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# What can abelian gauge theories teach us about kinematic algebras?

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Abstract: The phenomenon of BCJ duality implies that gauge theories possess an abstract kinematic algebra, mirroring the non-abelian Lie algebra underlying the colour information. Although the nature of the kinematic algebra is known in certain cases, a full understanding is missing for arbitrary non-abelian gauge theories, such that one typically works outwards from well-known examples. In this paper, we pursue an orthogonal approach, and argue that simpler abelian gauge theories can be used as a testing ground for clarifying our understanding of kinematic algebras. We first describe how classes of abelian gauge fields are associated with well-defined subalgebras of the diffeomorphism algebra. By considering certain special subalgebras, we show that one may construct interacting theories, whose kinematic algebras are inherited from those already appearing in a related abelian theory. Known properties of (anti-)self-dual Yang-Mills theory arise in this way, but so do new generalisations, including self-dual electromagnetism coupled to scalar matter. Furthermore, a recently obtained non-abelian generalisation of the Navier-Stokes equation fits into a similar scheme, as does Chern-Simons theory. Our results provide useful input to further conceptual studies of kinematic algebras.

Keywords: Duality in Gauge Field Theories, Gauge Symmetry, Scattering Amplitudes, Space-Time Symmetries

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## Contents



## 1 Introduction

(Non-)abelian gauge theories continue to be intensely studied, due to their role in describing our universe at its most fundamental level, as accessible in current experiments. In 2008, a remarkable new structure was discovered in the scattering amplitudes of non-abelian theories, which has become known as *BCJ duality*[1]. In simple terms, it states that certain kinematic parts of scattering amplitudes (i.e. which depend on momentum and / or polarisation information) can be made to obey similar identities to those satisfied by the colour charge information. Given that the latter constraints arise from the Lie algebra underlying the gauge symmetry of the theory, it appears to be the case that gauge theories have some sort of *kinematic algebra*, which had previously remained hidden. Quite what this new structure is trying to tell us — and how far-reaching its ultimate scope and consequences are — remain mysterious, not least due to the fact that we are ignorant of what the kinematic algebra actually is, for most non-abelian gauge theories of physical interest. It does not help that the BCJ duality property only usually shows up iteratively order-by-order in perturbation theory, so that it is not understood in a fully non-perturbative way.[1] Nevertheless, one very striking consequence of BCJ duality is already known: provided gauge theory amplitudes are written so that the kinematic algebra is manifest (*BCJ-dual form*), one may simply replace colour information by supplementary kinematic factors, as well as coupling constants, to obtain amplitudes in gravity theories. This is known as the *double copy* [3, 4], and is motivated

[1]Another possibility is that there is not in fact a non-perturbative realisation of colour-kinematics duality for arbitrary gauge theories, which may explain the difficulty of finding explicit realisations of it at higher orders in perturbation theory [2]. Here, we will remain optimistically agnostic.



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by earlier work in string theory [5]. In the field theory context, the double copy has been extended to classical solutions [6–15], leading to practical applications such as new techniques for gravitational wave physics (see e.g. refs. [16–19] for recent reviews). A parallel research frontier examines conceptual questions raised by BCJ duality and the double copy, both of which promise tantalising hints that our traditional ways of thinking about field theories might have obscured a deep underlying structure. We should thus leave no stone unturned in performing case studies relating to this structure, including looking at aspects of BCJ duality that may have been previously overlooked.

The first concrete example of a kinematic algebra being found for a (partial) theory was the case of self-dual Yang-Mills theory, in the by now well-known analysis of ref. [20]. This theory corresponds to keeping only one of the (circular) polarisation states of the gluon, and a convenient language for this theory exists in a particular choice of gauge (the lightcone gauge), such that the Lagrangian for the theory is manifestly cubic [21]. As the authors of ref. [20] showed, the cubic vertex can be seen to contain two sets of structure constants, one of which corresponds to the known non-abelian gauge group. The other structure constants correspond to an area-preserving diffeomorphism algebra, which is infinite-dimensional, and the fact that the structure constants appear alongside each other in the single interaction term for the theory means that any perturbative solutions will necessarily obey BCJ duality. Furthermore, a known equation for self-dual gravity (the *Plebanski equation* [22]) follows straightforwardly upon replacing the colour structure constants with a second set of kinematic ones. Deformations of both self-dual gauge / gravity theories, giving rise to generalisations of these structure constants, have been presented in refs. [23–25].

Another case in which the kinematic algebra is known is that of Chern-Simons theory in three spacetime dimensions, as examined in ref. [26]. The authors worked in Lorenz gauge, and showed that the cubic interaction vertex of the theory can be associated with the structure constants of a volume-preserving diffeomorphism algebra. Interestingly, this conclusion extended beyond the physical field itself. Working in Lorenz gauge necessitates the introduction of Faddeev-Popov ghost fields, whose role is to subtract the unphysical degree of freedom carried by the off-shell gauge field. The ghost fields can be combined with the gauge field to make a "superfield" living in superspace, where the latter possesses anti-commuting coordinates in addition to the spacetime ones [27, 28]. Using this formalism, ref. [26] showed that the full superfield 3-vertex gives rise to an extended volume-preserving diffeomorphism algebra, where the "volume" is separately preserved in the subspaces of (anti-)commuting coordinates. Both self-dual Yang-Mills and Chern-Simons theory (in Lorenz gauge) were argued to be special cases of a more general theory dubbed *semi-abelian Yang-Mills theory* in ref. [29]. We will revisit this theory in what follows.

The idea that off-shell degrees of freedom need to be explicitly included in kinematic algebras has been taken further in e.g. refs. [30–40], which also consider the idea that the latter may not be conventional Lie algebras. A Lie algebra is characterised by a vector space $V$ and a *Lie bracket*: $V \times V \to V$, that takes a pair of elements of $V$, and associates this with a third element. This bracket satisfies the well-known Jacobi identity, and the above references argue that this structure is insufficient to describe arbitrary field theories. Rather, these are expected to be built upon so-called $L_\infty$ or *strong homotopy* algebras. These can be viewed as generalisations of Lie algebras, where the Jacobi identity is satisfied only up to terms involving higher-order brackets. Each type of bracket obeys further identities that are satisfied only up to terms involving yet higher-order brackets, resulting in an intricate structure of constraints. A particular homotopy algebra known as a $\mathrm{BV}_\infty^\square$ algebra has been shown to be relevant for full Yang-Mills theory [30], itself a generalisation of the well-known Batalin-Vilkovisky formalism for gauge theory [41, 42]. Reference [35] (see also ref. [39]) showed how the kinematic algebra for Chern-Simons theory, previously identified in ref. [26], could be cast into this framework. More recently, ref. [43] reexamined the case of self-dual Yang-Mills theory using the $\mathrm{BV}_\infty^\square$ ideas, in particular addressing the question of whether the straightforward Lie algebra of area-preserving diffeomorphisms found for lightcone gauge in ref. [20] extends to more general gauges. The conclusion was that, in general, it is not expected that the $\mathrm{BV}_\infty^\square$ algebra reduces to a Lie algebra, even in the self-dual sector. A similar point was made in the earlier work of ref. [35]. Further work on kinematic algebras in a variety of contexts can be found in refs. [44–51].

Despite (or perhaps because of!) the above progress, a number of open questions remain: how do we find the kinematic algebras of particular theories, either as reductions of $L_\infty$ algebras or otherwise? Are kinematic algebras gauge-dependent in general? If so, is there an optimal choice of gauge, such that the kinematic algebra is somehow minimal? Is it always possible to reduce it to a Lie algebra? In this paper, we will explore some of these questions in the context of simple abelian gauge theories, and our motivations are as follows. For starters, it is often claimed — erroneously — that there is no manifestation of BCJ duality for linear (or linearised) gauge theories. That this is not in fact true rests on the fact that one may indeed associate a "kinematic algebra" with linear gauge fields, given that they are Lie-algebra valued in two different Lie algebras. The first of these is the usual gauge algebra of the theory, and the second is the algebra of diffeomorphisms generated by the (vector) gauge field. For self-dual linearised solutions that can be double-copied to make gravity solutions, one must replace the colour generators with a second set of diffeomorphism generators, which indeed amounts to a colour-kinematics duality, as argued in ref. [52]. In this paper, we wish to expand upon and clarify this point, by defining more precisely the above ideas, which were only briefly introduced in ref. [52]. In particular, we will see that certain gauge choices and / or solution types in abelian gauge theory pick out well-defined subgroups of the full diffeomorphism group, such that known cases of kinematic algebras correspond to some of these subgroups.

Admittedly, the diffeomorphism algebras that arise at linear level are not what people usually mean when they talk about kinematic algebras, which are instead associated with interactions between fields. However, we can then use abelian gauge theories to clarify aspects of more general kinematic algebras. Given that any interacting theory (including a non-abelian gauge theory) must have a non-interacting linearisation, we can ask which of our "special" subgroups of diffeomorphisms can be preserved by the inclusion of interactions. We will see that a particularly interesting case is when the gauge field generates so-called *symplectomorphisms* or, in other words, when the field itself is *Hamiltonian*. We will review the definition of these concepts below, but the presence of a Hamiltonian vector field allows us to define a kinematic Poisson bracket, which can in turn be used to construct interacting theories that contain a non-trivial kinematic algebra. We will show that the known kinematic algebra of (anti-)self dual Yang-Mills theory in lightcone gauge arises as a special case of this construction, but that there are also various generalisations of this story. As a novel byproduct, we will also see that the self-dual sector of QED coupled to scalar matter has a straightforward kinematic algebra, in terms of a similar Poisson bracket. There are also cases in which the Lie bracket of diffeomorphisms itself arises as part of an interaction term, and we will show that both Chern-Simons theory [26] and a recent non-abelian generalisation of the Navier-Stokes equation [53] in three spacetime dimensions arise in this way. In all cases, there is a geometric understanding of the kinematic algebra, in that it corresponds to the diffeomorphisms generated by the gauge field. We hope that our results are useful for further studies of such algebras, including guiding searches for higher geometric structures upon which they act.

The structure of our paper is as follows. In section 2, we review and expand the ideas of ref. [52], showing how sectors of linearised gauge theories can be classified according to their diffeomorphism algebras. In section 3, we describe how the kinematic algebras of interacting theories can be built upon the subgroups of diffeomorphisms one encounters at linearised level, giving a number of examples, some previously unknown. We discuss our results and conclude in section 4.

## 2 Diffeomorphisms and linearised gauge theories

### 2.1 Linearised self-dual fields

Reference [52] examined solutions of linearised non-abelian gauge theories, whose field equations constrain a field

$$\mathbf{A}_\mu = A_\mu^a \mathbf{T}^a. \tag{2.1}$$

Here Greek and Latin letters specify spacetime and adjoint (colour) indices respectively, and $\mathbf{T}^a$ is a generator of the gauge group. The set of all generators satisfies the Lie algebra

$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c, \tag{2.2}$$

with structure constants $f^{abc}$, such that the gauge field itself takes values in the Lie algebra. As pointed out in ref. [52] (see also ref. [45]), the field $\mathbf{A}_\mu$ is in fact valued in a second Lie algebra. To see this, we may recall that a given vector field $V^\mu$ on a manifold generates infinitesimal diffeomorphisms

$$V^\mu(x)\partial_\mu, \tag{2.3}$$

which can be visualised geometrically as follows. First, one may construct the *integral curves* (fieldlines) of the vector field $V^\mu$. These are an infinite set of non-intersecting curves, such that $V^\mu(x)$ is tangent to the integral curve passing through $x^\mu$. An example of these integral curves is shown in figure 1, and the action of eq. (2.3) is to effect an infinitesimal translation along each curve simultaneously. The set of all vector fields on a manifold then consists of the set of all such diffeomorphisms, and they form an algebra under the *Lie bracket*

$$[V^{(1)\mu}\partial_\mu, V^{(2)\nu}\partial_\nu] = V^{(3)\mu}\partial_\mu. \tag{2.4}$$

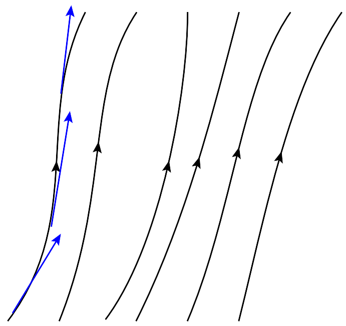

**Figure 1.** A vector field $V^\mu$ (shown in blue) generates integral curves, such that $V^\mu(x)$ is tangent to the integral curve passing through $x^\mu$.

As in eq. (2.2), one derives the right-hand side by forming the commutator of the two transformations on the left-hand side, and the algebra is closed given that the bracket of two vector fields is itself a vector field. The components of the latter turn out to be given by

$$V^{(3)\mu} = V^{(1)\nu}\partial_\nu V^{(2)\mu} - V^{(2)\nu}\partial_\nu V^{(1)\mu}. \tag{2.5}$$

The Lie bracket has a geometric interpretation as the *Lie derivative* of the vector field $V^{(1)\mu}$ along the vector field $V^{(2)\mu}$, and one may also interpret eq. (2.5) as representing the failure of a loop made of infinitesimal diffeomorphisms along two different vector fields to close in general.

Now let us consider an abelian-like solution of a non-abelian gauge theory, for which one may make the ansatz

$$A^a_\mu = c^a A_\mu, \tag{2.6}$$

for constant colour vector $c^a$, as has been done in the context of the double copy in e.g. ref. [6]. One may then consider $A_\mu$ to be a solution of an abelian gauge theory, and it will generate diffeomorphisms as described above. There are several known cases in which abelian-like gauge fields can be double-copied to make gravity solutions. An example relevant for the present paper is the case of self-dual linearised fields in lightcone gauge, examined in ref. [52], and which are such that the gauge field can be written as

$$A_\mu = \hat{k}_\mu \phi, \tag{2.7}$$

where $\phi(x)$ is a scalar field, and $\hat{k}$ a differential operator satisfying[2]

$$\hat{k}^2 = 0, \quad \partial \cdot \hat{k} = 0. \tag{2.8}$$

The general form of such an operator was found in ref. [52] to be expressible (in Euclidean spacetime signature) as[3]

$$\hat{k}_\mu = B_i \bar{\eta}^i_{\mu\nu}\partial_\nu, \tag{2.9}$$

[2] We use $A \cdot B$ to denote $A^\rho B_\rho$, where $A$ and $B$ could be either fields or operators.

[3] We do not raise or lower indices in eq. (2.9) and subsequent equations, as a reminder that we are in Euclidean signature. This choice is possible due to the explicit form of the Euclidean metric, and is a common convention in the physics literature.

where

$$\bar{\eta}^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^3_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.10}$$

are so-called *'t Hooft symbols*, which arise in the study of non-abelian (self-dual) instantons, and $B_i$ a constant vector such that $\boldsymbol{B}^2 = 0$. Then, the field

$$h_{\mu\nu} = \hat{k}_\mu \hat{k}_\nu \phi \tag{2.11}$$

is a self-dual gravity solution. A canonical example of this construction is the Eguchi-Hanson gravitational instanton, first considered from a double-copy point of view in ref. [54] (see also ref. [12]). We usually think of the double copy as replacing a colour Lie algebra with a second copy of the kinematic algebra. Thus, ref. [52] suggested identifying the Lie algebra of diffeomorphisms with the "kinematic algebra" of an abelian gauge field. It should be stressed again that this is not usually what we mean when we talk about kinematic algebras, which are instead associated with interaction terms in a theory. Hence, we shall continue to refer to the Lie algebra generated by abelian solutions as a diffeomorphism algebra (which of course it is) in what follows. Nevertheless, the particular diffeomorphisms generated by the self-dual abelian solutions of eq. (2.7) have a particularly elegant geometric interpretation. One may express the gauge field of eqs. (2.7), (2.9) as

$$A_\mu \partial_\mu = (\hat{k}_\mu \phi)\partial_\mu = \left( b^{(1)}_{[\mu} b^{(2)}_{\nu]} \partial_\nu \phi \right) \partial_\mu, \tag{2.12}$$

with[4]

$$b^{(1)}_\mu = (B_1, B_2, B_3, 0), \quad b^{(2)}_\mu = \left( 0, \frac{B_3}{B_1}, -\frac{B_2}{B_1}, -1 \right). \tag{2.13}$$

It then follows that the diffeomorphisms generated by eq. (2.12) take place in the family of null planes whose tangent bivectors are constructed from $b^{(1)}_\mu$ and $b^{(2)}_\mu$ (but which may have some displacement from the origin). Here the null property arises from the fact that

$$b^{(i)} \cdot b^{(j)} = 0, \quad \forall i \in \{1, 2\}. \tag{2.14}$$

Given also the condition $\partial \cdot \hat{k} = 0$, these diffeomorphisms will be area-preserving, such that the diffeomorphism algebra of self-dual abelian solutions is that of area-preserving diffeomorphisms.

As mentioned in the introduction, an area-preserving diffeomorphism algebra also arises in self-dual Yang-Mills theory in lightcone gauge, even when nonlinear interactions are included. Indeed, it is the *same* area-preserving diffeomorphism algebra as has been found in the case of abelian fields discussed above. One clue as to why this happens can be found in ref. [6], which showed that substituting the fully non-abelian ansatz

$$A^a_\mu = \hat{k}_\mu \Phi^a \tag{2.15}$$

[4]We have assumed $B_1 \neq 0$ in eq. (2.13), but similar solutions can be derived for $B_1 = 0$.



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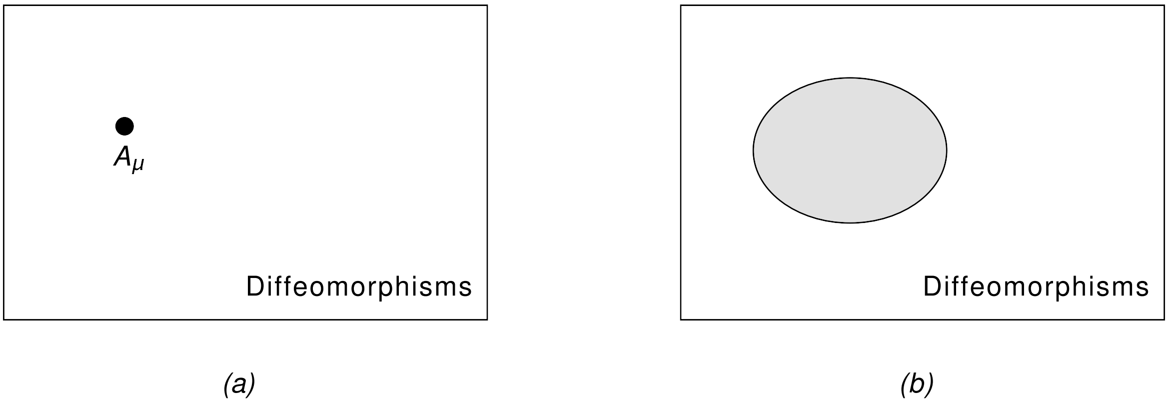


**Figure 2.** (a) A given gauge field $A_\mu$ constitutes a point in the space of all possible diffeomorphisms in spacetime; (b) closed subgroups of the general diffeomorphism algebra form well-defined blobs in the space of all possible diffeomorphisms, which can in turn be related to a special class of abelian gauge fields.

into the Yang-Mills equations, where $\hat{k}_\mu$ satisfies the conditions of eq. (2.8), leads to the known equation of self-dual Yang-Mills theory in lightcone gauge, whose underlying area-preserving diffeomorphism algebra was discovered in ref. [20]. This suggests that the kinematic algebra of self-dual Yang-Mills theory — a bona fide kinematic algebra of an interacting theory — is somehow related to the self-dual diffeomorphisms found in the abelian theory. In what follows, we will fully explain this connection, for which we first need to discuss abelian diffeomorphisms in more detail.

### 2.2 The space of abelian diffeomorphisms

A given abelian gauge field $A_\mu$ generates diffeomorphisms along a particular set of integral curves. We can represent this pictorially as in figure 2(a), which shows the set of all possible diffeomorphisms. Our given gauge field is then a point in this diagram. Note that there is no non-abelian algebra associated with this point: by eqs. (2.4), (2.5), the diffeomorphisms associated with $A_\mu$ are mutually commuting. This makes sense from figure 1, given that performing one simultaneous translation along all possible integral curves, followed by another, is clearly insensitive to the order in which these translations are carried out.

Non-trivial diffeomorphism algebras will consist of subgroups of the full diffeomorphism algebra, which can be represented pictorially as in figure 2(b). From figure 2, each point inside such a closed subgroup will correspond to an abelian gauge field $A_\mu$, and we can therefore ask the following question: given a particular subgroup of diffeomorphisms, can we identify a particular class of abelian gauge fields that this corresponds to? There are in fact two particular subgroups of the (infinitely dimensional) diffeomorphism group, whose physical interpretation is easy to appreciate.

#### 2.2.1 Volume-preserving diffeomorphisms

It is well-known that volume-preserving diffeomorphisms form a closed subgroup. Furthermore, a simple criterion for a diffeomorphism to be volume preserving is that the (multidimensional) divergence of the vector field vanishes:

$$\partial \cdot A = 0. \tag{2.16}$$

For an abelian gauge field, this is the Lorenz gauge condition. Thus, abelian gauge fields in the Lorenz gauge generate a volume-preserving diffeomorphism algebra. Interestingly, ref. [26] studied Chern-Simons theory in Lorenz gauge, finding that the kinematic algebra of the theory is indeed that of volume-preserving diffeomorphisms. As in the case of self-dual Yang-Mills theory, the kinematic algebra of Chern-Simons is a statement about the interaction vertex. However, the fact that it parallels the diffeomorphism algebra that already exists at linear level is reminiscent of how the kinematic algebra of self-dual Yang-Mills (area-preserving diffeomorphisms) is closely related to the diffeomorphism algebra of self-dual abelian solutions, in lightcone gauge.

Subgroups of the full spacetime volume-preserving diffeomorphism group also exist. In particular, one may consider taking a fixed hypervolume of lower dimension than the total spacetime dimension. Then, diffeomorphisms that preserve this lower-dimensional volume form a group by themselves. An example of this is the group of area-preserving diffeomorphisms seen in self-dual Yang-Mills theory.

### 2.2.2 Symplectomorphisms

A *symplectic manifold* is a manifold that is endowed with a particular two-form called the *symplectic form*, and whose existence allows us to define additional structures. A familiar example of a symplectic manifold in classical point particle mechanics is that of the phase space of a system, consisting of a set of generalised coordinates $\{(q^a)\}$, and momenta $\{(p_a)\}$. For $N$ position degrees of freedom, i.e. $a = 1, \ldots, N$, we may then write the total set of $2N$ phase space coordinates as $\{\xi^i\} \equiv \{q^a, p_a\}$, and the symplectic form is given by

$$\omega = \sum_a dq^a \wedge dp_a = \omega_{ij} d\xi^i \wedge d\xi^j, \tag{2.17}$$

where $\omega_{ij}$ consists of the $2N \times 2N$ matrix

$$\omega_{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{2.18}$$

and we have used the shorthand notation $\mathbf{0}$ and $\mathbf{I}$ for an $N \times N$ zero or identity matrix respectively. We will also need the inverse of $\omega$, which we denote by $\Omega$, such that

$$\Omega^{ij} \omega_{jk} = \delta^i_k, \tag{2.19}$$

and note we have

$$\Omega^{ij} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}. \tag{2.20}$$

Vector fields can be defined on phase space, whose integral curves associate particular values of the positions $\{q_i\}$ and momenta $\{p_i\}$ at any given value of some parameter. If we interpret this parameter as the time, then these integral curves represent possible histories



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of a classical system which, for a second-order equation of motion, can indeed be entirely specified by describing how the positions and momenta evolve as time progresses.

A *Hamiltonian vector field* is a vector field that preserves the symplectic form. Such fields can be written in general as

$$V^i = \Omega^{ij}\partial_j H(\{\xi^i\}), \tag{2.21}$$

where $\partial_j$ represents the partial derivative in the full set of generalised coordinates and momenta $\{\xi_j\}$. The function $H$ is called the *Hamiltonian*, and governs the time evolution of the system, in that it controls the diffeomorphisms along the vector field $V_i$, which we have already stated represents all possible histories. More precisely, the fact that $V_i$ is tangent to an integral curve parametrised by time $t$ implies

$$V^i = \frac{d\xi^i}{dt}, \tag{2.22}$$

such that the diffeomorphisms generated by $V^i$ are of the form

$$V^i\partial_i = \frac{d\xi^i}{dt}\partial_i \equiv \frac{d}{dt}. \tag{2.23}$$

This shows that $V_i$ generates time translations along each integral curve as required, and eq. (2.21) now yields

$$\frac{d}{dt} = \Omega^{ij}(\partial_j H)\partial_i. \tag{2.24}$$

In writing the equations of motion of the system, we may introduce the *Poisson bracket*, which is formally defined through the action of the symplectic form on two Hamiltonian vector fields. Denoting the latter by

$$X_f = \Omega^{ij}(\partial_j f)\partial_i, \quad X_g = \Omega^{ij}(\partial_j g)\partial_i \tag{2.25}$$

for two scalar functions $f$ and $g$, one then has

$$\{f, g\} = \omega(X_f, X_g) = \omega_{ij}(\Omega^{ik}\partial_k f)(\Omega^{jl}\partial_l g). \tag{2.26}$$

The right-hand side simplifies upon using (2.19) such that eq. (2.26) becomes

$$\{f, g\} \equiv \Omega^{ji}(\partial_i f)(\partial_j g) = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q^a}\frac{\partial f}{\partial p_a}. \tag{2.27}$$

Equation (2.24) then implies

$$\frac{d\xi^i}{dt} = \{\xi^i, H\}, \tag{2.28}$$

or

$$\frac{dq^a}{dt} = \frac{\partial H}{\partial p_a}, \quad \frac{dp_a}{dt} = -\frac{\partial H}{\partial q^a}, \tag{2.29}$$

which we recognise as *Hamilton's equations* of classical mechanics.



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Although the Hamiltonian formalism provides arguably the most frequently encountered application of symplectic manifolds — at least for the physicist — the language of Hamiltonian vector fields and Poisson brackets is used whenever a manifold is equipped with a symplectic form. Returning to the case of abelian gauge fields in Euclidean signature, we can define a symplectic form

$$\omega = \Omega_{\mu\nu} dx_\mu \wedge dx_\nu. \tag{2.30}$$

By definition, a symplectic form must be non-degenerate, meaning that the components $\Omega_{\mu\nu}$ are those of a non-singular matrix. We may then consider the special class of Hamiltonian gauge fields, which by analogy with eq. (2.21) are given by

$$A_\mu = \Omega_{\mu\nu}\partial_\nu\phi, \tag{2.31}$$

for some scalar field $\phi$. The definition of such fields is that they preserve the symplectic form, which amounts to the condition that the Lie derivative of $\omega$ along a Hamiltonian vector field $A$ is zero. In components this condition reads (see e.g. ref. [55])

$$\mathcal{L}_A\,\omega = (A_\rho\partial_\rho\Omega_{\mu\nu} + \Omega_{\mu\rho}\partial_\nu A_\rho + \Omega_{\rho\nu}\partial_\mu A_\rho) dx_\mu \wedge dx_\nu = 0. \tag{2.32}$$

Upon taking the coefficients $\{\Omega_{\mu\nu}\}$ to be constant and using eq. (2.31), eq. (2.32) implies

$$(\Omega_{\mu\rho}\Omega_{\rho\alpha}\partial_\alpha\partial_\nu\phi + \Omega_{\rho\nu}\Omega_{\rho\alpha}\partial_\mu\partial_\alpha\phi)\, dx_\mu \wedge dx_\nu = 0. \tag{2.33}$$

By analogy with eq. (2.19), we may take

$$\Omega_{\rho\mu}\Omega_{\rho\alpha} = \delta_{\mu\alpha}, \tag{2.34}$$

which indeed satisfies eq. (2.33). We may also introduce a Poisson bracket (cf. eq. (2.26)):

$$\{\phi_1, \phi_2\} = \Omega_{\mu\nu}(\Omega_{\mu\alpha}\partial_\alpha\phi_1)(\Omega_{\nu\beta}\partial_\beta\phi_2) = \Omega_{\mu\nu}(\partial_\mu\phi_1)(\partial_\nu\phi_2), \tag{2.35}$$

where the second equality follows from eq. (2.34).

The diffeomorphism generated by a Hamiltonian vector field is called a *symplectomorphism*, and to show that such transformations form a well-defined subgroup of the full diffeomorphism algebra, we must verify that the Lie bracket of two Hamiltonian gauge fields is itself Hamiltonian. This is indeed true, such that we may write

$$[A^{(1)}_\mu\partial_\mu, A^{(2)}_\nu\partial_\nu] = A^{(3)}_\mu\partial_\mu, \tag{2.36}$$

where all $A^{(i)}_\mu$ are Hamiltonian:

$$A^{(i)}_\mu = \Omega_{\mu\nu}\partial_\nu\phi_i. \tag{2.37}$$

A standard result of symplectic geometry then states that the “Hamiltonian” $\phi_3$ on the right-hand side of eq. (2.36) is related to the Poisson bracket of two Hamiltonians on the left-hand side:

$$\phi_3 = -\{\phi_1, \phi_2\}, \tag{2.38}$$

and we will make use of this result later on.

As for volume-preserving diffeomorphisms, one may consider closed subgroups of the symplectomorphism subgroup. Given that the space that symplectomorphisms act upon must be even-dimensional, there is only one additional possibility in four spacetime dimensions. That is, one may take a two-dimensional subspace of four-dimensional spacetime, and define a symplectic form in this space. Let us denote the relevant symplectic form coefficients by $\omega_{ij}$, where the indices $i, j \in \{1, 2\}$ span the independent coordinates in this subspace. The symplectic form coefficients are antisymmetric, and we can fix the normalisation such that these are equal to the two-dimensional Levi-Civita tensor:

$$\omega_{ij} = \epsilon_{ij}, \tag{2.39}$$

which is the two-dimensional analogue of eq. (2.20). By analogy with eq. (2.27), the Poisson bracket of two scalar fields will be given by

$$\{\phi_1, \phi_2\} = \omega_{ij}(\partial_i\phi_1)(\partial_j\phi_2), \tag{2.40}$$

where the difference with respect to eq. (2.35) is that the indices run only over the coordinates of the two-dimensional subspace acted on by symplectomorphisms, rather than the full four-dimensional spacetime volume. In what follows, it will nevertheless be convenient to work with 4-dimensional covariant notation, in which a Hamiltonian vector field defined with respect to a two-dimensional symplectic form is written as in eq. (2.31). In that case, the coefficients $\Omega_{\mu\nu}$ contain $\omega_{ij}$ as a submatrix, and the former do not strictly constitute the coefficients of a symplectic form due to the fact that the matrix $\Omega_{\mu\nu}$ is singular. However, it is straightforward to verify that

$$\Omega_{\mu\nu}(\partial_\mu\phi_1)(\partial_\nu\phi_2) = \omega_{ij}(\partial_i\phi_1)(\partial_j\phi_2). \tag{2.41}$$

That is, one may continue to use the final expression in eq. (2.35) for the Poisson bracket, as it correctly reduces to the appropriate two-dimensional result in eq. (2.40). Given that Hamiltonian fields involving a two-dimensional symplectic form satisfy $\partial_i A_i = 0$ ($i \in \{1, 2\}$), all such fields generate area-preserving diffeomorphisms. These will then act independently on a family of two-dimensional subspaces that foliate the four-dimensional spacetime.

Another type of symplectomorphism one may consider is the case in which the symplectic form coefficients $\Omega_{\mu\nu}$ become complex in Euclidean signature. An example has already been provided above in section 2.1, where eqs. (2.7), (2.9) define a Hamiltonian vector field with

$$\Omega_{\mu\nu} = B_i\bar{\eta}^i_{\mu\nu}, \quad \boldsymbol{B}^2 = 0. \tag{2.42}$$

In order to satisfy the second condition, the coefficients of $\boldsymbol{B}$ must become complex, but we will not need to use the full language of complex symplectic geometry in what follows. We will, however, need the following property for such symplectomorphisms:

$$\Omega_{\mu\alpha}\Omega_{\mu\beta} = 0, \tag{2.43}$$

which follows from the condition $\boldsymbol{B}^2 = 0$ as well as the known property of 't Hooft symbols

$$\bar{\eta}^i_{\mu\alpha}\bar{\eta}^j_{\mu\beta} = \delta^{ij}\delta_{\alpha\beta} + \epsilon^{ijk}\bar{\eta}^k_{\alpha\beta}. \tag{2.44}$$



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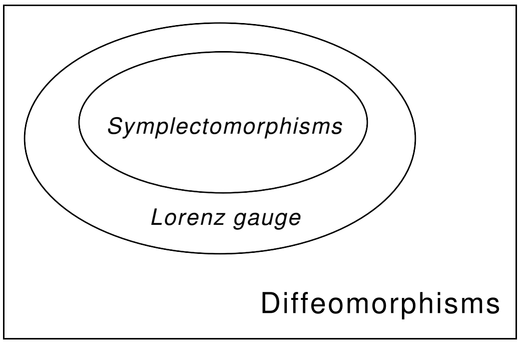


**Figure 3.** Schematic view of symplectomorphisms, as a subset of volume-preserving diffeomorphisms. The latter are associated with abelian gauge fields in the Lorenz gauge.



Once again, we can define a Poisson bracket for these complex symplectomorphisms, and it is given by the final expression in eq. (2.35) as before.

Equation (2.30) implies the antisymmetry property $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$. It then follows from eq. (2.31) that any Hamiltonian gauge field satisfies eq. (2.16), and hence is in the Lorenz gauge. However, symplectomorphisms are a smaller subgroup than mere volume-preserving diffeomorphisms, thus the class of Hamiltonian vector fields constitutes a special family of abelian solutions, that restricts to a subsector of abelian gauge theory, rather than simply being a gauge choice. Following figure 2, we represent this schematically as shown in figure 3.

Let us now ask what the special set of Hamiltonian gauge fields corresponds to physically. One example has already been given: the self-dual abelian fields of eqs. (2.7), (2.9) are all Hamiltonian, where the coefficients of the relevant symplectic form are given by eq. (2.42). This is clearly not the most general case, however, as one could also have abelian gauge fields based on two- or four-dimensional real symplectomorphisms. We will see an example of the former in what follows.

## 2.3 The diffeomorphism algebra of lightcone gauge electromagnetism

As a novel application of the ideas of this section, let us elucidate the diffeomorphism algebra of lightcone electromagnetism. If we restrict to real solutions of the gauge field in Lorentzian signature, we must analytically continue eq. (2.7) appropriately, and add a complex conjugate term as follows:

$$A_\mu = \hat{k}_\mu \phi + \hat{k}^\dagger_\mu \phi^\dagger. \tag{2.45}$$

The two terms consist of a self-dual and anti-self dual contribution respectively, and thus the diffeomorphism generated by $A_\mu$ consists of a sum of two area-preserving diffeomorphisms in a self-dual and anti-self-dual null plane respectively. These are known as $\alpha$- and $\beta$-planes respectively, and we may associate a $\beta$-plane with a given $\alpha$-plane by demanding that their respective tangent bivectors are related by complex conjugation in Lorentzian signature.[5]

[5]It would be interesting to connect the ideas of this section with those of twistor theory, which has appeared in a double-copy / kinematic algebra context in e.g. refs. [35, 56–60]. In particular, points in twistor space correspond to $\alpha$- and $\beta$-planes in spacetime.

Without loss of generality, we may choose a particular lightcone gauge defined through the lightcone coordinates

$$u = \frac{t-z}{\sqrt{2}}, \quad v = \frac{t+z}{\sqrt{2}}, \quad X = \frac{x+iy}{\sqrt{2}}, \quad Y = \frac{x-iy}{\sqrt{2}}, \tag{2.46}$$

where $(t, x, y, z)$ are Cartesian coordinates in Lorentzian signature, and the line elements in each coordinate system are given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 2dudv - 2dXdY. \tag{2.47}$$

For the differential operator appearing in eq. (2.45), we will take the explicit form (in the $(u, v, X, Y)$ system)

$$\hat{k}_\mu = (0, \partial_Y, \partial_u, 0) \quad \Rightarrow \quad \hat{k}^\mu = (\partial_Y, 0, 0, -\partial_u), \tag{2.48}$$

which corresponds to $(B_1, B_2, B_3) = (-i, 1, 0)$ in eq. (2.7), as shown in ref. [52]. The diffeomorphisms generated by the first term of eq. (2.45) are then area-preserving in the infinite family of $(u, Y)$ planes given parametrically by

$$x^\mu = x_0^\mu + \lambda_1(1, 0, 0, -1) + \lambda_2(0, 1, -i, 0) \tag{2.49}$$

in Cartesian coordinates. Here the first term on the right-hand side is a constant offset telling us which $\alpha$-plane we are on, and the remaining two terms contain vectors in the $u$- and $Y$-directions respectively. The imaginary piece in the final term corresponds to the well-known fact that null planes with real coordinate values cannot exist in Lorentzian signature, but would instead be real in (2,2) signature (in this case corresponding to the replacement $y \to iy$). The $\beta$-plane that is conjugate to eq. (2.49) is given by

$$x^\mu = x_0^\mu + \lambda_1(1, 0, 0, -1) + \lambda_2(0, 1, i, 0), \tag{2.50}$$

and corresponds to the differential operator

$$\hat{k}^\dagger_\mu = (0, \partial_X, 0, \partial_u) \quad \Rightarrow \quad \hat{k}^{\dagger\mu} = (\partial_X, 0, -\partial_u, 0) \tag{2.51}$$

in the lightcone coordinate system.[6] We can see that eq. (2.51) generates diffeomorphisms in the particular $(u, X)$ plane that is related to a given $(u, Y)$ plane by complex conjugation.

We have thus found that the general lightcone gauge field of eq. (2.45), which is neither self-dual nor anti-self-dual, generates a combination of two area-preserving diffeomorphisms, in $(u, Y)$ and $(u, X)$ planes respectively. These two diffeomorphisms are not arbitrary, but linked by the fact that the total gauge field must be real in Lorentzian signature. We therefore expect the "kinematic algebra" of lightcone gauge electromagnetism to be some subgroup of the product group

$$\mathrm{Diff}_{(u,Y)} \times \mathrm{Diff}_{(u,X)}, \tag{2.52}$$

[6]Note that the complex conjugate relation between $k^\mu$ and $k^{\dagger\mu}$ is required to hold in Lorentzian signature. That eq. (2.51) is indeed the complex conjugate of eq. (2.48) in the Lorentzian Cartesian coordinates $(t, x, y, z)$ follows from the fact that complex conjugation reverses the roles of $X$ and $Y$. Thus, $\partial_Y$ is replaced by $\partial_X$, and the $X$ and $Y$ coordinates interchanged, in going from eq. (2.48) to eq. (2.51).

where we denote the area-preserving diffeomorphism group associated with the family of planes $Z$ by $\mathrm{Diff}_Z$. To find this, we may substitute eqs. (2.48), (2.51) into eq. (2.45) to get (in lightcone coordinates)

$$(A^u, A^v, A^X, A^Y) = (\partial_Y \phi + \partial_X \phi^\dagger, 0, -\partial_u \phi^\dagger, -\partial_u \phi). \tag{2.53}$$

In order to understand which diffeomorphisms this generates, it is convenient to transform from $X$ and $Y$ to the Cartesian coordinates $x$ and $y$, keeping the lightcone coordinates $u$ and $v$ as is. The resulting transformed field is then given by

$$(A^u, A^v, A^x, A^y) = \frac{1}{\sqrt{2}}\Big(\partial_x(\phi+\phi^\dagger) + i\partial_y(\phi-\phi^\dagger), 0, -\partial_u(\phi+\phi^\dagger), -i\partial_u(\phi-\phi^\dagger)\Big). \tag{2.54}$$

Eq. (2.54) generates diffeomorphisms whose integral curves have components in the $u$ direction, but also the $x$ and $y$ directions. Furthermore, the $x$- and $y$-components are independent, given that they are governed by the real and imaginary parts of $\phi$ respectively. Thus, a general $A^\mu$ generates diffeomorphisms in the three-dimensional volume spanned by $(u, x, y)$. In fact, they preserve volume, given that eq. (2.54) implies

$$\partial_u A^u + \partial_x A^x + \partial_y A^y = 0. \tag{2.55}$$

In general then, the diffeomorphism algebra of lightcone gauge electromagnetism is a 3d-volume-preserving subgroup of the product group of eq. (2.52). Interestingly, something more special happens if the scalar field $\phi$ is itself real ($\phi \in \mathbb{R}$). Then eq. (2.53) simplifies to

$$(A^u, A^v, A^X, A^Y) = ((\partial_X + \partial_Y)\phi, 0, -\partial_u\phi, -\partial_u\phi). \tag{2.56}$$

Again transforming from $(X, Y)$ to $(x, y)$, one finds that the only non-zero components of the gauge field are

$$A^u = \sqrt{2}\partial_x\phi, \quad A^x = -\sqrt{2}\partial_u\phi. \tag{2.57}$$

We also find

$$\partial_u A^u + \partial_x A^x = 0, \tag{2.58}$$

so that $A^\mu$ generates area-preserving diffeomorphisms in the $(u, x)$ plane. There is a pleasing geometric interpretation of this, as shown in figure 4. Each individual term in eq. (2.45) generates area-preserving diffeomorphisms in a null plane, where the $y$ coordinates for points on the plane are pure imaginary, and thus do not show up in real Lorentzian coordinates. However, summing the two terms in eq. (2.45) for $\phi \in \mathbb{R}$ means that we keep only the projection into the $(u, x)$ plane. The fact that the resulting diffeomorphisms are then area-preserving means that this projection preserves the area-preserving property from the two original planes. Alternatively, one may consider the case in which the field $\phi$ is pure imaginary:

$$\phi = i\xi, \quad \xi \in \mathbb{R}. \tag{2.59}$$

Then eq. (2.53) becomes

$$(A^u, A^v, A^X, A^Y) = \Big(i(\partial_Y - \partial_X), 0, i\partial_u\xi, -i\partial_u\xi\Big) \tag{2.60}$$

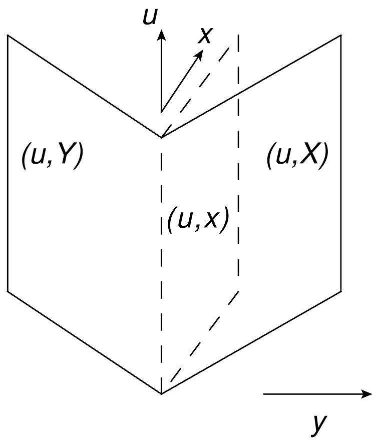


**Figure 4.** Each term in eq. (2.45) generates area-preserving diffeomorphisms in given $(u, Y)$ or $(u, X)$ planes, shown on the left and right respectively, where the $y$ components are pure imaginary. Summing these two terms projects these diffeomorphisms into the $(u, x)$ plane if $\phi \in \mathbb{R}$, and the projection preserves the area-preserving property.

from which one finds non-zero components after transforming to $(x, y)$

$$A^u = -\sqrt{2}\partial_y \xi, \quad A^y = \sqrt{2}\partial_u \xi, \tag{2.61}$$

and thus

$$\partial_u A^u + \partial_y A^y = 0. \tag{2.62}$$

We now have area-preserving diffeomorphisms in the $(u, y)$ plane, where again the area-preserving property is inherited, via a projection, from the original $\alpha$- and $\beta$-planes. To summarise, in both the pure real and imaginary $\phi$ cases, the kinematic algebra is one of real two-dimensional symplectomorphisms, i.e. the closed subgroup of full four-dimensional symplectomorphisms that we discussed in section 2.2.2.

### 2.4 Gauge dependence of the diffeomorphism algebra

Before moving on to discuss interacting theories, let us also note that the abelian / linearised context allows us to examine the issue of gauge-dependence of kinematic algebras. As mentioned in the introduction, precisely how kinematic algebras of a given theory depend upon the choice of gauge remains an open question. Indeed, this has only recently been explored for the best-known kinematic algebra, namely that of area-preserving diffeomorphisms for self-dual Yang-Mills theory in the lightcone gauge. As ref. [43] has shown, the kinematic algebra of self-dual Yang-Mills in other gauges is not expected to be a strict Lie algebra, but may instead involve a potentially infinite number of higher brackets in the $\mathrm{BV}_\infty^\square$ formalism.

In an abelian theory, all gauge fields are associated with the Lie algebra of diffeomorphisms, which does indeed constitute a strict Lie algebra. However, points in the space of diffeomorphisms, as shown in figure 2(a), constitute particular gauge fields, in a given gauge. If we instead vary the gauge and consider a so-called *gauge orbit* in the space of gauge fields, this will appear as a line in the space of diffeomorphisms, as exemplified in figure 5.

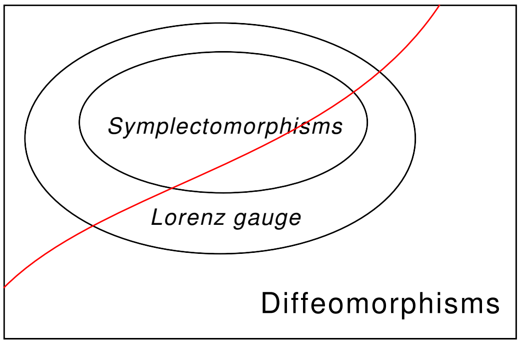


**Figure 5.** The set of all physically equivalent abelian gauge fields (related by a gauge transformation) shows up as a line — shown in red — in the space of all possible diffeomorphisms.

Let us now consider a gauge field that may be chosen to generate a symplectomorphism. The explicit form of arbitrary gauge fields in the same orbit is then obtainable as

$$A_\mu = \hat{k}_\mu \phi + \partial_\mu \chi, \tag{2.63}$$

for some function $\chi$. It is not true in general that

$$\partial_\mu \chi = \Omega_{\mu\nu} \partial_\nu \chi' \tag{2.64}$$

for some $\chi'$. Thus, a general gauge transformation will lead to a vector field that does not preserve the symplectic form. If $\chi$ is harmonic ($\partial^2 \chi = 0$), then the gauge field will at least remain in the Lorenz gauge, but this itself will no longer be true if we consider non-harmonic functions $\chi$.[7] Thus, by varying $\chi$, we will gradually move out of the special subgroups of the diffeomorphism algebra shown in figure 5. As we will see in the following section, in certain cases we can build kinematic algebras for interacting theories by relating these to the diffeomorphism algebra of an abelian theory. Thus, the gauge-dependence of the diffeomorphism algebra provides a direct analogue of the gauge-dependence of non-abelian kinematic algebras.

Note that the issue of gauge dependence has also been comprehensively explored in the $\mathrm{BV}_\infty^\square$ formalism, going back to the original analysis of Reiterer [30]. In that formalism, the kinematic algebra contains a derived bracket based on both the interaction vertices of the theory, and the propagator. The gauge-dependence of the latter thus feeds directly into the gauge-dependence of the kinematic algebra.

## 3 Interacting theories and kinematic algebras

In the previous section, we have seen that we can classify certain meaningful subgroups of diffeomorphism algebras of abelian gauge fields, which have a definite physical interpretation.

[7]In the self-dual sector, the maximal subset of residual symmetries which preserve the light-cone gauge, and hence the Lie kinematic algebra, was studied in [61, 62]. This was shown to double copy exactly to the residual subset of symmetries in self-dual gravity which preserve the Plebanski form of the action. More generally, the role of gauge choices and their residual symmetries in extending the double copy beyond flat backgrounds was explored in [63].

As we have stressed repeatedly throughout, however, this is not what is usually meant by the kinematic algebra of a gauge theory, which is instead associated with interaction terms. In this section, we argue that the ideas of the previous section remain useful, in allowing us to look for interacting theories that have straightforward kinematic algebras, and our starting point will be to reinterpret the well-known case of self-dual Yang-Mills theory.

### 3.1 Self-dual Yang-Mills theory and the Poisson bracket

As first presented in ref. [6], and discussed also above, we can obtain the field equation for self-dual Yang-Mills theory in lightcone gauge by making the ansatz of eq. (2.15), where $\hat{k}_\mu$ satisfies the conditions of eq. (2.8). Substituting this into the Yang-Mills equations yields

$$\hat{k}_\nu \left[\partial^2\phi^a + 2g f^{abc}(\hat{k}^\mu\phi^b)(\partial_\mu\phi^c)\right] = 0. \tag{3.1}$$

Introducing the matrix-valued scalar field

$$\mathbf{\Phi} = \phi^a \mathbf{T}^a, \tag{3.2}$$

eq. (3.1) assumes the form

$$\partial^2\mathbf{\Phi} + 2ig\,\{[\mathbf{\Phi},\mathbf{\Phi}]\} = 0, \tag{3.3}$$

where we have introduced the double bracket

$$\{[\mathbf{\Phi},\mathbf{\Phi}]\} = i f^{abc}\Omega^{\mu\nu}(\partial_\mu\phi^b)(\partial_\nu\phi^c)\mathbf{T}^a. \tag{3.4}$$

Comparison with eqs. (2.2), (2.35) reveals that this bracket combines a Poisson bracket in the kinematic variables, with the conventional Lie bracket of two colour generators. This in turn means that the interaction term of the theory carries structure constants of both the (colour) Lie algebra, and the kinematic Poisson algebra. Thus, these structure constants will appear alongside each other in all perturbative solutions of the theory, such that BCJ duality is manifest.

There is an interesting way to reinterpret this result, based on the ideas of the previous section. In particular, if we consider an abelian (or linearised non-abelian) gauge theory, we can focus on the subsector of the theory (in a particular gauge), such that the gauge field is Hamiltonian, and defined according to a differential operator $\hat{k}_\mu$ satisfying the conditions of eq. (2.8). We can then consider extending the theory to make it interacting, by using the building blocks that already exist at linear level. That is, we can make a Poisson bracket out of the symplectic form coefficients that already exist in the Hamiltonian gauge field, and then combine this with the colour Lie bracket to make an interaction term. There is then a sense in which the kinematic algebra of the interacting theory is inherited from the diffeomorphism algebra that already appears at linearised level. This link is made more precise by eq. (2.38), which expresses the fact that the Lie bracket of two Hamiltonian fields is itself Hamiltonian, but where the Hamiltonian of the resulting field is given by the Poisson bracket of the two original Hamiltonians. Thus, the Poisson bracket in the scalar formulation of the theory encodes the underlying Lie algebra of the Hamiltonian vector fields $A^a_\mu$.



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This suggests a general recipe for constructing non-linear extensions of linearised gauge theory, where the interaction term is characterised by a combined Poisson / Lie bracket. By choosing different symplectic form coefficients $\Omega_{\mu\nu}$, one may obtain different interacting theories. One may also replace $\Omega_{\mu\nu}$ with a more general antisymmetric matrix, which does not satisfy the symplectic form conditions, but nevertheless yields a closed algebra of abelian gauge fields. Interestingly, eq. (3.3), with a general antisymmetric matrix $\Omega_{\mu\nu}$ entering the Poisson bracket, was considered in e.g. refs. [23, 24] as a way of generalising (anti-)self-dual kinematic algebras. Here we provide a more systematic basis for this equation, and we are also able to obtain a direct geometric interpretation of the algebra: it corresponds to the diffeomorphism subgroup associated with the linearised gauge fields $A^a_\mu$. Unlike the (anti-)self-dual cases, diffeomorphisms associated with a general antisymmetric $\Omega_{\mu\nu}$ will not be area-preserving in general. To see this, note that a general antisymmetric matrix in four spacetime dimensions has rank $\leq 4$. Thus, the resulting diffeomorphisms, whilst still being volume-preserving, are not guaranteed to reduce to acting in lower-dimensional hypersurfaces.

### 3.2 Electromagnetism coupled to scalar matter

In the previous section, we have seen that one may construct non-linear theories whose kinematic algebras are based on the diffeomorphism algebra that already exists at linearised level. Those cases involved constructing a double bracket consisting of a colour Lie bracket, combined with a kinematic (Poisson) bracket. This is an intrinsically non-abelian construction, and our aim in this section is to show that a similar idea can be used, even if the gauge field is abelian. Let us start with an abelian gauge field $A_\mu$ and, in line with the examples of the previous section, we will restrict to the subset of Hamiltonian fields, such that

$$A_\mu = \Omega_{\mu\nu}\partial^\nu \phi, \tag{3.5}$$

for some scalar field $\phi$. The general vacuum field equation for $A_\mu$ is

$$\partial^2 A_\mu - \partial_\mu(\partial \cdot A) = 0 \tag{3.6}$$

which, in the case of Hamiltonian vector fields, yields

$$\partial^2 \phi = 0. \tag{3.7}$$

As before, considering Hamiltonian vector fields implies that there is a symplectic form, which can in turn be used to build a Poisson bracket. We can then look for non-linear extensions of eq. (3.7), and investigate whether any of them can be seen as truncations of some more complete theory. If we wish to preserve the fact that $A_\mu$ is abelian, then there is no non-zero Poisson bracket involving $\phi$ with itself:

$$\{\phi, \phi\} = 0. \tag{3.8}$$

Instead, we can consider an additional scalar field $\psi$, which at linear level satisfies the Klein-Gordon equation

$$\partial^2 \psi = 0. \tag{3.9}$$

This equation can then be extended non-linearly as

$$\partial^2\psi + c_1\{\psi, \phi\} = 0, \tag{3.10}$$

where we have used the fact that one may make a Poisson bracket out of the scalar field $\psi$, and the scalar that enters the gauge field via eq. (3.5). The question then naturally arises as to whether eq. (3.10) is a physically consistent theory, which may in turn depend on the value of the coefficient $c_1$. Indeed, in order to be talking about a gauge field interacting with $\psi$ at all, it must be the case that eq. (3.10) — or some generalisation of it — be gauge-covariant. Let us consider gauge transformations that preserve the Hamiltonian nature of $A_\mu$:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi, \tag{3.11}$$

where the corresponding gauge transformation for the scalar field is

$$\psi \to \psi' = e^{-ie\chi}\psi, \tag{3.12}$$

and where there will be a restriction on $\chi$:

$$\partial_\mu\chi = \Omega_{\mu\nu}\partial^\nu\alpha \tag{3.13}$$

for some $\alpha$. Using eq. (3.5), one may rewrite eq. (3.10) as

$$\partial^2\psi + c_1 A_\mu\partial^\mu\psi = 0, \tag{3.14}$$

which under a gauge transformation satisfies

$$\partial^2\psi' + A'_\mu\partial^\mu\psi' = 0 \quad \to \quad \partial^2\psi + A_\mu\partial^\mu\psi + \Delta = 0, \tag{3.15}$$

with

$$\Delta = (c_1 - 2ie)(\partial_\mu\chi)(\partial^\mu\psi) - iec_1 A_\mu(\partial^\mu\chi)\psi - (iec_1 + e^2)(\partial_\mu\chi)(\partial^\mu\chi)\psi. \tag{3.16}$$

There is no solution for $c_1$ that yields $\Delta = 0$, corresponding to the well-known fact that one must add a seagull vertex to scalar QED in order to make it gauge-invariant. Let us then correct eq. (3.14) to read

$$\partial^2\psi + c_1 A_\mu\partial^\mu\psi + c_2 A^\mu A_\mu\psi = 0. \tag{3.17}$$

Upon doing so and carrying through the above steps, the difference between field equations in different gauges of eq. (3.16) becomes instead

$$\Delta = (c_1 - 2ie)(\partial_\mu\chi)(\partial^\mu\psi) + (2c_2 - iec_1)A_\mu(\partial^\mu\chi)\psi + (c_2 - iec_1 - e^2)(\partial_\mu\chi)(\partial^\mu\chi)\psi. \tag{3.18}$$

The unique solution for $\Delta = 0$ is $(c_1, c_2) = (2ie, -e^2)$, so that the gauge-invariant scalar field equation is

$$\partial^2\psi + 2ie\{\psi, \phi\} - e^2 A^\mu A_\mu\psi = 0. \tag{3.19}$$

This has a cubic term, arising from a quartic interaction in the Lagrangian for the theory that gives rise to this equation of motion. It is not then true in general that there is a straightforward kinematic Lie algebra, i.e. such that there are up-to-quadratic terms in the field equation only.

We can of course find the subsector of solutions of eq. (3.19) for which the cubic term vanishes, and the criterion for this is straightforward. From eq. (3.5), the final term in eq. (3.19) will vanish provided

$$\Omega_{\mu\alpha}\Omega^{\mu}{}_{\beta} = 0. \tag{3.20}$$

This property is satisfied by self-dual field configurations in the light-cone gauge. In this case the gauge field must satisfy eq. (2.8), which for a Hamiltonian vector field written in terms of a symplectic form, amounts to eq. (3.20). As discussed previously in section 2.1, linearised self-dual solutions can be written in terms of a superposition of 't Hooft symbols, which in turn allows for a simple geometric interpretation of the generated diffeomorphisms. Thus, by the general arguments of section 2.1, such fields will generate area-preserving diffeomorphisms in either $\alpha$- or $\beta$-planes.

Let us therefore consider the case of abelian self-dual solutions in eq. (3.19), such that it straightforwardly reduces to

$$\partial^2\psi + 2ie\{\psi, \phi\} = 0. \tag{3.21}$$

We now wish to ask whether this equation can be obtained from a top-down approach, such that it corresponds to an equation of motion in a particular theory. As we have seen that in attempting to extend electromagnetism with a non-trivial Poisson bracket we arrive at a self-dual photon coupled to a scalar, let us consider the abelian gauge field to be self-dual. To this end, consider the following Lagrangian

$$\mathcal{L} = B_{\mu\nu}F_{-}^{\mu\nu} + (D^{\mu}\psi)^{\dagger}(D_{\mu}\psi), \tag{3.22}$$

where we have introduced the covariant derivative

$$D_{\mu} = \partial_{\mu} + ieA_{\mu}. \tag{3.23}$$

Here $B_{\mu\nu}$ are the components of an anti-self dual two-form and $F_{-}$ is the anti-self-dual part of the field strength. The field $B_{\mu\nu}$ acts as a Lagrange multiplier in the action, enforcing the self-duality of the abelian field strength. The theory thus corresponds to self-dual electromagnetism coupled to a complex scalar field.

By adopting a light-cone gauge and integrating out two of the three independent components of $B_{\mu\nu}$, we enforce a Hamiltonian form for the gauge field as well as the condition in eq. (3.20) (see e.g. ref. [64] for an example of this procedure in the non-abelian case). The result is an action only in terms of scalar degrees of freedom

$$\mathcal{L} = \bar{\phi}\,\partial^2\phi - \psi^{\dagger}\partial^2\psi - i\Omega^{\mu\nu}\partial_{\nu}\phi\left(\psi^{\dagger}\partial_{\mu}\psi - \psi\partial_{\mu}\psi^{\dagger}\right), \tag{3.24}$$

where $\bar{\phi}$ is the final component of $B_{\mu\nu}$ and the quartic interaction is not present as a consequence of eq. (3.20). We can recognise the first term as a linearised form of the self-dual Yang-Mills action in light-cone gauge [65], where $\phi$ and $\bar{\phi}$ are interpreted as the positive and negative helicity degrees of freedom of the gauge field respectively. By integrating by parts and making use of eqs. (2.40), (2.41) we obtain

$$\mathcal{L} = \bar{\phi}\,\partial^2\phi - \psi^{\dagger}\partial^2\psi + 2ie\phi\{\psi, \psi^{\dagger}\}. \tag{3.25}$$



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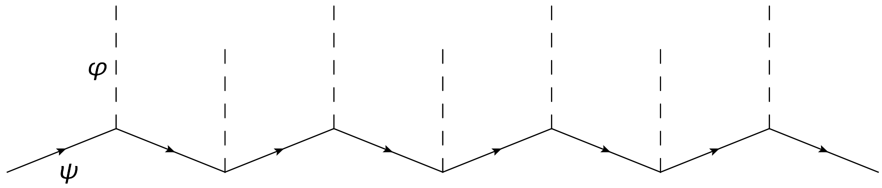


**Figure 6.** Example Feynman diagram generated by the theory of eq. (3.24).

Thus, we see that the symplectic form present in the linearised self-dual gauge field induces a Poisson bracket structure in the interaction vertex. The equations of motion for this theory are

$$\partial^2\phi = 0, \tag{3.26}$$
$$\partial^2\bar{\phi} + 2ie\{\psi, \psi^\dagger\} = 0, \tag{3.27}$$
$$\partial^2\psi + 2ie\{\psi, \phi\} = 0, \tag{3.28}$$
$$\partial^2\psi^\dagger - 2ie\{\psi^\dagger, \phi\} = 0. \tag{3.29}$$

Equation (3.28) corresponds precisely to eq. (3.21). In all cases, the non-linear terms inherit a Poisson bracket structure from the symplectic form present in the gauge field. This theory can be used to generate tree-level amplitudes in which a backbone of scalar $\psi$ exchanges radiates a series of photon states $\phi$, as exemplified in figure 6. This is clearly only a subset of the amplitudes contained in the full theory of electromagnetism coupled to a scalar field. However, it is interesting that a subset indeed exists, where the vertices can be associated with kinematic structure constants inherited from a Poisson bracket.

As in the Yang-Mills case, the kinematic algebra ceases to be straightforward once both self-dual and anti-self-dual degrees of freedom are included. To see this explicitly, let us view the action of eq. (3.22) as a sector of full scalar QED, where the action for this theory is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\psi)^\dagger(D_\mu\psi). \tag{3.30}$$

We now follow a standard procedure for reducing this action to one only in terms of the propagating degrees of freedom, as was done for Yang-Mills theory in ref. [66]. We choose the light-cone gauge $A_u = 0$. Then, in the scalar QED action, $A_v$ appears quadratically and can be functionally integrated out. Labelling the remaining components of the gauge field as $A_Y = A, A_X = \bar{A}$, we obtain an action

$$\begin{aligned}\mathcal{L} = &-\bar{A}\partial^2 A - \psi^\dagger\partial^2\psi + ieA\left[\psi^\dagger\partial_X\psi - \psi\partial_X\psi^\dagger - \frac{\partial_X}{\partial_u}\left(\psi^\dagger\partial_u\psi - \psi\partial_u\psi^\dagger\right)\right] \\ &+ ie\bar{A}\left[\psi^\dagger\partial_Y\psi - \psi\partial_Y\psi^\dagger - \frac{\partial_Y}{\partial_u}\left(\psi^\dagger\partial_u\psi - \psi\partial_u\psi^\dagger\right)\right] - 2e^2 A\bar{A}|\psi|^2 + \frac{e^2}{2}\left[\frac{1}{\partial_u}\left(\psi^\dagger\partial_u\psi - \psi\partial_u\psi^\dagger\right)\right]^2\end{aligned} \tag{3.31}$$

The fields $A$ and $\bar{A}$ are interpreted as the positive and negative helicity degrees of freedom of the gauge field. The action contains three-point $(A\,\psi\,\psi^\dagger)$ and $(\bar{A}\,\psi\,\psi^\dagger)$ vertices, a four-point $(A\,\bar{A}\,\psi\,\psi^\dagger)$ vertex, and a quartic scalar vertex. This theory will generate the amplitudes exemplified by figure 6, and more besides. However, the quartic vertices in eq. (3.31) will not enter the ladder amplitudes described above, which result upon keeping only the $(A\,\psi\,\psi^\dagger)$ vertex. To make contact with eq. (3.24) explicitly, we may perform the field redefinitions $A=\partial_u\phi, \bar{A}=\partial_u^{-1}\bar{\phi}$, such that keeping only the first cubic vertex in eq. (3.31) and disregarding the quartic vertices yields[8]

$$\mathcal{L}=\bar{\phi}\partial^2\phi-\psi^\dagger\partial^2\psi-ie\phi\left[\partial_u\left(\psi^\dagger\partial_X\psi-\psi\partial_X\psi^\dagger\right)-\partial_X\left(\psi^\dagger\partial_u\psi-\psi\partial_u\psi^\dagger\right)\right], \tag{3.32}$$

where we have integrated by parts. After a little algebra, the vertex structure reduces to a Poisson bracket and we obtain eq. (3.25).

Returning to eq. (3.31), we see that we lose any clear sign of the diffeomorphism algebra present in the linearised theory due to the non-vanishing of the quartic interaction. Similar to full Yang-Mills theory, it is the presence of this higher-order interaction that mixes the self-dual and anti-self dual degrees of freedom of the gauge field, and disrupts the possibility of identifying the kinematic algebra with a straightforward Lie algebra. This complication disappears in the self-dual sector, which constitutes a particular truncation of the theory.

In both the self-dual Yang-Mills and scalar QED examples, relevant equations of motion are "simple" in that their kinematic algebras terminate at cubic order, due to a judicious choice of subsector of the full theory. In both cases, this involves choosing Hamiltonian gauge fields, such that a Poisson bracket may be used to construct the interaction terms. It is worth asking whether one may instead use the Lie bracket of gauge fields, and to consider actions that are manifestly written in terms of vector gauge fields. That this will not lead to a physical theory in general is well-documented in the literature [45]. However, there are indeed special cases where this occurs, which have in fact already appeared in the literature. Let us take each case in turn.

### 3.3 Non-abelian Chern-Simons theory

Non-abelian Chern-Simons theory is a certain gauge theory in three spacetime dimensions, whose topologically non-trivial solutions have led to a variety of applications in mathematical physics, including connections to knot theory (see e.g. ref. [67] for a review). In ref. [26], the kinematic algebra of this theory was shown to be a simple Lie algebra of volume-preserving diffeomorphisms, if a particular gauge (the Lorenz gauge) was used for the field $A_\mu^a$. Here, we show that this conclusion naturally arises from the ideas of this paper, thus providing an alternative point of view on this result.

We start by considering the following action for abelian Chern-Simons theory:

$$S_{\rm CS}=\frac{k}{4\pi}\int d^3x\,\frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{3.33}$$

where $k$ is a constant parameter. The field equation for $A_\mu$ is

$$\epsilon^{\mu\nu\rho}\left[\partial_\nu A_\rho-\partial_\rho A_\nu\right]\equiv\epsilon^{\mu\nu\rho}F_{\nu\rho}=0, \tag{3.34}$$

where we have recognised the abelian field strength $F_{\mu\nu}$. This in turn implies $F_{\mu\nu}=0$, such that solutions of the theory are pure gauge.

[8]The theory of eq. (3.32) appears to be related to the bosonic truncation of $\mathcal{N}=4$ Super-Yang-Mills theory. We thank the anonymous referee for this suggestion.



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Similar to the previous sections, we may regard $A_\mu$ as generating diffeomorphisms, and then look to extend the theory by using the Lie bracket of diffeomorphisms in forming an interaction term. To do this, we can extend $A_\mu$ to make a non-abelian gauge field with components $A^a_\mu$. We may then consider the double bracket

$$[[\mathbf{A},\mathbf{A}]] = \mathbf{T}^a\, f^{abc}\left[A^{\mu\, b}\partial_\mu, A^{\nu\, c}\partial_\nu\right], \tag{3.35}$$

written in terms of the gauge field, contracted with colour and kinematic generators:

$$\mathbf{A} = A^{\mu\, a}\mathbf{T}^a\partial_\mu. \tag{3.36}$$

Equation (3.35) consists of a simultaneous colour Lie bracket, and kinematic Lie bracket (i.e. the latter corresponds to a commutator of diffeomorphisms). This is analogous to eq. (3.3), whose double bracket contains a colour Lie bracket and a Poisson (kinematic) bracket. For dimensional reasons, we cannot simply add this bracket to the non-abelian version of eq. (3.34). However, in line with ref. [26], we may instead contract eq. (3.34) with the combination $\epsilon_{\sigma\mu\alpha}\partial^\alpha$ to obtain

$$\partial^2 A_\sigma - \partial_\sigma(\partial\cdot A) = 0, \tag{3.37}$$

such that a suitable non-abelian generalisation is

$$\partial^2 A^a_\sigma - \partial_\sigma(\partial\cdot A^a) + \gamma f^{abc}\left[A^b\cdot\partial A^c_\sigma - A^c\cdot\partial A^b_\sigma\right] = 0, \tag{3.38}$$

where we have substituted the explicit form of the Lie bracket of two vector fields. In order for this to be a consistent (sub-)theory, the equation of motion must be gauge-covariant, or at least correspond to some suitable gauge-fixing. This will fix the undetermined parameter $\gamma$. To this end, one may use the product rule, and rearrange terms, to show that eq. (3.38) is equivalent to

$$\partial^\rho F^a_{\rho\sigma} + (2\gamma+1)f^{abc}A^b\cdot\partial A^c_\sigma + f^{abc}(\partial\cdot A^b)A^c_\sigma = 0, \tag{3.39}$$

where

$$F^a_{\rho\sigma} = \partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho - f^{abc}A^b_\rho A^c_\sigma \tag{3.40}$$

denotes a component of the non-abelian field strength. We can then find a suitably gauge-fixed field equation by setting

$$\gamma = -\frac{1}{2}, \quad \partial\cdot A^a = 0, \tag{3.41}$$

after which eq. (3.40) reduces to

$$\partial^\rho F^a_{\rho\sigma} = 0. \tag{3.42}$$

Given we have introduced an extra derivative above, we can then infer the field equation

$$F^a_{\rho\sigma} = 0, \tag{3.43}$$

which eq. (3.41) tells us is in Lorenz gauge. This is the known field equation of Chern-Simons theory, and substituting eq. (3.41) into eq. (3.38) reveals that this can be written as

$$\partial^2 A^a_\sigma - \frac{1}{2}f^{abc}\left[A^b\cdot\partial A^c_\sigma - A^c\cdot\partial A^b_\sigma\right] = 0 \tag{3.44}$$

or, when contracted with generators,

$$\partial^2 \mathbf{A} - \frac{1}{2}\left[[\mathbf{A}, \mathbf{A}]\right] = 0. \tag{3.45}$$

To see that this agrees with a standard derivation of the non-abelian Chern-Simons equation, note that the action for this theory can be written in components as[9]

$$S_{\text{CS,non-abel.}} = \frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\left(\frac{1}{2}A^a_\mu \partial_\nu A^a_\rho - \frac{1}{6}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right). \tag{3.46}$$

The field equation for $A^a_\mu$ is then

$$\epsilon^{\mu\nu\rho}\left[\partial_\nu A^a_\rho - \frac{1}{2}f^{abc}A^b_\nu A^c_\rho\right] = 0. \tag{3.47}$$

To compare with our above results, we must contract eq. (3.47) with $\epsilon_{\sigma\mu\alpha}\partial^\alpha$ to obtain

$$\partial^2 A^a_\sigma - \partial_\sigma(\partial \cdot A^a) - \frac{1}{2}f^{abc}\left[A^c_\sigma \partial \cdot A^b + A^b \cdot \partial A^c_\sigma - (b \leftrightarrow c)\right] = 0. \tag{3.48}$$

Then choosing the Lorenz gauge $\partial \cdot A^a = 0$ yields eq. (3.44) as required. Our construction of this theory using the double bracket above makes manifest that there is a Lie kinematic algebra. From the top-down point of view, however, we may see why a straightforward Lie kinematic algebra is not manifest at the level of the equation of motion if we are not in Lorenz gauge. Let us return to the full field equation of eq. (3.48), and define an alternative bracket whose vector components are

$$[A_1, A_2]^\mu_{\text{CS}} = a_1\left((\partial \cdot A_1)A^\mu_2 - (\partial \cdot A_2)A^\mu_1\right) + a_2\left(A_1 \cdot \partial A^\mu_2 - A_2 \cdot \partial A^\mu_1\right), \tag{3.49}$$

where $\{A_i\}$ are vector fields, and $\{a_i\}$ constant parameters. This bracket is skew-symmetric in its arguments, and reduces to the standard Lie bracket for $(a_1, a_2) = (0, 1)$. Furthermore, upon choosing the special case $(a_1, a_2) = (1, 1)$, we can express eq. (3.48) as

$$\partial^2 A^{a\sigma} - \partial^\sigma(\partial \cdot A^a) - \frac{1}{2}f^{abc}[A^b, A^c]^\sigma_{\text{CS}} = 0. \tag{3.50}$$

As in our previous examples, this contains a double bracket, this time consisting of a Lie bracket in the colour group, and the generalised kinematic bracket of eq. (3.49). What prevents the identification of a straightforward kinematic algebra, however, is the fact that the bracket of eq. (3.49) does not satisfy the Jacobi identity. Denoting a momentum-space gauge field by $A(p)$, an explicit calculation reveals that

$$\begin{aligned}&\Big[[A_1(p_1), A_2(p_2)], A_3(p_3)\Big]^\mu_{\text{CS}} + \Big[[A_2(p_1), A_3(p_2)], A_1(p_3)\Big]^\mu_{\text{CS}} + \Big[[A_3(p_1), A_1(p_2)], A_2(p_3)\Big]^\mu_{\text{CS}} = \\ &-a_1\Big\{\Big[(p_1 \cdot A_1)(p_1 \cdot A_2) - (p_2 \cdot A_2)(p_2 \cdot A_1)\Big]A^\mu_3 + \Big[(p_2 \cdot A_2)(p_2 \cdot A_3) - (p_3 \cdot A_3)(p_3 \cdot A_2)\Big]A^\mu_1 \\ &+ \Big[(p_3 \cdot A_3)(p_3 \cdot A_1) - (p_1 \cdot A_1)(p_1 \cdot A_3)\Big]A^\mu_2\Big\}.\end{aligned} \tag{3.51}$$

[9] It is more common to introduce the one-form $A \equiv A^a_\mu dx^\mu \mathbf{T}^a$, where $\{\mathbf{T}^a\}$ are the generators of the gauge group, and to consider the action $S = \frac{k}{4\pi}\int d^3x\, \text{Tr}\left(A \wedge dA + \frac{2i}{3}A \wedge A \wedge A\right)$. This reduces to eq. (3.33) after substituting components.



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The non-exact nature of the Jacobi identity can be directly traced to the coefficient $a_1$ appearing in eq. (3.49), and thus to the additional contribution that supplements the strict Lie bracket of vector fields. This contribution vanishes only in Lorenz gauge in general, such that we indeed see that the kinematic algebra must be a more complicated mathematical structure than a Lie algebra if we go to arbitrary gauges. Similar considerations were applied to four-dimensional Yang-Mills theory in ref. [45], which proposed certain generalisations of Lie algebras as underlying kinematic algebras. An extended discussion of Chern-Simons theory has been given in the $\mathrm{BV}^{\square}_{\infty}$ approach in refs. [35, 39]. In the latter, rather than contracting eq. (3.47) with $\epsilon_{\sigma\mu\alpha}\partial^{\alpha}$, a more general procedure is given for extracting a kinematic bracket of gauge fields, based on the $\mathrm{BV}^{\square}_{\infty}$ algebra underlying the theory. This then turns out to yield a simple Lie bracket after all.

Next, we examine a second case in which the Lie bracket of vector fields appears in an interacting theory, and which includes the theory of this section as a special case.

### 3.4 Semi-abelian Yang-Mills theory

Recently, ref. [29] introduced an interesting field theory, aimed at unifying diverse examples of theories obeying colour-kinematics duality, as well as developing systematic procedures for constructing BCJ-dual kinematic numerators in scattering amplitudes. The authors refer to this as *semi-abelian Yang-Mills theory*, and it has the following Lagrangian:

$$\mathcal{L}^{\mathrm{semi-YM}} = -\frac{1}{2}\mathrm{Tr}\left[\bar{\mathbf{F}}_{\mu\nu}\mathbf{F}^{\mu\nu}\right]. \tag{3.52}$$

Here $\mathbf{F}_{\mu\nu}$ is the field strength for a non-abelian gauge field $\mathbf{A}_{\mu}$ valued in the Lie algebra of $\mathrm{U}(N)$. The additional field strength $\bar{\mathbf{F}}_{\mu\nu}$ corresponds to a field $\bar{\mathbf{A}}_{\mu}$ associated with the gauge group $\mathrm{U}(1)^{N^2}$. In components, both gauge fields will carry a "colour index" $a$ taking values in the range $\{1,\ldots N^2\}$, and the Lagrangian takes the form

$$\mathcal{L}^{\mathrm{semi-YM}} = \bar{A}^{\nu\, a}\left[\partial^2 A^a_{\nu} - \partial_{\nu}(\partial\cdot A^a) - f^{abc}(\partial\cdot A^b)A^c_{\nu} + \frac{1}{2}f^{abc}\left(A^b\cdot\partial A^c_{\nu} - A^c\cdot\partial A^b_{\nu}\right)\right]. \tag{3.53}$$

Upon constructing the Euler-Lagrange equation for $\bar{A}^a_{\mu}$, one straightforwardly obtains that the field $\mathbf{A}\equiv A^{\mu\, a}\mathbf{T}^a\partial_{\mu}$ satisfies eq. (3.45) in the Lorenz gauge. There is thus again a double-bracket combining the colour Lie bracket with the Lie algebra of diffeomorphisms. Furthermore, the latter are volume-preserving, given the Lorenz gauge condition. As argued in ref. [29], both self-dual Yang-Mills theory and Chern-Simons theory can be seen as special cases of semi-abelian Yang-Mills theory. From the perspective of this paper, we can perhaps regard semi-abelian Yang-Mills theory as the theory one arrives at upon starting with linearised Yang-Mills theory, and demanding a well-defined kinematic algebra by looking for a double bracket based on the Lie algebra of diffeomorphisms. It would then be interesting to know if this conclusion is unique. We provide a final novel example of an interacting theory containing a similar double bracket in the following section.

### 3.5 Fluid mechanics and kinematic algebras

Since its original incarnation involving gauge and gravity theories, the study of the double copy and its related kinematic algebras has considerably broadened. Useful for our purposes



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is ref. [53], which considered a non-abelian generalisation of the Navier-Stokes equation of fluid mechanics:

$$(\partial_0 - \nu\nabla^2)u_i^a + f^{abc}u_j^b\partial_j u_i^c = J_i^a. \tag{3.54}$$

The quantity $u_i^a$ ($i \in \{1,2,3\}$) is the velocity field of a fluid of viscosity $\nu$, and satisfies the solenoidal requirement $\partial_i u_i^a = 0$. The velocity also carries an adjoint index $a$ associated with a non-abelian colour group, with structure constants $f^{abc}$. Finally, there is a source current $J_i^a$ on the right-hand side of eq. (3.54). This theory was used in ref. [53] for various purposes, including elucidating infrared properties of its scattering amplitudes, examining its kinematic algebra, and exploring its double copy to a bifluid theory, whose velocity field $u_{i\bar{i}}$ carries two independent spatial indices. Here we draw attention to the fact, already noted in ref. [53], that eq. (3.54) may be rewritten as

$$(\partial_0 - \nu\nabla^2)u_i^a + \frac{1}{2}f^{abc}f_{ijk}u_j^b u_k^c = J_i^a, \tag{3.55}$$

where

$$f_{ijk}v_j w_k = v_j\partial_j w_i - w_j\partial_j v_i \tag{3.56}$$

for two arbitrary vectors $v_j$ and $w_j$. Recognising the components of the Lie bracket of two vector fields (cf. eq. (2.5) in the relativistic case), we may instead write the field equation as

$$(\partial_0 - \nu\nabla^2)\mathbf{u} + \frac{1}{2}[[\mathbf{u},\mathbf{u}]] = \mathbf{J}. \tag{3.57}$$

We have here introduced the velocity field

$$\mathbf{u} = u_i^a\mathbf{T}^a\partial_i \tag{3.58}$$

contracted with the appropriate generators. We have also introduced a double bracket (cf. eq. (3.4)) consisting of simultaneous Lie brackets in both the colour and diffeomorphism algebras:

$$[[\mathbf{u},\mathbf{u}]] \equiv f^{abc}f_{ijk}u_j^b u_k^c. \tag{3.59}$$

This theory fits into the scheme outlined in the rest of this paper for Hamiltonian vector fields in Yang-Mills and abelian gauge theory coupled to a scalar. That is, one may take the diffeomorphism algebra associated with the linear theory of $u_j^a$, and use it to construct a bracket for use in a cubic interaction term. As in the case of Chern-Simons theory, the Lie bracket of diffeomorphisms in the linearised theory survives in the interaction term. Due to the solenoidal requirement, these are volume-preserving diffeomorphisms, which is directly analogous to the use of the Lorenz gauge in Chern-Simons theory. It is worthwhile to note that the presence of Lie brackets in fluid mechanics has a direct physical interpretation: as we reviewed earlier, the Lie bracket of two vector fields can be interpreted in terms of the Lie derivative of one vector field along the other. In fluid mechanics, Lie derivatives naturally arise from convection: the Lie derivative of a vector $\boldsymbol{w}$ along a vector field $\boldsymbol{v}$ compares the change in the vector field $\boldsymbol{w}$ along the direction of $\boldsymbol{v}$ with the form $\boldsymbol{w}$ would take if it were simply dragged along the flow of $\boldsymbol{v}$. Thus, the kinematic algebra of this theory is directly traceable to the convection properties of fluid flows.



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Another potential example of these ideas appears to be ref. [68], which looks at a gauge theory for shallow water waves first presented in ref. [69]. The authors note that this description contains an area-preserving diffeomorphism algebra. This is interpreted as a residual gauge symmetry corresponding to the continuum limit of the freedom in relabelling discrete elements of the two-dimensional surface of the fluid. It would be interesting to study this theory in more detail, in order to see if one can indeed interpret the area-preserving diffeomorphisms as a kinematic algebra. For other work connecting diffeomorphism algebras with gauge theory, see ref. [70].

## 4 Conclusion

Kinematic algebras are relatively new structures underlying gauge, gravity and related field theories, and so remain somewhat mysterious. For a general gauge theory, the kinematic algebra is expected to be complicated, and not reducible to a straightforward Lie algebra. However, in certain theories — or sectors of theories — we are able to define a definite kinematic Lie algebra, without necessarily understanding its precise origin.

In this paper, we have gained insights into this phenomenon by studying simple abelian gauge theories. We argue that many known cases of kinematic algebras for nonlinear (sub-)theories can be obtained by taking well-defined subalgebras of the diffeomorphism algebra of gauge (vector) fields. In the particular case of the symplectomorphism subgroup, one may build a Poisson bracket involving the scalar field entering the gauge field, and use this to generate interaction terms. The case of self-dual Yang-Mills theory arises in this way, as do its various generalisations that have been previously explored in the literature. We obtain novel new examples of kinematic algebras involving area-preserving diffeomorphism algebras, such as the case of electromagnetism coupled to a complex scalar field. Interestingly, we have not found examples of interacting theories containing Poisson brackets based on full real four-dimensional symplectomorphisms, and it would be interesting to know if such cases exist. Furthermore, the question of exactly which (types of) theories have simple kinematic algebras remains open, and we hope that our insights prove useful in further exploring this issue.

We are also able to shed light on the issue of the gauge dependence of kinematic algebras, given that it is only for certain gauge choices (even at abelian level) that the subgroup of diffeomorphisms takes a minimal form. Finally, we noted that Lorenz-gauge Chern-Simons theory [26] and the non-abelian Navier-Stokes equation formulated in ref. [53] are yet more cases in which the diffeomorphism algebra already appearing at linear level can be used to formulate a consistent interacting theory, thereby furnishing the kinematic algebra with a direct geometric interpretation.

The study of kinematic algebras in recent years has provided tantalising hints of a hidden structure underlying gauge theories. Interestingly, making gauge invariance manifest makes for a simple Lagrangian, but obscures the kinematic algebra. On the other hand, making the kinematic algebra visible appears to lead to a much more complicated Lagrangian involving complex mathematical structures, thus obscuring the role of gauge invariance. We hope that our results may provide useful insights into how to navigate this quandary going forwards.



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