $D \geq 4$.

We now wish to relate these gravitational solutions to solutions of the self-dual Yang-Mills equations in four dimensions in a physically meaningful way. In the above discussion we have omitted the role played by the sources that source the gravitational solutions and the solutions to the self-dual Yang-Mills equations. We demand that the nature of the sources be the same when mapping gravitational to non-gravitational solutions, that is, monopoles should be mapped to monopoles, and so on. This will serve as a guiding principle for constructing the single copy[1] of a gravitational solution that satisfies (1.1). To this end, in section 6 we focus on solutions to the PDEs (1.1) that have both a class I and a class II description. We remark that one could set up the single copy prescription using only a class I description, but additionally putting the solution in class II allows us to directly recover the scalar version of the self-dual Yang-Mills solution. Any such solution corresponds to a gravitational solution in space-time dimensions $D \geq 4$, and we describe how to construct a single copy description of these gravitational solutions. For illustrative purposes let us consider a gravitational solution whose class I description is of the form $A = df\, C$, as before. Given that we may construct new functions from f by integrating f over v, as explained above, which of the resulting 1-forms should we use to construct the single copy of the original gravitational solution encoded in $A = df\, C$? This will depend on the nature of the sources of the gravitational solution. In most of the examples that we consider in this paper, the single copy is constructed using the function G that is obtained by integrating f once over v. This is summarized in table 1 for the case that the single copy gauge field $\mathcal{A}$ is a solution to the self-dual Yang-Mills equations in four-dimensional Minkowski space-time with line element $ds^2 = -dUdV + dwd\bar{w}$, where the four-dimensional coordinates $(U, V, w, \bar{w})$ are expressed in terms of the Weyl coordinates (ρ, v) by $U = \tau - v$, $V = \tau + v$, $w = \rho\, e^{i\phi}$, $\bar{w} = \rho\, e^{-i\phi}$. Thus, the role of the associated solution G is to serve as a bridge for obtaining the single copy description of the gravitational solution $A = df\, C$.

Among the explicit examples that we discuss, a notable exception to integrating f only once over v in order to obtain the single copy description of the gravitational solution is the Lorentzian version of the Eguchi-Hanson solution. We find that in this example we have to integrate f twice over v in order to obtain the single copy given in the literature [10–12]. This is discussed in section 6.3.

Let us summarise some of the salient features of our procedure. Our approach is based on the field equations (1.1), which are given in Weyl coordinates. We assume that a solution

[1]The term single copy is taken from the literature on the classical double copy.

of (1.1) depends on a parameter m which, when switched off, results in a gravitational solution that denotes a gravitational background. We then work to first order in the parameter m and consider the gravitational 1-form A at this order. We further assume that at first order in m, A has both a class I and a class II description, and we follow the steps outlined above for obtaining a single copy description of the gravitational solution. We note that since our construction relates a gravitational solution that is (in general) not self-dual, to a self-dual solution of the Yang-Mills equations in four dimensions, half of the latter solution is not required when matching with results in the classical double copy literature.[2] We confirm that the above procedure reproduces the single copy for various gravitational solutions discussed in the literature, such as the Schwarzschild solution [16], the non-extremal Kerr solution [16], the (Lorenzian version of the) self-dual Taub-NUT solution [17] and the (Lorenzian version of the) Eguchi-Hanson solution [10–12]. We further illustrate our single copy procedure in the following examples: the extremal Reissner-Nordstrom solution, a solution describing a black hole in $AdS_2 \times S^2$, the Kaluza-Klein rotating attractor solution (which is supported by a non-constant dilaton field), the Einstein-Rosen wave solution and the self-dual Kleinian Taub-NUT solution, which is a four-dimensional gravitational solution with signature $(2, 2)$ [18].

A noteworthy consequence of describing gravitational solutions in higher dimensions from a two-dimensional point of view through (1.1) is that gravitational solutions that look distinct in higher dimensions may turn out to be related to one another by solution generating transformations acting on A (see for instance [19]). This in turn implies that their single copy descriptions are also related. We will discuss two such examples, see sections 3 and 6.

The paper is organized as follows. In section 2 we briefly review how the PDEs (1.1) result from the two-step dimensional reduction of gravitational theories in $D \geq 4$. These PDEs are the compatibility conditions for a Lax pair, the Breitenlohner-Maison linear system. We give this Lax pair in terms of differential operators $(\mathcal{L}, \mathcal{M})$ in two dimensions. In appendix A we give new Lax pairs in three dimensions that also give rise to the field equations (1.1) and which provide an intermediate step between the previously known Lax pairs in four and two dimensions. We then proceed and discuss two classes of solutions to the PDEs (1.1), which we denote by class I and class II, respectively. Given a solution in either one of these two classes, we construct further solutions in these classes by integration over the Weyl coordinate v. In section 3 we discuss various known gravitational solutions that possess both a class I and a class II description. In section 4 we discuss the class I and class II descriptions of the (Lorenzian version of the) Eguchi-Hanson solution. In section 5 we discuss the mapping of gravitational solutions belonging to either class I or II (or both) to solutions of the self-dual Yang-Mills equations in four dimensions. We do so by starting from the self-dual Yang-Mills equations and performing dimensional reductions of them to two dimensions. In appendix D we present a different approach to this mapping, by starting on the gravitational side in two dimensions and expressing the 1-form A in two dimensions as a 1-form in four space-time dimensions. In section 6 we turn to the single copy description of gravitational solutions that have both a class I and a class II description, and we discuss the non-trivial role of the gauge group. We present our conclusions in section 7. In appendix B we discuss the

[2]We note that a similar step appears in the so called Newman-Penrose map, which relates self-dual solutions of Maxwell's equations to non-self-dual gravitational solutions, see [13–15].

self-dual Yang-Mills equations in four dimensions in different space-time signatures, while in appendix C we give the anti self-dual Yang-Mills equations in signature $(1, 3)$.

2 The dimensionally reduced gravitational field equations

We consider gravitational theories in $D \geq 4$ dimensions, at the two-derivative level, and without a cosmological constant term. We focus on solutions to the associated field equations that only depend on two of the D space-time coordinates. We assume that these gravitational theories, when dimensionally reduced to two dimensions using the two-step procedure discussed in [1–4], take the following form as discussed below. Namely, in the first step, the theory is reduced to three dimensions, and subsequently, the 2-form fields are dualised into scalar fields. The resulting theory in three dimensions describes the coupling of a non-linear sigma model, whose target space is a symmetric space G/H, to three-dimensional gravity. Then, a further reduction to two dimensions yields the following non-linear field equations,

$$d\,(\rho \star A) = 0\,, \quad \text{with } \; A = M^{-1} dM\,, \tag{2.1}$$

where $M \in G/H$ is a coset representative of the symmetric space G/H. The latter is invariant under an involution $\natural$ called generalized transposition, i.e. $M^\natural = M$, and M depends on two coordinates, denoted here by $\rho > 0$ and $v \in \mathbb{R}$, also called Weyl coordinates. In (2.1), $\star$ denotes the Hodge star operator in two dimensions,

$$\star d\rho = -\lambda\, dv\,, \quad \star dv = d\rho\,, \quad (\star)^2 = -\lambda\, \mathrm{id}\,, \tag{2.2}$$

where $\lambda = \pm 1$ depends on the spacetime signature, as detailed below.

The matrix 1-form A takes values in the Lie algebra of G/H. Denoting its generators by $\mathfrak{t}^a$, we have

$$A = M^{-1} dM = A_a \mathfrak{t}^a\,, \quad a = 1, \ldots, n\,, \tag{2.3}$$

where the dependence on (ρ, v) is encoded in the 1-forms $A_a = A_a(\rho, v)$.

Let us discuss the case when $D = 4$. Then, a solution M to (2.1) yields a gravitational solution, whose four-dimensional space-time metric in Weyl-Lewis-Papapetrou coordinates is a Weyl metric given by

$$ds_4^2 = -\lambda\, \Delta (dy + B d\phi)^2 + \Delta^{-1} \left(e^{\psi}\, ds_2^2 + \rho^2 d\phi^2 \right) . \tag{2.4}$$

The reduction from four down to two dimensions is performed along the directions (y, ϕ). If $\lambda = 1$, the two-dimensional line element ds_2^2 is space-like ($ds_2^2 = d\rho^2 + dv^2$), whereas if $\lambda = -1$, ds_2^2 is time-like (either $ds_2^2 = -d\rho^2 + dv^2$ or $ds_2^2 = d\rho^2 - dv^2$). Δ, B are functions of (ρ, v) determined by the solution $M(\rho, v)$ of (2.1) and $\psi(\rho, v)$ is a scalar function determined from $M(\rho, v)$ by integration [3, 4],

$$\partial_\rho \psi = \tfrac{1}{4}\, \rho\, \mathrm{Tr} \left(A_\rho^2 - \lambda\, A_v^2 \right) , \qquad \partial_v \psi = \tfrac{1}{2}\, \rho\, \mathrm{Tr} \left(A_\rho A_v \right) . \tag{2.5}$$

As an example, consider the two-step reduction of the four-dimensional Einstein-Hilbert action to two dimensions. The resulting coset is $G/H = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, the involution $\natural$

is matrix transposition and the coset representative M takes the form

$$M = \begin{pmatrix} \Delta + \frac{\tilde{B}^2}{\Delta} & \frac{\tilde{B}}{\Delta} \\ \frac{\tilde{B}}{\Delta} & \frac{1}{\Delta} \end{pmatrix}, \tag{2.6}$$

where $\tilde{B}$ is related to B through [5]

$$\rho \star d\tilde{B} = \Delta^2 \, dB \,. \tag{2.7}$$

Writing out the matrix 1-form A as $A = A_\rho d\rho + A_v dv$, the field equations (2.1) read

$$\partial_v A_v + \lambda \partial_\rho A_\rho + \frac{\lambda}{\rho} A_\rho = 0, \tag{2.8}$$

with

$$F_{\rho v} = \partial_\rho A_v - \partial_v A_\rho + [A_\rho, A_v] = 0 \tag{2.9}$$

(i.e. $F = dA + A \wedge A = 0$) by virtue of $A = M^{-1} dM$.

The field equations (2.8) and (2.9) are the compatibility conditions for a Lax pair, the Breitenlohner-Maison linear system [4, 5]. This Lax pair can be written in the form of a pair of differential operators $(\mathcal{L}, \mathcal{M})$ acting on matrix functions X,

$$\mathcal{L} X = 0 \,, \quad \mathcal{M} X = 0 \,. \tag{2.10}$$

These differential operators are given by

$$\begin{aligned} \mathcal{L} &= -\partial_v + \tau \, D_\rho, \\ \mathcal{M} &= \lambda \, \partial_\rho + \tau \, D_v \,, \quad \lambda = \pm 1, \end{aligned} \tag{2.11}$$

where

$$D_{\cdot} = \partial_{\cdot} + A_{\cdot} \,, \quad A = M^{-1} dM, \tag{2.12}$$

and τ is a function of ρ and v as well as of a complex parameter ω, which satisfies

$$\tau^2 - \lambda + \frac{2\lambda\tau}{\rho} (\omega - v) = 0 \,, \quad \omega \in \mathbb{C}. \tag{2.13}$$

Likewise, the matrix functions X in (2.10) are functions of ρ, v, ω. The compatibility condition for the linear system (2.10),

$$[\mathcal{L}, \mathcal{M}] X = 0 \,, \tag{2.14}$$

then implies the field equations (2.8) and (2.9).

The field equations (2.8) and (2.9) can also be obtained as compatibility conditions for Lax pairs formulated in three and four dimensions. The Lax pair in $D = 4$ is determined by the self-dual Yang-Mills equations in four dimensions, as we will discuss in section 5. In appendix A we give two Lax pairs in $D = 3$ dimensions. In $D = 4$ dimensions the spectral parameter is constant, while in $D = 2$ dimensions the spectral parameter is non-constant. Interestingly, in $D = 3$ dimensions one can describe the integrable system using either a constant or a non-constant spectral parameter. We summarize our findings in table 2.

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D	spectral parameter	coordinates	L	M
4	Ω	$(U,V,w,\bar{w})$	$\partial_U - \Omega\mathcal{D}_{\bar{w}}$	$\partial_w + \Omega\mathcal{D}_V$
3	Ω	(ρ,v,ϕ)	$-e^{-2i\phi}\,\partial_v + \frac{1}{2}\rho\,\Omega\,\left(D_\rho + \frac{i}{\rho}\,\partial_\phi\right)$	$e^{-i\phi}\,\lambda\,\partial_\rho + \frac{1}{2}\rho\,\Omega\,e^{i\phi}\,D_v$
3	$\tau(\rho,v,\omega)$	(ρ,v,ϕ)	$e^{i\phi}\left(-\partial_v + \tau\,\left(D_\rho + \frac{i}{\rho}\,\partial_\phi\right)\right)$	$e^{-i\phi}\,(\lambda\,\partial_\rho + \tau\,D_v)$
2	$\tau(\rho,v,\omega)$	(ρ,v)	$-\partial_v + \tau\,D_\rho$	$\lambda\,\partial_\rho + \tau\,D_v$

Table 2. Lax pairs (L,M) in various space-time dimensions D. Ω and ω denote constant spectral parameters, whereas τ is a non-constant spectral parameter that satisfies (2.13).

A noteworthy consequence of describing gravitational solutions in higher dimensions through a coset representative M is that gravitational solutions that look distinct in higher dimensions may turn out to be related to one another by solution generating transformations acting on M (see [20], for instance). This is done by acting on M with a suitably chosen constant matrix $g \in G/H$, to obtain the matrix

$$\tilde{M} = g^\natural M g\,, \tag{2.15}$$

which results in a matrix 1-form

$$\tilde{A} = g^{-1} A g\,. \tag{2.16}$$

An example thereof, discussed in [19], is the solution generating transformation that maps the 3×3 matrix M describing the near-horizon region of the extremal Reissner-Nordstrom black hole solution to the 3×3 matrix $\tilde{M}$ describing the full interpolating extremal Reissner-Nordstrom black hole solution in four space-time dimensions. We refer to section 3 for details on this solution generating technique. Then, as we will see in section 6, the Harrison transformation is mapped to a non-abelian rotation of the Yang-Mills gauge field via our single copy procedure.

2.1 Two classes of solutions

In the following, we will consider two classes of solutions to (2.8) and (2.9). Solutions belonging to the first class are of the gradient type, thus possessing a structure that is suggestive of the form of A, i.e. $A = M^{-1}dM$. Solutions in the second class take a form that results in a Plebanski type equation.

As we shall see, we define the two classes such that the role of (2.8) and of (2.9) is interchanged between the two: (2.9) is automatically satisfied by the solutions in class I, while (2.8) is automatically satisfied by the solutions in class II.

2.1.1 Class I

The first class consists of solutions of the form

$$A = (df_i)C_i\,, \tag{2.17}$$

where the f_i are functions of (ρ, v), and where the C_i are constant commuting matrices that reside in a subsector of the Lie algebra of G/H, so as to ensure that (2.17) satisfies the field strength condition (2.9). Inserting this into (2.8) and using that the constant matrices C_i are linearly independent, gives

$$\partial_v^2 f_i + \lambda \partial_\rho^2 f_i + \frac{\lambda}{\rho}\partial_\rho f_i = 0\,. \tag{2.18}$$

When $\lambda = 1$, this is the harmonic equation in three-dimensional Euclidean space, written in cylindrical coordinates (ρ, v, ϕ), acting on functions f_i that are independent of ϕ. When $\lambda = -1$, this is the harmonic equation in three-dimensional Minkowski space.

2.1.2 Class II

The second class of solutions consists of solutions of the form

$$A = -\frac{\lambda}{2} \star dP + \frac{P}{2\rho}\, dv\,, \tag{2.19}$$

that is,

$$A_\rho = -\frac{\lambda}{2}\partial_v P(\rho, v)\,, \quad A_v = \frac{1}{2}\left(\partial_\rho + \frac{1}{\rho}\right) P(\rho, v), \tag{2.20}$$

where

$$P(\rho, v) = p_a(\rho, v)\, \mathfrak{t}^a. \tag{2.21}$$

This satisfies (2.8), while the condition (2.9) imposes[3]

$$\left(\partial_\rho^2 + \lambda\partial_v^2 + \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\right) P + \frac{\lambda}{2}\left[\left(\partial_\rho + \frac{1}{\rho}\right) P, \partial_v P\right] = 0, \tag{2.23}$$

[3]We remark that the interaction term in (2.23) is of the form $[\{P, P\}_*]$, where $\{,\}_*$ is the modified Poisson bracket defined in [21] in the context of self-dual gravity on an AdS background. Specifically, in [21] we have a 4-dimensional theory in lightcone coordinates $(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \bar{\mathfrak{w}})$, with

$$\{f, g\}_* = \partial_\mathfrak{w} f \partial_\mathfrak{u} g - \partial_\mathfrak{u} f \partial_\mathfrak{w} g + \frac{2}{\mathfrak{u}-\mathfrak{v}}\left(f \partial_\mathfrak{w} g - \partial_\mathfrak{w} f\, g\right)\,.$$

If f and g are matrices, following [21], we define the combined Poisson-commutator bracket via

$$[\{f, g\}_*] = [\partial_\mathfrak{w} f, \partial_\mathfrak{u} g] - [\partial_\mathfrak{u} f, \partial_\mathfrak{w} g] + \left[\frac{2}{\mathfrak{u}-\mathfrak{v}} f, \partial_\mathfrak{w} g\right] - \left[\partial_\mathfrak{w} f, \frac{2}{\mathfrak{u}-\mathfrak{v}} g\right]\,.$$

Note that because $\{,\}_*$ and $[,]$ are both anti-commuting, the combine bracket $[\{,\}_*]$ commutes, therefore we can take

$$f = g = \frac{\sqrt{\lambda}P}{2}$$

to get

$$\left[\left\{\frac{\sqrt{\lambda}P}{2}, \frac{\sqrt{\lambda}P}{2}\right\}_*\right] = -\frac{\lambda}{2}[\partial_\mathfrak{u} P, \partial_\mathfrak{w} P] + \frac{\lambda}{2}\left[\frac{2}{\mathfrak{u}-\mathfrak{v}} P, \partial_\mathfrak{w} P\right]\,.$$

The commutator terms in (2.23) will be obtained by setting

$$\mathfrak{u} = -\frac{1}{2}\rho + \frac{1}{2}\Sigma\,, \quad \mathfrak{v} = -\frac{5}{2}\rho + \frac{1}{2}\Sigma\,, \quad \mathfrak{w} = \frac{1}{2}v + \frac{i}{2}\hat{\Sigma}\,, \quad \bar{\mathfrak{w}} = \frac{1}{2}v - \frac{i}{2}\hat{\Sigma}\,,$$

so that

$$\mathfrak{u} - \mathfrak{v} = 2\rho, \quad \mathfrak{w} + \bar{\mathfrak{w}} = v\,, \tag{2.22}$$

and using that P only depends on the two combinations (2.22). It would be interesting to explore the significance of this, particularly in light of recent deformations of the Breitenlohner-Maison system presented in [22]. We leave this for future work.

which can be written as

$$\left(\partial_\rho^2 + \lambda\partial_v^2\right) P + \frac{\lambda}{2}[\partial_\rho P, \partial_v P] + \partial_\rho \left(\frac{P}{\rho}\right) + \frac{\lambda}{2}\left[\frac{P}{\rho}, \partial_v P\right] = 0. \tag{2.24}$$

Let us focus on the case when the commutator terms in (2.24) vanish, in which case we obtain

$$\left(\partial_\rho^2 + \lambda\partial_v^2 + \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\right) P(\rho, v) = 0\,. \tag{2.25}$$

Then, introducing

$$\Psi(\rho, v, \phi) = \frac{1}{2}\, e^{i\phi}\, P(\rho, v)\,, \tag{2.26}$$

we find that $\Psi(\rho, v, \phi)$ satisfies

$$\left(\partial_\rho^2 + \lambda\partial_v^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2\right) \Psi(\rho, v, \phi) = 0\,. \tag{2.27}$$

When $\lambda = 1$, this is the harmonic equation in three-dimensional Euclidean space, written in cylindrical coordinates (ρ, v, ϕ). When $\lambda = -1$, this is the harmonic equation in three-dimensional Minkowski space.

2.1.3 Solutions belonging to both classes

There is a subspace of solutions to (2.8) and (2.9) that allows for both a class I and a class II description. The examples that we will be discussing in the following sections all belong to this subspace of solutions.

Consider a solution of class I of the form

$$A = (df)\, C\,, \tag{2.28}$$

with C a constant matrix. Such a solution can also be written in the form (2.20), with P given by

$$P(\rho, v) = p(\rho, v)\, C\,, \tag{2.29}$$

provided that p satisfies (2.25). This is so, because the equations

$$\begin{aligned}\partial_\rho f &= -\frac{\lambda}{2}\, \partial_v p\,,\\ \partial_v f &= \frac{1}{2}\left(\partial_\rho + \frac{1}{\rho}\right) p\end{aligned} \tag{2.30}$$

are compatible with one another provided p satisfies (2.25). Similarly, given a solution of class II of the form (2.20), with P given by (2.29) and with p satisfying (2.25), it can be written in the form (2.28) by using (2.30).

A straightforward generalization of the above shows that solutions of class I of the form $A = (df_i)C_i$, where the C_i are commuting matrices, also admit a description as class II solutions.

2.2 Constructing further solutions

Given a solution f in class I or a solution P in class II, we can construct further solutions of (2.18) and of (2.25), respectively, by integration with respect to v, as follows.

Let f denote a function that satisfies (2.18). Then we construct a function G that also solves (2.18), namely

$$G(\rho, v) = \int_a^v f(\rho, \tilde{v})\, d\tilde{v} + g(\rho)\,, \tag{2.31}$$

where $g(\rho)$ has to satisfy

$$\lambda\, \rho\, g'(\rho) = -\int_b^\rho \tilde{\rho}\, \left(\partial_v f(v, \tilde{\rho})\right)_{|v=a}\, d\tilde{\rho}\,. \tag{2.32}$$

Here a and b denote constants. $G - g$ is an indefinite integral in v, but also a parametric integral with parameter ρ. We assume that f possesses the necessary continuity properties for applying the Leibniz integral rule for differentiation under the integral sign. Then,

$$A = (dG)\, C\,, \tag{2.33}$$

with C a constant matrix, is a solution of (2.8) and (2.9), as can be easily verified by using (2.18) and (2.32).

Similarly, given a solution Ψ of the form (2.26) that satisfies (2.27), we construct a function H that also solves (2.27), namely

$$H(\rho, v, \phi) = \frac{1}{2} e^{i\phi} \left(\int_a^v P(\rho, \tilde{v})\, d\tilde{v} + h(\rho) \right)\,, \tag{2.34}$$

where $h(\rho)$ has to satisfy

$$h'' + \frac{h'}{\rho} - \frac{h}{\rho^2} = -\lambda \left(\partial_v P\right)_{|v=a}\,. \tag{2.35}$$

$e^{-i\phi}\, H - h$ is an indefinite integral in v, but also a parametric integral with parameter ρ. We assume that P possesses the necessary continuity properties for applying the Leibniz integral rule for differentiation under the integral sign. It can be easily verified that H satisfies (2.27) by using (2.25) and (2.35). Then,

$$A_\rho = -\lambda\, e^{-i\phi}\, \partial_v H = -\lambda\, e^{-i\phi}\, \Psi\,, \quad A_v = e^{-i\phi} \left(\partial_\rho + \frac{1}{\rho} \right) H \tag{2.36}$$

is a solution of (2.8) and (2.9).

Let us now consider a solution (2.28) that can also be written in the form (2.30). We then introduce the functions G and H as above. The associated solution (2.33) can then also be expressed in the form (2.36), since the relations

$$\begin{aligned} e^{i\phi}\, \partial_\rho G\, C &= -\lambda \partial_v H = -\frac{\lambda}{2}\, e^{i\phi}\, P\,, \\ e^{i\phi}\, \partial_v G\, C &= e^{i\phi}\, f\, C = \left(\partial_\rho + \frac{1}{\rho} \right) H \end{aligned} \tag{2.37}$$

are compatible by virtue of H satisfying (2.25).

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An example thereof is provided by $\lambda = 1$ with

$$f(\rho, v) = -2\frac{m}{\sqrt{\rho^2+v^2}}\,, \quad P(\rho, v) = -4m\left(\frac{v}{\rho\,\sqrt{\rho^2+v^2}}\right) C\,. \tag{2.38}$$

Using (2.31) and (2.34), we obtain

$$\begin{aligned} G(\rho, v) &= m\,\log\left[\frac{\sqrt{\rho^2+v^2}-v}{\sqrt{\rho^2+v^2}+v}\right] = -2m\ \tanh^{-1}\left(\frac{v}{\sqrt{\rho^2+v^2}}\right)\,, \\ H(\rho, v, \phi) &= -2m\,e^{i\phi}\,\frac{\sqrt{\rho^2+v^2}}{\rho}\,C\,, \end{aligned} \tag{2.39}$$

which satisfy the differential equations (2.18) and (2.27), respectively.

3 Examples

We now give examples of gravitational solutions in four dimensions that have both a class I and a class II description.

3.1 Class I description

Let us consider solutions of the form (2.17) with one or two matrices C_i,

$$A = df_i\, C_i\,, \tag{3.1}$$

where the C_i are constant commuting matrices and the $f_i = f_i(\rho, v)$ satisfy (2.18). Using (2.31), to each f_i we associate a function G_i that satisfies the differential equation (2.18).

In all the examples discussed in this section, the solutions depend on various parameters which, when they are switched off, ensure the vanishing of A.

There are two cases to consider, either $\lambda = 1$ or $\lambda = -1$ (cf. (2.4)). When $\lambda = 1$, the examples that we consider are the Schwarzschild solution, the extremal Reissner-Nordstrom black hole in four space-time dimensions, a black hole in $AdS_2 \times S^2$, the (Lorentzian version of the) self-dual Taub-NUT solution, the non-extremal Kerr black hole, the rotating attractor solution for a Kaluza-Klein black hole. When $\lambda = -1$, one example that we consider is the Einstein-Rosen wave solution. In addition, we also discuss the self-dual Kleinian Taub-NUT solution, which is a four-dimensional gravitational solution with signature $(2, 2)$ [18].

3.1.1 Exterior region of the Schwarzschild black hole

The line element describing the exterior region of the Schwarzschild black hole solution can be written in the form

$$ds_4^2 = -e^{-\varphi} dt^2 + e^{\varphi} ds_3^2 \tag{3.2}$$

with

$$ds_3^2 = dr^2 + (r^2 - m^2)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{3.3}$$

and

$$e^{-\varphi(r)} = \frac{r-m}{r+m}\,, \quad r > m\,. \tag{3.4}$$

This solution carries one parameter, the mass parameter m.

Introducing Weyl coordinates

$$\rho = \sqrt{r^2 - m^2}\,\sin\theta\,, \quad v = r\cos\theta\,, \tag{3.5}$$

the line element (3.2) can be brought to the form (2.4) with (see, for instance, [23])

$$\begin{aligned}
\Delta(\rho, v) &= \frac{v + m - \sqrt{(v+m)^2 + \rho^2}}{v - m - \sqrt{(v-m)^2 + \rho^2}}\,, \quad B = 0\,, \\
\psi(\rho, v) &= \log\left[\frac{1}{2}\frac{v^2 + \rho^2 - m^2}{\sqrt{(v+m)^2 + \rho^2}\sqrt{(v-m)^2 + \rho^2}} + \frac{1}{2}\right]\,.
\end{aligned} \tag{3.6}$$

The associated matrix $M(\rho, v)$ (with $M \in G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$) is given by (2.6), with $\tilde{B} = 0$,

$$M(\rho, v) = \begin{pmatrix} \Delta(\rho, v) & 0 \\ 0 & \frac{1}{\Delta(\rho, v)} \end{pmatrix}\,, \quad \Delta(\rho, v) = \frac{v + m - \sqrt{(v+m)^2 + \rho^2}}{v - m - \sqrt{(v-m)^2 + \rho^2}}\,, \tag{3.7}$$

which results in

$$A = M^{-1} dM = df\, C \tag{3.8}$$

with

$$f(\rho, v) = \log\Delta(\rho, v)\,, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3.9}$$

To first order in the parameter m, f reads

$$f(\rho, v) = -2\frac{m}{\sqrt{\rho^2 + v^2}}\,, \tag{3.10}$$

which describes a monopole type potential. Since (3.10) is a solution of (2.18) with $\lambda = 1$, it will give rise to a solution (2.4). To it we associate the function

$$G(\rho, v) = m\left[\log\left(\sqrt{\rho^2 + v^2} - v\right) - \log\left(\sqrt{\rho^2 + v^2} + v\right)\right]\,. \tag{3.11}$$

3.1.2 Extremal Reissner-Nordstrom black hole

The static extremal Reissner-Nordstrom black hole solution is a charged black hole solution of the Einstein+Maxwell theory in four space-time dimensions. It carries one electric charge q and one magnetic charge p. We introduce the combination $Q = q + ip$. In the near-horizon limit, the line element describes the product geometry $AdS_2 \times S^2$, and the solution is supported by electro-/magnetostatic potentials χ_e and χ_m,

$$\begin{aligned}
ds_4^2 &= -\frac{r^2}{|Q|^2}dt^2 + |Q|^2\frac{dr^2}{r^2} + |Q|^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \\
\chi_e + i\chi_m &= \frac{Q}{|Q|^2}\, r\,.
\end{aligned} \tag{3.12}$$

Introducing Weyl coordinates

$$\rho = r\,\sin\theta\,, \quad v = r\cos\theta\,, \tag{3.13}$$

the line element (3.12) can be brought to the form (2.4) with $e^{\psi} = 1$, $B = 0$ and $\Delta = (\rho^2 + v^2)/|Q|^2$. This near-horizon solution is encoded in a 3×3 matrix $M \in G/H = \mathrm{SU}(2,1)/(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1))$ given by [19]

$$M = \begin{pmatrix} e^{\varphi} & \sqrt{2}e^{\varphi} Z & 1 \\ -\sqrt{2}e^{\varphi}\bar{Z} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e^{-\varphi} = \Delta, \quad Z = \chi_e + i\chi_m = c\, e^{-\varphi/2}, \tag{3.14}$$

where

$$c = \frac{Q}{|Q|}, \quad |c| = 1. \tag{3.15}$$

We note that the matrix M can also be written as

$$M = \frac{1}{|Z|^2} \begin{pmatrix} 1 & \sqrt{2}Z & |Z|^2 \\ -\sqrt{2}\bar{Z} & -|Z|^2 & 0 \\ |Z|^2 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

The associated matrix 1-form $A = M^{-1}dM$ reads $A = df\, C$ with

$$f(\rho, v) = e^{\varphi(\rho,v)/2} = \frac{|Q|}{\sqrt{\rho^2 + v^2}}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\bar{c} & 0 & 0 \\ 0 & \sqrt{2}c & 0 \end{pmatrix}. \tag{3.17}$$

Note that f is linear in the parameter Q and that it describes a monopole type potential. The associated function G is

$$G(\rho, v) = -\frac{|Q|}{2} \left[\log\left(\sqrt{\rho^2 + v^2} - v\right) - \log\left(\sqrt{\rho^2 + v^2} + v\right) \right]. \tag{3.18}$$

The constant matrix C is a linear combination of two Lie algebra generators (given in (A.6) and (A.12) in [19]),

$$C = \sqrt{2}\,(\mathrm{Re}\, c)\,(F_1 + F_2) - \sqrt{2}\,(\mathrm{Im}\, c)\; i(F_2 - F_1). \tag{3.19}$$

When the black hole is electrically charged ($p = 0$) we have $c = 1$, while when it is magnetically charged ($q = 0$) we have $c = i$.

The interpolating solution describing an extremal Reissner-Nordstrom black hole can be obtained from (3.16) by applying a solution generating transformation called Harrison transformation to M, as follows [19]. We act on M with the transformation $g(\hat{c})$,

$$g(\hat{c}) = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}\bar{\hat{c}} & 1 & 0 \\ |\hat{c}|^2 & -\sqrt{2}\hat{c} & 1 \end{pmatrix}, \quad \hat{c} = -c/2. \tag{3.20}$$

The action on M is given by $\tilde{M} = g^{\natural}(\hat{c}) M g(\hat{c})$. Here the involution $\natural$ is defined by $g^{\natural}(\hat{c}) = \eta g(\hat{c})^{\dagger}\eta^{-1}$ with $\eta = \mathrm{diag}(1, -1, 1)$,

$$g^{\natural}(\hat{c}) = \begin{pmatrix} 1 & \sqrt{2}\hat{c} & |\hat{c}|^2 \\ 0 & 1 & \sqrt{2}\bar{\hat{c}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.21}$$

The resulting matrix $\tilde{M}$ takes the form (3.16), with Z replaced by $\tilde{Z}$ [19],

$$Z \to \tilde{Z} = \frac{Z}{1 - 2\bar{\hat{c}} Z} = \frac{Z}{1 + \bar{c} Z} \,. \tag{3.22}$$

Under (3.20), the matrix 1-form A transforms as

$$A = M^{-1} dM \longrightarrow \tilde{A} = \tilde{M}^{-1} d\tilde{M} = g^{-1}(\hat{c})\, A\, g(\hat{c}), \tag{3.23}$$

and we obtain

$$\tilde{A} = A \,. \tag{3.24}$$

Thus, both the interpolating and the near-horizon solution have the same matrix 1-form A.

3.1.3 Black hole in $AdS_2 \times S^2$

It is well known that the solution describing the near-horizon near-extremal limit of a Reissner-Nordstrom black hole in four space-time dimensions is a black hole in $AdS_2 \times S^2$ [24]. Such a solution can be obtained by applying a Harrison transformation (see table 2 in [20]) to the Schwarzschild solution, as discussed in [19]. Namely, we embed the 2×2 matrix M given in (3.7) into a 3×3 matrix $M \in G/H = \mathrm{SU}(2,1)/(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1))$,

$$M = \begin{pmatrix} e^{-\varphi} & 0 \\ 0 & e^{\varphi} \end{pmatrix} \Longrightarrow M = \begin{pmatrix} e^{\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\varphi} \end{pmatrix} , \tag{3.25}$$

where $e^{-\varphi} = \Delta$. The associated matrix one-form $A = M^{-1} dM$ reads

$$A = df\, C \,, \quad f = -\varphi \,, \quad C = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{3.26}$$

Now we act on M with a transformation $g(c)$, as described below (3.20), i.e. $\tilde{M} = g^{\natural}(c) M g(c)$ with $g^{\natural}(c) = \eta g(c)^{\dagger} \eta^{-1}$ with $\eta = \mathrm{diag}(1, -1, 1)$ and

$$g(c) = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}\bar{c} & 1 & 0 \\ |c|^2 & -\sqrt{2} c & 1 \end{pmatrix} , \quad c \in \mathbb{C} \,. \tag{3.27}$$

We obtain

$$\tilde{M} = \begin{pmatrix} \left(e^{\varphi/2} - |c|^2 e^{-\varphi/2}\right)^2 & \sqrt{2}\, c \left(1 - |c|^2 e^{-\varphi}\right) & |c|^2 e^{-\varphi} \\ -\sqrt{2}\, \bar{c} \left(1 - |c|^2 e^{-\varphi}\right) & 1 - 2|c|^2 e^{-\varphi} & \sqrt{2}\bar{c}\, e^{-\varphi} \\ |c|^2 e^{-\varphi} & -\sqrt{2} c\, e^{-\varphi} & e^{-\varphi} \end{pmatrix} = \tilde{M}^{\natural}. \tag{3.28}$$

Under (3.27), the 1-form A in (3.26) transforms into

$$A = M^{-1} dM \longrightarrow \tilde{A} = \tilde{M}^{-1} d\tilde{M} = g^{-1}(c)\, A\, g(c). \tag{3.29}$$

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Hence we obtain

$$\tilde{A} = d\tilde{f}\,\tilde{C}\,, \quad \tilde{f} = f\,, \tag{3.30}$$

with $\tilde{C} = g^{-1}(c)\,C\,g(c)$, where

$$\tilde{C} = -\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}\bar{c} & 0 & 0 \\ 0 & \sqrt{2}c & -1 \end{pmatrix}. \tag{3.31}$$

Note that the dependence on the parameter c is entirely contained in the matrix $\tilde{C}$, and that this dependence is linear in c. The matrix $\tilde{C}$ is a linear combination of Lie algebra generators,

$$\tilde{C} = -H_2 - \sqrt{2}\,(\mathrm{Re}\,c)\,\left(F_1 + F_2\right) + \sqrt{2}\,(\mathrm{Im}\,c)\,\,i(F_2 - F_1)\,, \tag{3.32}$$

where $H_2, F_1 + F_2, i(F_2 - F_1)$ are the generators given in (A.6) and (A.12) in [19].

If we now set $c = e^{i\alpha}$, we get

$$\tilde{M} = \begin{pmatrix} \left(e^{\varphi/2} - e^{-\varphi/2}\right)^2 & \sqrt{2}\,e^{i\alpha}\,(1 - e^{-\varphi}) & e^{-\varphi} \\ -\sqrt{2}\,e^{-i\alpha}\,(1 - e^{-\varphi}) & 1 - 2e^{-\varphi} & \sqrt{2}e^{-i\alpha}e^{-\varphi} \\ e^{-\varphi} & -\sqrt{2}e^{i\alpha}e^{-\varphi} & e^{-\varphi} \end{pmatrix} = \tilde{M}^{\natural}. \tag{3.33}$$

The resulting solution has a line element of the form (3.2), with e^{φ} replaced by

$$e^{\tilde{\varphi}} = \left(e^{\varphi/2} - e^{-\varphi/2}\right)^2 = \frac{4m^2}{r^2 - m^2}\,. \tag{3.34}$$

This line element describes a product geometry given by a black hole in $AdS_2 \times S^2$ (cf. (9.19) in [19]),

$$ds_4^2 = 4m^2\left(-(r^2 - m^2)dt^2 + \frac{dr^2}{r^2 - m^2}\right) + 4m^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\,, \tag{3.35}$$

where we have rescaled the time-like coordinate t by a factor $4m^2$. This solution is supported by electric-/magnetostatic potentials χ_e/χ_m,

$$\chi_e + i\chi_m = \frac{c}{2m}\,(r - m)\,. \tag{3.36}$$

To lowest order in the parameter m, we infer using (3.30) that

$$\tilde{f}(\rho, v) = -2\frac{m}{\sqrt{\rho^2 + v^2}}\,. \tag{3.37}$$

The associated function G is

$$G(\rho, v) = m\left[\log\left(\sqrt{\rho^2 + v^2} - v\right) - \log\left(\sqrt{\rho^2 + v^2} + v\right)\right]\,. \tag{3.38}$$

3.1.4 Self-dual Taub-NUT solution

In Lorentzian signature, the Taub-NUT solution is given by [25, 26] (see also (12.1) in [27])

$$ds^2 = -f(r)\left(d\bar{t} - 2\ell\cos\theta d\phi\right)^2 + \frac{dr^2}{f(r)} + (r^2+\ell^2)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{3.39}$$

with

$$f(r) = \frac{r^2 - 2mr - \ell^2}{r^2+\ell^2}. \tag{3.40}$$

Here, m and ℓ are parameters, and ℓ is called NUT parameter. Setting $\ell = +im$, we obtain the self-dual Taub-NUT solution, for which

$$f(r) = \frac{r-m}{r+m}. \tag{3.41}$$

We restrict to $r > m$ to ensure that $f(r) > 0$. Introducing Weyl coordinates (see (12.6) in [27])

$$\rho = (r-m)\sin\theta\,, \quad v = (r-m)\cos\theta, \tag{3.42}$$

the line element for the self-dual Taub-NUT solution becomes

$$ds^2 = -\Delta(d\bar{t} + B\, d\phi)^2 + \Delta^{-1}\left[e^{\psi}\left(d\rho^2 + dv^2\right) + \rho^2\, d\phi^2\right], \tag{3.43}$$

where

$$\Delta = \frac{\sqrt{\rho^2+v^2}}{\sqrt{\rho^2+v^2}+2m}\,, \quad e^{\psi} = 1\,, \quad B = -2im\frac{v}{\sqrt{\rho^2+v^2}}\,. \tag{3.44}$$

Using (2.7) we infer

$$\begin{aligned}
\partial_v \tilde{B} &= \frac{\Delta^2}{\rho}\,\partial_\rho B = 2im\frac{v}{\sqrt{\rho^2+v^2}\left(\sqrt{\rho^2+v^2}+2m\right)^2}, \\
\partial_\rho \tilde{B} &= -\frac{\Delta^2}{\rho}\,\partial_v B = 2im\frac{\rho}{\sqrt{\rho^2+v^2}\left(\sqrt{\rho^2+v^2}+2m\right)^2},
\end{aligned} \tag{3.45}$$

which can be integrated and gives

$$\tilde{B} = i\left(\alpha - \frac{2m}{\sqrt{\rho^2+v^2}+2m}\right)\,, \quad \alpha \in \mathbb{C}. \tag{3.46}$$

The matrix $M(\rho, v)$ belonging to a complexification of $M \in G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is obtained using (2.6),

$$\begin{aligned}
M(\rho,v) &= \begin{pmatrix} -(\alpha^2-1) - \frac{2m(\alpha-1)^2}{\Phi} & i\alpha + \frac{2i(\alpha-1)m}{\Phi} \\ i\alpha + \frac{2i(\alpha-1)m}{\Phi} & \frac{\Phi+2m}{\Phi}\end{pmatrix} = \begin{pmatrix} -(\alpha^2-1) & i\alpha \\ i\alpha & 1\end{pmatrix} + \frac{2m}{\Phi}\,\hat{C}\,, \\
\hat{C} &= \begin{pmatrix} -(\alpha-1)^2 & i(\alpha-1) \\ i(\alpha-1) & 1\end{pmatrix}\,, \quad \Phi = \sqrt{\rho^2+v^2}\,.
\end{aligned} \tag{3.47}$$

Note that $\det\hat{C} = 0$. We obtain for M^{-1},

$$M^{-1}(\rho,v) = \begin{pmatrix} 1 & -i\alpha \\ -i\alpha & -(\alpha^2-1)\end{pmatrix} + \frac{2m}{\Phi}\,\tilde{C}\,, \; \tilde{C} = \begin{pmatrix} 1 & -i(\alpha-1) \\ -i(\alpha-1) & -(\alpha-1)^2\end{pmatrix}\,, \tag{3.48}$$

with $\hat{C}\tilde{C} = 0$. This yields an exact class I description,

$$A = M^{-1}dM = df\, C\,, \quad f(\rho,v) = -\frac{2m}{\sqrt{\rho^2+v^2}}\,, \; C = \begin{pmatrix} 1-\alpha & i \\ i(1-\alpha)^2 & \alpha-1 \end{pmatrix}\,, \; C^2 = 0\,. \tag{3.49}$$

The expression for f describes a monopole type potential. The associated function G is

$$G(\rho,v) = m\left[\log\left(\sqrt{\rho^2+v^2}-v\right) - \log\left(\sqrt{\rho^2+v^2}+v\right)\right]. \tag{3.50}$$

3.1.5 The non-extremal Kerr black hole

The matrix $M(\rho,v)$ (with $M \in G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$) that describes the exterior region of the non-extremal Kerr black hole with mass m and angular momentum a is obtained using (2.6) and is given by [19, 28]

$$M(\rho,v) = \frac{1}{c^2(u^2-1)+a^2(y^2-1)}\begin{pmatrix} (c\,u-m)^2+a^2y^2 & 2amy \\ 2amy & (c\,u+m)^2+a^2y^2 \end{pmatrix}, \tag{3.51}$$

where $c = \sqrt{m^2-a^2}$, $u > 1$, $|y| < 1$, with the prolate spheroidal coordinates (u,y) expressed in terms of the Weyl coordinates (ρ,v) by (cf. (G.6) in [29])

$$u = \frac{\sqrt{\rho^2+(v+c)^2}+\sqrt{\rho^2+(v-c)^2}}{2c}\,, \quad y = \frac{\sqrt{\rho^2+(v+c)^2}-\sqrt{\rho^2+(v-c)^2}}{2c}\,. \tag{3.52}$$

Computing $A = M^{-1}dM$ and expanding the result up to fifth order in the angular momentum parameter a and to first order in the mass parameter m gives

$$A = df_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + df_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.53}$$

with

$$\begin{aligned} f_1(\rho,v) &= m\left[-\frac{2}{\sqrt{\rho^2+v^2}} - a^2\,\frac{\rho^2-2v^2}{(\rho^2+v^2)^{5/2}} - a^4\,\frac{3\rho^4-24\rho^2v^2+8v^4}{4\,(\rho^2+v^2)^{9/2}}\right], \\ f_2(\rho,v) &= m\left[a\,\frac{2v}{(\rho^2+v^2)^{3/2}} + a^3\,\frac{v(3\rho^2-2v^2)}{(\rho^2+v^2)^{7/2}} + a^5\,\frac{v(15\rho^4-40\rho^2v^2+8v^4)}{4\,(\rho^2+v^2)^{11/2}}\right]. \end{aligned} \tag{3.54}$$

We note that f_1 carries even powers of the rotation parameter a, while f_2 carries odd powers of a. When switching off a, f_1 reduces to the function f of the Schwarzschild solution, to first order in m.

Since the two matrices in (3.53) do not commute, we have to choose which one to keep in order to obtain a class I description. Since f_2 vanishes when switching off a, we keep the term proportional to df_1, which reduces to the function f of the Schwarzschild solution when $a = 0$.

We note that f_1 is the expansion of the function $-\phi$ given in eq. (52) of [16],

$$f_1(\rho,v) = -2m\,\frac{R^3(\rho,v)}{R^4(\rho,v)+a^2v^2}\,, \quad R = \frac{\sqrt{\rho^2+v^2-a^2+\sqrt{(\rho^2+v^2-a^2)^2+4a^2v^2}}}{\sqrt{2}}\,, \tag{3.55}$$

where R is a solution of the equation (cf. eq. (54) of [16])

$$\frac{\rho^2}{R^2+a^2}+\frac{v^2}{R^2}=1\,, \tag{3.56}$$

and where f_1 satisfies the harmonic equation (2.18) with $\lambda=1$. The associated function G is

$$G_1(\rho,v)=-2m\tanh^{-1}\left(\frac{\sqrt{-a^2+\sqrt{a^4+2a^2\left(v^2-\rho^2\right)+\left(\rho^2+v^2\right)^2}+\rho^2+v^2}}{\sqrt{2}v}\right)\,. \tag{3.57}$$

When expanded to fourth order in the angular momentum parameter a, G_1 reads

$$G_1=-m\left[\log\left(\sqrt{\rho^2+v^2}+v\right)-\log\left(\sqrt{\rho^2+v^2}-v\right)+\frac{a^2v}{(\rho^2+v^2)^{3/2}}+\frac{a^4v(3\rho^2-2v^2)}{4(\rho^2+v^2)^{7/2}}\right]\,. \tag{3.58}$$

3.1.6 Kaluza-Klein black hole: rotating attractor

Rotating Kaluza-Klein black hole solutions are solutions of the four-dimensional theory that is obtained by dimensional reduction of five-dimensional pure gravity on a circle. The resulting fields in four space-time dimensions are the metric, the scalar field Φ and the Kaluza-Klein Maxwell field $\mathcal{A}_\mu$. We consider extremal rotating black hole solutions in this theory. The near-horizon behaviour of these solutions has been thoroughly studied in [30] and it is determined by the attractor mechanism in the presence of rotation (hence the name rotating attractors). Let us focus on the near-horizon behaviour of slowly rotating extremal solutions (this is also referred to as the ergo-free branch). The near-horizon behaviour of the four-dimensional metric, the dilaton field $e^{-4\Phi/\sqrt{3}}$ and the Maxwell field $\mathcal{F}=d\mathcal{A}$ is given in eqs. (5.36)-(5.44) of [30], and is determined in terms of the electric-magnetic charges (Q,P) (which, for definiteness, were taken to be positive in [30]) and in terms of the angular momentum J. In particular, in the near-horizon region and in the presence of rotation, the dilaton field is not any longer constant,

$$e^{-4\Phi/\sqrt{3}}=\left(\frac{P}{Q}\right)^{2/3}\frac{PQ-J\cos\theta}{PQ+J\cos\theta}\,. \tag{3.59}$$

The four-dimensional near-horizon metric can be written in the form (2.4) with $\lambda=1$,

$$ds_4^2=-\Delta(dt+Bd\phi)^2+\Delta^{-1}\left(e^{\psi}\,ds_2^2+\rho^2d\phi^2\right)\,, \tag{3.60}$$

with [19]

$$\rho=r\sin\theta\,,\quad v=r\cos\theta\,,\quad \Delta=\frac{8\pi(\rho^2+v^2)}{\sqrt{P^2Q^2-J^2\frac{v^2}{\rho^2+v^2}}}\,,\quad B=-\frac{J}{8\pi}\,\frac{\rho^2}{(\rho^2+v^2)^{3/2}}\,,\quad e^{\psi}=1. \tag{3.61}$$

The matrix $M\in G/H$ that describes the near-horizon solution takes values in the coset $G/H=\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$ and is given by [19]

$$M(\rho,v)=\frac{1}{\rho^2+v^2}\begin{pmatrix}-\frac{\mathcal{B}^2}{2\mathcal{D}}+\frac{2\mathcal{AD}+\mathcal{B}^2}{2\mathcal{D}}\frac{v}{\sqrt{\rho^2+v^2}} & \mathcal{B}\sqrt{\rho^2+v^2} & \mathcal{C}(\rho^2+v^2)\\ -\mathcal{B}\sqrt{\rho^2+v^2} & \mathcal{D}(\rho^2+v^2) & 0\\ \mathcal{C}(\rho^2+v^2) & 0 & 0\end{pmatrix}\,, \tag{3.62}$$

where

$$\mathcal{B} = \frac{1}{2\sqrt{\pi}}\, P^{1/3}\, Q^{2/3}\,, \quad \mathcal{C} = -\left(\frac{P}{Q}\right)^{1/3}\,, \quad \mathcal{D} = -\left(\frac{Q}{P}\right)^{2/3}\,, \quad 2\mathcal{A}\mathcal{D} + \mathcal{B}^2 = \frac{1}{4\pi}\left(\frac{Q}{P}\right)^{1/3} J\,, \tag{3.63}$$

with $\mathcal{B}/\mathcal{D} \propto P$, $\mathcal{B}/\mathcal{C} \propto Q$ and $(2\mathcal{A}\mathcal{D}+\mathcal{B}^2)/(\mathcal{C}\mathcal{D}) \propto J$. These constants satisfy $-\mathcal{C}^2\mathcal{D} = 1$.

The associated matrix $A = M^{-1}dM$ reads

$$A = df_1\, C_1 + df_2\, C_2\,, \tag{3.64}$$

where C_1 and C_2 denote commuting matrices given by

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\mathcal{B}}{\mathcal{D}} & 0 & 0 \\ 0 & \frac{\mathcal{B}}{\mathcal{C}} & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2\mathcal{A}\mathcal{D}+\mathcal{B}^2}{2\mathcal{C}\mathcal{D}} & 0 & 0 \end{pmatrix}, \tag{3.65}$$

and where the functions f_1 and f_2 are given by

$$f_1(\rho, v) = \frac{1}{\sqrt{\rho^2+v^2}}\,, \quad f_2(\rho, v) = -\frac{v}{(\rho^2+v^2)^{3/2}}\,. \tag{3.66}$$

The associated functions G_1 and G_2 take the form

$$G_1(\rho, v) = \frac{1}{2}\left[\log\left(\sqrt{\rho^2+v^2}+v\right) - \log\left(\sqrt{\rho^2+v^2}-v\right)\right]\,, \quad G_2(\rho, v) = \frac{1}{\sqrt{\rho^2+v^2}}\,. \tag{3.67}$$

Note that (3.64) is an example of the form (2.17) based on two commuting matrices. The matrices C_1 and C_2 are linear in Q, P, J. When $J = 0$, the matrix C_2 vanishes, and (3.64) reduces to the case of a static attractor solution based on $AdS_2 \times S^2$.

3.1.7 Einstein-Rosen wave

The Einstein-Rosen wave describes a cylindrical gravitational wave. In Weyl coordinates, its line element takes the form (2.4) with $\lambda = -1$ (see, for instance, [31, 32]) and

$$\begin{aligned} &\Delta(\rho, v) = e^{f(\rho,v)}\,, \quad f(\rho, v) = 2\cos(kv)\, J_0(k\rho) - 2\,, \quad B = 0\,, \\ &\psi(\rho, v) = k^2\rho^2 J_0^2(k\rho) + k^2\rho^2 J_1^2(k\rho) - 2k\cos^2(kv)\,\rho\, J_0(k\rho) J_1(k\rho)\,, \end{aligned} \tag{3.68}$$

where J_0 and J_1 denote Bessel functions of the first kind. The parameter k takes values in $\mathbb{R}_0^+$. When $k = 0$ we recover flat space-time, since $J_0(0) = 1$.

The matrix $M(\rho, v)$ (with $M \in G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$) is given by (2.6) with $\tilde{B} = 0$,

$$M(\rho, v) = \begin{pmatrix} \Delta(\rho, v) & 0 \\ 0 & \frac{1}{\Delta(\rho,v)} \end{pmatrix}, \tag{3.69}$$

and results in $A = df\, C$ with

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.70}$$

The associated function G is

$$G(\rho, v) = \frac{2}{k}\sin(kv)\, J_0(k\rho) - 2v\,. \tag{3.71}$$

To lowest order in k we obtain

$$\begin{aligned} f(\rho, v) &= k^2\left(-v^2 J_0(0) + \rho^2 J_0''(0)\right) = k^2\left(-v^2 - \frac{1}{2}\rho^2\right)\,, \\ G(\rho, v) &= k^2\left(-\frac{1}{3}v^3 - \frac{1}{2}v\rho^2\right)\,, \end{aligned} \tag{3.72}$$

where we used $J_0(0) = 1, J_0''(0) = -\frac{1}{2}$.

3.1.8 Self-dual Kleinian Taub-NUT solution

All the four-dimensional gravitational solutions described above have signature $(1,3)$ and they have a line element of the form given in (2.4). Here we consider a four-dimensional gravitational solution with signature $(2,2)$, namely the self-dual Kleinian Taub-NUT solution discussed in [18], whose line element takes the form

$$ds^2 = f(r)\left(d\bar{t} - 2m\cosh\theta d\phi\right)^2 + \frac{dr^2}{f(r)} - (r^2 - m^2)\left(d\theta^2 + \sinh^2 d\phi^2\right)\,, \tag{3.73}$$

where we have set $\ell = +im$ and

$$f(r) = \frac{r-m}{r+m}\,. \tag{3.74}$$

We restrict to $r > m$ to ensure that $f(r) > 0$. Introducing Weyl coordinates

$$\rho = (r-m)\sinh\theta\,, \quad v = (r-m)\cosh\theta, \tag{3.75}$$

the line element for the self-dual Kleinian Taub-NUT solution takes a form that differs from (2.4) by signs, namely

$$ds^2 = \Delta(d\bar{t} + B\, d\phi)^2 + \Delta^{-1}\left[e^{\psi}\left(dv^2 - d\rho^2\right) - \rho^2\, d\phi^2\right], \tag{3.76}$$

where

$$\Delta = \frac{\sqrt{v^2-\rho^2}}{\sqrt{v^2-\rho^2}+2m}\,, \quad e^{\psi} = 1\,, \quad B = -2m\frac{v}{\sqrt{v^2-\rho^2}}\,. \tag{3.77}$$

Note that $v^2 - \rho^2 > 0$ in view of $r > m$. Using (2.7) with $\lambda = -1$,

$$\begin{aligned} \partial_v \tilde{B} &= \frac{\Delta^2}{\rho}\,\partial_\rho B, \\ \partial_\rho \tilde{B} &= \frac{\Delta^2}{\rho}\,\partial_v B, \end{aligned} \tag{3.78}$$

we obtain

$$\tilde{B} = \alpha + \frac{2m}{\sqrt{v^2-\rho^2}+2m}\,, \quad \alpha \in \mathbb{C}. \tag{3.79}$$

We define the matrix $M(\rho, v)$ by

$$M = \begin{pmatrix} \Delta - \frac{\tilde{B}^2}{\Delta} & i\frac{\tilde{B}}{\Delta} \\ i\frac{\tilde{B}}{\Delta} & \frac{1}{\Delta} \end{pmatrix}, \tag{3.80}$$

which satisfies $M = M^T$ as well as

$$\det M = 1\,. \tag{3.81}$$

Then we obtain,

$$\begin{aligned} M(\rho, v) &= \begin{pmatrix} 1-\alpha^2 & i\alpha \\ i\alpha & 1 \end{pmatrix} + \frac{2m}{\tilde{\Phi}}\,\hat{C}\,, \\ \hat{C} &= \begin{pmatrix} -(1+\alpha)^2 & i(1+\alpha) \\ i(1+\alpha) & 1 \end{pmatrix}, \quad \tilde{\Phi} = \sqrt{v^2-\rho^2}\,. \end{aligned} \tag{3.82}$$

Note that $\det \hat{C} = 0$. We obtain for M^{-1},

$$M^{-1}(\rho, v) = \begin{pmatrix} 1 & -i\alpha \\ -i\alpha & 1-\alpha^2 \end{pmatrix} + \frac{2m}{\tilde{\Phi}}\,\tilde{C}\,, \quad \tilde{C} = \begin{pmatrix} 1 & -i(1+\alpha) \\ -i(1+\alpha) & -(1+\alpha)^2 \end{pmatrix}, \tag{3.83}$$

with $\det \tilde{C} = 0$, $\hat{C}\tilde{C} = 0$. It follows that

$$A = M^{-1}dM = df\,C\,, \quad f(\rho, v) = -\frac{2m}{\sqrt{v^2-\rho^2}}\,, \quad C = \begin{pmatrix} 1+\alpha & -i \\ -i(1+\alpha)^2 & -(1+\alpha) \end{pmatrix}, \quad C^2 = 0\,. \tag{3.84}$$

This expression is exact. The associated function G is

$$G(\rho, v) = -m\left[\log\left(\sqrt{v^2-\rho^2}+v\right) - \log\left(\sqrt{v^2-\rho^2}-v\right)\right]. \tag{3.85}$$

It satisfies (2.18) with $\lambda = -1$.

3.2 Class II description

Let us now turn to the class II description of the solutions given above.

We bring these solutions into the form (2.20) and (2.21) with one or two matrices C_i. Using (2.34), to each P_i we associate a function H_i that satisfies the differential equation (2.27).

3.2.1 Exterior region of the Schwarzschild black hole

Using (2.20), we associate the following matrix P to A,

$$P(\rho, v) = \frac{2}{\rho}\left(\sqrt{(v-m)^2+\rho^2} - \sqrt{(v+m)^2+\rho^2}\right) C\,, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.86}$$

To first order in m we obtain

$$P(\rho, v) = -4m\,\frac{v}{\rho\sqrt{\rho^2+v^2}}\,C\,, \tag{3.87}$$

and the associated function H reads

$$H(\rho, v, \phi) = -2m\,e^{i\phi}\,\frac{\sqrt{\rho^2+v^2}}{\rho}\,C\,. \tag{3.88}$$

3.2.2 Extremal Reissner-Nordstrom black hole

We associate a matrix P to $\tilde{A}$ using (2.20),

$$\begin{aligned}\tilde{A}_\rho &= g^{-1}(c)\, A_\rho\, g(c) = A_\rho = -\frac{1}{2}\partial_v P(\rho, v)\,,\\ \tilde{A}_v &= g^{-1}(c)\, A_v\, g(c) = A_v = \frac{1}{2}\left(\partial_\rho + \frac{1}{\rho}\right) P(\rho, v)\,,\end{aligned} \tag{3.89}$$

which results in

$$P(\rho, v) = 2\,|Q| \frac{v}{\rho\sqrt{v^2+\rho^2}}\, C\,, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\bar{c} & 0 & 0 \\ 0 & \sqrt{2}c & 0 \end{pmatrix}\,, \quad c = \frac{Q}{|Q|}\,. \tag{3.90}$$

The associated function H reads

$$H(\rho, v, \phi) = |Q|\, e^{i\phi}\, \frac{\sqrt{\rho^2+v^2}}{\rho}\, C\,. \tag{3.91}$$

3.2.3 Black hole in $AdS_2 \times S^2$

The associated matrix P is

$$P(\rho, v) = \frac{2}{\rho}\left(\sqrt{(v-m)^2+\rho^2} - \sqrt{(v+m)^2+\rho^2}\right) C\,, \quad C = -\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}e^{-i\alpha} & 0 & 0 \\ 0 & \sqrt{2}e^{i\alpha} & -1 \end{pmatrix}\,, \tag{3.92}$$

where $\alpha \in \mathbb{R}$. To first order in m we obtain

$$P(\rho, v) = -4m\, \frac{v}{\rho\,\sqrt{\rho^2+v^2}}\, C\,, \tag{3.93}$$

as well as

$$H(\rho, v, \phi) = -2m\, e^{i\phi}\, \frac{\sqrt{\rho^2+v^2}}{\rho}\, C\,. \tag{3.94}$$

3.2.4 Self-dual Taub-NUT solution

The associated matrix P is

$$P(\rho, v) = -4m\, \frac{v}{\rho\,\sqrt{\rho^2+v^2}}\, C\,, \quad C = \begin{pmatrix} 1-\alpha & i \\ i(1-\alpha)^2 & \alpha-1 \end{pmatrix}\,, \quad C^2 = 0\,, \tag{3.95}$$

where $\alpha \in \mathbb{C}$, and

$$H(\rho, v, \phi) = -2m\, e^{i\phi}\, \frac{\sqrt{\rho^2+v^2}}{\rho}\, C\,. \tag{3.96}$$

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3.2.5 The non-extremal Kerr black hole

To first order in m and to fourth order in the rotation parameter a, the function P associated to f_1 given in (3.54) reads

$$P_1(\rho,v) = -m\left[\frac{4v}{\rho\sqrt{(\rho^2+v^2)}} + \frac{6a^2\rho v}{(\rho^2+v^2)^{5/2}} + \frac{5a^4\rho v(3\rho^2-4v^2)}{2(\rho^2+v^2)^{9/2}}\right] C\,, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3.97}$$

The associated function H_1 is

$$H_1(\rho,v,\phi) = -m\, e^{i\phi}\left(2\frac{\sqrt{\rho^2+v^2}}{\rho} - a^2\frac{\rho}{(\rho^2+v^2)^{3/2}} - \frac{a^4}{4}\rho\frac{(\rho^2-4v^2)}{(\rho^2+v^2)^{7/2}}\right) C\,. \tag{3.98}$$

3.2.6 Kaluza-Klein black hole: rotating attractor

The associated matrix P (cf. (2.21)) takes the form

$$P(\rho,v) = p_1(\rho,v)\, C_1 + p_2(\rho,v)\, C_2 \tag{3.99}$$

with

$$p_1(\rho,v) = 2\frac{v}{\rho\sqrt{\rho^2+v^2}}\,, \quad p_2(\rho,v) = \frac{2\rho}{(\rho^2+v^2)^{3/2}}\,. \tag{3.100}$$

The matrices C_1 and C_2 are given in (3.65). The associated functions H_1 and H_2 are

$$H_1(\rho,v,\phi) = e^{i\phi}\,\frac{\sqrt{\rho^2+v^2}}{\rho}\, C_1\,, \quad H_2(\rho,v,\phi) = e^{i\phi}\frac{v}{\rho\sqrt{\rho^2+v^2}}\, C_2\,. \tag{3.101}$$

3.2.7 Einstein-Rosen wave

The associated matrix P is

$$P(\rho,v) = \frac{4}{k}\,\sin(kv)\, J_0'(k\rho)\, C\,, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.102}$$

and

$$H(\rho,v,\phi) = e^{i\phi}\left(-\frac{2}{k^2}\cos(kv)\, J_0'(k\rho) - \rho\right) C\,. \tag{3.103}$$

To lowest order in k we obtain

$$H(\rho,v,\phi) = -k^2\, e^{i\phi}\left(\frac{1}{2}v^2\rho + \frac{1}{8}\rho^3\right)\,. \tag{3.104}$$

P satisfies

$$\left(\partial_\rho^2 - \partial_v^2 + \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\right) P(\rho,v) = 0 \tag{3.105}$$

by virtue of the differential equation for the Bessel function $J_0(\rho)$,

$$J_0''(\rho) + \frac{J_0'(\rho)}{\rho} + J_0(\rho) = 0\,. \tag{3.106}$$

3.2.8 Kleinian self-dual Taub-NUT solution

The associated matrix P is

$$P(\rho, v) = 4m \, \frac{v}{\rho \sqrt{v^2 - \rho^2}} \, C \,, \quad C = \begin{pmatrix} 1+\alpha & -i \\ -i(1+\alpha)^2 & -(1+\alpha) \end{pmatrix} \,, \quad C^2 = 0 \,, \tag{3.107}$$

where $\alpha \in \mathbb{C}$, and

$$H(\rho, v, \phi) = 2m \, e^{i\phi} \, \frac{\sqrt{v^2 - \rho^2}}{\rho} \, C \,. \tag{3.108}$$

The latter satisfies (2.27) with $\lambda = -1$.

4 Another example: the Eguchi-Hanson metric

In this section we discuss the Eguchi-Hanson metric in Weyl coordinates. Subsequently we determine the associated matrix 1-form $A = M^{-1} dM$.

The Eguchi-Hanson metric depends on one parameter, a. When switching off a, the line element describes four-dimensional flat space-time. We have verified that there does not exist a coordinate change for the Eguchi-Hanson metric which takes us to Weyl coordinates in such a way that (i) the background metric is mapped to the vacuum version of the Weyl metric with $A_\rho = A_v = 0$, and (ii) the linear fluctuation (i.e. $\mathcal{O}(a^4)$) is in the standard Weyl form. In order to proceed, we will relax condition (i), while still requiring that (ii) is satisfied, and describe the corresponding coordinate change below. We will then provide a single copy prescription for the linear part, as follows. We expand A in powers of the parameter a^4, i.e. $A = A^{(0)} + A^{(1)} + \mathcal{O}(a^8)$, where $A^{(0)}$ (which is non-vanishing) describes the background. We focus on $A^{(1)}$ and show that it admits both a class I and a class II description, based on functions f_1 and p_1, respectively. In section 6 we will show that the single copy description for the Eguchi-Hanson solution is constructed using p_1 as a starting point and integrating it twice with respect to the coordinate v. We denote the resulting function by $\hat{H}_1$. This differs from the examples discussed in the previous section, since for these the single copy description is based on a function H that is obtained from the function p by integrating once with respect to v. The number of integrations over v that one has to perform to obtain a single copy description differs from solution to solution and is related to the nature of the sources supporting the gravitational solution.

The Eguchi-Hanson metric is a self-dual Euclidean solution described by [33]

$$ds_4^2 = \frac{1}{4} r^2 \, h(r) \, (d\gamma + \cos\theta \, d\phi)^2 + \frac{dr^2}{h(r)} + \frac{1}{4} r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \,, \quad 0 \leq \gamma < 4\pi, \tag{4.1}$$

with

$$h(r) = 1 - \frac{a^4}{r^4} > 0. \tag{4.2}$$

We now write the Eguchi-Hanson metric in Weyl form

$$ds^2 = \Delta (d\gamma + B \, d\phi)^2 + \Delta^{-1} [e^{\psi} \left(d\rho^2 + dv^2 \right) + \rho^2 \, d\phi^2]. \tag{4.3}$$

To do so, we first compare (4.3) with (4.1) and infer

$$\begin{aligned}\Delta &= \frac{1}{4}\, r^2\, h(r)\\ B &= \cos\theta\\ \rho^2 &= \frac{1}{4}\, r^2\, \Delta\, \sin^2\theta.\end{aligned} \tag{4.4}$$

Hence,

$$\rho = \frac{1}{4}\, r^2\, \sqrt{h}\, \sin\theta. \tag{4.5}$$

For convenience, we define

$$F(r) = \frac{1}{4}\, r^2\, \sqrt{h(r)}, \tag{4.6}$$

so that

$$\rho = F(r)\, \sin\theta. \tag{4.7}$$

We introduce

$$v = G(r)\, \cos\theta, \tag{4.8}$$

and calculate

$$\begin{aligned}d\rho^2 + dv^2 = dr^2 &\left((F')^2\, \sin^2\theta + (G')^2\, \cos^2\theta\right) + d\theta^2 \left(F^2\, \cos^2\theta + G^2\, \sin^2\theta\right)\\ &+ 2dr d\theta\, (F'\, F - G'\, G) \sin\theta\, \cos\theta.\end{aligned} \tag{4.9}$$

Since there are no terms proportional to $\sin\theta\, \cos\theta$ in (4.1), we demand

$$F'\, F - G'\, G = 0 \longleftrightarrow (F^2)' = (G^2)' \longrightarrow G^2 = F^2 + b\,, \quad b \in \mathbb{R}\,. \tag{4.10}$$

Hence

$$G(r) = \sqrt{F^2(r) + b}. \tag{4.11}$$

Comparing (4.9) with (4.1) results in

$$\begin{aligned}dr^2 :&\quad \left((F')^2\, \sin^2\theta + (G')^2\, \cos^2\theta\right) \frac{e^\psi}{\Delta} = \frac{1}{h},\\ d\theta^2 :&\quad \left(F^2\, \cos^2\theta + G^2\, \sin^2\theta\right) \frac{e^\psi}{\Delta} = \frac{r^2}{4}.\end{aligned} \tag{4.12}$$

By combining these two equations we infer

$$\left(\frac{4}{r^2}\, F^2 - h(G')^2\right) \cos^2\theta = \left(h(F')^2 - \frac{4}{r^2}\, G^2\right) \sin^2\theta, \tag{4.13}$$

and hence

$$\begin{aligned}\frac{4}{r^2}\, F^2 - h(G')^2 &= 0,\\ h(F')^2 - \frac{4}{r^2}\, G^2 &= 0.\end{aligned} \tag{4.14}$$

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Using

$$G' = \frac{F' F}{G}, \tag{4.15}$$

and inserting this into the first equation of (4.14) we obtain

$$F^2 \left(\frac{4}{r^2} - h \frac{(F')^2}{G^2} \right) = 0 \Rightarrow \frac{4}{r^2} - h \frac{(F')^2}{G^2} = 0, \tag{4.16}$$

which is the second equation of (4.14). Thus, both equations in (4.14) are consistent with one another. It suffices to solve the second equation. Using (4.6) and (4.11), we obtain

$$h(F')^2 - \frac{4}{r^2} G^2 = 0 \Rightarrow b = \frac{a^4}{16}. \tag{4.17}$$

Thus,

$$G(r) = \sqrt{F^2(r) + \frac{a^4}{16}} = \frac{r^2}{4}, \tag{4.18}$$

and

$$\begin{aligned} \rho &= \frac{1}{4} r^2 \sqrt{h} \sin\theta, \\ v &= \frac{1}{4} r^2 \cos\theta. \end{aligned} \tag{4.19}$$

We determine e^ψ from the second equation in (4.12),

$$e^\psi = \frac{r^4}{16} \frac{h}{F^2 \cos^2\theta + G^2 \sin^2\theta} = \frac{1 - \frac{a^4}{r^4}}{1 - \frac{a^4}{r^4} \cos^2\theta} = \frac{r^4 - a^4}{r^4 - a^4 \cos^2\theta}. \tag{4.20}$$

Summarizing, we have

$$\begin{aligned} \Delta &= \frac{1}{4} r^2 \left(1 - \frac{a^4}{r^4}\right), \\ B &= \cos\theta, \\ e^\psi &= \frac{r^4 - a^4}{r^4 - a^4 \cos^2\theta}. \end{aligned} \tag{4.21}$$

We still have to express Δ, B, e^ψ in terms of the Weyl coordinates (ρ, v) given in (4.19). To this end we write

$$\rho^2 = \frac{1}{16}(r^4 - a^4)\left(1 - 16\frac{v^2}{r^4}\right), \tag{4.22}$$

which equals

$$r^8 - r^4(a^4 + 16(\rho^2 + v^2)) = -16a^4 v^2. \tag{4.23}$$

This yields

$$r^4 = \frac{1}{2}\left[a^4 + 16(\rho^2 + v^2) \pm \sqrt{(a^4 + 16(\rho^2 + v^2))^2 - 64a^4 v^2}\right]. \tag{4.24}$$

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We choose the plus sign, so as to ensure that when $a = 0$, we get $r^4 = 16(\rho^2 + v^2)$,

$$r^4 = \frac{1}{2}\left[a^4 + 16(\rho^2 + v^2) + \sqrt{(a^4 + 16(\rho^2 + v^2))^2 - 64a^4v^2}\right]. \tag{4.25}$$

We also have

$$\cos\theta = \frac{4v}{r^2} = \frac{4\sqrt{2}\,v}{\sqrt{a^4 + 16(\rho^2 + v^2) + \sqrt{(a^4 + 16(\rho^2 + v^2))^2 - 64a^4v^2}}}. \tag{4.26}$$

We have thus expressed r^2 and $\cos\theta$ in terms of ρ and v. Since the resulting expressions for (4.21) are rather long, we do not present them here.

Let us now convert back to Minkowski signature through the analytic continuation $\gamma = -i\bar{t}$, i.e. to

$$ds^2 = -\Delta(d\bar{t} + B\,d\phi)^2 + \Delta^{-1}\left[e^{\psi}\left(d\rho^2 + dv^2\right) + \rho^2\,d\phi^2\right], \tag{4.27}$$

with

$$\begin{aligned}
\Delta &= \frac{1}{4}r^2\left(1 - \frac{a^4}{r^4}\right) = \frac{r^4 - a^4}{4r^2}, \\
B &= i\cos\theta, \\
e^{\psi} &= \frac{r^4 - a^4}{r^4 - a^4\cos^2\theta}.
\end{aligned} \tag{4.28}$$

Let us for convenience define the quantity Q by

$$Q = \frac{1}{2}\left(a^4 + 16(\rho^2 + v^2)\right), \tag{4.29}$$

and express

$$r^4 = Q + \sqrt{Q^2 - 16a^4v^2}. \tag{4.30}$$

Then,

$$\begin{aligned}
\Delta &= \frac{Q + \sqrt{Q^2 - 16a^4v^2} - a^4}{4\sqrt{Q + \sqrt{Q^2 - 16a^4v^2}}} \\
B &= i\frac{4v}{\sqrt{Q + \sqrt{Q^2 - 16a^4v^2}}},
\end{aligned} \tag{4.31}$$

and

$$\tilde{B} = -\frac{i}{4}\frac{Q + \sqrt{Q^2 - 16a^4v^2} + a^4}{\sqrt{Q + \sqrt{Q^2 - 16a^4v^2}}}, \tag{4.32}$$

where $\tilde{B}$ is the solution to

$$\star d\tilde{B} = \frac{\Delta^2}{\rho}\,dB, \tag{4.33}$$

i.e. to

$$\partial_v \tilde{B} = \frac{\Delta^2}{\rho}\,\partial_\rho B,$$
$$\partial_\rho \tilde{B} = -\frac{\Delta^2}{\rho}\,\partial_v B. \tag{4.34}$$

The matrix M in (2.6) takes the form

$$\begin{aligned} M &= \begin{pmatrix} -a^4\frac{\sqrt{Q+\sqrt{Q^2-16a^4v^2}}}{Q+\sqrt{Q^2-16a^4v^2}-a^4} & -i\frac{Q+\sqrt{Q^2-16a^4v^2}+a^4}{Q+\sqrt{Q^2-16a^4v^2}-a^4} \\ -i\frac{Q+\sqrt{Q^2-16a^4v^2}+a^4}{Q+\sqrt{Q^2-16a^4v^2}-a^4} & 4\frac{\sqrt{Q+\sqrt{Q^2-16a^4v^2}}}{Q+\sqrt{Q^2-16a^4v^2}-a^4} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} + \frac{\sqrt{Q+\sqrt{Q^2-16a^4v^2}}}{Q+\sqrt{Q^2-16a^4v^2}-a^4}\begin{pmatrix} -a^4 & 0 \\ 0 & 4 \end{pmatrix} \\ &\quad + \frac{a^4}{Q+\sqrt{Q^2-16a^4v^2}-a^4}\begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix}. \end{aligned} \tag{4.35}$$

Evaluating $A = M^{-1}dM$ we obtain

$$A = df\begin{pmatrix} 0 & 1 \\ -\frac{a^4}{4} & 0 \end{pmatrix} \tag{4.36}$$

with

$$f(\rho,v) = 4i\frac{\tanh^{-1}\left(\frac{\sqrt{a^4+16(\rho^2+v^2)+\sqrt{a^8+32a^4(\rho^2-v^2)+256(\rho^2+v^2)^2}}}{\sqrt{2}a^2}\right)}{a^2} - \frac{2\pi}{a^2}\,, \tag{4.37}$$

which satisfies the harmonic equation (2.18) with $\lambda = 1$. The integration constant $-\frac{2\pi}{a^2}$ was chosen such that when expanding f in powers of a, the expansion starts with a zeroth order contribution given by

$$f_0(\rho,v) = \frac{i}{\sqrt{\rho^2+v^2}}\,. \tag{4.38}$$

This is the function that is associated to the background metric, which is described by (4.27) with $a = 0$.

Expanding f in powers of a^4, i.e. $f = f_0 + f_1 + \mathcal{O}(a^8)$, we obtain at first order in a^4,

$$f_1(\rho,v) = \frac{ia^4}{96}\,\frac{2v^2-\rho^2}{(\rho^2+v^2)^{5/2}}\,. \tag{4.39}$$

Collecting all the terms of order a^4 in (4.36), we obtain

$$-a^4 df_0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + df_1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.40}$$

Since both matrices do not commute, we have to decide which term to keep in order to obtain a class I description at order a^4. We discard the term proportional to df_0, since it only carries information about the background. We write the term proportional to df_1 as $df_1\, C$, where

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{4.41}$$

The function p_1 associated to f_1 is (cf. (2.29))

$$p_1(\rho, v) = -\frac{ia^4}{16} \frac{\rho v}{(\rho^2 + v^2)^{5/2}} . \tag{4.42}$$

Differently from all the other examples discussed in this paper, the single copy description of the Eguchi-Hanson solution requires integrating p_1 twice over v. Integrating p_1 once with respect to v (cf. (2.34)) gives

$$H_1(\rho, v, \phi) = \frac{ia^4}{96}\, e^{i\phi} \frac{\rho}{(\rho^2 + v^2)^{3/2}}\, C . \tag{4.43}$$

Integrating once more over v gives

$$\hat{H}_1(\rho, v, \phi) = \frac{ia^4}{96}\, e^{i\phi} \frac{v}{\rho\, (\rho^2 + v^2)^{1/2}}\, C . \tag{4.44}$$

5 Mapping gravitational solutions to solutions of the self-dual Yang-Mills equations in four dimensions

As already mentioned, the PDEs (2.1) that arise by dimensionally reducing gravitational theories in space-time dimensions $D \geq 4$ also arise by performing a suitable reduction of the self-duality equations for Yang-Mills fields in four dimensions down to two dimensions [6–9]. Using this link, we may therefore associate solutions of the self-dual Yang-Mills equations in four-dimensional flat space-time to the gravitational solutions belonging to either class I or to class II. This is what we will do next.

We note that in these constructions there are two cases to consider, either $\lambda = 1$ or $\lambda = -1$. When $\lambda = 1$, the flat space-time signature is $(1, 3)$, whereas when $\lambda = -1$ the signature is $(2, 2)$.

5.1 $\lambda = 1$: signature $(1, 3)$

We consider the self-dual sector of Yang-Mills (YM) theory in four-dimensional Minkowski space-time, whose description we briefly review following [34], and perform a suitable dimensional reduction of it to two dimensions.

We introduce coordinates

$$U = \tau - v\,, \quad V = \tau + v\,, \quad X = -w\,, \quad Y = -\bar{w}, \tag{5.1}$$

where

$$w = x + iy\,, \quad \bar{w} = x - iy\,, \tag{5.2}$$

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so that

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + dv^2 = -dUdV + dwd\bar{w}. \tag{5.3}$$

We set

$$w = \rho\, e^{i\phi}\,. \tag{5.4}$$

We then have

$$\partial_U = -\frac{1}{2}\partial_v + \frac{1}{2}\partial_\tau\,, \quad \partial_V = \frac{1}{2}\partial_v + \frac{1}{2}\partial_\tau \tag{5.5}$$

as well as

$$\partial_w = \frac{1}{2}e^{-i\phi}\left(\partial_\rho - \frac{i}{\rho}\,\partial_\phi\right)\,, \quad \partial_{\bar{w}} = \frac{1}{2}e^{i\phi}\left(\partial_\rho + \frac{i}{\rho}\,\partial_\phi\right). \tag{5.6}$$

We introduce the Yang-Mills connection

$$\mathcal{D}_\mu = \partial_\mu - ig\mathcal{A}_\mu\,, \quad \mathcal{A}_\mu = \mathcal{A}_{\mu a}\, T_a\,, \quad T_a \in \mathfrak{g}\,. \tag{5.7}$$

Note that we are using calligraphic $\mathcal{A}_\mu$ to denote the genuine YM fields, in order to differentiate them from the vectors appearing in the Breitenlohner-Maison procedure, cf. (2.3). Then,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -ig\mathcal{F}_{\mu\nu} \tag{5.8}$$

with

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{5.9}$$

Self-dual solutions of the Yang-Mills equations of motion satisfy (see appendix B)

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}i\epsilon_{\mu\nu\rho\sigma}\,\mathcal{F}^{\rho\sigma} \tag{5.10}$$

with $\epsilon_{UVw\bar{w}} = \frac{i}{4} = \sqrt{-g}$. In the coordinate system $(U, V, w, \bar{w})$, the self-duality condition (5.10) becomes (see [34])

$$\mathcal{F}_{Uw} = 0, \tag{5.11}$$

$$\mathcal{F}_{V\bar{w}} = 0, \tag{5.12}$$

$$\mathcal{F}_{UV} - \mathcal{F}_{w\bar{w}} = 0. \tag{5.13}$$

These 3 equations are the compatibility conditions for the following linear system (see eqs. (4) and (5) in [7]),

$$LY = (\mathcal{D}_U - \Omega\mathcal{D}_{\bar{w}})\, Y = 0\,, \quad MY = (\mathcal{D}_w - \Omega\mathcal{D}_V)\, Y = 0. \tag{5.14}$$

Here, Y is an invertible matrix function and $\mathcal{D}$ denotes the covariant derivative introduced in (5.7), and $\Omega \in \mathbb{C}$ denotes a *constant* complex parameter, called the spectral parameter. We take $Y \in C^2$ with respect to $(U, V, w, \bar{w})$. The compatibility condition

$$[L, M]Y = 0 \quad \forall \quad Y \in C^2\,, \quad \Omega \in \mathbb{C}\,, \tag{5.15}$$

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gives

$$0=[L,M]Y=\Big([\mathcal{D}_U,\mathcal{D}_w]-\Omega\left([\mathcal{D}_U,\mathcal{D}_V]-[\mathcal{D}_w,\mathcal{D}_{\bar w}]\right)+\Omega^2\,[\mathcal{D}_{\bar w},\mathcal{D}_V]\Big)Y$$
$$=\Big(-i\mathcal{F}_{Uw}+i\Omega\left(\mathcal{F}_{UV}-\mathcal{F}_{w\bar w}\right)+i\Omega^2\,\mathcal{F}_{V\bar w}\Big)Y. \tag{5.16}$$

Then, multiplying by Y^{-1} on the right gives

$$-i\mathcal{F}_{Uw}+i\Omega\left(\mathcal{F}_{UV}-\mathcal{F}_{w\bar w}\right)+i\Omega^2\,\mathcal{F}_{V\bar w}=0, \tag{5.17}$$

which yields (5.11)–(5.13) as vanishing coefficients of this quadratic polynomial in Ω.

Note that differently from the Breitenlohner-Maison linear system (2.11), the linear system (5.14) uses a constant spectral parameter.

Now we make the gauge choice $\mathcal{A}_U=0$. Then, (5.11) will force $\mathcal{A}_w=0$, subject to appropriate boundary conditions,

$$\mathcal{A}_U=\mathcal{A}_w=0, \tag{5.18}$$

in which case the linear system (L,M) becomes

$$L=\partial_U-\Omega\mathcal{D}_{\bar w}\,,\quad M=\partial_w-\Omega\mathcal{D}_V\,. \tag{5.19}$$

Note that $\mathcal{A}$ has been complexified; hence $\mathcal{A}_{\bar w}$ is not the complex conjugate of $\mathcal{A}_w$.

Then, (5.12) and (5.13) become

$$0=\mathcal{F}_{V\bar w}=\partial_V\mathcal{A}_{\bar w}-\partial_{\bar w}\mathcal{A}_V-ig[\mathcal{A}_V,\mathcal{A}_{\bar w}],$$
$$0=\mathcal{F}_{UV}-\mathcal{F}_{w\bar w}=\partial_U\mathcal{A}_V-\partial_w\mathcal{A}_{\bar w}. \tag{5.20}$$

Next, we restrict to self-dual solutions that are *static*, i.e. $\mathcal{A}_V$ and $\mathcal{A}_{\bar w}$ are taken to be independent of τ. Using (5.5) and (5.6), the relations (5.20) become

$$0=\mathcal{F}_{V\bar w}=\frac{1}{2}\left(\partial_v\mathcal{A}_{\bar w}-e^{i\phi}\left(\partial_\rho\mathcal{A}_V+\frac{i}{\rho}\,\partial_\phi\mathcal{A}_V\right)-2ig[\mathcal{A}_V,\mathcal{A}_{\bar w}]\right),$$
$$0=\mathcal{F}_{UV}-\mathcal{F}_{w\bar w}=-\frac{1}{2}\left(\partial_v\mathcal{A}_V+e^{-i\phi}\left(\partial_\rho\mathcal{A}_{\bar w}-\frac{i}{\rho}\,\partial_\phi\mathcal{A}_{\bar w}\right)\right). \tag{5.21}$$

Next, let us further specialize to solutions of the form

$$\mathcal{A}_{\bar w}=\frac{1}{2}\,e^{i\phi}\,A_\rho(\rho,v)\,,\quad \mathcal{A}_V=\frac{1}{2}\,A_v(\rho,v)\,, \tag{5.22}$$

where the relation between the gauge algebra of the YM theory and the algebra in which the Breitenlohner-Maison vectors A_ρ and A_v are valued will we made explicit in section 6. For notational simplicity, we will generally not be writing the algebra generators explicitly. Then, (5.21) become

$$0=\mathcal{F}_{V\bar w}=\frac{1}{4}e^{i\phi}\Big(\partial_vA_\rho-\partial_\rho A_v-ig[A_v,A_\rho]\Big),$$
$$0=\mathcal{F}_{UV}-\mathcal{F}_{w\bar w}=-\frac{1}{4}\left(\partial_vA_v+\partial_\rho A_\rho+\frac{1}{\rho}\,A_\rho\right). \tag{5.23}$$

Finally, performing the rescaling

$$\mathcal{A} \to \frac{i}{g}\,\mathcal{A}, \tag{5.24}$$

the relations (5.23) become

$$\begin{aligned} 0 &= \mathcal{F}_{V\bar{w}} = \frac{i}{4g} e^{i\phi}\Big(\partial_v A_\rho - \partial_\rho A_v + [A_v, A_\rho]\Big), \\ 0 &= \mathcal{F}_{UV} - \mathcal{F}_{w\bar{w}} = -\frac{i}{4g}\left(\partial_v A_v + \partial_\rho A_\rho + \frac{1}{\rho} A_\rho\right). \end{aligned} \tag{5.25}$$

These are precisely the field equations (2.8) and (2.9) with $\lambda = 1$.

Therefore, we conclude that when $\lambda = 1$, *any* solution to the two-dimensional PDEs (2.8) lifts to a static solution

$$\mathcal{A}_U = \mathcal{A}_w = 0\,, \quad \mathcal{A}_{\bar{w}} = \frac{1}{2}\, e^{i\phi}\, A_\rho(\rho, v)\,, \quad \mathcal{A}_V = \frac{1}{2}\, A_v(\rho, v) \tag{5.26}$$

of the self-duality conditions for a Yang-Mills field in four flat space-time dimensions. Note that the solutions of (2.8) will, in general, correspond to gravitational solutions in D space-time dimensions that are not self-dual.

In the following, we specialise to gravitational solutions belonging either to class I or to class II.

5.1.1 Class I

Solutions in class I satisfy (2.17). Inserting (2.17) into (5.26) gives

$$\mathcal{A}_{\bar{w}} = \frac{1}{2}\, e^{i\phi}\, \partial_\rho f_i(\rho, v)\, C^i\,, \quad \mathcal{A}_V = \frac{1}{2}\, \partial_v f_i(\rho, v)\, C^i \tag{5.27}$$

with C^i commuting matrices, which, using (5.5) and (5.6), yields

$$\mathcal{A}_{\bar{w}} = \partial_{\bar{w}} f_i(\rho, v)\, C^i\,, \quad \mathcal{A}_V = \partial_V f_i(\rho, v)\, C^i. \tag{5.28}$$

Inserting this into the self-dual Yang-Mills equations (5.20) gives

$$\left(\partial_U \partial_V - \partial_w \partial_{\bar{w}}\right) f_i = 0\,, \tag{5.29}$$

in accordance with (2.18).

5.1.2 Class II

To a solution (2.20) in class II, we associate the matrix function Ψ defined in (2.26). We may then express A_ρ and A_v as

$$A_\rho = -e^{-i\phi}\partial_v \Psi(\rho, v, \phi)\,, \quad A_v = e^{-i\phi}\left(\partial_\rho - \frac{i}{\rho}\partial_\phi\right)\Psi(\rho, v, \phi)\,. \tag{5.30}$$

Then, using (5.5) and (5.6), we infer that

$$\mathcal{A}_{\bar{w}} = \partial_U \Psi\,, \quad \mathcal{A}_V = \partial_w \Psi\,. \tag{5.31}$$

Inserting (5.31) into (5.20) and performing the rescaling (5.24) gives

$$(\partial_U \partial_V - \partial_w \partial_{\bar{w}})\, \Psi + [\partial_w \Psi, \partial_U \Psi] = 0, \tag{5.32}$$

which, defining [21, 35]

$$\begin{aligned} \{f, g\} &= \partial_w f \partial_U g - \partial_U f \partial_w g = \varepsilon^{\alpha\beta} \partial_\alpha f \partial_\beta g\,, \quad \alpha, \beta = w, U \\ [\{f, g\}] &= \varepsilon^{\alpha\beta} [\partial_\alpha f, \partial_\beta g] = [\partial_w f, \partial_U g] - [\partial_U f, \partial_w g], \end{aligned} \tag{5.33}$$

and using

$$[\partial_w \Psi, \partial_U \Psi] = \frac{1}{2} [\{\Psi, \Psi\}], \tag{5.34}$$

can also be written as

$$(\partial_U \partial_V - \partial_w \partial_{\bar{w}})\, \Psi + \frac{1}{2} [\{\Psi, \Psi\}] = 0, \tag{5.35}$$

which is a Plebanski type equation.

5.2 $\lambda = -1$: signature $(2, 2)$

We now change the signature of flat space-time in four dimensions from $(1, 3)$ to $(2, 2)$. To this end, we consider two different coordinate systems $(U, V, X, Y) = (U, V, -w, -\bar{w})$, namely

$$U = \tau - v\,, \quad V = \tau + v\,, \quad w = \rho\, e^{-\phi}\,, \quad \bar{w} = -\rho\, e^{\phi}, \tag{5.36}$$

and

$$U = \tau + iv\,, \quad V = \tau - iv\,, \quad w = \rho\, e^{i\phi}\,, \quad \bar{w} = \rho\, e^{-i\phi}. \tag{5.37}$$

Then the line element $ds^2 = -dUdV + dwd\bar{w}$ becomes

$$ds^2 = -d\tau^2 + dv^2 - d\rho^2 + \rho^2 d\phi^2 \tag{5.38}$$

and

$$ds^2 = -d\tau^2 - dv^2 + d\rho^2 + \rho^2 d\phi^2\,, \tag{5.39}$$

respectively. Note that the associated two-dimensional line element is $dv^2 - d\rho^2$ in the first case, and $-dv^2 + d\rho^2$ in the second case. The Yang-Mills self-duality condition (5.10) again takes the form given in (5.11)–(5.13) in both these coordinate systems (see appendix B).

Let us first consider the coordinate system (5.36), in which case

$$\begin{aligned} \partial_U &= -\frac{1}{2}\partial_v + \frac{1}{2}\partial_\tau\,, & \partial_V &= \frac{1}{2}\partial_v + \frac{1}{2}\partial_\tau\,, \\ \partial_w &= \frac{1}{2} e^{\phi} \left(\partial_\rho - \frac{1}{\rho}\, \partial_\phi\right)\,, & \partial_{\bar{w}} &= -\frac{1}{2} e^{-\phi} \left(\partial_\rho + \frac{1}{\rho}\, \partial_\phi\right). \end{aligned} \tag{5.40}$$

Similar to the discussion given above for signature $(1, 3)$, we specialise to *static* solutions, of the form

$$\mathcal{A}_U = \mathcal{A}_w = 0\,, \quad \mathcal{A}_{\bar{w}} = -\frac{1}{2}\, e^{-\phi}\, A_\rho(\rho, v)\,, \quad \mathcal{A}_V = \frac{1}{2} A_v(\rho, v)\,. \tag{5.41}$$

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Then, inserting this into the self-dual Yang-Mills equations (5.20) gives, upon performing the rescaling (5.24),

$$
\begin{aligned}
&0=\mathcal{F}_{V\bar{w}} \Longrightarrow -\frac{1}{4}\,e^{-\phi}\Big(\partial_v A_\rho-\partial_\rho A_v+[A_v,A_\rho]\Big)=0\,,\\
&0=\mathcal{F}_{UV}-\mathcal{F}_{w\bar{w}}=\partial_U\mathcal{A}_V-\partial_w\mathcal{A}_{\bar{w}} \Longrightarrow -\frac{1}{4}\left(\partial_v A_v-\partial_\rho A_\rho-\frac{A_\rho}{\rho}\right)=0.
\end{aligned}
\tag{5.42}
$$

These are precisely the field equations (2.8) and (2.9) with $\lambda=-1$.

Next, let us consider the coordinate system (5.37), in which case

$$
\begin{aligned}
\partial_U&=-\frac{i}{2}\partial_v+\frac{1}{2}\partial_\tau\,, & \partial_V&=\frac{i}{2}\partial_v+\frac{1}{2}\partial_\tau,\\
\partial_w&=\frac{1}{2}e^{-i\phi}\left(\partial_\rho-\frac{i}{\rho}\,\partial_\phi\right)\,, & \partial_{\bar{w}}&=\frac{1}{2}e^{i\phi}\left(\partial_\rho+\frac{i}{\rho}\,\partial_\phi\right).
\end{aligned}
\tag{5.43}
$$

We again specialise to *static solutions*, of the form

$$
\mathcal{A}_U=\mathcal{A}_w=0\,,\quad \mathcal{A}_{\bar{w}}=\frac{1}{2}\,e^{i\phi}\,A_\rho(\rho,v)\,,\quad \mathcal{A}_V=\frac{i}{2}A_v(\rho,v)\,.
\tag{5.44}
$$

Then, inserting this into the self-dual Yang-Mills equations (5.20) gives, upon performing the rescaling (5.24),

$$
\begin{aligned}
&0=\mathcal{F}_{V\bar{w}} \Longrightarrow \frac{i}{4}\,e^{i\phi}\Big(\partial_v A_\rho-\partial_\rho A_v+[A_v,A_\rho]\Big)=0\,,\\
&0=\mathcal{F}_{UV}-\mathcal{F}_{w\bar{w}}=\partial_U\mathcal{A}_V-\partial_w\mathcal{A}_{\bar{w}} \Longrightarrow \frac{1}{4}\left(\partial_v A_v-\partial_\rho A_\rho-\frac{A_\rho}{\rho}\right)=0.
\end{aligned}
\tag{5.45}
$$

Again, these are precisely the field equations (2.8) and (2.9) with $\lambda=-1$.

Finally, we note that we could also have started from the $(2,2)$ line element

$$
ds^2=dU dV-dw d\bar{w}
\tag{5.46}
$$

instead. Then, using the coordinate system (5.37), we get

$$
ds^2=d\tau^2+dv^2-d\rho^2-\rho^2 d\phi^2\,.
\tag{5.47}
$$

The self-duality condition (5.10) with $\epsilon_{\tau\rho v\phi}=\sqrt{-g}$ yields the set of equations (5.11)–(5.13). Then, specialising to static solutions of the form (5.44) results again in the field equations (2.8) and (2.9) with $\lambda=-1$.

In the following, we specialise to gravitational solutions belonging either to class I or to class II.

5.2.1 Class I

Solutions in class I satisfy (2.17). Inserting (2.17) into either (5.41) or (5.44) gives

$$
\mathcal{A}_{\bar{w}}(\rho,v,\phi)=\partial_{\bar{w}}f_i(\rho,v)\,T^i\,,\quad \mathcal{A}_V(\rho,v,\phi)=\partial_V f_i(\rho,v)\,T^i.
\tag{5.48}
$$

Inserting this into the self-dual Yang-Mills equations (5.20) gives

$$
\left(\partial_U\partial_V-\partial_w\partial_{\bar{w}}\right)f_i=0\,,
\tag{5.49}
$$

in accordance with (2.18).

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5.3 Class II

Let us consider solutions of the form (5.44). Using (2.20) as well as Ψ defined in (2.26), we obtain

$$\mathcal{A}_{\bar{w}} = i\partial_U \Psi \,, \quad \mathcal{A}_V = i\partial_w \Psi \,, \tag{5.50}$$

where we used (5.43). Inserting (5.50) into (5.20) and performing the rescaling (5.24) gives

$$(\partial_U \partial_V - \partial_w \partial_{\bar{w}})\, \Psi + i[\partial_w \Psi, \partial_U \Psi] = 0, \tag{5.51}$$

which, defining [21, 35]

$$\begin{aligned} \{f, g\} &= \partial_w f \partial_U g - \partial_U f \partial_w g = \varepsilon^{\alpha\beta} \partial_\alpha f \partial_\beta g \,, \quad \alpha, \beta = w, U \\ [\{f, g\}] &= \varepsilon^{\alpha\beta} [\partial_\alpha f, \partial_\beta g] = [\partial_w f, \partial_U g] - [\partial_U f, \partial_w g], \end{aligned} \tag{5.52}$$

and using

$$[\partial_w \Psi, \partial_U \Psi] = \frac{1}{2} [\{\Psi, \Psi\}], \tag{5.53}$$

can also be written as

$$(\partial_U \partial_V - \partial_w \partial_{\bar{w}})\, \Psi + \frac{i}{2} [\{\Psi, \Psi\}] = 0. \tag{5.54}$$

On the other hand, for solutions of the form (5.41), for which the coordinates $(U, V, w, \bar{w})$ are all real, we define Ψ to be

$$\Psi(\rho, v, \phi) = -\frac{1}{2}\, e^{-\phi}\, P(\rho, v)\,. \tag{5.55}$$

Then, using (2.20), together with (5.55) and (5.41), we have

$$\mathcal{A}_{\bar{w}} = -\partial_U \Psi \,, \quad \mathcal{A}_V = -\partial_w \Psi \,. \tag{5.56}$$

Inserting these into (5.20) and performing the rescaling (5.24), we get

$$(\partial_U \partial_V - \partial_w \partial_{\bar{w}})\, \Psi - \frac{1}{2} [\{\Psi, \Psi\}] = 0\,. \tag{5.57}$$

6 Single copy prescription

Let us consider gravitational solutions in four dimensions that admit both a class I and a class II description. We remark that one could set up the single copy prescription using only a class I description, but additionally putting the solution in class II allows us to directly recover the scalar version of the self-dual Yang-Mills solution, cf. (6.5).

Let us assume that these solutions depend on certain parameters, and that when switching off these parameters the solution describes a flat space-time solution, with all other fields vanishing. This, however, does not necessarily imply that the gravitational 1-form $A = M^{-1} dM$ vanishes when switching off these parameters. An example thereof is provided by the Eguchi-Hanson solution. We will therefore have to distinguish between two cases: either A vanishes or it does not vanish when switching off these parameters. In the following, we will first consider the case when A vanishes in this limit. The other case will be discussed in section 6.3.

Let us consider working to first order in one of the parameters, which we call m. We assume that when m vanishes, A also vanishes. Then the gravitational 1-form $A = M^{-1}dM$ is of order m, whereas the field ψ in the line element (2.4) is order m^2 in view of (2.5). Therefore, at first order in m, all the information about the perturbed solution is encoded in the gravitational 1-form $A = M^{-1}dM$.

Let us assume that to first order in m, the gravitational 1-form A takes the form

$$A = df_i\, C_i\,, \tag{6.1}$$

where the C_i are constant commuting matrices and where the $f_i = f_i(\rho, v)$ satisfy (2.18). Further, let us assume that A can be brought to the form (2.20), with P given by $P(\rho, v) = p_i(\rho, v)\, C_i$ and satisfying (2.25),

$$A_\rho = -\frac{\lambda}{2}\partial_v p_i(\rho, v)\, C_i\,, \quad A_v = \frac{1}{2}\left(\partial_\rho + \frac{1}{\rho}\right) p_i(\rho, v)\, C_i\,. \tag{6.2}$$

This is the case in all the examples discussed in this paper, see section 3.

To such a gravitational solution we would now like to associate a solution of the self-dual Yang-Mills equations, as described in the previous section, which is the single copy of (6.1). Since the associated Yang-Mills solution is self-dual, whereas the gravitational solution is in general not, to match with the single copy literature we will have to discard half of the content of the former to obtain the single copy of the gravitational solution.

We will first consider the case $\lambda = 1$ and subsequently discuss the case $\lambda = -1$.

6.1 $\lambda = 1$

We introduce the coordinates $(U, V, w, \bar{w})$ given in (5.1). In these coordinates, we introduce the operators[4]

$$(\hat{\Pi}_U, \hat{\Pi}_V, \hat{\Pi}_w, \hat{\Pi}_{\bar{w}}) = (0, \partial_w, 0, \partial_U)\,, \tag{6.3}$$

which satisfy

$$\hat{\Pi} \cdot \hat{\Pi} = 0\,, \quad \hat{\Pi} \cdot \partial = 0\,, \tag{6.4}$$

where the inner product is taken with respect to the flat metric in (5.3). Then, we may associate to (6.1) the following static Yang-Mills solution (cf. (5.31))

$$\mathcal{A}_\mu = \hat{\Pi}_\mu \Psi\,, \quad \mu = U, V, w, \bar{w}\,, \tag{6.5}$$

where Ψ is determined in terms of P via (2.26).

Now recall that given a solution (6.1), we may construct a solution $A = dG_i\, C_i$, where the functions G_i are constructed as in (2.31) by integration over v, and a solution $H = H_i\, C_i$,

[4]The operators $\hat{\Pi}$ have also appeared in [21, 36–38].

where the functions H_i are constructed as in (2.34) by integration over v. Continuing this procedure, we may in principle integrate G_i over v to produce yet another solution, and so forth. To each of these gravitational solutions we then may associate a Yang-Mills solution as described above. Which of these Yang-Mills solutions will correspond to the single copy associated with the gravitational solution (6.1)?

Our proposal for the single copy associated with the gravitational solution (6.1) is as follows. Rather than starting from the gravitational solution (6.1), we instead start from the associated gravitational solution $A = (dG_i)\, C_i$. This is motivated by physical considerations. Even though we have not presented the sources explicitly, we have used them as a guide for a physically reasonable single copy: the guiding principle is that we would like to map sources of the same type, i.e. monopole to monopole, dipole to dipole etc.

We construct the static Yang-Mills solution in two steps. In the first step we write the vectors

$$A_U = A_w = 0\,, \tag{6.6}$$

and for class I

$$\begin{aligned} A_{\bar{w}} &= \frac{1}{2}\, e^{i\phi}\, \partial_\rho G_i(\rho, v)\, C_i\,, \\ A_V &= \frac{1}{2}\, \partial_v G_i(\rho, v)\, C_i\,, \end{aligned} \tag{6.7}$$

alternatively for class II

$$\begin{aligned} A_{\bar{w}} = &\qquad\qquad -\frac{1}{2}\partial_v H = \;\; \partial_U H\,, \\ A_V = &\;\; \frac{1}{2}\, e^{-i\phi} \left(\partial_\rho + \frac{1}{\rho}\right) H = \;\; \partial_w H\,. \end{aligned} \tag{6.8}$$

Note that (6.8) is automatically of the form (6.5).

In the above, to get to (6.7), we used the uplift (5.26). One can map from class I to class II by an application of (2.37), and this gives the first equality in (6.8). The second equality in (6.8) is reached via the coordinate change (5.5), (5.6). We note that, using (2.37), (2.29) and $f_i = \partial_v G_i$ we can alternatively write the non-vanishing gauge fields in terms of the original functions describing class I and II solutions, as:

$$\begin{aligned} A_{\bar{w}} &= -\frac{1}{4}\, e^{i\phi}\, p_i(\rho, v)\, C_i\,, \\ A_V &= \frac{1}{2}\, f_i(\rho, v) C_i\,. \end{aligned} \tag{6.9}$$

For the non-vanishing components of the field strength we obtain, making use of (6.8), (6.9) and (2.37)

$$\begin{aligned} F_{UV} = F_{w\bar{w}} = -\frac{1}{2}\partial_v A_V = -\frac{1}{4}\partial_v f_i\, C_i = -\frac{1}{8}\left(\partial_\rho + \frac{1}{\rho}\right) p_i\, C_i\,, \\ F_{U\bar{w}} = -\frac{1}{2}\partial_v A_{\bar{w}} = -\frac{1}{4} e^{i\phi} \partial_\rho f_i\, C_i = \frac{1}{8} e^{i\phi}\, \partial_v p_i\, C_i\,, \\ F_{Vw} = -\partial_w A_V = -\frac{1}{4} e^{-i\phi} \partial_\rho f_i\, C_i = \frac{1}{8} e^{-i\phi}\, \partial_v p_i\, C_i\,. \end{aligned} \tag{6.10}$$

Note that the field strength components are given in terms of single derivatives of f_i (or p_i). We now proceed to the second step, and promote the vectors in (6.6) and (6.8), to proper Yang-Mills (YM) vectors via

$$A_\mu = A_{\mu i} C_i \quad \rightarrow \quad \mathcal{A}_\mu = \mathcal{A}_{\mu a} T_a\,, \tag{6.11}$$

where T_a are the generators of some non-abelian gauge algebra $\mathfrak{g}$, such that C_i are equal to a commuting subset of the T_a's. We see that, unlike in the usual single copy story, this places certain restrictions of the type and size of the YM gauge group. This will turn out to have interesting consequences, as we will detail below. We can make the map in (6.11) precise via

$$\mathcal{A}_{\mu a} = \begin{cases} A_{\mu i}, & a = i \\ 0, & \text{otherwise} \end{cases} \tag{6.12}$$

Thus we have 'enlarged' the matrix-valued vectors in (6.6) and (6.8) to proper Yang-Mills vectors.

As an aside, we note that under the procedure above, we also have

$$H = H_i C_i \quad \rightarrow \quad \mathcal{H} = \mathcal{H}_a T_a \tag{6.13}$$

with $\mathcal{H}_a$ constructed from H_i as in (6.12). We then see that the role of Ψ in (6.5) is played by our integrated function H_i, and thus we can think of H_i as a 'seed' for both the double and 0th copy procedure, in the self-dual sector:

$$\mathcal{H}^{(0)} \xleftarrow{\text{0th copy}} \mathcal{A}_\mu = \hat{\Pi}_\mu \mathcal{H} \xrightarrow{\text{double copy}} h_{\mu\nu} = \hat{\Pi}_\mu \hat{\Pi}_\nu \mathcal{H}^{(2)}\,, \tag{6.14}$$

where $\mathcal{H}^{(2)}$ is a function, and $\mathcal{H}^{(0)} = \mathcal{H}^{(0)}_{a\tilde{a}} T_a \tilde{T}_{\tilde{a}}$ is a bi-adjoint scalar.

All the examples of gravitational solutions with $\lambda = 1$ discussed in section 3 have both a class I and a class II description, and to each of them we associated functions G_i and H_i. We may therefore associate a single copy to each of these gravitational solutions using (6.8) and (6.11). As an illustrative example, let us consider the example in (2.38), based on a single function f, for which we obtain using (5.1),

$$\begin{aligned}
\mathcal{F}_{\tau v a} &= 2\mathcal{F}_{UVa} = -m\,\frac{v}{(\rho^2+v^2)^{3/2}}\,T_a\,, \\
\mathcal{F}_{\tau x a} &= \mathcal{F}_{U\bar{w}a} + \mathcal{F}_{Vwa} = -m\,\cos\phi\,\frac{\rho}{(\rho^2+v^2)^{3/2}}\,T_a\,, \\
\mathcal{F}_{\tau y a} &= -i\left(\mathcal{F}_{U\bar{w}a} - \mathcal{F}_{Vwa}\right) = -m\,\sin\phi\,\frac{\rho}{(\rho^2+v^2)^{3/2}}\,T_a
\end{aligned} \tag{6.15}$$

which, when converting to spherical coordinates using $v = r\cos\theta$, $\rho = r\sin\theta$, gives

$$\begin{aligned}
\mathcal{F}_{\tau v a} &= -m\,\frac{\cos\theta}{r^2}\,T_a\,, \\
\mathcal{F}_{\tau x a} &= -m\,\frac{\cos\phi\,\sin\theta}{r^2}\,T_a\,, \\
\mathcal{F}_{\tau y a} &= -m\,\frac{\sin\phi\,\sin\theta}{r^2}\,T_a\,,
\end{aligned} \tag{6.16}$$

which describes the electric field produced by a point charge. Since the solution is self-dual, there is also an associated magnetic monopole field. If we discard the magnetic monopole solution, then this describes the single copy associated to the gravitational solution (2.38) discussed in [16].

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6.1.1 The nontrivial role of the gauge group

Let us compare our single copy prescription with what is done in the context of the double copy. In the double copy framework, a gravitational solution is constructed as a "product" of two gauge theory solutions. Depending on the complexity of the gravity solution, the two gauge theory factors might be identical (e.g. for Schwarzschild, in which case we have a unique single copy) or different (this is often the case for gravity solutions supported by additional fields). For example the black hole in section 3.1.3 is supported by a Maxwell field. Thus it will arise from an asymmetric double copy, where one of the factors is a pure Yang-Mills theory, and the other is Yang-Mills coupled with a scalar.[5] Generally in the double copy procedure, the gauge algebra $\mathfrak{g}$ is unspecified, and independent of the specific kinematics or profile of the solutions considered.

Our single copy proposal is drastically different, in that we have a unique single copy for which the (non-compact) Yang-Mills gauge group is determined by the Breitenlohner-Maison procedure itself. Thus various aspects of the gravitational solution will be encoded on the Yang-Mills side in the Yang-Mills gauge group generators supporting the solution.

Let us summarise a few interesting cases below, all of which share the kinematics described in (6.15) and (6.16), i.e. they are all of the form

$$A = df\, C \tag{6.17}$$

with the same

$$f = -2\frac{m}{\sqrt{\rho^2 + v^2}}\,, \tag{6.18}$$

but with distinct gauge groups (i.e. different C's), encoding additional information about the gravity solution.

- Exterior region of the Schwarzschild black hole, linearised in m (see section 3.1.1 for details). The Breitenlohner-Maison procedure dictates that the vector field is valued in $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The matrix M is given by

$$M^{(S)} = \begin{pmatrix} 1+f & 0 \\ 0 & 1-f \end{pmatrix}. \tag{6.19}$$

Then for the single copy procedure we will construct $A = M^{-1}dM = df C$, and we will take the gauge group to be

$$G^{(S)}_{\mathrm{YM}} = \mathrm{SL}(2,\mathbb{R})\,. \tag{6.20}$$

We have a single non-vanishing component in (6.12) with the corresponding generator given by

$$T^{(S)}_1 \equiv C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.21}$$

[5]For some explicit examples of asymmetric double copies for black hole solutions see [39, 40].

Then, taking into account (6.6) and (6.8), the YM gauge field is, explicitly

$$\begin{aligned} \mathcal{A}_U &= 0\,, \\ \mathcal{A}_V &= \frac{1}{2} f\, T_1^{(S)}\,, \\ \mathcal{A}_w &= 0\,, \\ \mathcal{A}_{\bar{w}} &= \frac{1}{2}\, e^{i\phi}\, \partial_\rho G\, T_1^{(S)}\,, \end{aligned} \tag{6.22}$$

with f as (6.18), and G obtained by integration from f as in (2.31) and (2.32) (recall that the coordinates $(U, V, w, \bar{w})$ are related to the coordinates (ρ, v, ϕ) through (5.1) and (5.4)).

- Black hole in $AdS_2 \times S^2$ (see section 3.1.3). This can be obtained from the Schwarzschild solution in two steps. First we embed $M^{(S)}$ into $G/H = \mathrm{SU}(2,1)/(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1))$ via

$$M^{(emb)} = \begin{pmatrix} 1-f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+f \end{pmatrix}. \tag{6.23}$$

Then, we obtain the matrix M for a black hole in $AdS_2 \times S^2$, namely

$$\tilde{M} = g^{\natural}(c) M g(c)\,, \tag{6.24}$$

where

$$g(c) = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}\bar{c} & 1 & 0 \\ |c|^2 & -\sqrt{2}c & 1 \end{pmatrix}, \quad c \in \mathbb{C}\,, \tag{6.25}$$

and $g^{\natural}(c) = \eta g(c)^{\dagger} \eta^{-1}$ with $\eta = \mathrm{diag}(1, -1, 1)$. Then we have

$$\tilde{M} = \begin{pmatrix} \left(|c|^2-1\right)^2 - f\left(1-|c|^4\right) & \sqrt{2}c\left(1-|c|^2\,(1+f)\right) & |c|^2\,(1+f) \\ \sqrt{2}\bar{c}\left(|c|^2\,(1+f)-1\right) & 1-2|c|^2\,(1+f) & \sqrt{2}\bar{c}\,(1+f) \\ |c|^2\,(1+f) & -\sqrt{2}c\,(1+f) & 1+f \end{pmatrix}. \tag{6.26}$$

Again we construct $A = \tilde{M}^{-1} d\tilde{M} = df\tilde{C}$, obtaining

$$\tilde{C} = -\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}\bar{c} & 0 & 0 \\ 0 & \sqrt{2}c & -1 \end{pmatrix} \tag{6.27}$$

and we will take the gauge group to be

$$G_{\mathrm{YM}}^{(AdS_2\times S^2)} = \mathrm{SU}(2,1)\,. \tag{6.28}$$

We have

$$\tilde{C} = -H_2 - \sqrt{2}\,(\mathrm{Re}\, c)\;(F_1 + F_2) + \sqrt{2}\,(\mathrm{Im}\, c)\; i(F_2 - F_1)\,, \tag{6.29}$$

where $H_2, F_1 + F_2, i(F_2 - F_1)$ denote a subset of generators of $\mathrm{SU}(2,1)$ given in (A.6) and (A.12) in [19]. Then we identify our gauge group via

$$T_1^{(AdS_2\times S^2)} \equiv H_2, \qquad T_2^{(AdS_2\times S^2)} \equiv F_1 + F_2, \qquad T_3^{(AdS_2\times S^2)} \equiv i(F_2 - F_1) \tag{6.30}$$

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and the YM gauge field components are given by

$$\begin{aligned}
\mathcal{A}_U &= 0\,, \\
\mathcal{A}_V &= \frac{1}{2} f \left(-T_1^{(AdS_2\times S^2)} - \sqrt{2}\,(\mathrm{Re}\,c)\; T_2^{(AdS_2\times S^2)} + \sqrt{2}\,(\mathrm{Im}\,c)\; T_3^{(AdS_2\times S^2)} \right) \,, \\
\mathcal{A}_w &= 0\,, \\
\mathcal{A}_{\bar{w}} &= \frac{1}{2}\, e^{i\phi}\, \partial_\rho G \left(-T_1^{(AdS_2\times S^2)} - \sqrt{2}\,(\mathrm{Re}\,c)\; T_2^{(AdS_2\times S^2)} + \sqrt{2}\,(\mathrm{Im}\,c)\; T_3^{(AdS_2\times S^2)} \right) \,,
\end{aligned} \tag{6.31}$$

where f and G are the same as in (6.22). Thus we already see that the gravitational information is not lost when taking the single copy, but rather repackaged in the matrices associated with generators of the gauge group.

Finally, we remark that the YM generators associated to the black hole in $AdS_2 \times S^2$ can be constructed via a global non-abelian transformation generated by the very same matrix that gave us the Harrison transformation. Explicitly, the embedded 1-form A is given by

$$A^{(emb)} = \left[M^{(emb)} \right]^{-1} dM^{(emb)} = df C^{(emb)}, \quad C^{(emb)} = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6.32}$$

and indeed the gauge field is obtained via

$$\mathcal{A}_\mu^{(AdS_2\times S^2)} = g(c)^{-1} \mathcal{A}_\mu^{(emb)} g(c)\,. \tag{6.33}$$

- Self-dual Taub-NUT (see section 3.1.4). The matrix $M(\rho, v)$ belonging to a complexification of $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is given by

$$M^{(TN)} = \begin{pmatrix} -(\alpha^2-1) + (\alpha-1)^2 f & i\alpha + i(1-\alpha) f \\ i\alpha + i(1-\alpha) f & 1-f \end{pmatrix} \,, \quad \alpha \in \mathbb{C}\,. \tag{6.34}$$

We have $A^{(TN)} = \left[M^{(TN)} \right]^{-1} dM^{(TN)} = df\, C^{(TN)}$, with

$$C^{(TN)} = \begin{pmatrix} 1-\alpha & i \\ i(1-\alpha)^2 & \alpha - 1 \end{pmatrix} \,. \tag{6.35}$$

We then pick the gauge group to be

$$G_{\mathrm{YM}}^{(TN)} = \mathrm{SL}(2,\mathbb{C})\,. \tag{6.36}$$

A set of Lie algebra generators for $\mathrm{SL}(2,\mathbb{C})$ is given by

$$T_1^{(TN)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2^{(TN)} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad T_3^{(TN)} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \,, \tag{6.37}$$

such that

$$C^{(TN)} = (1-\alpha)\, T_1^{(TN)} + T_2^{(TN)} + (1-\alpha)^2\, T_3^{(TN)} \tag{6.38}$$

and thus we have

$$\begin{aligned}
&\mathcal{A}_U = 0\,,\\
&\mathcal{A}_V = \frac{1}{2} f\left((1-\alpha)\, T_1^{(TN)} + T_2^{(TN)} + (1-\alpha)^2\, T_3^{(TN)}\right)\,,\\
&\mathcal{A}_w = 0\,,\\
&\mathcal{A}_{\bar w} = \frac{1}{2}\, e^{i\phi}\, \partial_\rho G\left((1-\alpha)\, T_1^{(TN)} + T_2^{(TN)} + (1-\alpha)^2\, T_3^{(TN)}\right)\,,
\end{aligned} \tag{6.39}$$

again with the same f and G as in (6.22) and (6.31).

6.1.2 Solutions with multiple generators

An example of the form (6.1) based on two commuting matrices C_1 and C_2 is provided by the Kaluza-Klein black hole described in section 3.1.6. Its single copy description is then based on two functions G_1 and G_2 (respectively H_1 and H_2) with

$$\mathcal{A}_V = \frac{1}{2}\left(f_1(\rho, v)\, C_1 + f_2(\rho, v)\, C_2\right)\,, \tag{6.40}$$

where f_1 and f_2 are given in (3.66).

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\mathcal{B}}{\mathcal{D}} & 0 & 0 \\ 0 & \frac{\mathcal{B}}{\mathcal{C}} & 0 \end{pmatrix}\,, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2\mathcal{A}\mathcal{D}+\mathcal{B}^2}{2\mathcal{C}\mathcal{D}} & 0 & 0 \end{pmatrix}\,. \tag{6.41}$$

Note that this example does not require an expansion in a parameter m, since the field e^{ψ} in the line element (2.4) satisfies $e^{\psi} = 1$ [19]. The coset is $G/H = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$, so following the prescription from the previous subsections, we define

$$G_{\mathrm{YM}}^{(KK)} = \mathrm{SL}(3,\mathbb{R})\,. \tag{6.42}$$

The following subset of Lie algebra generators of $\mathrm{SL}(3,\mathbb{R})$ will be relevant

$$T_1^{(KK)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad T_2^{(KK)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\,, \quad T_3^{(KK)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,. \tag{6.43}$$

We can write C_1 and C_2 in terms of the above as

$$C_1 = -\frac{\mathcal{B}}{\mathcal{D}} T_1^{(KK)} - \frac{\mathcal{B}}{\mathcal{C}} T_2^{(KK)}, \quad C_2 = -\frac{2\mathcal{A}\mathcal{D}+\mathcal{B}^2}{2\mathcal{C}\mathcal{D}} T_3^{(KK)} \tag{6.44}$$

and then the components of the gauge field are given explicitly as

$$\begin{aligned}
&\mathcal{A}_U = 0\,,\\
&\mathcal{A}_V = -\frac{1}{2} f_1\left(\frac{\mathcal{B}}{\mathcal{D}} T_1^{(KK)} + \frac{\mathcal{B}}{\mathcal{C}} T_2^{(KK)}\right) - \frac{1}{2} f_2 \frac{2\mathcal{A}\mathcal{D}+\mathcal{B}^2}{2\mathcal{C}\mathcal{D}} T_3^{(KK)}\,,\\
&\mathcal{A}_w = 0\,,\\
&\mathcal{A}_{\bar w} = -\frac{1}{2}\, e^{i\phi}\, \partial_\rho G_1\left(\frac{\mathcal{B}}{\mathcal{D}} T_1^{(KK)} + \frac{\mathcal{B}}{\mathcal{C}} T_2^{(KK)}\right) - \frac{1}{2}\, e^{i\phi}\, \partial_\rho G_2 \frac{2\mathcal{A}\mathcal{D}+\mathcal{B}^2}{2\mathcal{C}\mathcal{D}} T_3^{(KK)}\,,
\end{aligned} \tag{6.45}$$

with f_1, f_2 given in (3.66) and G_1, G_2 given in (3.67).

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6.2 $\lambda = -1$

For $\lambda = -1$, the procedure is completely analogous to the one for $\lambda = 1$. In particular (6.6) remains the same, whereas (6.8) and (6.10) need to be slightly modified to account for the new coordinate systems. We have two possibilities: we may consider either the coordinate system (5.36) or (5.37). Let us first consider the latter. Then, using (5.44), the first step in our single copy prescription is

$$\begin{aligned} A_U &= A_w = 0\,, \\ A_{\bar{w}} &= \partial_{\bar{w}} G_i\, C_i = \frac{1}{2}\, e^{i\phi}\, \partial_\rho G_i(\rho, v)\, C_i = i\partial_U H = \frac{1}{4}\, e^{i\phi}\, p_i(\rho, v)\, C_i\,, \\ A_V &= \partial_V G_i\, C_i = \frac{i}{2}\, f_i(\rho, v)\, C_i = i\partial_w H = \frac{i}{2}\, e^{-i\phi} \left(\partial_\rho + \frac{1}{\rho}\right) H\,. \end{aligned} \tag{6.46}$$

For the non-vanishing components of the field strength we obtain

$$\begin{aligned} F_{UV} &= F_{w\bar{w}} = -\frac{i}{2}\partial_v A_V = \frac{1}{4}\partial_v f_i\, C_i = \frac{1}{8}\left(\partial_\rho + \frac{1}{\rho}\right) p_i\, C_i\,, \\ F_{U\bar{w}} &= -\frac{i}{2}\partial_v A_{\bar{w}} = -\frac{i}{4} e^{i\phi} \partial_\rho f_i\, C_i = -\frac{i}{8} e^{i\phi}\, \partial_v p_i\, C_i\,, \\ F_{Vw} &= -\partial_w A_V = -\frac{i}{4} e^{-i\phi} \partial_\rho f_i\, C_i = -\frac{i}{8} e^{-i\phi}\, \partial_v p_i\, C_i\,. \end{aligned} \tag{6.47}$$

Note that the field strength components are given in terms of single derivatives of f_i (or p_i). In the second step we promote the above to YM fields, following the same procedure as for $\lambda = 1$, see eq. (6.11) and (6.12).

Using this prescription, we may associate a single copy to the Einstein-Rosen wave solution (3.69). The Breitenlohner-Maison procedure dictates that the vector field is valued in $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The matrix M is given by

$$M^{(ER)} = \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix} \tag{6.48}$$

with

$$f(\rho, v) = 2\cos(kv)\, J_0(k\rho) - 2\,, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{6.49}$$

Then for the single copy procedure we will construct $A = M^{-1}dM = df C$, and we will take the gauge group to be

$$G_{\mathrm{YM}}^{(ER)} = \mathrm{SL}(2,\mathbb{R})\,. \tag{6.50}$$

We have a single non-vanishing component in (6.12), with the corresponding generator given by

$$T_1^{(ER)} \equiv C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{6.51}$$

i.e. the YM gauge field is, explicitly

$$\begin{aligned}
\mathcal{A}_U &= 0\,, \\
\mathcal{A}_V &= \frac{i}{2} f\; T_1^{(ER)}\,, \\
\mathcal{A}_w &= 0\,, \\
\mathcal{A}_{\bar{w}} &= \frac{1}{2}\, e^{i\phi}\, \partial_\rho G\; T_1^{(ER)}\,.
\end{aligned} \tag{6.52}$$

To lowest order in k, f and G are given by (3.72).

As a further example, let us consider the Kleinian Taub-NUT solution[6] described in section 3.1.8. The coset is a complexification of $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and the Breitenlohner-Maison vector is given by

$$A = df\, C\,, \qquad f(\rho, v) = -\frac{2m}{\sqrt{v^2-\rho^2}}\,, \qquad C = \begin{pmatrix} 1+\alpha & -i \\ -i(1+\alpha)^2 & -(1+\alpha) \end{pmatrix}\,, \qquad C^2 = 0\,. \tag{6.53}$$

We take the gauge group to be

$$G_{\mathrm{YM}}^{(KTN)} = \mathrm{SL}(2,\mathbb{C})\,. \tag{6.54}$$

Let us use the same subset of Lie algebra generators as in (6.37):

$$T_1^{(KTN)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2^{(KTN)} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad T_3^{(KTN)} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}\,. \tag{6.55}$$

Then we have

$$C = (1+\alpha) T_1^{(KTN)} - T_2^{(KTN)} - (1+\alpha)^2 T_3^{(KTN)}\,, \tag{6.56}$$

so that the YM field components are given by

$$\begin{aligned}
\mathcal{A}_U &= 0\,, \\
\mathcal{A}_V &= \frac{i}{2} f \left((1+\alpha) T_1^{(KTN)} - T_2^{(KTN)} - (1+\alpha)^2 T_3^{(KTN)} \right)\,, \\
\mathcal{A}_w &= 0\,, \\
\mathcal{A}_{\bar{w}} &= \frac{1}{2}\, e^{i\phi}\, \partial_\rho G \left((1+\alpha) T_1^{(KTN)} - T_2^{(KTN)} - (1+\alpha)^2 T_3^{(KTN)} \right)\,.
\end{aligned} \tag{6.57}$$

Finally, let us also consider the coordinate system (5.36). Then, using (5.41), the first step in our single copy prescription is

$$\begin{aligned}
A_U &= A_w = 0\,, \\
A_{\bar{w}} &= \partial_{\bar{w}} G_i\, C_i = -\frac{1}{2}\, e^{-\phi}\, \partial_\rho G_i(\rho, v)\, C_i = -\partial_U H = -\frac{1}{4}\, e^{-\phi}\, p_i(\rho, v)\, C_i\,, \\
A_V &= \partial_V G_i\, C_i = \frac{1}{2}\, f_i(\rho, v)\, C_i = -\partial_w H = -\frac{1}{2}\, e^{\phi} \left(\partial_\rho + \frac{1}{\rho} \right) H\,,
\end{aligned} \tag{6.58}$$

[6]A single copy description of this Kleinian space-time has also been discussed recently in [41, 42] using a different approach based on Kerr-Schild coordinates.

where we used

$$A_{\bar{w}} = -\partial_U H \,, \quad A_V = -\partial_w H \tag{6.59}$$

with

$$H(\rho, v, \phi) = -e^{-\phi} \left(\frac{1}{2} \int_a^v P(\rho, \tilde{v})\, d\tilde{v} + h(\rho) \right) \,, \quad P(\rho, v) = p_i(\rho, v)\, C_i \,. \tag{6.60}$$

For the non-vanishing components of the field strength we obtain

$$\begin{aligned} F_{UV} &= F_{w\bar{w}} = -\frac{1}{2}\partial_v A_V = -\frac{1}{4}\partial_v f_i\, C_i = -\frac{1}{8}\left(\partial_\rho + \frac{1}{\rho}\right) p_i\, C_i \,, \\ F_{U\bar{w}} &= -\frac{1}{2}\partial_v A_{\bar{w}} = \frac{1}{4} e^{-\phi}\partial_\rho f_i\, C_i = \frac{1}{8} e^{-\phi}\, \partial_v p_i\, C_i \,, \\ F_{Vw} &= -\partial_w A_V = -\frac{1}{4} e^{\phi}\partial_\rho f_i\, C_i = -\frac{1}{8} e^{\phi}\, \partial_v p_i\, C_i \,. \end{aligned} \tag{6.61}$$

In the second step we promote the above to YM fields, following the same procedure as for $\lambda = 1$, see eq. (6.11) and (6.12).

6.3 The case of the Eguchi-Hanson metric

In this subsection we consider the Eguchi-Hanson metric (4.27). As already remarked, when switching off the parameter a, the associated matrix 1-form A does not vanish (cf. (4.36)), even though the metric is flat when written in lightcone coordinates, see (6.65). This differs from all the other examples considered in this paper. To deal with this situation, we expand the matrix 1-form A in powers of the parameter a^4,

$$A = A^{(0)} + A^{(1)} + \dots \,. \tag{6.62}$$

We call $A^{(0)}$ the background 1-form, while $A^{(1)}$ denotes its perturbation at first order in the perturbation parameter a^4.

Now we focus on $A^{(1)}$ and invoke the construction given in section D. We express $A^{(1)}$ in terms of the four-dimensional lightcone coordinates (U, V, X, Y) given below, and subsequently we focus on the component set $(0, A_V^{(1)}, 0, A_Y^{(1)})$. We use the four-dimensional lightcone coordinates introduced in [10]. Performing the analytic continuation $\gamma = -i\bar{t}$, we obtain

$$\begin{aligned} U &= r\, \sin\frac{\theta}{2}\, e^{\frac{i}{2}\,\phi - \frac{1}{2}\bar{t}} \,, \\ V &= -r\, \sin\frac{\theta}{2}\, e^{-\frac{i}{2}\,\phi + \frac{1}{2}\bar{t}} \,, \\ X &= -w = -r\, \cos\frac{\theta}{2}\, e^{\frac{i}{2}\,\phi + \frac{1}{2}\bar{t}} \,, \\ Y &= -\bar{w} = -r\, \cos\frac{\theta}{2}\, e^{-\frac{i}{2}\,\phi - \frac{1}{2}\bar{t}} \,, \end{aligned} \tag{6.63}$$

which satisfy

$$r^2 = XY - UV \,. \tag{6.64}$$

In these lightcone coordinates, and to first order in the parameter a^4, the Eguchi-Hanson metric (4.27) takes the form

$$ds_4^2 = -dUdV + dXdY + \frac{a^4}{2}\frac{(VdU - XdY)^2 + (UdV - YdX)^2}{(XY - UV)^3}\,. \tag{6.65}$$

The component set $(0, A_V^{(1)}, 0, A_Y^{(1)})$ is expressed in terms of a matrix function (cf. (D.2)), which in this example is the matrix function $\hat{H}_1$ given in (4.44),

$$A_V^{(1)} = -\partial_X \hat{H}_1\,, \quad A_Y^{(1)} = -\partial_U \hat{H}_1\,. \tag{6.66}$$

Using

$$e^{2i\phi} = -\frac{U\,X}{V\,Y} \Rightarrow e^{i\phi} = \sqrt{-\frac{U\,X}{V\,Y}} \tag{6.67}$$

as well as

$$\begin{aligned}\sqrt{\rho^2 + v^2} &= \frac{1}{4}\left(XY - UV\right)\,,\\ \rho &= \frac{1}{2}\sqrt{-XYUV}\,,\\ v &= \frac{1}{4}\left(XY + UV\right)\,,\end{aligned} \tag{6.68}$$

we get

$$\hat{H}_1(U,V,X,Y) = \frac{ia^4}{48}\left(\frac{1}{YV} + \frac{2}{XY - UV}\frac{U}{Y}\right) C\,. \tag{6.69}$$

The first term, proportional to $1/(YV)$ gets projected out when computing $A_V^{(1)}, A_Y^{(1)}$,

$$\begin{aligned}A_V^{(1)} &= -\partial_X \hat{H}_1 = \frac{ia^4}{24}\frac{U}{(XY - UV)^2}\,C\,,\\ A_Y^{(1)} &= -\partial_U \hat{H}_1 = -\frac{ia^4}{24}\frac{X}{(XY - UV)^2}\,C\,.\end{aligned} \tag{6.70}$$

The associated matrix field strength $F^{(1)}$ reads

$$\begin{aligned}F^{(1)} = {}& \frac{ia^4}{24(UV - XY)^3}\\ &\times\left[-(UV + XY)\left(dU \wedge dV + dX \wedge dY\right) + 2VXdU \wedge dY + 2UYdX \wedge dV\right] C\,.\end{aligned} \tag{6.71}$$

This is the expression for the field strength is the Lorentzian space-time (6.65). To compare with the expressions for the gauge field and its field strength in Euclidean space with line element $ds_4^2 = dUdV - dXdY$, given in eq. (5.9) in [11] and in eq. (114) in [12] respectively, we first interchange U and V in $F^{(1)}$, which corresponds to passing from the self-dual to the anti self-dual sector, cf. appendix C. Subsequently we send $U \to iU, V \to iV, X \to iX, Y \to iY$. The resulting expressions for $A^{(1)}$ and $F^{(1)}$ agree with the expressions given in [11, 12]. The field strength $F^{(1)}$ can be promoted to a field strength of a non-abelian Yang-Mills gauge group as in (6.11).

7 Conclusions

In this paper we discussed the mapping of the integrable sector of certain gravitational theories in $D \geq 4$ to the integrable sector of Yang-Mills theories in four dimensions. This mapping is formulated in terms of Weyl coordinates, since these are the natural coordinates used in the Lax pair description of the integrable sector of these gravitational theories. The gauge group on the Yang-Mills side is specified by the dimensional reduction performed on the gravitational side. Let us summarize our results.

We considered two classes of solutions (class I and class II) to the two-dimensional PDEs (1.1). These PDEs arise by dimensional reduction of the field equations of certain gravitational theories in $D \geq 4$ dimensions, and therefore solutions belonging to these two classes correspond to solutions of the gravitational field equations in D space-time dimensions. The two-dimensional PDEs (1.1) are the compatibility conditions for a Lax pair, the Breitenlohner- Maison linear system. This Lax pair can be described by a pair of differential operators $(\mathcal{L}, \mathcal{M})$ in two dimensions, which have been given in section 2.

The PDEs (1.1) also arise as compatibility conditions for certain Lax pairs in three and in four dimensions, see appendix A. That there is a Lax pair in four dimensions giving rise to (1.1) is not surprising, since it is known [8, 9] that a certain dimensional reduction of the self-duality equations for Yang-Mills fields in four dimensions gives rise to the two-dimensional PDEs (1.1). Solutions to the PDEs (1.1) can thus be regarded as either describing gravitational solutions in $D \geq 4$ or describing solutions of the self-dual Yang-Mills equations in four dimensions.

The single copy prescription associates a Yang-Mills solution to a gravitational solution. To ensure that this mapping is physically meaningful, we demand that the nature of the sources of both types of solutions be the same. Our single copy prescription is as follows. We focus on gravitational solutions that have both a class I and a class II description. These gravitational solutions are constructed from the 1-form A which we assume to depend on a parameter which, when set to zero, gives rise to a background solution $A^{(0)}$. Subsequently we work at first order in this parameter. By expanding A in powers of this parameter, $A = A^{(0)} + A^{(1)} + \ldots$, we obtain the first-order perturbation $A^{(1)}$. Then, given $A^{(1)}$, we construct a new gravitational solution by integrating over the Weyl coordinate v, as depicted in table 3 for the case $\lambda = 1$ and in table 4 for the case $\lambda = -1$. We then use this new solution as a device for constructing the single copy $\mathcal{A}$ of the original gravitational solution, that is, for ensuring that the nature of the sources supporting the gravitational and the Yang-Mills solutions is the same (monopole to monopole, etc.). We illustrated our single copy construction in various examples and confirmed that it correctly reproduces the single copy for various gravitational solutions discussed in the literature. In all but one of the examples the background $A^{(0)}$ vanishes and we integrated once over v. The exception is the Eguchi-Hanson solution, which we discussed in section 6.3: its background $A^{(0)}$ is non-vanishing, and the single copy requires integrating twice over v.

$\lambda = 1$	$A^{(1)}$		signature $(1,3)$: $\mathcal{A}$
Class I	$df\,C$	$G(\rho,v) = \int_a^v f(\rho,\tilde{v})\,d\tilde{v} + g(\rho)$	$\mathcal{A}_U = \mathcal{A}_w = 0,$ $\mathcal{A}_{\bar{w}} = \partial_{\bar{w}} G\,C,\ \mathcal{A}_V = \partial_V G\,C$
Class II	$-\frac{1}{2} \star dP + \frac{P}{2\rho}\,dv$	$H(\rho,v,\phi) = \frac{1}{2} e^{i\phi} \left(\int_a^v P(\rho,\tilde{v})\,d\tilde{v} + h(\rho)\right)$	$\mathcal{A}_U = \mathcal{A}_w = 0,$ $\mathcal{A}_{\bar{w}} = \partial_U H,\ \mathcal{A}_V = \partial_w H$

Table 3. Single copy construction for $\lambda = 1$. The coordinates $(U, V, w, \bar{w})$ are given in (5.1) and (5.4). The Hodge star $\star$ is defined in (2.2).

$\lambda = -1$	$A^{(1)}$		signature $(2,2)$: $\mathcal{A}$
Class I	$df\,C$	$G(\rho,v) = \int_a^v f(\rho,\tilde{v})\,d\tilde{v} + g(\rho)$	$\mathcal{A}_U = \mathcal{A}_w = 0,$ $\mathcal{A}_{\bar{w}} = \partial_{\bar{w}} G\,C,\ \mathcal{A}_V = \partial_V G\,C$
Class II	$\frac{1}{2} \star dP + \frac{P}{2\rho}\,dv$	$H(\rho,v,\phi) = \frac{1}{2} e^{i\phi} \left(\int_a^v P(\rho,\tilde{v})\,d\tilde{v} + h(\rho)\right)$	$\mathcal{A}_U = \mathcal{A}_w = 0,$ $\mathcal{A}_{\bar{w}} = i\partial_U H,\ \mathcal{A}_V = i\partial_w H$
Class II	$\frac{1}{2} \star dP + \frac{P}{2\rho}\,dv$	$H(\rho,v,\phi) = -e^{-\phi} \left(\frac{1}{2} \int_a^v P(\rho,\tilde{v})\,d\tilde{v} + h(\rho)\right)$	$\mathcal{A}_U = \mathcal{A}_w = 0,$ $\mathcal{A}_{\bar{w}} = -\partial_U H,\ \mathcal{A}_V = -\partial_w H$

Table 4. Single copy construction for $\lambda = -1$. The coordinates $(U, V, w, \bar{w})$ are given in (5.37) and in (5.36). The Hodge star $\star$ is defined in (2.2).

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A Various Lax pairs

The PDEs (2.1) can be viewed as being the compatibility conditions for a Lax pair of differential operators. Here we discuss 4 such pairs.

The first Lax pair (L, M), formulated in $D = 4$ in coordinates $(U, V, w, \bar{w})$, is given in (5.19),

$$L = \partial_U - \Omega\left(\partial_{\bar{w}} + \mathcal{A}_{\bar{w}}\right), \quad M = \partial_w + \Omega\left(\partial_V + \mathcal{A}_V\right), \tag{A.1}$$

with $\mathcal{A}_{\bar{w}}, \mathcal{A}_V$ expressed in terms of A_ρ, A_v as in (5.26) when $\lambda = 1$, and as in (5.41) and in (5.44) when $\lambda = -1$. Here, Ω is a constant spectral parameter.

The second Lax pair $(\hat{L}, \hat{M})$ is formulated in $D = 3$ in coordinates (ρ, v, ϕ), and again uses a constant spectral parameter Ω,

$$\begin{aligned} \hat{L} &= -e^{-2i\phi}\,\partial_v + \frac{1}{2}\rho\,\Omega\left(D_\rho + \frac{i}{\rho}\,\partial_\phi\right), \\ \hat{M} &= e^{-i\phi}\,\lambda\,\partial_\rho + \frac{1}{2}\rho\,\Omega\,e^{i\phi}\,D_v\,, \quad \lambda = \pm 1, \end{aligned} \tag{A.2}$$

where the covariant derivatives D are given by

$$\begin{aligned} D_\rho &= \partial_\rho + A_\rho, \\ D_v &= \partial_v + A_v, \\ [D_\rho, D_v] &= F_{\rho v}. \end{aligned} \tag{A.3}$$

Note the presence of the factor ρ multiplying the constant spectral parameter Ω. The compatibility condition

$$\hat{L}Y = 0 \,\wedge\, \hat{M}Y = 0 \Longrightarrow [\hat{L}, \hat{M}]Y = 0 \quad \forall \quad Y \in C^2\,, \quad \Omega \in \mathbb{C} \tag{A.4}$$

gives

$$0 = [\hat{L}, \hat{M}]Y = \left[-\frac{1}{2}\rho\,\Omega\,e^{-i\phi}\left(\partial_v A_v + \lambda\,\partial_\rho A_\rho + \frac{\lambda}{\rho}A_\rho\right) + \left(\frac{1}{2}\rho\,\Omega\right)^2 e^{i\phi}\,F_{\rho v}\right]Y. \tag{A.5}$$

Taking Y to be an invertible matrix function and multiplying by Y^{-1} on the right gives (2.8) and (2.9) as vanishing coefficients of this quadratic polynomial in Ω.

The third Lax pair $(\check{L}, \check{M})$ is also formulated in $D = 3$ in coordinates (ρ, v, ϕ), but now it uses the non-constant spectral parameter τ specified by (2.13), which is a function of ρ, v and of $\omega \in \mathbb{C}$ and satisfies

$$\begin{aligned} \partial_v \tau &= \frac{2\lambda\tau^2}{\rho(\tau^2 + \lambda)}, \\ \partial_\rho \tau &= \frac{\tau(\lambda - \tau^2)}{\rho(\tau^2 + \lambda)}. \end{aligned} \tag{A.6}$$

The Lax pair $(\check{L}, \check{M})$ reads

$$\begin{aligned} \check{L} &= e^{i\phi}\left(-\partial_v + \tau\left(D_\rho + \frac{i}{\rho}\,\partial_\phi\right)\right), \\ \check{M} &= e^{-i\phi}\left(\lambda\,\partial_\rho + \tau\,D_v\right), \quad \lambda = \pm 1, \end{aligned} \tag{A.7}$$

where the covariant derivatives D are given by (A.3). The compatibility condition

$$\check{L}Y = 0 \,\wedge\, \check{M}Y = 0 \Longrightarrow [\check{L}, \check{M}]Y = 0 \quad \forall \quad Y \in C^2\,, \quad \Omega \in \mathbb{C} \tag{A.8}$$

yields (2.8) and (2.9), as follows. We compute

$$\begin{aligned} [\check{L}, \check{M}] = &-\tau\left(\partial_v A_v + \lambda\partial_\rho A_\rho + \frac{\lambda}{\rho}A_\rho\right) + \tau^2 F_{\rho v} \\ &+ \frac{\tau}{\rho}\left(\lambda D_\rho + \tau D_v\right) - \partial_v\tau\,D_v - \lambda\partial_\rho\tau\left(D_\rho + \frac{i}{\rho}\,\partial_\phi\right) \\ &+ \tau\partial_\rho\tau\,D_v - \tau\partial_v\tau\left(D_\rho + \frac{i}{\rho}\,\partial_\phi\right) + i\frac{\lambda\tau}{\rho^2}\,\partial_\phi. \end{aligned} \tag{A.9}$$

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Using (A.6), we find that all the terms in the second and third lines cancel out, resulting in

$$0 = [\check{L}, \check{M}]Y = \left[-\tau \left(\partial_v A_v + \lambda \partial_\rho A_\rho + \frac{\lambda}{\rho} A_\rho \right) + \tau^2 F_{\rho v} \right] Y. \tag{A.10}$$

Assuming that the matrix function Y is invertible and multiplying by Y^{-1} on the right gives, using $\tau \neq 0$,

$$-\left(\partial_v A_v + \lambda\, \partial_\rho A_\rho + \frac{\lambda}{\rho} A_\rho \right) + \tau\, F_{\rho v} = 0. \tag{A.11}$$

Now we note that $F_{\rho v}$ has to vanish, because otherwise we would obtain

$$\tau = -\frac{\left(\partial_v A_v + \lambda\, \partial_\rho A_\rho + \frac{\lambda}{\rho} A_\rho \right)}{F_{\rho v}}, \tag{A.12}$$

which cannot hold, since the right hand side is independent of ω, whereas τ is a function of ω. Hence we infer that (2.8) and (2.9) hold.

The fourth Lax pair $(\mathcal{L}, \mathcal{M})$, formulated in $D = 2$ in coordinates (ρ, v), is given in (2.11),

$$\begin{aligned} \mathcal{L} &= -\partial_v + \tau\, D_\rho = -\partial_v + \tau\, (\partial_\rho + A_\rho)\,, \\ \mathcal{M} &= \lambda\, \partial_\rho + \tau\, D_v = \lambda\, \partial_\rho + \tau\, (\partial_v + A_v)\,, \quad \lambda = \pm 1, \end{aligned} \tag{A.13}$$

where τ is a non-constant spectral parameter, namely a function of ρ, v and of $\omega \in \mathbb{C}$ as specified by (2.13). The compatibility condition

$$\mathcal{L}X = 0 \;\wedge\; \mathcal{M}X = 0 \Longrightarrow [\mathcal{L}, \mathcal{M}]X = 0 \quad \forall \quad X \in C^2\,, \;\; \omega \in \mathbb{C} \tag{A.14}$$

yields (2.8) and (2.9), as follows. Using

$$[\mathcal{L}, \mathcal{M}] = -\left([\partial_v, \tau\, D_v] + \lambda\, [\partial_\rho, \tau D_\rho] \right) + [\tau\, D_\rho, \tau\, D_v] \tag{A.15}$$

as well as

$$[D_\rho, D_v] = F_{\rho v}\,, \quad [\partial_v, D_v] = \partial_v A_v\,, \quad [\partial_\rho, D_\rho] = \partial_\rho A_\rho, \tag{A.16}$$

we obtain

$$0 = [\mathcal{L}, \mathcal{M}]X = -\frac{\lambda}{\tau} \left(-\partial_v \tau + \tau \partial_\rho \tau \right) \partial_\rho - \left(\lambda \partial_\rho \tau + \tau \partial_v \tau \right) D_\rho - \tau \left(\partial_v A_v + \lambda \partial_\rho A_\rho \right) + \tau^2\, F_{\rho v}, \tag{A.17}$$

where we used $\mathcal{M}X = 0$ once. Next, using the relations (A.6) in the form

$$\begin{aligned} -\partial_v \tau + \tau \partial_\rho \tau &= -\frac{\tau^2}{\rho}\,, \\ \lambda \partial_\rho \tau + \tau \partial_v \tau &= \lambda \frac{\tau}{\rho}\,, \end{aligned} \tag{A.18}$$

and inserting them into (A.17) gives

$$0 = [\mathcal{L}, \mathcal{M}]X = \left[-\tau \left(\partial_v A_v + \lambda \partial_\rho A_\rho + \frac{\lambda}{\rho}\, A_\rho \right) + \tau^2\, F_{\rho v} \right] X. \tag{A.19}$$

Assuming that the matrix function X is invertible and multiplying by X^{-1} on the right gives, using $\tau \neq 0$,

$$-\left(\partial_v A_v + \lambda\,\partial_\rho A_\rho + \frac{\lambda}{\rho} A_\rho\right) + \tau\, F_{\rho v} = 0. \tag{A.20}$$

By an argument analogous to the one used below (A.11), we conclude that (2.8) and (2.9) both hold.

B Self-dual Yang-Mills equations in different space-time signatures

B.1 Signature $(1, 3)$

We consider the line element in signature $(1, 3)$,

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2\,. \tag{B.1}$$

Given a two-form $\mathcal{F}$, the self-duality condition $\mathcal{F} = \star\mathcal{F}$ implies the relations

$$\mathcal{F}_{01} = i\mathcal{F}_{23}\,,\; \mathcal{F}_{02} = -i\mathcal{F}_{13}\,,\; \mathcal{F}_{03} = i\mathcal{F}_{12}\,. \tag{B.2}$$

Setting

$$x_0 = \tau\,,\; x_1 = v\,,\; x_2 = \rho\,\cos\phi\,,\; x_3 = \rho\,\sin\phi\,, \tag{B.3}$$

and introducing the coordinates

$$U = \tau - v\,, \quad V = \tau + v\,, \quad w = \rho\, e^{i\phi}\,, \quad \bar{w} = \rho\, e^{-i\phi}, \tag{B.4}$$

we obtain the line element (5.3),

$$ds^2 = -d\tau^2 + dv^2 + d\rho^2 + \rho^2 d\phi^2 = -dU dV + dw d\bar{w}\,. \tag{B.5}$$

Using

$$\begin{aligned} dx_0 \wedge dx_1 &= \frac{1}{2} dU \wedge dV\,, \\ dx_2 \wedge dx_3 &= \rho\, d\rho \wedge d\phi = \frac{i}{2} dw \wedge d\bar{w}\,, \end{aligned} \tag{B.6}$$

we obtain

$$\begin{aligned} \mathcal{F}_{01}\, dx_0 \wedge dx_1 &= \frac{1}{2}\mathcal{F}_{01}\, dU \wedge dV = \mathcal{F}_{UV}\, dU \wedge dV\,, \\ \mathcal{F}_{23}\, dx_2 \wedge dx_3 &= \frac{i}{2}\mathcal{F}_{23}\, dw \wedge d\bar{w} = \mathcal{F}_{w\bar{w}}\, dw \wedge d\bar{w}\,, \end{aligned} \tag{B.7}$$

and we infer the relations

$$\mathcal{F}_{UV} = \frac{1}{2}\mathcal{F}_{01}\,, \quad \mathcal{F}_{w\bar{w}} = \frac{i}{2}\mathcal{F}_{23}\,. \tag{B.8}$$

The, using (B.2), we obtain

$$\mathcal{F}_{UV} = \frac{1}{2}\mathcal{F}_{01} = \frac{i}{2}\mathcal{F}_{23} = \mathcal{F}_{w\bar{w}}\,, \tag{B.9}$$

in agreement with (5.13). Similarly, one infers the relations (5.11) and (5.12).

B.2 Signature $(2,2)$

We consider the line element in signature $(2,2)$,

$$ds^2 = dx_0^2 + dx_1^2 - dx_2^2 - dx_3^2 \,. \tag{B.10}$$

Given a two-form $\mathcal{F}$, the self-duality condition $\mathcal{F} = \star\mathcal{F}$ implies the relations [43]

$$\mathcal{F}_{01} = \mathcal{F}_{23}\,,\ \mathcal{F}_{02} = \mathcal{F}_{13}\,,\ \mathcal{F}_{03} = -\mathcal{F}_{12}\,. \tag{B.11}$$

Now let us perform various coordinate changes. First, consider setting

$$x_0 = \rho\,\cos\phi\,,\ x_1 = \rho\,\sin\phi\,,\ x_2 = \tau\,,\ x_3 = v\,, \tag{B.12}$$

in which case we obtain the line element (5.39),

$$ds^2 = -d\tau^2 - dv^2 + d\rho^2 + \rho^2 d\phi^2 \,. \tag{B.13}$$

Then, introducing the coordinates

$$U = \tau + iv\,,\quad V = \tau - iv\,,\quad w = \rho\, e^{i\phi}\,,\quad \bar{w} = \rho\, e^{-i\phi}, \tag{B.14}$$

the line element becomes expressed as

$$ds^2 = -dUdV + dwd\bar{w}\,. \tag{B.15}$$

Using

$$\begin{aligned} dx_0 \wedge dx_1 &= \rho\, d\rho \wedge d\phi = \frac{i}{2} dw \wedge d\bar{w}\,,\\ dx_2 \wedge dx_3 &= \frac{i}{2} dU \wedge dV\,, \end{aligned} \tag{B.16}$$

we obtain

$$\begin{aligned} \mathcal{F}_{01}\, dx_0 \wedge dx_1 &= \frac{i}{2}\mathcal{F}_{01}\, dw \wedge d\bar{w} = \mathcal{F}_{w\bar{w}}\, dw \wedge d\bar{w}\,,\\ \mathcal{F}_{23}\, dx_2 \wedge dx_3 &= \frac{i}{2}\mathcal{F}_{23}\, dU \wedge dV = \mathcal{F}_{UV}\, dU \wedge dV, \end{aligned} \tag{B.17}$$

and we infer the relations

$$\mathcal{F}_{UV} = \frac{i}{2}\mathcal{F}_{23}\,,\quad \mathcal{F}_{w\bar{w}} = \frac{i}{2}\mathcal{F}_{01}\,. \tag{B.18}$$

The, using (B.11), we obtain

$$\mathcal{F}_{UV} = \frac{i}{2}\mathcal{F}_{23} = \frac{i}{2}\mathcal{F}_{01} = \mathcal{F}_{w\bar{w}}\,, \tag{B.19}$$

in agreement with (5.13). Similarly, one infers the relations (5.11) and (5.12).

Next, let us consider setting

$$x_0 = \rho\,\sinh\phi\,,\quad x_1 = v\,,\ x_2 = \tau\,,\quad x_3 = \rho\,\cosh\phi\,, \tag{B.20}$$

in which case we obtain the line element (5.38),

$$ds^2 = -d\tau^2 + dv^2 - d\rho^2 + \rho^2 d\phi^2 \,. \tag{B.21}$$

Then, introducing the coordinates

$$U = \tau - v \,, \quad V = \tau + v \,, \quad w = \rho\, e^{-\phi} \,, \quad \bar{w} = -\rho\, e^{\phi}, \tag{B.22}$$

the line element becomes expressed as

$$ds^2 = -dU dV + dw d\bar{w} \,. \tag{B.23}$$

Using

$$\begin{aligned} dx_0 \wedge dx_3 &= -\rho\, d\rho \wedge d\phi = \frac{1}{2} dw \wedge d\bar{w} \,, \\ dx_1 \wedge dx_2 &= -\frac{1}{2} dU \wedge dV \,, \end{aligned} \tag{B.24}$$

we obtain

$$\begin{aligned} \mathcal{F}_{03}\, dx_0 \wedge dx_3 &= \frac{1}{2} \mathcal{F}_{03}\, dw \wedge d\bar{w} = \mathcal{F}_{w\bar{w}}\, dw \wedge d\bar{w} \,, \\ \mathcal{F}_{12}\, dx_1 \wedge dx_2 &= -\frac{1}{2} \mathcal{F}_{12}\, dU \wedge dV = \mathcal{F}_{UV}\, dU \wedge dV, \end{aligned} \tag{B.25}$$

and we infer the relations

$$\mathcal{F}_{UV} = -\frac{1}{2} \mathcal{F}_{12} \,, \quad \mathcal{F}_{w\bar{w}} = \frac{1}{2} \mathcal{F}_{03} \,. \tag{B.26}$$

Then, using (B.11), we obtain

$$\mathcal{F}_{UV} = -\frac{1}{2} \mathcal{F}_{12} = \frac{1}{2} \mathcal{F}_{03} = \mathcal{F}_{w\bar{w}} \,, \tag{B.27}$$

in agreement with (5.13). Similarly, one infers the relations (5.11) and (5.12).

Finally, we note that the line element (5.47) differs from (B.13) by an overall sign, and hence the relation (B.19) applies to this case as well.

C Anti self-dual Yang-Mills equations in signature $(1, 3)$

We consider the line element in signature $(1, 3)$,

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \,. \tag{C.1}$$

Given a two-form $\mathcal{F}$, the anti self-duality condition $\mathcal{F} = - \star \mathcal{F}$ implies the relations

$$\mathcal{F}_{01} = -i \mathcal{F}_{23} \,, \; \mathcal{F}_{02} = i \mathcal{F}_{13} \,, \; \mathcal{F}_{03} = -i \mathcal{F}_{12} \,. \tag{C.2}$$

Setting

$$x_0 = \tau \,, \; x_1 = v \,, \; x_2 = \rho \, \cos\phi \,, \; x_3 = \rho \, \sin\phi \,, \tag{C.3}$$

and introducing the coordinates

$$U = \tau - v\,, \quad V = \tau + v\,, \quad w = \rho\, e^{i\phi}\,, \quad \bar{w} = \rho\, e^{-i\phi}, \tag{C.4}$$

we obtain the line element (5.3),

$$ds^2 = -d\tau^2 + dv^2 + d\rho^2 + \rho^2 d\phi^2 = -dUdV + dwd\bar{w}\,. \tag{C.5}$$

Using

$$\begin{aligned} dx_0 \wedge dx_1 &= \frac{1}{2} dU \wedge dV\,, \\ dx_2 \wedge dx_3 &= \rho\, d\rho \wedge d\phi = \frac{i}{2} dw \wedge d\bar{w}\,, \end{aligned} \tag{C.6}$$

we obtain

$$\begin{aligned} \mathcal{F}_{01}\, dx_0 \wedge dx_1 &= \frac{1}{2} \mathcal{F}_{01}\, dU \wedge dV = \mathcal{F}_{UV}\, dU \wedge dV\,, \\ \mathcal{F}_{23}\, dx_2 \wedge dx_3 &= \frac{i}{2} \mathcal{F}_{23}\, dw \wedge d\bar{w} = \mathcal{F}_{w\bar{w}}\, dw \wedge d\bar{w}\,, \end{aligned} \tag{C.7}$$

and we infer the relations

$$\mathcal{F}_{UV} = \frac{1}{2} \mathcal{F}_{01}\,, \quad \mathcal{F}_{w\bar{w}} = \frac{i}{2} \mathcal{F}_{23}\,. \tag{C.8}$$

Then, using (C.2), we obtain

$$\mathcal{F}_{UV} = \frac{1}{2} \mathcal{F}_{01} = -\frac{i}{2} \mathcal{F}_{23} = -\mathcal{F}_{w\bar{w}}\,. \tag{C.9}$$

Similarly, one infers the relations

$$\mathcal{F}_{U\bar{w}} = \mathcal{F}_{Vw} = 0\,. \tag{C.10}$$

If we now interchange U and V we obtain

$$\mathcal{F}_{UV} = \mathcal{F}_{w\bar{w}}\,, \quad \mathcal{F}_{Uw} = \mathcal{F}_{V\bar{w}} = 0\,. \tag{C.11}$$

Thus, by interchanging U and V the anti self-dual equations get mapped to the self-dual equations and vice-versa.

D From $2D$ to $4D$

In section 5 we constructed the mapping of gravitational solutions to solutions of the self-dual Yang-Mills equations in four dimensions by starting from the latter equations and performing dimensional reductions of them to two dimensions. Here we present a different approach to this mapping, by starting on the gravitational side in two dimensions and expressing the 1-form A in two dimensions as a 1-form in four space-time dimensions, as follows.

Let us consider a solution M to the gravitational field equations (2.1). The associated matrix 1-form $A = A_\rho d\rho + A_v dv$ satisfies the field equation (2.8) as well as $F_{\rho v} = 0$, cf. (2.9).

Next, we view A as a 1-form in a four-dimensional flat space-time with lightcone coordinates (U, V, X, Y),

$$A = A_U dU + A_V dV + A_X dX + A_Y dY\,. \tag{D.1}$$

The map between the lightcone coordinates (U, V, X, Y) and the Weyl coordinates (ρ, v) depends on the space-time signature. For example, for the case of signature $(1, 3)$, it is given by (5.1) and (5.4). The four components (A_U, A_V, A_X, A_Y) are expressed in terms of (A_ρ, A_v). Since the latter have to satisfy $F_{\rho v} = 0$, this will impose conditions on the field strength components constructed from (D.1), which will be discussed below. Let us now assume that these four components are given in terms of first-order derivatives of two matrix functions $\Psi(U, V, X, Y)$ and $\tilde{\Psi}(U, V, X, Y)$,

$$A_U = \partial_Y \tilde{\Psi}\,, \quad A_V = -\partial_X \Psi\,, \quad A_X = \partial_V \tilde{\Psi}\,, \quad A_Y = -\partial_U \Psi\,, \tag{D.2}$$

and that $\Psi(U, V, X, Y)$ and $\tilde{\Psi}(U, V, X, Y)$ satisfy

$$\begin{aligned} (\partial_U \partial_V - \partial_X \partial_Y)\,\Psi + [\partial_U \Psi, \partial_X \Psi] = 0\,, \\ (\partial_U \partial_V - \partial_X \partial_Y)\,\tilde{\Psi} + [\partial_Y \tilde{\Psi}, \partial_V \tilde{\Psi}] = 0\,. \end{aligned} \tag{D.3}$$

Now we consider the following two sets, $(0, A_V, 0, A_Y)$ and $(A_U, 0, A_X, 0)$. We observe that each of these two sets, when viewed by itself, satisfies the self-duality conditions

$$F_{UV} = F_{XY}\,, \quad F_{UX} = 0\,, \quad F_{VY} = 0 \tag{D.4}$$

by virtue of (D.2) and (D.3). For instance, if we consider the set $(0, A_V, 0, A_Y)$, then $F_{UX} = 0$ is trivially satisfied, since for this set $A_U = A_X = 0$. The condition $F_{VY} = 0$ holds by virtue of the first equation in (D.3), while the condition $F_{UV} = F_{XY}$ holds by virtue of (D.2). Thus, there are two non-trivial conditions, which precisely equals the number of equations (namely (2.8) and (2.9)) that (A_ρ, A_v) have to satisfy. It is thus tempting to identify the condition $F_{VY} = 0$ with $F_{\rho v} = 0$, and to identify the condition $F_{UV} = F_{XY}$ with the field equation (2.8). This turns out to be indeed the case in all the examples considered in this paper. Similar considerations hold for the set $(A_U, 0, A_X, 0)$.

Since the two sets $(0, A_V, 0, A_Y)$ and $(A_U, 0, A_X, 0)$ are constructed from the same data (namely from (A_ρ, A_v) and the coordinate transformation relating the lightcone coordinates (U, V, X, Y) to the coordinates used in the four-dimensional line element (2.4)), they carry the same information, and hence the matrix functions $\Psi(U, V, X, Y)$ and $\tilde{\Psi}(U, V, X, Y)$ should be related. This expectation has been borne out in all the examples studied in this paper. In these examples we find that

$$\tilde{\Psi} = e^{-2i\phi}\,\Psi\,. \tag{D.5}$$

We also note that in these examples Ψ and $\tilde{\Psi}$ are complex conjugates of one another.

As an illustrative example, let us consider the Schwarzschild solution discussed in section 3. Using (3.87) and (2.26), we obtain the following expressions for Ψ and $\tilde{\Psi}$ when expressed in terms of the lightcone coordinates given in (5.1) and (5.4),

$$\Psi = m\,\frac{V-U}{Y\,\sqrt{\frac{1}{4}(U-V)^2+XY}}\,,$$
$$\tilde{\Psi} = m\,\frac{V-U}{X\,\sqrt{\frac{1}{4}(U-V)^2+XY}}\,. \tag{D.6}$$

Using (D.2), we obtain

$$A_U = 4m\,\frac{U-V}{((U-V)^2+4XY)^{3/2}}\,,$$
$$A_X = 8m\,\frac{Y}{((U-V)^2+4XY)^{3/2}}\,,$$
$$A_V = 4m\,\frac{V-U}{((U-V)^2+4XY)^{3/2}}\,,$$
$$A_Y = 8m\,\frac{X}{((U-V)^2+4XY)^{3/2}}\,. \tag{D.7}$$

It is straightforward to check that each of the two sets $(0, A_V, 0, A_Y)$ and $(A_U, 0, A_X, 0)$ satisfies the self-duality conditions (D.4). Both sets are related by interchanging (U, X) with (V, Y). We note that were we to pick any of the sets $(A_U, 0, 0, A_Y)$ and $(0, A_V, A_X, 0)$, each of them would satisfy anti self-duality conditions, cf. appendix C.

Since the two sets $(0, A_V, 0, A_Y)$ and $(A_U, 0, A_X, 0)$ carry the same information, we pick the set $(0, A_V, 0, A_Y)$ and declare it to be the solution to the self-duality conditions (D.4) that we associate to the gravitational solution encoded in (A_ρ, A_v).

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