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Percolation transition for random forests in $d \geqslant 3$

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Abstract

The arboreal gas is the probability measure on (unrooted spanning) forests of a graph in which each forest is weighted by a factor $\beta > 0$ per edge. It arises as the $q \to 0$ limit of the q-state random cluster model with $p = \beta q$. We prove that in dimensions $d \geqslant 3$ the arboreal gas undergoes a percolation phase transition. This contrasts with the case of $d = 2$ where no percolation transition occurs.

The starting point for our analysis is an exact relationship between the arboreal gas and a non-linear sigma model with target space the fermionic hyperbolic plane $\mathbb{H}^{0|2}$. This latter model can be thought of as the 0-state Potts model, with the arboreal gas being its random cluster representation. Unlike the standard Potts models, the $\mathbb{H}^{0|2}$ model has continuous symmetries. By combining a renormalisation group analysis with Ward identities we prove that this symmetry is spontaneously broken at low temperatures. In terms of the arboreal gas, this symmetry breaking translates into the existence of infinite trees in the thermodynamic limit. Our analysis also establishes massless free field correlations at low temperatures and the existence of a macroscopic tree on finite tori.

Contents

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1 Introduction

This paper has two distinct motivations. The first is to study the percolative properties of the *arboreal gas*, and the second is to understand *spontaneously broken continuous symmetries*. We first present our results from the percolation perspective, and then turn to continuous symmetries.

1.1 Main results for the arboreal gas

The arboreal gas is the uniform measure on (unrooted spanning) forests of a weighted graph. More precisely, given an undirected graph $G = (\Lambda, E)$, a forest $F = (\Lambda, E(F))$ is an acyclic subgraph of G having the same vertex set as G. Given an edge weight $\beta > 0$ (inverse temperature) and a vertex weight $h \geqslant 0$ (external field), the probability of a forest F under the arboreal gas measure is

$$\mathbb{P}^G_{\beta,h}[F] = \frac{1}{Z^G_{\beta,h}} \beta^{|E(F)|} \prod_{T \in F} (1 + h|V(T)|) \tag{1.1}$$

where $T \in F$ denotes that T is a tree in the forest, i.e., a connected component of F, $|E(F)|$ is the number of edges in F, and $|V(T)|$ is the number of vertices in T. The arboreal gas is also known as the (weighted) uniform forest model, as Bernoulli bond percolation conditioned to be acyclic, and as the $q \to 0$ limit of the q-state random cluster model with p/q converging to β, see [57].

We study the arboreal gas on a sequence of tori $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ with L fixed and $N \to \infty$. To simplify notation, we will use Λ_N to denote both the graph and its vertex set. From the percolation point of view, the most fundamental question concerns whether a typical forest F under the law (1.1) contains a giant tree. In all dimensions, elementary arguments show that giant trees can exist only if $h = 0$ and if β is large enough, in the sense that connection probabilities decay exponentially whenever $h > 0$ or β is small; see Appendix A.2.

The existence of a percolative phase for $h = 0$ and β large does not, however, follow from standard techniques. The subtlety of the existence of a percolative phase is perhaps best evidenced by considering the case $d = 2$: in this case giant trees do not exist for any $\beta > 0$ [20]. Our main result is that for $d \geqslant 3$ giant trees do exist for β large and $h = 0$, and that truncated correlations have massless free field decay. To state our result precisely, let $\{0 \leftrightarrow x\}$ denote the event that 0 and x are connected, i.e., in the same tree.

Theorem 1.1 *Let $d \geqslant 3$ and $L \geqslant L_0(d)$. Then there is $\beta_0 \in (0, \infty)$ such that for $\beta \geqslant \beta_0$ there exist $\zeta_d(\beta) = 1 - O(1/\beta)$, $\mathsf{c}(\beta) = \mathsf{c} + O(1/\beta)$ with $\mathsf{c} > 0$, and $\kappa > 0$ such that*

$$\mathbb{P}^{\Lambda_N}_{\beta,0}[0 \leftrightarrow x] = \zeta_d(\beta) + \frac{\mathsf{c}(\beta)}{\beta |x|^{d-2}} + O\left(\frac{1}{\beta |x|^{d-2+\kappa}}\right) + O\left(\frac{1}{\beta L^{\kappa N}}\right), \tag{1.2}$$

where $|x|$ denotes the Euclidean norm.

Numerical evidence for this phase transition of the arboreal gas was given in [40]. More broadly our work was inspired by [21, 37, 38, 40, 60, 61], and we discuss further motivation later.

Although both the arboreal gas and Bernoulli bond percolation have phase transitions for $d \geqslant 3$, the supercritical phases of these models behave very differently: (1.2) shows that the arboreal gas behaves like a critical model even in the supercritical phase, in the sense that it has massless free field truncated correlation decay.

While this behaviour looks unusual when viewed through the lens of supercritical percolation, it is natural from the viewpoint of broken continuous symmetries. We will return to this point in Sect. 1.2.

Theorem 1.1 concerns the arboreal gas on large finite tori in zero external field (i.e., $h = 0$). Another limit to consider the arboreal gas in is the weak infinite volume limit. To this end, we consider the limit obtained by first taking $N \to \infty$ with $h > 0$ and then taking $h \downarrow 0$. In a manner similar to that for Bernoulli bond percolation in [47, Sect. 5] and [2, Sect. 2.2], the external field is equivalent to considering the arboreal gas on an extended graph $G^{\mathfrak{g}} = (\Lambda \cup \{\mathfrak{g}\}, E \cup E^{\mathfrak{g}})$ where $E^{\mathfrak{g}} = \Lambda \times \{\mathfrak{g}\}$ and each edge in $E^{\mathfrak{g}}$ has weight h. The additional vertex $\mathfrak{g}$ is called the *ghost* vertex. The measure (1.1) is then obtained by forgetting the connections to the ghost. This rephrases that the product in (1.1) is equivalent to connecting a uniformly chosen vertex in each tree T to $\mathfrak{g}$ with probability $h|V(T)|/(1+h|V(T)|)$. For vertices $x, y \in \Lambda$, we continue to denote by $\{x \leftrightarrow y\}$ the event that x and y are connected in the random forest subgraph of G with law (1.1), i.e., $\{x \leftrightarrow y\}$ denotes the event that there is a path from x to y in the random subgraph, and that this (necessarily unique) path does not contain $\mathfrak{g}$. We write $\{x \leftrightarrow \mathfrak{g}\}$ to denote the event that x is connected to $\mathfrak{g}$.

The event $\{0 \leftrightarrow \mathfrak{g}\}$ is a finite volume proxy for the event that the tree T_0 containing 0 becomes infinite in the infinite volume limit when $h \downarrow 0$. Indeed, let us define

$$\theta_d(\beta) = \lim_{h\downarrow 0} \lim_{N\to\infty} \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow \mathfrak{g}], \tag{1.3}$$

and let $\mathbb{P}_\beta^{\mathbb{Z}^d}$ be any (possibly subsequential) weak infinite volume limit $\lim_{h\downarrow 0} \lim_{N\to\infty} \mathbb{P}_{\beta,h}^{\Lambda_N}$. Then

$$\theta_d(\beta) = \mathbb{P}_\beta^{\mathbb{Z}^d}[|T_0| = \infty], \tag{1.4}$$

see Proposition A.6. By a stochastic domination argument it is straightforward to show that

$$\theta_d(\beta) = 0 \qquad \text{for } 0 \leqslant \beta < p_c(d)/(1 - p_c(d)) < \infty, \tag{1.5}$$

where $p_c(d)$ is the critical probability for Bernoulli bond percolation on $\mathbb{Z}^d$, see Proposition A.3. When $d = 2$, $\theta_2(\beta) = 0$ for all $\beta > 0$ by [20, Sect. 4.2]. The next theorem shows that for $d \geqslant 3$ the arboreal gas also has a phase transition in this infinite volume limit.

Theorem 1.2 *Let $d \geqslant 3$ and $L \geqslant L_0(d)$. Then there is $\beta_0 \in (0, \infty)$ such that for $\beta \geqslant \beta_0$ the limit (1.3) exists and*

$$\theta_d(\beta)^2 = \zeta_d(\beta) = 1 - O(1/\beta), \tag{1.6}$$

where $\zeta_d(\beta)$ is the finite volume density of the tree containing 0 from Theorem 1.1.

In fact, our proof shows that $\theta_d(\beta) \sim 1 - c/\beta$ with $c = (-\Delta^{\mathbb{Z}^d})^{-1}(0, 0) > 0$ the expected time a simple random walk spends at the origin. This behaviour is different from that of Bernoulli bond percolation and more generally that of the random cluster model with $q > 0$. For these models the percolation probability is governed by Peierls' contours and is $1 - O((1-p)^{2d})$ by [70, Remark 5.10].

That the arboreal gas behaves critically within its supercritical phase can be further quantified in terms of the following truncated two-point functions:

$$\tau_\beta(x) = \lim_{h\downarrow 0} \tau_{\beta,h}(x), \qquad \tau_{\beta,h}(x) = \lim_{N\to\infty} \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}], \tag{1.7}$$

$$\sigma_\beta(x) = \lim_{h\downarrow 0} \sigma_{\beta,h}(x), \qquad \sigma_{\beta,h}(x) = \lim_{N\to\infty} \Big(\mathbb{P}_{\beta,h}^{\Lambda_N}[0 \not\leftrightarrow \mathfrak{g}]^2 - \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \not\leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}, x \not\leftrightarrow \mathfrak{g}] \Big). \tag{1.8}$$

Theorem 1.3 *Under the assumptions of Theorem 1.2, for $\beta \geqslant \beta_0$, the limits (1.7)–(1.8) exist and there exist constants $\mathsf{c}_i(\beta) = \mathsf{c}_i + O(1/\beta)$ and $\kappa > 0$ such that*

$$\tau_\beta(x) = \frac{\mathsf{c}_1(\beta)}{\beta|x|^{d-2}} + O(\frac{1}{\beta|x|^{d-2+\kappa}}), \tag{1.9}$$

$$\sigma_\beta(x) = \frac{\mathsf{c}_2(\beta)}{\beta^2|x|^{2d-4}} + O(\frac{1}{\beta^2|x|^{2d-4+\kappa}}). \tag{1.10}$$

The constants satisfy $(\mathsf{c}_2(\beta)/\mathsf{c}_1(\beta)^2)\theta_d(\beta)^2 = 1$ and $\mathsf{c}(\beta) = 2\mathsf{c}_1(\beta)$, $\mathsf{c}(\beta)$ from Theorem 1.1.

Further results could be deduced from our analysis, but to maintain focus we have not carried these out in detail. We mention some of them below in Sect. 1.4 when discussing our results and open problems.

1.2 The $\mathbb{H}^{0|2}$ model and its continuous symmetries

In [37, 38], the arboreal gas was related to a fermionic field theory and a supersymmetric non-linear sigma model with target space one half of the degenerate supersphere $\mathbb{S}^{0|2}$. In [20] this was reinterpreted as a non-linear sigma model with hyperbolic target space $\mathbb{H}^{0|2}$, which we refer to as the $\mathbb{H}^{0|2}$ model for short. The reinterpretation was essential in [20]; it is less essential for the present work, but nevertheless we continue to use the $\mathbb{H}^{0|2}$ formulation of the model.

Briefly, the $\mathbb{H}^{0|2}$ model is defined as follows, see [20, Sect. 2] for further details. For every vertex $x \in \Lambda$, there are two (anticommuting) Grassmann variables ξ_x and η_x and we then set

$$z_x = \sqrt{1 - 2\xi_x\eta_x} = 1 - \xi_x\eta_x. \tag{1.11}$$

Thus the z_x commute with each other and with the odd elements ξ_x and η_x. The formal triples $u_x = (\xi_x, \eta_x, z_x)$ are supervectors with two odd components ξ_x, η_x

and an even component z_x. These supervectors satisfy the sigma model constraint $u_x \cdot u_x = -1$ for the super inner product

$$u_x \cdot u_y = -\xi_x \eta_y - \xi_y \eta_x - z_x z_y. \tag{1.12}$$

In analogy with the tetrahedral representation of the q-state Potts model, see [26, Sect. 2.2], the sigma model constraint can be thought of as $u_x \cdot u_x = q-1$ with $q=0$. The constraint is also reminiscent of the embedding of the hyperbolic space $\mathbb{H}^2$ in $\mathbb{R}^3$ equipped with the standard quadratic form with Lorentzian signature $(1,1,-1)$. Indeed, $-\xi_x \eta_y - \xi_y \eta_x$ is the fermionic analogue of the Euclidean inner product on $\mathbb{R}^2$.

Let F be a (non-commutative) polynomial in the variables $\{\xi_x, \eta_x\}_{x\in\Lambda}$. The expectation of F with respect to the $\mathbb{H}^{0|2}$ model is

$$\langle F\rangle_{\beta,h} = \frac{1}{Z_{\beta,h}} \int (\prod_{x\in\Lambda} \partial_{\eta_x}\partial_{\xi_x} \frac{1}{z_x}) e^{\frac{\beta}{2}(u,\Delta u) - h(1,z-1)} F. \tag{1.13}$$

In this expression, $\int \prod_{x\in\Lambda} \partial_{\eta_x}\partial_{\xi_x}$ denotes the Grassmann integral (i.e., the coefficient of the top degree monomial of the integrand), $Z_{\beta,h}$ is a normalising constant, and

$$\begin{aligned} \frac{1}{2}(u,\Delta u) &= -\frac{1}{2}\sum_{xy\in E(\Lambda)} (u_x - u_y)\cdot(u_x - u_y) = \sum_{xy\in E(\Lambda)} (u_x\cdot u_y + 1), \\ (1,z) &= \sum_{x\in\Lambda} z_x, \end{aligned} \tag{1.14}$$

where $xy \in E(\Lambda)$ denotes that x and y are nearest neighbours (counting every pair once), and the inner products are given by (1.12). The factors $1/z_x$ in (1.13) are the canonical fermionic volume form invariant under the symmetries associated with (1.12) as discussed further below.

As explained in [20, Sect. 2.1] (see also [37] where such relations were first observed) connection and edge probabilities of the arboreal gas are equivalent to correlation functions of the $\mathbb{H}^{0|2}$ model. The following proposition summarises the relations we need, see Appendix A for the proof.

Proposition 1.4 *For any finite graph G, any $\beta \geqslant 0$ and $h \geqslant 0$,*

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow \mathfrak{g}] = \langle z_0\rangle_{\beta,h}, \tag{1.15}$$

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}] = \langle \xi_0\eta_x\rangle_{\beta,h}, \tag{1.16}$$

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow x] + \mathbb{P}_{\beta,h}[0 \not\leftrightarrow x, 0 \leftrightarrow \mathfrak{g}, x \leftrightarrow \mathfrak{g}] = -\langle u_0\cdot u_x\rangle_{\beta,h}, \tag{1.17}$$

and the normalising constants in (1.1) and (1.13) are equal. In particular,

$$\mathbb{P}_{\beta,0}[0 \leftrightarrow x] = -\langle u_0\cdot u_x\rangle_{\beta,0} = -\langle z_0 z_x\rangle_{\beta,0} = \langle \xi_0\eta_x\rangle_{\beta,0} = 1 - \langle \xi_0\eta_0\xi_x\eta_x\rangle_{\beta,0}. \tag{1.18}$$

These relations resemble those between the Potts model and the random cluster model, giving further credence to our proposal that the $\mathbb{H}^{0|2}$ model may be interpreted as the 0-state Potts model, with the arboreal gas playing the role of the 0-state random cluster model. Nevertheless, there are important differences from the q-state Potts model with $q \geqslant 2$. Chief amongst them is that the $\mathbb{H}^{0|2}$ model has continuous symmetries. To make this precise, let

$$T = \sum_{x\in\Lambda} z_x \partial_{\xi_x}, \qquad \bar{T} = \sum_{x\in\Lambda} z_x \partial_{\eta_x}. \tag{1.19}$$

One way to understand the significance of T, $\bar{T}$ is via the identities $\langle TF\rangle_{\beta,0} = \langle \bar{T}F\rangle_{\beta,0} = 0$ for any polynomial F in the variables ξ and η; see [20, Sect. 2.2]. For example, $\langle T\xi_0\rangle_{\beta,0} = \langle z_0\rangle_{\beta,0} = 0$. Identities derived in this way are conventionally called Ward identities.

The maps T and $\bar{T}$ are infinitesimal generators of two global internal supersymmetries of the $\mathbb{H}^{0|2}$ model. These supersymmetries are explicitly broken if $h \neq 0$. They are analogues of infinitesimal Lorentz boosts or infinitesimal rotations. Together with a further internal symmetry corresponding to rotations in the ξ, η plane, these operators generate the symmetry algebra $\mathfrak{osp}(1|2)$ of the $\mathbb{H}^{0|2}$ model. For details and further explanations we again refer to [20, Sect. 2.2]. As generators of continuous symmetries, T and $\bar{T}$ imply Ward identities that are not available for the Potts model with $q \geqslant 2$. These identities are crucial for our analysis and will be discussed below.

The phase transition of the arboreal gas corresponds to a spontaneous breaking of the above supersymmetries in the infinite volume limit. By spontaneous symmetry breaking we mean that there is an observable F for which $\lim_{N\to\infty}\lim_{h\downarrow 0}\langle F\rangle_{\beta,h} \neq \lim_{h\downarrow 0}\lim_{N\to\infty}\langle F\rangle_{\beta,h}$. Indeed, this is shown in our next theorem for the $\mathbb{H}^{0|2}$ model from which Theorems 1.2 and 1.3 follow immediately by (1.15)–(1.17) (except for the same statements relating the constants, which we omitted here). A similar reformulation applies to Theorem 1.1.

Theorem 1.5 *Let $d \geqslant 3$ and $L \geqslant L_0(d)$. There exists $\beta_0 \in (0,\infty)$ and constants $\theta_d(\beta) = 1 + O(1/\beta)$ and $\mathsf{c}_i(\beta) = \mathsf{c}_i + O(1/\beta)$ and $\kappa > 0$ (all dependent on d) such that for $\beta \geqslant \beta_0$,*

$$\lim_{h\downarrow 0}\lim_{N\to\infty}\langle z_0\rangle_{\beta,h} = \theta_d(\beta) \tag{1.20}$$

$$\lim_{h\downarrow 0}\lim_{N\to\infty}\langle \xi_0\eta_x\rangle_{\beta,h} = \frac{\mathsf{c}_1(\beta)}{\beta|x|^{d-2}} + O\Big(\frac{1}{\beta|x|^{d-2+\kappa}}\Big) \tag{1.21}$$

$$\lim_{h\downarrow 0}\lim_{N\to\infty}\Big(\langle z_0 z_x\rangle_{\beta,h} - \langle z_0\rangle_{\beta,h}\langle z_x\rangle_{\beta,h}\Big) = -\frac{\mathsf{c}_2(\beta)}{\beta^2|x|^{2d-4}} + O\Big(\frac{1}{\beta^2|x|^{2d-4+\kappa}}\Big). \tag{1.22}$$

In particular,

$$\lim_{h\downarrow 0}\lim_{N\to\infty}\langle u_0\cdot u_x\rangle_{\beta,h} = -\theta_d(\beta)^2 - \frac{2\mathsf{c}_1(\beta)}{\beta|x|^{d-2}} + O\Big(\frac{1}{\beta|x|^{d-2+\kappa}}\Big). \tag{1.23}$$

In fact, the constants $\mathsf{c}_i(\beta)$ both satisfy $\mathsf{c}_i(\beta) = (c_d)^i + O(1/\beta)$, where c_d is the leading constant in the asymptotics of the Green function of the Laplacian $-\Delta^{\mathbb{Z}^d}$ on $\mathbb{Z}^d$:

$$(-\Delta^{\mathbb{Z}^d})^{-1}(0,x) = \frac{c_d}{|x|^{d-2}} + O(|x|^{-(d-2)-1}). \tag{1.24}$$

Our proof of Theorem 1.5 is by a rigorous renormalisation group analysis aided by Ward identities. We set $\psi = \sqrt{\beta}\eta$ and $\bar\psi = \sqrt{\beta}\xi$. Algebra then shows the fermionic density in (1.13) is equivalent to

$$\exp\left[-(\psi, -\Delta\bar\psi) - \frac{1}{\beta}(1+h)\sum_{x\in\Lambda}\psi_x\bar\psi_x - \frac{1}{2\beta}\sum_{x\in\Lambda}\psi_x\bar\psi_x\sum_{e\in\mathcal{E}_d}\psi_{x+e}\bar\psi_{x+e}\right], \tag{1.25}$$

where the 1 in the quadratic term arises from putting the $\mathbb{H}^{0|2}$ volume form (see (1.13)) into the exponential, i.e.,

$$\prod_{x\in\Lambda}\frac{1}{z_x} = \prod_{x\in\Lambda}e^{+\xi_x\eta_x} = \prod_{x\in\Lambda}e^{-\eta_x\xi_x} = \exp\left[-\frac{1}{\beta}\sum_{x\in\Lambda}\psi_x\bar\psi_x\right], \tag{1.26}$$

and $\mathcal{E}_d = \{e_1, \dots, e_{2d}\}$ are the standard unit vectors (where $e_{d+j} = -e_j$). The reformulation (1.25) looks very much like a fermionic version of the φ^4 spin model. However, the following differences are important:

(1) Due to the fermionic nature of the field, and because the fermionic field only has two components (different for example from the case of Dirac fermions with four components), the quartic term actually has gradients in it: denoting the discrete gradient in direction $e \in \mathcal{E}_d$ by $(\nabla_e\psi)_x = \psi_{x+e} - \psi_x$, the quartic term can be written as

$$\frac{1}{2}\psi_x\bar\psi_x\sum_{e\in\mathcal{E}_d}\psi_{x+e}\bar\psi_{x+e} = \frac{1}{2}\psi_x\bar\psi_x\sum_{e\in\mathcal{E}_d}(\nabla_e\psi)_x(\nabla_e\bar\psi)_x = \psi_x\bar\psi_x(\nabla\psi)_x(\nabla\bar\psi)_x, \tag{1.27}$$

where we introduced the shorthand notation

$$(\nabla\psi)_x(\nabla\bar\psi)_x = \frac{1}{2}\sum_{e\in\mathcal{E}_d}(\nabla_e\psi)_x(\nabla_e\bar\psi)_x.$$

(2) The coupling constants $\frac{1}{\beta}(1+h)$ of the quadratic and $\frac{1}{\beta}$ of the quartic terms are related, and they are equal in the case $h = 0$ of ultimate interest. This relation is due to the geometric origin of the model as a non-linear sigma model and analogous relations are present in intrinsic coordinates for other sigma models like the vector $O(n)$ model. We remark that if the coupling constant of the quartic term was much smaller than that of the quadratic term (so $h \gg 0$) the study of the model would reduce to an exercise in fermionic cluster expansions (but see Appendix A.2 for even simpler arguments in this case).

To study the case of equal coupling constants, we will first consider their renormalisation group trajectories as a one parameter family among the set of all renormalisation group trajectories obtained by allowing the initial quadratic and quartic couplings to vary independent of one another. We will then place the equal-initial-coupling trajectories on the critical manifold of the renormalisation group dynamical system using the following Ward identity for the $\mathbb{H}^{0|2}$ model:

$$\langle z_0\rangle_{\beta,h} = \langle T\xi_0\rangle_{\beta,h} = -\sum_{x\in\Lambda}h\langle\xi_0 T z_x\rangle_{\beta,h} = h\sum_{x\in\Lambda}\langle\xi_0\eta_x\rangle_{\beta,h}, \tag{1.28}$$

where T is the symmetry generator (1.19); in the second equality we have used [20, Lemma 2.3].

After taking into account the two points above (in particular that the flow of the expanding quadratic term is constrained by the Ward identity), power counting heuristics predict that the lower critical dimension for spontaneous symmetry breaking with free field low temperature fluctuations is two for the $\mathbb{H}^{0|2}$ model. We expect that these considerations generalise to all non-linear sigma models with continuous symmetry, in agreement with the Goldstone mechanism. In conjunction with [20], our results rigorously establish that the lower critical dimension is two for the $\mathbb{H}^{0|2}$ model.

1.3 Background on non-linear sigma models and renormalisation

The low temperature renormalisation group analysis of non-linear sigma models with non-abelian continuous symmetry is a notorious problem that was famously considered by Balaban for the case of $O(n)$ symmetry, see [10, 11] and references therein. Our comparatively simple analysis of the $\mathbb{H}^{0|2}$ model, which is a non-linear sigma model with non-abelian continuous $OSp(1|2)$ symmetry, is made possible mainly by the fact that it does not suffer a large field problem because it has a fermionic representation. Our approach to the $\mathbb{H}^{0|2}$ model differs from Balaban's approach to the $O(n)$ model on a conceptual level, in that it is based on *intrinsic* coordinates as opposed to *extrinsic* ones. In the extrinsic approach of Balaban the sphere $\mathbb{S}^{n-1}$ is embedded into $\mathbb{R}^n$ and the renormalised action evolves as a function of $\mathbb{R}^n$-valued fields, manifestly preserving $O(n)$ symmetry. The intrinsic approach we use, which is similar to the one taken in the physics literature (see [79]), is based on local coordinates for the target space $\mathbb{H}^{0|2}$. The renormalised action does not have the $OSp(1|2)$ symmetry of the model, and Ward identities are used *a posteriori* to enforce the essential constraints (relations between couplings) due to the symmetry. It is unclear to us how to implement an extrinsic approach in our situation of $OSp(1|2)$ symmetry, and more generally for noncompact symmetries.

Somewhat remarkably, despite its simplicity, the $\mathbb{H}^{0|2}$ model has all of the main features present in the non-abelian $O(n)$ models, including: absence of spontaneous symmetry breaking in 2d (proven in [20]); mass generation in 2d (conjectured in [38]); and a spontaneous symmetry breaking phase transition with massless low temperature fluctuations in $d \geqslant 3$ (the main result of this work).

The $\mathbb{H}^{0|2}$ model is a member of the family of hyperbolic sigma models with target spaces $\mathbb{H}^{n|2m}$, see [39] for a discussion of some aspects of this. By supersymmetric localisation the observables of the $\mathbb{H}^{0|2}$ model considered in Theorem 1.5 are equivalent to the analogous ones of the non-linear sigma model with target $\mathbb{H}^{2|4}$. While this relation does not play a role in this paper, it leads to a more direct representation of the continuous symmetry breaking observed here. In brief, in the $\mathbb{H}^{2|4}$ model each vertex comes equipped with two real and four Grassmann fields. By expressing these fields in horospherical coordinates one of the real fields and the four Grassmann fields can be integrated out. The marginal distribution of the remaining real field, which is called the t-field, may be viewed as a '$\nabla\phi$' random surface model, albeit with a nonconvex and nonlocal Hamiltonian. By this we mean that the potential is invariant under the global translation $t_x \mapsto t_x + r$ for $r \in \mathbb{R}$. See [20] for more details, where this perspective was used to prove the absence of symmetry breaking in $d = 2$. The full $\mathbb{H}^{n|2m}$ family has been important for advancing our understanding of other aspects of these models [20, 39]. Of particular note, we mention that the $\mathbb{H}^{2|2}$ model has received substantial prior attention due to its exact connection to linearly reinforced random walks and its motivation from random matrix theory, see [42, 72, 78, 80, 81].

For hyperbolic sigma models with target $\mathbb{H}^n$, $n \geqslant 1$, spontaneous symmetry breaking for all $\beta > 0$ was shown in [78], and with target $\mathbb{H}^{2|2}$ for β large in [42] (see also [43]). For motivation from random matrix theory and the Anderson transition see [76, 77]. These proofs make essential use of the horospherical coordinates mentioned above. Moreover, the proof of symmetry breaking for the $\mathbb{H}^{2|2}$ model in [42] relies on an infinite number of Ward identifies resulting from supersymmetric localisation. These identities are absent in the $\mathbb{H}^{0|2}$ model, limiting the applicability of the methods of [42] to our setting. At the same time, the $\mathbb{H}^{2|2}$ model has no purely fermionic representation, and so our methods do not apply there, at least without significant further developments.

Introductions to fermionic renormalisation include [22, 66, 73], see also [53]. Recent probabilistic applications of these approaches to fermionic renormalisation include the study of interacting dimers [51, 52] and two-dimensional finite range Ising models [7, 8, 49, 50]. Our organisation of the renormalisation group is instead based on a finite range decomposition and polymer coordinates, and follows [28] and its further developments in [12, 16, 17, 29–32, 36]. This approach has its origins in [33]. For an introduction to this approach in a hierarchical bosonic context see [18]. Previous applications of this approach include the study of 4d weakly self-avoiding walks [14, 15]; the nearest-neighbour critical 4d $|\varphi|^4$ model [13, 75] and long-range versions thereof [63, 74]; the ultraviolet φ^4_3 problem [34, 35]; analysis of the Kosterlitz–Thousless transition of the 2d Coulomb gas [41, 45]; the Cauchy–Born problem [1]; and others.

While the construction of the bulk renormalisation group flow is simpler for the intrinsic representation of the $\mathbb{H}^{0|2}$ model than in many of the previous references, a crucial novelty of our present work is the combination of the finite range renormalisation group approach with Ward identities, together with a precise analysis of a nontrivial zero mode. This has enabled us to apply these methods to a non-linear sigma model in the phase of broken symmetry. It would be extremely interesting to understand this approach for bosonic non-linear sigma models where, while ‘large fields’ cause serious complications, the formal perturbative analysis is very much in parallel to the fermionic version we study in this paper. Ward identities of a different type have previously been used in the renormalisation group analyses in [9] and [23] and many follow-up works including [51, 52]. Finally, we mention that Theorem 1.1 yields quantitative finite volume statements. The proof implements a rigorous finite size analysis along the lines of that proposed in [27]. It would be very interesting to extend this to even higher precision as discussed in Sect. 1.4 below.

1.4 Future directions for the arboreal gas

In this section we discuss several interesting open directions, including the geometric structure of the weak infinite volume limits of the arboreal gas and its relation to the uniform spanning tree, and a conjectural finite size universality similar to Wigner–Dyson universality from random matrix theory.

1.4.1 Finite volume behaviour

The detailed finite volume behaviour of the arboreal gas would be very interesting to understand beyond the precision of Theorem 1.1. On the complete graph at supercritical temperatures it is known that there is a unique macroscopic cluster, and that there are an unbounded number of clusters whose sizes are of order $|\Lambda|^{2/3}$ [64]. The fluctuations of the macroscopic cluster are non-Gaussian of scale $|\Lambda|^{2/3}$ and the distribution of the ordered cluster sizes of the mesoscopic clusters has been determined [64]. The joint law of the mesoscopic clusters can be characterised [65, Sect. 1.4.3]. Intriguingly, $|\Lambda|^{2/3}$ is the size of the largest tree at criticality on the complete graph, and the order statistics of the supercritical mesoscopic clusters can be related to the order statistics at the critical point [65, Sect. 1.4.3].

Going beyond the complete graph, is this distribution of ordered cluster sizes universal, at least in sufficiently high dimensions? This would be similar to the conjectured universality of Wigner–Dyson statistics from random matrix theory [67] or the conjectured universality of the distribution of macroscopic loops in loop representations of $O(n)$ (and other) spin systems [55, 68]. More generally it would be an instance of the universality of low temperature fluctuations in finite volume in models with continuous symmetries.

Finally, we mention that on expander graphs the existence of a phase transition for the arboreal gas is not difficult to show by using a natural split–merge dynamics [54]. It would be interesting if this dynamical approach could also be used to obtain information about the cluster size distribution.

1.4.2 Infinite volume behaviour and relation to the uniform spanning tree

As mentioned previously, the arboreal gas is also known as the *uniform forest model* [57]. We emphasise that the arboreal gas is *not* what is typically known as the *uniform spanning forest* (USF), which is in fact the weak limit as $\Lambda_N \uparrow \mathbb{Z}^d$ of a uniform spanning tree (UST) [69]. On a finite graph, the UST is the $\beta \to \infty$ limit of the arboreal gas. The correct scaling of the external field for this limit is $h = \beta\kappa$ and we thus write $\mathbb{P}_{\mathrm{UST},\kappa} = \lim_{\beta\to\infty} \mathbb{P}_{\beta,\beta\kappa}$ for the UST on a finite graph (plus ghost vertex if $\kappa > 0$). For $\kappa > 0$, this measure is also known as the *rooted* spanning forest, because disregarding the connections to the ghost vertex disconnects the tree of the UST, with vertices previously connected to the ghost becoming roots. The distributions of rooted and unrooted forests are not the same. To help prevent confusion we will refer to the rooted spanning forests as (a special case of) the UST.

It is trivial that $\mathbb{P}^{\Lambda_N}_{\mathrm{UST},0}[0 \leftrightarrow x] = 1$. Nevertheless, the behaviour of the UST in the weak infinite volume limit depends on the dimension d. This limit can be defined as $\mathbb{P}^{\mathbb{Z}^d}_{\mathrm{UST}} = \lim_{\kappa\downarrow 0} \lim_{N\to\infty} \mathbb{P}^{\Lambda_N}_{\mathrm{UST},\kappa}$ and is independent of the finite volume boundary conditions (e.g. free, wired, or periodic as above) imposed on Λ_N, see [69]. Even though the function $1_{0\leftrightarrow x}$ is not continuous with respect to the topology of weak convergence, it is still true that

$$\mathbb{P}^{\mathbb{Z}^d}_{\text{UST}}[0 \leftrightarrow x] = \lim_{\kappa \downarrow 0} \lim_{N \to \infty} \mathbb{P}^{\Lambda_N}_{\text{UST},\kappa}[0 \leftrightarrow x]. \tag{1.29}$$

The order of limits here is essential. In this infinite volume limit the UST disconnects into infinitely many infinite trees if $d > 4$, but remains a single connected tree if $d \leqslant 4$, see [69]. Moreover,

$$\mathbb{P}^{\mathbb{Z}^d}_{\text{UST}}[0 \leftrightarrow x] + \mathbb{P}^{\mathbb{Z}^d}_{\text{UST}}[0 \not\leftrightarrow x, |T_0| = \infty, |T_x| = \infty] = 1. \tag{1.30}$$

On the left-hand side, the second term vanishes if $d \leqslant 4$ whereas the first term tends to 0 as $|x| \to \infty$ if $d > 4$. Furthermore, the geometric structure of the trees under $\mathbb{P}^{\mathbb{Z}^d}_{\text{UST}}$ is well understood. In particular, all trees are one-ended, meaning that removing one edge from a tree results in two trees, of which one is finite [24, 69].

For the arboreal gas, the existence and uniqueness of infinite volume limits is an open question. Nonetheless, subsequential limits exist, and in such an infinite volume limit all trees are finite almost surely when β is small, while Theorem 1.2 implies the existence of an infinite tree for β large. Moreover, by Theorem 1.3,

$$\begin{aligned}\lim_{h \downarrow 0} \lim_{N \to \infty} &\left(\mathbb{P}^{\Lambda_N}_{\beta,h}[0 \leftrightarrow x] + \mathbb{P}^{\Lambda_N}_{\beta,h}[0 \not\leftrightarrow x, 0 \leftrightarrow \mathfrak{g}, x \leftrightarrow \mathfrak{g}] \right) \\ &= \theta_d(\beta)^2 + \frac{2\mathsf{c}_1(\beta)}{\beta |x|^{d-2}} + O\big(\frac{1}{\beta |x|^{d-2+\kappa}}\big).\end{aligned} \tag{1.31}$$

By analogy with the UST, we expect that only the first term on the left-hand side contributes for $d \leqslant 4$ and that only the second term contributes asymptotically as $|x| \to \infty$ for $d > 4$. The tempting conjecture that the UST stochastically dominates the arboreal gas on the torus is consistent with these expectations. The analogue of the left-hand side of (1.31) plays an important role in the proof of uniqueness of the infinite cluster in Bernoulli percolation in [4]; this is related to the vanishing of the second term. As already mentioned, for the arboreal gas we only expect this to be true in $d \leqslant 4$. Significant progress towards this statement has been obtained in [58], where it is shown that translation-invariant infinite volume limits of the arboreal gas have a unique infinite tree in $d = 3, 4$. More precisely, [58] makes use of the existence results of the present paper and establishes uniqueness.

Beyond the questions above, it would be interesting to analyse more detailed geometric aspects of the arboreal gas. For example, can one construct scaling limits as has been done for some spanning tree models [3, 5, 6, 48]?

Finally, we mention that a detailed analysis of the infinite volume behaviour of the arboreal gas on regular trees with wired boundary conditions has been carried out [44, 71]. This infinite volume behaviour is consistent with the finite volume behaviour of the complete graph, e.g., at all supercritical temperatures the sizes of finite clusters have the same distribution as those of critical percolation.

1.4.3 Order of phase transition

Our analysis could be extended to a detailed study of the approach $h \downarrow 0$. To keep the length of this paper within bounds, we do not carry this out, but here briefly comment on what we expect can be shown by extensions of our analysis. As discussed above, a natural object is the magnetisation

$$M(\beta, h) = \lim_{N\to\infty} M_N(\beta, h), \qquad M_N(\beta, h) = \mathbb{P}^{\Lambda_N}_{\beta,h}[0 \leftrightarrow \mathfrak{g}], \tag{1.32}$$

and the corresponding susceptibility (neglecting questions concerning the order of limits)

$$\chi(\beta, h) = \frac{\partial}{\partial h} M(\beta, h) = \sum_x \sigma_{\beta,h}(x). \tag{1.33}$$

Thus for the arboreal gas, the susceptibility is not the sum over $\tau_{\beta,h}(x)$ as is the case for Bernoulli bond percolation, but the sum over $\sigma_{\beta,h}(x)$. In terms of the sigma model, χ maybe viewed as the longitudinal susceptibility, often denoted $\chi_{||}$. In this interpretation, the sum over $\tau_{\beta,h}(x)$ is the transversal susceptibility $\chi_\perp$ and satisfies the Ward identity $\chi_\perp(\beta, h) = \sum_x \tau_{\beta,h}(x) = h^{-1} M(\beta, h)$ which is crucial in our analysis. For the longitudinal susceptibility, we expect that it would be possible to extend our analysis to show

$$\chi(\beta, h) \sim \begin{cases} C(\beta) h^{-1/2} & (d = 3) \\ C(\beta) |\log h| & (d = 4) \\ C(\beta) & (d > 4). \end{cases} \tag{1.34}$$

Defining the *free energy* $f(\beta, h) = \lim_{N\to\infty} |\Lambda_N|^{-1} \log Z^{\Lambda_N}_{\beta,h}$, for $\beta \geqslant \beta_0$ the previous asymptotics suggest that $h \mapsto f(\beta, h)$ is C^2 in $d > 4$ but only C^1 for $d = 3, 4$. In fact, extrapolating from our renormalisation group analysis we believe that for $\beta \geqslant \beta_0$ the free energy is C^n but not C^{n+1} as a function of $h \geqslant 0$ for $n = \lfloor \frac{d-1}{2} \rfloor$. It is unclear how this is connected to the geometry of the component graph of the UST, which also changes as the dimension is varied [25, 59].

1.4.4 Critical behaviour

The critical behaviour of the $\mathbb{H}^{0|2}$ model and its generalisations (the $\mathbb{H}^{0|2M}$ models) were studied in [46, 62], using ε-expansions formally continued from the $O(n)$ models, with the motivation of being candidates for the CFTs relevant for a dS-CFT correspondence. Rigorous results about the critical behaviour of the arboreal gas on $\mathbb{Z}^d$ for $d \geqslant 3$ would be very interesting.

1.5 Organisation and notation

This paper is organised as follows. In Sect. 2, we show how Theorem 1.5 is reduced to renormalisation group results with the help of the Ward identity (1.28). The main renormalisation group input is Theorem 2.1 and 2.3. Sections 3–7 then prove these renormalisation group results. Section 3 is concerned with the construction of the bulk renormalisation group flow, and Sect. 4 uses this analysis to compute the susceptibility. In Sect. 5 we extend this construction to include observables. The renormalisation group flow for observables is then used in Sect. 6 to compute pointwise correlation functions. These computations involve a precise analysis of the zero mode. The short Sect. 7 then collects the results and establishes Theorems 2.1 and 2.3. Finally, in Appendix A we collect relations between the arboreal gas and the $\mathbb{H}^{0|2}$ model as well as basic percolation and high temperature properties of the arboreal gas, and in Appendix B we include some background material about the finite range decomposition that we use.

Throughout we use $a_n \sim b_n$ to denote $\lim_{n\to\infty} a_n/b_n = 1$, $a_n \asymp b_n$ to denote the existence of $c, C > 0$ such that $ca_n \leqslant b_n \leqslant Ca_n$, $a_n \lesssim b_n$ if $a_n \leqslant Cb_n$, and $a_n = O(b_n)$ if $|a_n| \lesssim |b_n|$. We write $a \wedge b = \min\{a, b\}$. We consider the dimension $d \geqslant 3$ to be fixed, and hence allow implicit constants to depend on d. In Sects. 1 and 2 we allow implicit constants to depend on L as well, as this dependence does not play a role. In subsequent sections L-dependence is made explicit, though uniformity in L is only crucial in the contractive estimate of Theorem 3.13. Our main theorems hypothesise $L = L(d)$ is large, and for geometric convenience we will assume throughout that L is at least 2^{d+2}.

2 Consequences of combining renormalisation and Ward identities

In our renormalisation group analysis, which provides the foundation for the proofs of the theorems stated in Sect. 1, we will not assume any relation between the coupling constants of the quadratic and quartic terms in (1.25) (except that they are small). The equality of the quadratic and quartic couplings is restored with the help of the Ward identity (1.28), i.e.,

$$\langle z_0\rangle_{\beta,h} = h \sum_{x\in\Lambda} \langle \xi_0 \eta_x\rangle_{\beta,h}, \qquad \text{and in particular } \langle z_0\rangle_{\beta,0} = 0. \tag{2.1}$$

This application of the Ward identity is the subject of this section.

In our analysis we distinguish between two orders of limits. We first analyse the 'infinite volume' limit $\lim_{h\downarrow 0}\lim_{N\to\infty}$, and prove Theorem 1.5 (and thus Theorems 1.2–1.3). Using results of this analysis (and with several applications of the Ward identity), we then also analyse the much more delicate 'finite volume' limit $\lim_{N\to\infty}\lim_{h\downarrow 0}$ in order to prove Theorem 1.1.

2.1 Infinite volume correlation functions

For $m^2 > 0$ arbitrary and coupling constants s_0, a_0, b_0, which eventually will be taken small, we consider the model with fermionic Gaussian reference measure with covariance

$$C = (-\Delta + m^2)^{-1} \tag{2.2}$$

on Λ_N and interaction

$$V_0 = V_0(\Lambda_N) = \sum_{x\in\Lambda_N} \Big[s_0 (\nabla\psi)_x (\nabla\bar\psi)_x + a_0 \psi_x \bar\psi_x + b_0 \psi_x \bar\psi_x (\nabla\psi)_x (\nabla\bar\psi)_x \Big], \tag{2.3}$$

where we recall the squared gradient notation from (1.27). Thus the corresponding expectation is

$$\langle F \rangle_{m^2,s_0,a_0,b_0} = \frac{1}{Z_{m^2,s_0,a_0,b_0}} \frac{1}{\det(-\Delta + m^2)} \int \partial_\psi \partial_{\bar\psi}\, e^{-(\psi,(-\Delta+m^2)\bar\psi)-V_0} F, \quad (2.4)$$

where $\int \partial_\psi \partial_{\bar\psi}$ denotes the Grassmann integral, and Z_{m^2,s_0,a_0,b_0} is defined such that $\langle 1 \rangle_{m^2,s_0,a_0,b_0} = 1$. We emphasise the connection to the arboreal gas arises only if m^2, s_0, a_0, b_0 are chosen to agree with (1.25), c.f. (2.14)–(2.15) below.

The following result states that for correctly chosen a_0 the correlation functions of the fields ψ and $\bar\psi\psi$ are to leading order multiples of those of the free (Grassmann Gaussian) case $b_0 = 0$ if first $N \to \infty$ and then $m^2 \downarrow 0$. The result resembles those in [14, 15, 75] for weakly self-avoiding walks in dimension 4. Compared to the latter results, our analysis is substantially simplified since the $\mathbb{H}^{0|2}$ model can be studied in terms of only fermionic variables with a quartic interaction that is irrelevant in dimensions $d > 2$. However, in Sect. 2.2, we state an improvement of the following result that captures the full zero mode of the low temperature phase and goes beyond the analysis of [14, 15, 75].

Theorem 2.1 *Let $d \geqslant 3$ and $L \geqslant L_0(d)$. For b_0 sufficiently small and $m^2 \geqslant 0$, there are $s_0 = s_0^c(b_0, m^2)$ and $a_0 = a_0^c(b_0, m^2)$ independent of N so that the following hold: The functions s_0^c and a_0^c are continuous in both variables, differentiable in b_0 with uniformly bounded b_0-derivatives, and satisfy the estimates*

$$s_0^c(b_0, m^2) = O(b_0), \qquad a_0^c(b_0, m^2) = O(b_0) \quad (2.5)$$

uniformly in $m^2 \geqslant 0$. There exists $\kappa > 0$ such that if the torus sidelength satisfies $L^{-N} \leqslant m$,

$$\sum_{x\in\Lambda_N} \langle \bar\psi_0 \psi_x \rangle_{m^2,s_0,a_0,b_0} = \frac{1}{m^2} + \frac{O(b_0 L^{-(2+\kappa)N})}{m^4}. \quad (2.6)$$

Moreover, there are functions

$$\lambda = \lambda(b_0, m^2) = 1 + O(b_0), \qquad \gamma = \gamma(b_0, m^2) = (-\Delta^{\mathbb{Z}^d} + m^2)^{-1}(0,0) + O(b_0), \quad (2.7)$$

having the same continuity properties as s_0^c and a_0^c such that

$$\langle \bar\psi_0 \psi_0 \rangle_{m^2,s_0,a_0,b_0} = \gamma + O(b_0 L^{-\kappa N}), \quad (2.8)$$

$$\langle \bar\psi_0 \psi_x \rangle_{m^2,s_0,a_0,b_0} = (-\Delta + m^2)^{-1}(0,x) + O(b_0 |x|^{-(d-2)-\kappa}) + O(b_0 L^{-\kappa N}), \quad (2.9)$$

$$\langle \bar\psi_0 \psi_0; \bar\psi_x \psi_x \rangle_{m^2,s_0,a_0,b_0} = -\lambda^2 (-\Delta + m^2)^{-1}(0,x)^2 + O(b_0 |x|^{-2(d-2)-\kappa}) + O(b_0 L^{-\kappa N}). \quad (2.10)$$

Here $\langle A; B \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle$.

The proof of this theorem is given in Sects. 3–7. We now show how to derive Theorem 1.5 for the $\mathbb{H}^{0|2}$ model from it together with the Ward identity (1.28). To this end, assuming $s_0 > -1$ we further rescale ψ by $1/\sqrt{1+s_0}$ (and likewise for $\bar\psi$) in (1.25), and thus set

$$\xi = \sqrt{\frac{1+s_0}{\beta}}\,\bar\psi, \qquad \eta = \sqrt{\frac{1+s_0}{\beta}}\,\psi. \tag{2.11}$$

Up to a normalisation constant, the fermionic density (1.25) becomes, see also (1.27),

$$\exp\Bigg[-\sum_{x\in\Lambda_N}\Bigg((1+s_0)(\nabla\psi)_x(\nabla\bar\psi)_x + \frac{1+s_0}{\beta}(1+h)\psi_x\bar\psi_x + \frac{(1+s_0)^2}{\beta}\psi_x\bar\psi_x(\nabla\psi)_x(\nabla\bar\psi)_x\Bigg)\Bigg]. \tag{2.12}$$

For any $m^2 \geqslant 0$ and $s_0 > -1$, (2.12) is of the form (2.4) with

$$a_0 = \frac{1+s_0}{\beta}(1+h) - m^2, \qquad b_0 = \frac{(1+s_0)^2}{\beta}. \tag{2.13}$$

To use Theorem 2.1 to study the arboreal gas we need to invert this implicit relation between (β, h) and (m^2, s_0, a_0, b_0). This is achieved by the following corollary. A key observation is that the Ward identity (1.28) allows us to identify the critical point with $h = 0$. To make this precise, with s_0^c and a_0^c as in Theorem 2.1, define the functions

$$\beta(b_0, m^2) = \frac{(1+s_0^c(b_0, m^2))^2}{b_0}, \tag{2.14}$$

$$h(b_0, m^2) = -1 + \frac{a_0^c(b_0, m^2) + m^2}{b_0}(1 + s_0^c(b_0, m^2)). \tag{2.15}$$

By Theorem 2.1, both functions are continuous in $b_0 > 0$ small enough and $m^2 \geqslant 0$.

Corollary 2.2 (*i*) *Assume $b_0 > 0$ is small enough. Then*

$$h(b_0, m^2) = m^2\beta(b_0, m^2)(1 + O(b_0)). \tag{2.16}$$

In particular, $h(b_0, 0) = 0$ and $h(b_0, m^2) > 0$ if $m^2 > 0$.

(*ii*) *For β large enough and $h \geqslant 0$, there are functions $\tilde b_0(\beta, h) > 0$ and $\tilde m^2(\beta, h) \geqslant 0$ such that $h(\tilde b_0, \tilde m^2) = h$ and $\beta(\tilde b_0, \tilde m^2) = \beta$. Both functions are right-continuous as $h \downarrow 0$ when β is fixed.*

Proof To prove (i), we use the Ward identity (2.1) with (β, h) given by (2.14)–(2.15). The left- and right-hand sides of (2.1) are, respectively,

$$\langle z_0\rangle_{\beta,h} = 1 - \frac{1+s_0^c(b_0, m^2)}{\beta}\langle\bar\psi_0\psi_0\rangle_{m^2,s_0,a_0,b_0}, \tag{2.17}$$

$$h \sum_{x\in\Lambda_N} \langle \xi_0\eta_x\rangle_{\beta,h} = \frac{(1+s_0^c(b_0,m^2))h(b_0,m^2)}{\beta(b_0,m^2)} \sum_{x\in\Lambda_N} \langle \bar\psi_0\psi_x\rangle_{m^2,s_0,a_0,b_0}. \tag{2.18}$$

By Theorem 2.1, in the limit $N\to\infty$, we obtain from (2.1) that if $m^2>0$, the identity

$$1-\frac{1+s_0^c(b_0,m^2)}{\beta(b_0,m^2)}\gamma(b_0,m^2) = \frac{(1+s_0^c(b_0,m^2))h(b_0,m^2)}{\beta(b_0,m^2)m^2} \tag{2.19}$$

holds. Solving for h, we have

$$h(b_0,m^2) = m^2\left[\frac{\beta(b_0,m^2)}{1+s_0^c(b_0,m^2)} - \gamma(b_0,m^2)\right]. \tag{2.20}$$

Since $s_0^c(b_0,m^2)=O(b_0)$, $\beta(b_0,m^2)\asymp 1/b_0$, and $\gamma(b_0,m^2)=O(1)$, all uniformly in $m^2\geqslant 0$, we obtain $h(b_0,m^2)=m^2\beta(b_0,m^2)(1+O(b_0))$. In particular, $h(b_0,0)=0$.

Claim (ii) follows from an implicit function theorem argument that uses that s_0^c and a_0^c are continuous in $m^2\geqslant 0$ and differentiable in b_0 if $m^2>0$ with b_0-derivatives uniformly bounded in $m^2>0$. This argument is the same as the proof of [15, Proposition 4.2] (with our notation s_0 instead of z_0, a_0 instead of ν_0, b_0 instead of g_0, and with $1/\beta$ instead of g and h instead of ν) and is omitted here. □

Assuming Theorem 2.1, the proof of Theorem 1.5 is immediate from the last corollary. The main statements of Theorems 1.2 and 1.3 then follow immediately, except for the identifications $\theta_d(\beta)^2=\zeta_d(\beta)$, $(\mathsf{c}_2(\beta)/\mathsf{c}_1(\beta)^2)\theta_d(\beta)^2=1$, and $\mathsf{c}(\beta)=2\mathsf{c}_2(\beta)$ which we will obtain in Sect. 2.2.

Proof of Theorem 1.5 Given $\beta\geqslant\beta_0$ and $h>0$ we choose $b_0>0$ and $m^2>0$ as in Corollary 2.2 (ii). Since $z_x=1-\xi_x\eta_x$ and using (2.11) we then have

$$\langle z_0\rangle_{\beta,h} = 1-\langle\xi_0\eta_0\rangle_{\beta,h} = 1-\frac{1+s_0}{\beta}\langle\bar\psi_0\psi_0\rangle_{m^2,s_0,a_0,b_0}, \tag{2.21}$$

$$\langle\xi_0\eta_x\rangle_{\beta,h} = \frac{1+s_0}{\beta}\langle\bar\psi_0\psi_x\rangle_{m^2,s_0,a_0,b_0}, \tag{2.22}$$

$$\begin{aligned}\langle z_0z_x\rangle_{\beta,h}-\langle z_0\rangle_{\beta,h}^2 &= \langle\xi_0\eta_0\xi_x\eta_x\rangle_{\beta,h}-\langle\xi_0\eta_0\rangle_{\beta,h}^2\\ &= \frac{(1+s_0)^2}{\beta^2}\langle\bar\psi_0\psi_0;\bar\psi_x\psi_x\rangle_{m^2,s_0,a_0,b_0}.\end{aligned} \tag{2.23}$$

Taking $N\to\infty$ and then $h\downarrow 0$, the results follow from Corollary 2.2 (i) and Theorem 2.1 with

$$\theta_d(\beta) = 1-\frac{b_0\gamma}{1+s_0^c}, \qquad \mathsf{c}_1(\beta)=(1+s_0^c)c_d, \qquad \mathsf{c}_2(\beta)=\lambda^2(1+s_0^c)^2c_d^2, \tag{2.24}$$

where the functions λ and γ are evaluated at $m^2=0$ and b_0 given as above, c_d is the constant in the asymptotics of the free Green's function on $\mathbb{Z}^d$, see (1.24), and we have used the simplification of the error terms $O(|x|^{-(d-2)-1}) + O(b_0|x|^{-(d-2+\kappa)}) = O(|x|^{-(d-2+\kappa)})$ and $O(|x|^{-2(d-2)-1}) + O(b_0|x|^{-2(d-2)-\kappa}) = O(|x|^{-2(d-2)-\kappa})$. □

2.2 Finite volume limit

The next theorem extends Theorem 2.1 by more precise estimates valid in the limit $m^2 \downarrow 0$ with Λ_N fixed. In these estimates $t_N \in (0, 1/m^2)$ is a continuous function of $m^2 > 0$ that satisfies

$$t_N - \frac{1}{m^2} = O(L^{2N}) \qquad \text{and} \tag{2.25}$$

$$\lim_{m \downarrow 0} \left[(-\Delta + m^2)^{-1}(0, x) - \frac{t_N}{|\Lambda_N|} \right] = (-\Delta^{\mathbb{Z}^d})^{-1}(0, x) + O(L^{-(d-2)N}), \tag{2.26}$$

where on the right-hand side $\Delta^{\mathbb{Z}^d}$ is the Laplacian on $\mathbb{Z}^d$, on the left-hand side Δ is the Laplacian on Λ_N, and $|\Lambda_N| = L^{dN}$ denotes the volume of the torus Λ_N. We define

$$W_N(x) = W_{N,m^2}(x) = (-\Delta + m^2)^{-1}(0, x) - \frac{t_N}{|\Lambda_N|}, \tag{2.27}$$

so that $W_N(x)$ is essentially the torus Green's function $(-\Delta + m^2)^{-1}$ with the zero mode omitted.

In the following theorem, and throughout this section, Λ_N is fixed and the parameters (β, h) are related to (m^2, s_0, a_0, b_0) as in Corollary 2.2. We will write $\langle \cdot \rangle_{m^2,b_0} = \langle \cdot \rangle_{m^2, s_0^c(b_0,m^2), a_0^c(b_0,m^2), b_0}$ for the corresponding expectation and similarly for the partition function Z_{m^2,b_0}.

Theorem 2.3 *Under the conditions of Theorem 2.1 except that we no longer restrict $L^{-N} \leqslant m$, in addition to the functions a_0^c, s_0^c, λ, and γ, there are functions $\tilde{a}_{N,N}^c = \tilde{a}_{N,N}^c(b_0, m^2)$ and $u_N^c = u_N^c(b_0, m^2)$, both continuous in b_0 small and $m^2 \geqslant 0$, as well as*

$$\tilde{u}_{N,N}^c = \tilde{u}_{N,N}^c(b_0, m^2) = t_N \tilde{a}_{N,N}^c(b_0, m^2) + O(b_0 L^{-\kappa N}), \tag{2.28}$$

continuous in b_0 small and $m^2 > 0$, such that, for $x \in \Lambda_N$,

$$\sum_{x \in \Lambda_N} \langle \bar{\psi}_0 \psi_x \rangle_{m^2,b_0} = \frac{1}{m^2} - \frac{1}{m^4} \frac{\tilde{a}_{N,N}^c}{1 + \tilde{u}_{N,N}^c}, \tag{2.29}$$

$$\langle \bar{\psi}_0 \psi_0 \rangle_{m^2,b_0} = \gamma + \frac{\lambda t_N |\Lambda_N|^{-1}}{1 + \tilde{u}_{N,N}^c} + E_{00}, \tag{2.30}$$

$$\langle \bar{\psi}_0 \psi_0 \bar{\psi}_x \psi_x \rangle_{m^2,b_0} = -\lambda^2 W_N(x)^2 + \gamma^2 + \frac{-2\lambda^2 W_N(x) + 2\lambda\gamma}{1 + \tilde{u}_{N,N}^c} t_N |\Lambda_N|^{-1} + E_{00xx}, \tag{2.31}$$

and

$$Z_{m^2,b_0} = e^{-u_N^c|\Lambda_N|}(1 + \tilde{u}_{N,N}^c). \tag{2.32}$$

The remainder terms satisfy

$$E_{00} = \frac{O(b_0 L^{-(d-2+\kappa)N} + b_0 L^{-\kappa N}(m^2|\Lambda_N|)^{-1})}{1 + \tilde{u}_{N,N}^c}, \tag{2.33}$$

$$\begin{aligned} E_{00xx} &= O(b_0|x|^{-2(d-2)-\kappa}) + O(b_0 L^{-(d-2+\kappa)N}) \\ &\quad + (O(b_0|x|^{-(d-2+\kappa)}) + O(b_0 L^{-\kappa N}))\frac{(m^2|\Lambda_N|)^{-1}}{1 + \tilde{u}_{N,N}^c}. \end{aligned} \tag{2.34}$$

The proof of this theorem is again given in Sects. 3–7. This proof also gives a bound on $\tilde{a}_{N,N}^c$ of order $b_0 L^{-(2+\kappa)N}$ for a small $\kappa > 0$. We did not state this bound above because (by using the Ward identity (2.1)) the existence of $\tilde{a}_{N,N}^c$ with its relation to the correlation functions as stated in the theorem is, in fact, sufficient to determine its precise asymptotic value of order $b_0/|\Lambda_N| = b_0 L^{-dN} \ll b_0 L^{-(2+\kappa)N}$, see Lemmas 2.4–2.5 below. Using this precise asymptotic information on $\tilde{a}_{N,N}^c$, Theorem 1.1 then follows from Theorem 2.3. The key computation occurs in Lemma 2.6, where the asymptotic value of $\tilde{a}_{N,N}^c$ is used to exhibit important cancelations between the terms on the right-hand side of (2.31).

Lemma 2.4 *Under the conditions of Theorem 2.3,*

$$\mathbb{E}_{\beta,0}^{\Lambda_N}|T_0| = \frac{b_0}{1 + s_0^c(b_0, 0)} \frac{1 + O(b_0 L^{-\kappa N})}{\tilde{a}_{N,N}^c(b_0, 0)} + O(b_0 L^{2N}). \tag{2.35}$$

In particular, if $b_0 > 0$ this implies $1/\tilde{a}_{N,N}^c(b_0, 0) = O(|\Lambda_N|/b_0)$ and $\tilde{a}_{N,N}^c(b_0, 0) > 0$.

Proof From (1.18), we have that

$$\mathbb{E}_{\beta,0}^{\Lambda_N}|T_0| = \sum_{x \in \Lambda_N} \mathbb{P}_{\beta,0}[0 \leftrightarrow x] = \sum_{x \in \Lambda_N} \langle \xi_0 \eta_x \rangle_{\beta,0} = \lim_{h \to 0} \sum_{x \in \Lambda_N} \langle \xi_0 \eta_x \rangle_{\beta,h}. \tag{2.36}$$

Changing variables,

$$\sum_{x \in \Lambda_N} \langle \xi_0 \eta_x \rangle_{\beta,h} = \frac{b_0}{1 + s_0^c(b_0, m^2)} \sum_{x \in \Lambda_N} \langle \bar{\psi}_0 \psi_x \rangle_{m^2,b_0}, \tag{2.37}$$

where (β, h) and (b_0, m^2) are related as in (2.14) and (2.15). To evaluate the right-hand side we use (2.29). Note that

$$\frac{1}{m^2} - \frac{1}{m^4} \frac{\tilde{a}_{N,N}^c}{1 + \tilde{u}_{N,N}^c} = \frac{1}{m^2} \frac{1 + \tilde{u}_{N,N}^c - \tilde{a}_{N,N}^c m^{-2}}{1 + \tilde{u}_{N,N}^c}$$

$$= \frac{1 + \tilde{a}^c_{N,N}(t_N - m^{-2}) + O(b_0 L^{-\kappa N})}{m^2 + \tilde{a}^c_{N,N} t_N m^2 + O(b_0 m^2 L^{-\kappa N})}$$

$$= \frac{1 + \tilde{a}^c_{N,N} O(L^{2N}) + O(b_0 L^{-\kappa N})}{m^2 + \tilde{a}^c_{N,N}(1 + O(m^2 L^{2N})) + O(b_0 m^2 L^{-\kappa N})}, \qquad (2.38)$$

where the second equality is due to (2.28) and the third follows from (2.25). As $m^2 \downarrow 0$, the right-hand side of the third equality behaves asymptotically as

$$\frac{1 + \tilde{a}^c_{N,N} O(L^{2N}) + O(b_0 L^{-\kappa N})}{\tilde{a}^c_{N,N}} = \frac{1 + O(b_0 L^{-\kappa N})}{\tilde{a}^c_{N,N}} + O(L^{2N}). \qquad (2.39)$$

Since $s^c_0(b_0, 0) = O(b_0)$ by Theorem 2.1 we therefore obtain the first claim:

$$\mathbb{E}^{\Lambda_N}_{\beta,0} |T_0| = \frac{b_0}{1 + s^c_0(b_0, 0)} \lim_{m^2 \downarrow 0} \sum_{x \in \Lambda_N} \langle \bar{\psi}_0 \psi_x \rangle_{m^2, b_0}$$

$$= \frac{b_0}{1 + s^c_0(b_0, 0)} \frac{1 + O(b_0 L^{-\kappa N})}{\tilde{a}^c_{N,N}(b_0, 0)} + O(b_0 L^{2N}). \qquad (2.40)$$

For the second claim, let us observe that, on the one hand,

$$Z_{\beta,h} = \left(\frac{\beta}{1 + s^c_0}\right)^{|\Lambda_N|} (\det(-\Delta + m^2)) Z_{m^2, b_0}$$

$$= \left(\frac{\beta e^{-u^c_N}}{1 + s^c_0}\right)^{|\Lambda_N|} (\det(-\Delta + m^2))(1 + \tilde{u}^c_{N,N}), \qquad (2.41)$$

where the first equality is by Proposition 1.4 and (2.4), (2.11), and (2.12), and the second equality is (2.32). On the other hand, by (1.1),

$$\lim_{h \to 0} Z_{\beta,h} = Z_{\beta,0} > 0. \qquad (2.42)$$

Since, by Theorem 2.3, u^c_N and s^c_0 remain bounded as $m^2 \downarrow 0$ with Λ_N fixed (and thus also β which is given by (2.14)), from $\det(-\Delta + m^2) \downarrow 0$, we conclude that $1 + \tilde{u}^c_{N,N}$ diverges as $m^2 \downarrow 0$. By (2.28), this implies $\tilde{a}^c_{N,N}(b_0, 0) > 0$. The upper bound on $1/\tilde{a}^c_{N,N}(b_0, 0)$ follows by re-arranging (2.35) and using the trivial bound $|T_0| \leqslant |\Lambda_N|$. □

Using that $\tilde{a}^c_{N,N}$ is at least of order $b_0/|\Lambda_N|$ as established in the previous lemma, the following lemma gives an asymptotic representation of $\tilde{a}^c_{N,N}$ of order $b_0/|\Lambda_N|$ in terms of γ from Theorem 2.3.

Lemma 2.5 *Under the conditions of Theorem 2.3 and if $b_0 > 0$,*

$$1 = \frac{b_0}{1 + s^c_0(b_0, 0)} \left[\gamma(b_0, 0) + \frac{\lambda(b_0, 0)}{|\Lambda_N| \tilde{a}^c_{N,N}(b_0, 0)} (1 + O(b_0 L^{-\kappa N})) \right]. \qquad (2.43)$$

Proof The Ward identity $\langle z_0\rangle_{\beta,0} = 0$ implies

$$0 = \langle z_0\rangle_{\beta,0} = 1 - \langle \xi_0\eta_0\rangle_{\beta,0} = 1 - \lim_{m^2\downarrow 0} \frac{1+s_0^c(b_0,m^2)}{\beta(b_0,m^2)}\langle\bar\psi_0\psi_0\rangle_{m^2,b_0}$$

$$= 1 - \lim_{m^2\downarrow 0} \frac{b_0}{1+s_0^c(b_0,m^2)}\langle\bar\psi_0\psi_0\rangle_{m^2,b_0}, \qquad (2.44)$$

where we used (2.11) and that $\beta = \beta(b_0, m^2)$ is as in (2.14). To compute $\langle\bar\psi_0\psi_0\rangle_{m^2,b_0}$, we apply (2.30). Since $\tilde u^c_{N,N} = \tilde a^c_{N,N} t_N + O(b_0 L^{-\kappa N})$ and $t_N = m^{-2} + O(L^{2N})$,

$$\lim_{m^2\downarrow 0}\langle\bar\psi_0\psi_0\rangle_{m^2,b_0} = \gamma(b_0,0) + \lim_{m^2\downarrow 0}\frac{\lambda(b_0,m^2)t_N|\Lambda_N|^{-1}}{1+\tilde a^c_{N,N}(b_0,m^2)t_N + O(b_0L^{-\kappa N})} + \lim_{m^2\downarrow 0} E_{00}$$

$$= \gamma(b_0,0) + \frac{\lambda(b_0,0)|\Lambda_N|^{-1}}{\tilde a^c_{N,N}(b_0,0)} + \lim_{m^2\downarrow 0} E_{00}. \qquad (2.45)$$

The limits in the second line exist by Theorem 2.3 and Lemma 2.4, which in particular implies $\tilde a^c_{N,N}(b_0,0) > 0$ since $b_0 > 0$. As $m^2 \downarrow 0$, the error term E_{00} is bounded by $O(b_0 L^{-\kappa N}/(|\Lambda_N|\tilde a^c_{N,N})) = (\lambda(b_0,0)|\Lambda_N|^{-1}/\tilde a^c_{N,N})O(b_0L^{-\kappa N})$ since $\lambda(b_0,0) = 1 - O(b_0) \geqslant 1/2$, finishing the proof. □

Given Theorem 2.3, the following lemma is the main step in the proof of Theorem 1.1. It uses the asymptotic representation of $\tilde a^c_{N,N}$ to exhibit cancelations in expressions in Theorem 2.3.

Lemma 2.6 *Under the conditions of Theorem 2.3 and if $b_0 > 0$,*

$$\mathbb{P}^{\Lambda_N}_{\beta,0}[0 \leftrightarrow x] = \theta_d(\beta)^2 + 2\frac{b_0}{1+s_0^c}\lambda\theta_d(\beta)(-\Delta^{\mathbb{Z}^d})^{-1}(0,x)$$

$$+ O(b_0^2|x|^{-(d-2)-\kappa}) + O(b_0L^{-(d-2)N}) + O(b_0^2L^{-\kappa N}), \qquad (2.46)$$

where $\theta_d(\beta)$ is defined in (2.24).

Proof By the last expression for $\mathbb{P}_{\beta,0}[0 \leftrightarrow x]$ in (1.18) and (2.11), (2.14):

$$\mathbb{P}^{\Lambda_N}_{\beta,0}[0 \leftrightarrow x] = 1 - \lim_{h\downarrow 0}\langle \xi_0\eta_0\xi_x\eta_x\rangle_{\beta,h}$$

$$= 1 - \lim_{m^2\downarrow 0}\left[\frac{b_0^2}{(1+s_0^c)^2}\langle\bar\psi_0\psi_0\bar\psi_x\psi_x\rangle_{m^2,b_0}\right]. \qquad (2.47)$$

To compute $\lim_{m^2\downarrow 0}\langle\bar\psi_0\psi_0\bar\psi_x\psi_x\rangle_{m^2,b_0}$ we start from (2.31). By Lemma 2.4, as $m^2 \downarrow 0$ with Λ_N fixed,

$$\frac{1}{1+\tilde u^c_{N,N}} \sim \frac{1}{m^{-2}\tilde a^c_{N,N}(b_0,0)} = O(\frac{m^2|\Lambda_N|}{b_0}). \qquad (2.48)$$

This implies the error term in (2.31) is, as $m^2 \downarrow 0$ with Λ_N fixed,

$$|E_{00xx}| \leqslant O(|x|^{-(d-2)-\kappa}) + O(L^{-\kappa N}). \tag{2.49}$$

For the main term we have (recall $W_N(x) = W_{N,m^2}(x)$, see (2.27))

$$\begin{aligned}
&\lim_{m^2\downarrow 0} \langle \bar\psi_0\psi_0\bar\psi_x\psi_x\rangle_{m^2,b_0} - \lim_{m^2\downarrow 0} |E_{00xx}| \\
&= -\lambda^2 W_{N,0}(x)^2 + \gamma^2 + \lim_{m^2\downarrow 0} \frac{-2\lambda^2 W_N(x) + 2\lambda\gamma}{1+\tilde u^c_{N,N}} t_N |\Lambda_N|^{-1} \\
&= -\lambda^2 W_{N,0}(x)^2 + \gamma^2 + 2(-\lambda W_{N,0}(x) + \gamma)\frac{\lambda}{\tilde a^c_{N,N}|\Lambda_N|},
\end{aligned} \tag{2.50}$$

where on the right-hand side the functions λ, γ, and $\tilde a^c_{N,N}$ are evaluated at $m^2 = 0$. By Lemma 2.5,

$$\frac{b_0}{1+s_0^c}\frac{\lambda}{\tilde a^c_{N,N}|\Lambda_N|} = \left(1 - \frac{b_0\gamma}{1+s_0^c}\right)(1 + O(b_0 L^{-\kappa N})) \tag{2.51}$$

so that

$$-\left(\frac{b_0}{1+s_0^c}\right)^2 \frac{2\lambda^2 W_{N,0}(x)}{\tilde a^c_{N,N}|\Lambda_N|}(1 + O(b_0 L^{-\kappa N})) = -\frac{2b_0}{1+s_0^c}(1 - \frac{b_0\gamma}{1+s_0^c})\lambda W_{N,0}(x) \tag{2.52}$$

$$\left(\frac{b_0}{1+s_0^c}\right)^2 \frac{2\lambda\gamma}{\tilde a^c_{N,N}|\Lambda_N|}(1 + O(b_0 L^{-\kappa N})) = 2\gamma\frac{b_0}{1+s_0} - 2\gamma^2\left(\frac{b_0}{1+s_0^c}\right)^2. \tag{2.53}$$

Substituting these bounds into (2.50) and then (2.47) we obtain

$$\begin{aligned}
\mathbb{P}^{\Lambda_N}_{\beta,0}[0 \leftrightarrow x] &= 1 - \left(\frac{b_0}{1+s_0^c}\right)^2 \lim_{m^2\downarrow 0} \langle \bar\psi_0\psi_0\bar\psi_x\psi_x\rangle_{m^2,b_0} \\
&= (1 - \frac{\gamma b_0}{1+s_0})^2 + \frac{2b_0\lambda}{1+s_0^c}(1 - \frac{b_0\gamma}{1+s_0^c})W_{N,0}(x) + \left(\frac{b_0\lambda W_{N,0}(x)}{1+s_0^c}\right)^2 \\
&\quad + O(b_0^2 L^{-\kappa N} W_{N,0}(x)) + O(b_0^2 L^{-\kappa N}) + O(b_0^2 |E_{00xx}|).
\end{aligned} \tag{2.54}$$

Using the definition (2.24) of $\theta_d(\beta)$, that $W_{N,0}(x) = (-\Delta^{\mathbb{Z}^d})^{-1}(0,x) + O(L^{-(d-2)N})$ by (2.26), and in particular $W_{N,0}(x) = O(|x|^{-(d-2)})$, the claim follows. □

The next (and final) lemma is inessential for the main conclusions, but will allow us to identify the constants from the infinite volume and the finite volume analyses.

Lemma 2.7 *Under the conditions of Theorem 2.3 and if $b_0 > 0$, then $\lambda\theta_d(\beta) = 1$.*

Proof Let

$$w_N = \frac{b_0}{1+s_0^c} \frac{1}{\tilde{a}_{N,N}^c |\Lambda_N|} = \mathbb{E}_{\beta,0}^{\Lambda_N} \frac{|T_0|}{|\Lambda_N|} + O(b_0 L^{-\kappa N} + b_0 L^{-(d-2)N}), \qquad (2.55)$$

where the second equality is due to Lemma 2.4. The density $\mathbb{E}_{\beta,0}^{\Lambda_N}|T_0|/|\Lambda_N|$ can also be computed by summing the estimate in Lemma 2.6 and dividing by $|\Lambda_N|$. Subtracting this result from (2.55) gives

$$w_N - \theta_d(\beta)^2 = O(b_0 L^{-\kappa N}). \qquad (2.56)$$

On the other hand, (2.43) shows that

$$\lambda w_N - \theta_d(\beta) = O(b_0 L^{-\kappa N}). \qquad (2.57)$$

The limit $w = \lim_{N\to\infty} w_N$ thus exists and satisfies $\lambda w = \theta_d(\beta)$ and $w = \theta_d(\beta)^2$. Since $\theta_d(\beta) = 1 - O(1/\beta) \neq 0$ this implies $\lambda\theta_d(\beta) = 1$. □

Proof of Theorem 1.1 The proof follows by rewriting Lemma 2.6. Let c_d be the constant in the Green function asymptotics of (1.24), and recall the constants $\theta_d(\beta)$ and $\mathsf{c}_i(\beta)$ from (2.24). Theorem 1.1 then follows from Lemma 2.6 by setting

$$\zeta_d(\beta) = \theta_d(\beta)^2, \qquad \mathsf{c}(\beta) = (1+s_0^c)2\lambda\theta_d(\beta)c_d, \qquad (2.58)$$

and simplifying the error terms using $O(b_0|x|^{-(d-2)-1}) + O(b_0^2|x|^{-(d-2+\kappa)}) = O(\beta^{-1}|x|^{-(d-2+\kappa)})$ and $O(b_0 L^{-(d-2)N}) + O(b_0^2 L^{-\kappa N}) = O(\beta^{-1}L^{-\kappa N})$. □

Completion of proof of Theorems 1.2 and 1.3 For Theorem 1.2, $\zeta_d(\beta) = \theta_d(\beta)^2$ was established in the previous proof. For Theorem 1.3, the identity $(\mathsf{c}_2(\beta)/\mathsf{c}_1(\beta)^2)\theta_d(\beta)^2 = 1$ is equivalent (by (2.24)) to $\theta_d(\beta)\lambda = 1$, i.e., Lemma 2.7. Similarly, $\mathsf{c}(\beta) = 2\lambda\theta_d(\beta)\mathsf{c}_1(\beta) = 2\mathsf{c}_1(\beta)$. □

Remark 2.8 To compute $\mathbb{P}_{\beta,0}[0 \leftrightarrow x]$ we started from the expression $1 - \langle \xi_0\eta_0\xi_x\eta_x\rangle_{\beta,0}$ in (1.18). An alternative route would have been to start from $\langle \xi_0\eta_x\rangle_{\beta,0}$. For technical reasons arising in Sect. 5 it is, however, easier to obtain sufficient precision when working with $\langle \xi_0\eta_0\xi_x\eta_x\rangle_{\beta,0}$.

3 The bulk renormalisation group flow

We will prove Theorems 2.1 and 2.3 by a renormalisation group analysis that is set up following [28, 32] and [14, 15]; see also [18] for a conceptual introduction. Our proof is largely self-contained. The exceptions to self-containment concern general properties about finite range decomposition, norms, and approximation by local polynomials that were developed systematically in [12, 29, 30]. The properties we need are all reviewed in this section. The first six subsections set up the framework of the analysis, and the remaining three define and analyse the renormalisation group flow.

Throughout $\Lambda = \Lambda_N$ is the discrete torus of side length L^N. We leave L implicit; it will eventually be chosen large. We sometimes omit the N when it does not play a role.

3.1 Finite range decomposition

Let Δ denote the lattice Laplacian on Λ_N, and let $m^2 > 0$. Our starting point for the analysis is the decomposition

$$C = (-\Delta + m^2)^{-1} = C_1 + \cdots + C_{N-1} + C_{N,N} \tag{3.1}$$

where the C_j (with $j < N$) and $C_{N,N}$ are positive semidefinite m^2-dependent matrices indexed by Λ_N. These covariances can be chosen with the following properties, see [18, Proposition 3.3.1 and Sect. 3.4] and Appendix B. The notation $C_{N,N}$ for the last covariance is explained below.

3.1.1 Finite range property

For $j < N$, the covariances C_j satisfy the finite range property

$$C_j(x, y) = 0 \qquad \text{if } |x - y|_\infty \geqslant \frac{1}{2} L^j. \tag{3.2}$$

Moreover, they are invariant under lattice symmetries and independent of Λ_N in the sense that $C_j(x, y)$ can be identified as a function of $x - y$ that is independent of the torus Λ_N. They are defined and continuous for $m^2 \geqslant 0$ including the endpoint $m^2 = 0$ (and in fact smooth).

3.1.2 Scaling estimates

The covariances satisfy estimates consistent with the decay of the Green function:

$$|\nabla^\alpha C_{j+1}(x, y)| \leqslant O_{\alpha,s}(\vartheta_j(m^2)) L^{-(d-2+|\alpha|)j}, \tag{3.3}$$

where for an arbitrary fixed constant s,

$$\vartheta_j(m^2) = \frac{1}{2d + m^2} \left(1 + \frac{m^2 L^{2j}}{2d + m^2}\right)^{-s}. \tag{3.4}$$

The discrete gradient in (3.3) can act on either the x or the y variable, and is defined as follows. Recalling that $e_1, \ldots, e_d$ denote the standard unit vectors generating $\mathbb{Z}^d$, that $e_{d+j} = -e_j$, and that $\mathcal{E}_d = \{e_1, \ldots, e_{2d}\}$, for any multiindex $\alpha \in \mathbb{N}_0^{\mathcal{E}_d}$, we define the discrete derivative in directions α with order $|\alpha| = \sum_{i=1}^{2d} \alpha(e_i)$ by:

$$\nabla^\alpha = \prod_{i=1}^{2d} \nabla_{e_i}^{\alpha(e_i)}, \qquad \nabla_e f = f(x + e) - f(x), \tag{3.5}$$

with $\nabla_{e_i}^k = \nabla_{e_i} \cdots \nabla_{e_i}$, where there are k terms on the right-hand side.

3.1.3 Zero mode

By the above independence of the covariances C_j with $j < N$ from Λ_N, all finite volume torus effects are concentrated in the last covariance $C_{N,N}$. We further separate this covariance into a bounded part and the zero mode:

$$C_{N,N} = C_N + t_N Q_N, \tag{3.6}$$

where t_N is an m^2-dependent constant and Q_N is the projection onto the zero mode, i.e., the matrix with all entries equal to $1/|\Lambda_N|$. The bounded contribution C_N (which does depend on Λ_N) satisfies the estimates (3.3) with $j = N$ and also extends continuously to $m^2 = 0$. The constant t_N satisfies

$$t_N > 0, \qquad t_N - \frac{1}{m^2} = O(L^{2N}). \tag{3.7}$$

In this section, we only consider the effect of C_N (which is parallel to that of the C_j with $j < N$) while the nontrivial finite volume effect of t_N will be analysed in Sects. 4–6.

The above properties imply (2.26) and $W_N(x)$ in (2.27) is given by $W_N(x) = C_1(x) + \cdots + C_N(x)$.

3.2 Grassmann Gaussian integration

For $X \subset \Lambda = \Lambda_N$, we denote by $\mathcal{N}(X)$ the Grassmann algebra generated by ψ_x, $\bar\psi_x$, $x \in X$ with the natural inclusions $\mathcal{N}(X) \subset \mathcal{N}(X')$ for $X \subset X'$. Moreover, we denote by $\mathcal{N}(X \sqcup X)$ the doubled algebra with generators ψ_x, $\bar\psi_x$, ζ_x, $\bar\zeta_x$ and by $\theta \colon \mathcal{N}(X) \to \mathcal{N}(X \sqcup X)$ the doubling homomorphism acting on the generators of $\mathcal{N}(X)$ by

$$\theta\psi_x = \psi_x + \zeta_x, \qquad \theta\bar\psi_x = \bar\psi_x + \bar\zeta_x. \tag{3.8}$$

For a covariance matrix C the associated Gaussian expectation $\mathbb{E}_C$ acts on $\mathcal{N}(X \sqcup X)$ on the ζ, $\bar\zeta$ variables. Explicitly, when C is positive definite, $F \in \mathcal{N}(X \sqcup X)$ maps to $\mathbb{E}_C F \in \mathcal{N}(X)$ given by

$$\mathbb{E}_C F = \mathbb{E}_C[F] = (\det C) \int \partial_\zeta \partial_{\bar\zeta}\, e^{-(\zeta, C^{-1}\bar\zeta)} F. \tag{3.9}$$

Thus $\mathbb{E}_C\theta \colon \mathcal{N}(\Lambda) \to \mathcal{N}(\Lambda)$ is the fermionic convolution of $F \in \mathcal{N}(\Lambda)$ with the fermionic Gaussian measure with covariance C. Recall the following well-known facts about $\mathbb{E}_C\theta$; elementary proofs can be found in, e.g., [29]. First, this convolution operator can be written as

$$\mathbb{E}_C\theta F = \mathbb{E}_C[\theta F] = e^{\mathcal{L}_C} F \tag{3.10}$$

where $\mathcal{L}_C = \sum_{x,y\in\Lambda} C_{xy} \partial_{\psi_y} \partial_{\bar\psi_x}$. In particular, it follows that $\mathbb{E}_C\theta$ has the semigroup property

$$\mathbb{E}_{C_2}\theta \circ \mathbb{E}_{C_1}\theta = \mathbb{E}_{C_1+C_2}\theta. \tag{3.11}$$

This formula also holds for C positive *semi*definite if we take (3.10) as the definition of $\mathbb{E}_C\theta F$, which we will in the sequel. The identity (3.10) is a fermionic version of the relation between Gaussian convolution and the heat equation, and (3.11), which follows from (3.10), is the analogue of the fact that the sum of two independent Gaussian processes is Gaussian with covariance given by the sum of the covariances. The identity (3.10) allows for the evaluation of moments, e.g., $\mathbb{E}_C\theta\bar{\psi}_x\psi_y = \bar{\psi}_x\psi_y + C_{xy}$. An important consequence of the finite range property (3.2) of C_j is that if $F_i \in \mathcal{N}(X_i)$ with $\mathrm{dist}_\infty(X_1, X_2) > \frac{1}{2}L^j$ then, by (3.10),

$$\mathbb{E}_{C_j}\Big[\theta(F_1F_2)\Big] = \Big(\mathbb{E}_{C_j}[\theta F_1]\Big)\Big(\mathbb{E}_{C_j}[\theta F_2]\Big). \tag{3.12}$$

3.3 Symmetries

We briefly discuss symmetries, which are important in extracting the relevant and marginal contributions in each renormalisation group step (see Sect. 3.6 below). We call an element $F \in \mathcal{N}(\Lambda)$ *symplectically invariant* or $U(1)$ *invariant* if every monomial in its representation has the same number of factors of $\bar{\psi}$ and ψ. We remark that in [29, 30], to which we will sometimes refer, this property is called (global) gauge invariance. Similarly, $F \in \mathcal{N}(\Lambda \sqcup \Lambda)$ is $U(1)$ invariant if the combined number of factors of $\bar{\psi}$ and $\bar{\zeta}$ is the same as the combined number of factors of ψ and ζ. We denote by $\mathcal{N}_{\mathrm{sym}}(X)$ the subalgebra of $\mathcal{N}(X)$ of $U(1)$ invariant elements and likewise for $\mathcal{N}_{\mathrm{sym}}(\Lambda \sqcup \Lambda)$. The maps θ and $\mathbb{E}_C$ preserve $U(1)$ symmetry.

A bijection $E: \Lambda \to \Lambda$ is an *automorphism* of the torus Λ if it maps nearest neighbours to nearest neighbours. Bijections act as homomorphisms on the algebra $\mathcal{N}(\Lambda)$ by $E\psi_x = \psi_{Ex}$ and $E\bar{\psi}_x = \bar{\psi}_{Ex}$ and similarly for $\mathcal{N}(\Lambda \sqcup \Lambda)$. If C is invariant under lattice symmetries, i.e., $C(Ex, Ey) = C(x, y)$ for all automorphisms E, then the convolution $\mathbb{E}_C\theta$ commutes with automorphisms of Λ, i.e., $E\mathbb{E}_C\theta F = \mathbb{E}_C\theta EF$. In particular $E\mathbb{E}_{C_j}\theta F = \mathbb{E}_{C_j}\theta EF$ for the covariances of the finite range decomposition (3.1). An important consequence of this discussion is that if $X \subset \Lambda$, $F \in \mathcal{N}_{\mathrm{sym}}(X)$ and F is invariant under lattice symmetries that fix X, then $\mathbb{E}_{C_j}\theta F \in \mathcal{N}_{\mathrm{sym}}(X)$ is also invariant under such lattice symmetries.

3.4 Polymer coordinates

We will use (3.11) and the decomposition (3.1) to study the progressive integration

$$Z_{j+1} = \mathbb{E}_{C_{j+1}}\Big[\theta Z_j\Big], \tag{3.13}$$

for a given $Z_0 \in \mathcal{N}(\Lambda)$. To be concrete here, the reader may keep $Z_0 = e^{-V_0(\Lambda)}$ with $V_0(\Lambda)$ from (2.3) in mind, but to compute correlation functions we will consider generalisations of this choice of Z_0 in Sect. 6. The analysis is performed by defining suitable coordinates *(polymer coordinates)* and norms (on polymer coordinates) that enable the progressive integration to be treated as a dynamical system: this is the *renormalisation group*. Towards this end, this section defines local polymer coordinates as in [28, 32]. Section 3.5 then defines relevant norms, and norms on polymer coordinates are introduced in Sect. 3.8 after other preliminary material is introduced.

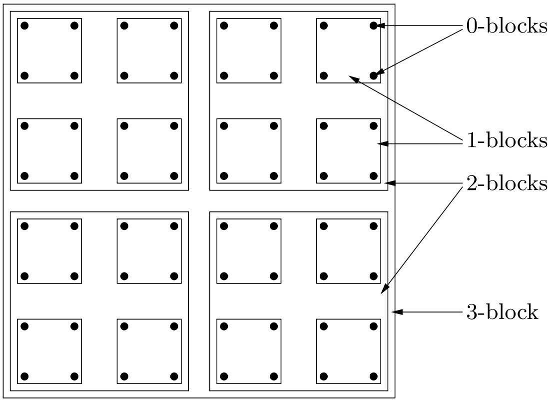

Fig. 1 Illustration of j-blocks when $L = 2$

3.4.1 Blocks and polymers

Recall $\Lambda = \Lambda_N$ denotes a torus of side length L^N. Partition Λ_N into nested scale-j blocks $\mathcal{B}_j$ of side lengths L^j where $j = 0, \dots, N$. Thus scale-0 blocks are simply the points in Λ, while the only scale-N block is Λ itself, see Fig. 1. The set of j-*polymers* $\mathcal{P}_j = \mathcal{P}_j(\Lambda)$ consists of finite unions of blocks in $\mathcal{B}_j$. To define a notion of connectedness, say $X, Y \in \mathcal{P}_j$ *do not touch* if $\inf_{x \in X, y \in Y} |x - y|_\infty > 1$. A polymer is *connected* if it is not empty and there is a path of touching blocks between any two blocks of the polymer. The subset of connected j-polymers is denoted $\mathcal{C}_j$. We will drop j- prefixes when the scale is clear.

For a fixed j-polymer X, let $\mathcal{B}_j(X)$ denote the set of j-blocks contained in X and let $|\mathcal{B}_j(X)|$ be the number of such blocks. Connected polymers X with $|\mathcal{B}_j(X)| \leqslant 2^d$ are called *small sets* and the collection of all small sets is denoted $\mathcal{S}_j$. Polymers which are not small will be called *large*. Finally, for $X \in \mathcal{P}_j$ we define its *small set neighbourhood* $X^\square \in \mathcal{P}_j$ as the union of all small sets containing a block in $\mathcal{B}_j(X)$, and its *closure* $\bar{X}$ as the smallest $Y \in \mathcal{P}_{j+1}$ such that $X \subset Y$.

3.4.2 Coordinates

We will write Z_j in the form

$$Z_j = e^{-u_j |\Lambda_N|} \sum_{X \in \mathcal{P}_j} e^{-V_j(\Lambda_N \setminus X)} K_j(X), \tag{3.14}$$

where the u_j are constants (essentially the free energy), the $V_j(X)$ are functions of the fields $\bar{\psi}_x$, ψ_x for x in a neighbourhood of X, parametrised by finitely many *coupling constants* which require special attention (and are independent of X), and everything else is organised into the functions $K_j(X)$, which will be called *polymer activities*. Unlike the V_j, the polymer activities track quantities whose precise value is not important. Explicit (somewhat complicated) formulas for the evolution of K_j will be given below. An essential point will be that they can be tracked in terms of estimates. The tuple (V_j, K_j) together with the representation (3.14) will be referred to as *polymer coordinates*. In the remainder of this subsection, we will discuss some structural properties of these coordinates.

Coupling constants. We will always identify V_j with the coupling constants which parametrise it. Explicitly, for coupling constants $V_j = (z_j, y_j, a_j, b_j) \in \mathbb{C}^4$ and a set $X \subset \Lambda_N$, let

$$V_j(X) = \sum_{x\in X} \Big[y_j (\nabla\psi)_x (\nabla\bar\psi)_x + \frac{z_j}{2}((-\Delta\psi)_x \bar\psi_x + \psi_x(-\Delta\bar\psi)_x) + a_j \psi_x\bar\psi_x + b_j \psi_x\bar\psi_x (\nabla\psi)_x(\nabla\bar\psi)_x \Big]. \tag{3.15}$$

For the scale $j = 0$, if we set $Z_0 = e^{-V_0(\Lambda_N)}$, then the polymer coordinates take the simple form

$$Z_0 = e^{-V_0(\Lambda_N)} = e^{-u_0|\Lambda_N|} \sum_{X\subset\Lambda_N} e^{-V_0(\Lambda_N\setminus X)} K_0(X), \tag{3.16}$$

with

$$K_0(X) = 1_{X=\varnothing}, \qquad u_0 = 0. \tag{3.17}$$

To study the recursion $Z_{j+1} = \mathbb{E}_{C_{j+1}}\theta Z_j$ at a general scale $j = 1, \dots, N$, we will make a choice of coupling constants V_j and of polymer activities $K_j = (K_j(X))_{X\in\mathcal{P}_j(X)}$ such that

$$Z_j = e^{-u_j|\Lambda_N|} \sum_{X\in\mathcal{P}_j} e^{-V_j(\Lambda_N\setminus X)} K_j(X). \tag{3.18}$$

Polymer activities. The K_j will be defined in such a way that they satisfy the locality and symmetry property $K_j(X) \in \mathcal{N}_{\text{sym}}(X^\square)$ and the following important component factorisation property: for $X, Y \in \mathcal{P}_j$ that do not touch,

$$K_j(X \cup Y) = K_j(X)K_j(Y). \tag{3.19}$$

Note that since they are $U(1)$ symmetric, the $K_j(X)$ are even elements of $\mathcal{N}$, so they commute and the product on the right-hand side is unambiguous. Using the previous identity,

$$K_j(X) = \prod_{Y\in\text{Comp}(X)} K_j(Y), \tag{3.20}$$

where $\text{Comp}(X)$ denotes the set of connected components of the polymer X. In particular, each $K_j = (K_j(X))_{X\in\mathcal{P}_j(\Lambda_N)}$ satisfying (3.19) can be identified with its restriction $K_j = (K_j(X))_{X\in\mathcal{C}_j(\Lambda_N)}$. We say that K_j is automorphism invariant if $EK_j(X) = K_j(E(X))$ for all $X \in \mathcal{P}_j(\Lambda_N)$ and all torus automorphisms $E \in \text{Aut}(\Lambda_N)$ that map blocks in $\mathcal{B}_j$ to blocks in $\mathcal{B}_j$.

Definition 3.1 Let $\mathcal{K}_j^{\varnothing}(\Lambda_N)$ be the linear space of automorphism invariant $K_j = (K_j(X))_{X \in \mathcal{C}_j(\Lambda_N)}$ with $K_j(X) \in \mathcal{N}_{\mathrm{sym}}(X^{\square})$ for every $X \in \mathcal{C}_j$.

Polymer coordinates at scale j are thus a choice of (the coupling constants) V_j together with a choice of (polymer activities) K_j from the space $\mathcal{K}_j^{\varnothing}$. The renormalisation group map is a particular choice of a map $(V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$.

For a given Z_j, the above conditions do *not* determine K_j uniquely given V_j (see the proof of Proposition 3.11, where the non-uniqueness is apparent). We will state our specific choice of such a map in Sect. 3.7 below. The goal is to choose V_j such that the size of the K_j decrease rapidly as j increases when the sizes of V_j and K_j are measured in appropriate norms. Thus K_j will capture the *irrelevant* (or contracting) directions of the renormalisation group dynamics, while the *relevant* (or expanding) and *marginal* directions will be captured by the V_j coordinates. The next section defines the norms we will use.

3.5 Norms

We now define the $T_j(\ell)$ norms we will use on the Grassmann algebras $\mathcal{N}(\Lambda)$. General properties of these norms were systematically developed in [29], to which we will refer for some proofs. To help the reader, in places where we specialise the definitions of [29] we indicate the more general notation that is used in [29].

We start with some notation. For any set S, we write S^* for the set of finite sequences in S. We write $\Lambda_f = \Lambda \times \{\pm 1\}$ and for $(x, \sigma) \in \Lambda_f$ we write $\psi_{x,\sigma} = \psi_x$ if $\sigma = +1$ and $\psi_{x,\sigma} = \bar{\psi}_x$ if $\sigma = -1$. Then every element $F \in \mathcal{N}(\Lambda)$ can be written in the form

$$F = \sum_{z \in \Lambda_f^*} \frac{1}{z!} F_z \psi^z \tag{3.21}$$

where $\psi^z = \psi_{z_1} \cdots \psi_{z_n}$ if $z = (z_1, \dots, z_n)$. We are using the notation that $z! = n!$ if the sequence z has length n. The representation in (3.21) is in general not unique. To obtain a unique representation we require that the F_z are antisymmetric with respect to permutations of the components of z (this is possible due to the antisymmetry of the Grassmann variables). Antisymmetry implies that $F_z = 0$ if z has length exceeding $2|\Lambda|$ or if z has any repeated entries.

Definition 3.2 Let $p_\Phi = 2d$. The space of test functions $\Phi_j(\ell)$ is defined as the set of functions $g\colon \Lambda_f^* \to \mathbb{R}$, $z \mapsto g_z$ together with norm

$$\|g\|_{\Phi_j(\ell)} = \sup_{n \geqslant 0} \sup_{z \in \Lambda_f^n} \sup_{|\alpha_i| \leqslant p_\Phi} \ell^{-n} L^{j(|\alpha_1| + \cdots |\alpha_n|)} |\nabla_{z_1}^{\alpha_1} \cdots \nabla_{z_n}^{\alpha_n} g_z|. \tag{3.22}$$

In this definition, $\nabla_{z_i}^{\alpha_i}$ denotes the discrete derivative ∇^{α_i} with multiindex α_i acting on the spatial part of the ith component of the finite sequence z.

The $\Phi_j(\ell)$ norm measures spatial smoothness of test functions, which act as substitutes for fields. Restricted to sequences of fixed length, it is a lattice C^{p_Φ} norm at spatial scale L^j and field scale ℓ. We will mainly use the following choice of ℓ when using the $\Phi_j(\ell)$ norm:

$$\ell_j = \ell_0 L^{-\frac{1}{2}(d-2)j} \tag{3.23}$$

for a large constant ℓ_0, and ℓ_j will always be as in (3.23). This choice captures the size of the covariances in the decomposition (3.1). Indeed, regarding the covariances C_j as functions of sequences of length 2 (i.e., as the coefficient in (3.21) of $F = \sum_{x,y} \bar\psi_x \psi_y C_j(x,y)$), the bounds (3.3) imply

$$\|C_j\|_{\Phi_j(\ell_j)} \leqslant 1, \tag{3.24}$$

when ℓ_0 is chosen as a large (L-dependent, due to the index $j+1$ on the left-hand side of (3.3)) constant relative to the constants in (3.3) with $|\alpha| \leqslant 2p_\Phi$. From now on, we will always assume that ℓ_0 is fixed in this way.

Definition 3.3 We define $T_j(\ell)$ to be the algebra $\mathcal{N}(\Lambda)$ together with the dual norm

$$\|F\|_{T_j(\ell)} = \sup_{\|g\|_{\Phi_j(\ell)} \leqslant 1} |\langle F, g\rangle|, \qquad \text{where } \langle F, g\rangle = \sum_{z \in \Lambda_f^*} \frac{1}{z!} F_z g_z \tag{3.25}$$

when $F \in \mathcal{N}(\Lambda)$ is expressed as in (3.21).

An analogous definition applies to $\mathcal{N}(\Lambda \sqcup \Lambda)$, and we then write $T_j(\ell \sqcup \ell) = T_j(\ell)$ for this norm (with the first notation to emphasise the doubled algebra), where we recall that $\mathcal{N}(\Lambda \sqcup \Lambda)$ is defined above (3.8).

The $T_j(\ell)$ norm measures smoothness of field functionals $F \in \mathcal{N}(\Lambda)$ with respect to fields whose size is measured by $\Phi_j(\ell)$. They therefore implement the power counting on which renormalisation relies. Important, but relatively straightforwardly verified, properties of these norms are systematically developed in [29]; we summarise the ones we need now.

Product property. First, the $T_j(\ell)$ norm defines a Banach algebra, i.e., the following product property holds (see [29, Proposition 3.7]): for $F_1, F_2 \in \mathcal{N}(\Lambda)$,

$$\|F_1 F_2\|_{T_j(\ell)} \leqslant \|F_1\|_{T_j(\ell)} \|F_2\|_{T_j(\ell)}. \tag{3.26}$$

Using the product property, we may gain some intuition regarding these norms by considering the following simple examples:

$$\|\psi_x \bar\psi_x\|_{T_j(\ell)} \leqslant \|\psi_x\|_{T_j(\ell)} \|\bar\psi_x\|_{T_j(\ell)} = \ell^2, \tag{3.27}$$

$$\|(\nabla_e \psi)_x \bar\psi_x\|_{T_j(\ell)} \leqslant \|\nabla_e \psi_x\|_{T_j(\ell)} \|\bar\psi_x\|_{T_j(\ell)} = \ell^2 L^{-j}. \tag{3.28}$$

The following more subtle example relies on $\psi_x^2 = \bar\psi_x^2 = 0$ and plays an important role for our model:

$$\|\psi_x \bar\psi_x \psi_{x+e} \bar\psi_{x+e}\|_{T_j(\ell)} = \|\psi_x \bar\psi_x (\nabla_e \psi)_x (\nabla_e \bar\psi)_x\|_{T_j(\ell)} \asymp \ell^4 L^{-2j}. \tag{3.29}$$

In general each factor of the fields contributes a factor ℓ and each derivative a factor L^{-j}.

Monotonicity. Second, as follows immediately from the definition, the following monotonicity properties hold: for $\ell \leqslant \ell'$ and $F \in \mathcal{N}(\Lambda)$,

$$\|F\|_{T_j(\ell)} \leqslant \|F\|_{T_j(\ell')}, \qquad \|F\|_{T_{j+1}(\ell)} \leqslant \|F\|_{T_j(\ell')}. \tag{3.30}$$

Doubling map. Third, the doubling map satisfies (see [29, Proposition 3.12]): for $F \in \mathcal{N}(\Lambda)$,

$$\|\theta F\|_{T_j(\ell)} \leqslant \|F\|_{T_j(2\ell)} \tag{3.31}$$

where the norm on the left-hand side is the $T_j(\ell) = T_j(\ell \sqcup \ell)$ norm on $\mathcal{N}(\Lambda \sqcup \Lambda)$.

Gram inequality. Finally, the following contraction bound for the fermionic Gaussian expectation is an application of the Gram inequality whose importance is well-known in fermionic renormalisation. It is proved in [29, Proposition 3.19].

Proposition 3.4 *Assume C is a covariance matrix with $\|C\|_{\Phi_j(\ell)} \leqslant 1$. For $F \in \mathcal{N}(\Lambda \sqcup \Lambda)$, then*

$$\|\mathbb{E}_C F\|_{T_j(\ell)} \leqslant \|F\|_{T_j(\ell)}. \tag{3.32}$$

In particular, for $F \in \mathcal{N}(\Lambda)$, by (3.31) the fermionic Gaussian convolution satisfies

$$\|\mathbb{E}_C \theta F\|_{T_j(\ell)} \leqslant \|F\|_{T_j(2\ell)}. \tag{3.33}$$

For our choices of ℓ_j and of the finite range covariance matrices C_j, the inequalities (3.30) and (3.33) in particular imply

$$\begin{aligned} \|F\|_{T_{j+1}(\ell_{j+1})} \leqslant \|F\|_{T_{j+1}(2\ell_{j+1})} \leqslant \|F\|_{T_j(\ell_j)}, \\ \|\mathbb{E}_{C_{j+1}} \theta F\|_{T_{j+1}(\ell_{j+1})} \leqslant \|F\|_{T_j(\ell_j)}. \end{aligned} \tag{3.34}$$

We remark that the existence of this contraction estimate for the expectation combined with (3.40) below is what makes renormalisation of fermionic fields much simpler than that of bosonic ones.

3.6 Localisation

To define the renormalisation group map we need one more important ingredient: the *localisation operators* Loc_X and $\mathrm{Loc}_{X,Y}$ that will be used to extract the relevant and marginal terms from the K_j coordinate to incorporate them in the renormalisation from V_j into V_{j+1}. These operators are generalised Taylor approximations which take as inputs $F \in \mathcal{N}(X)$ and produce best approximations of F in a finite dimensional space of *local field polynomials*.

3.6.1 Local field polynomials

By *formal local field polynomials* we refer to formal polynomials in the symbols $\psi, \bar\psi, \nabla\psi, \nabla\bar\psi, \Delta\psi, \Delta\bar\psi, \nabla^2\psi, \dots$ (without spatial index). The *dimension* of a formal local field monomial is given by $(d-2)/2$ times the number of factors of ψ or $\bar\psi$ plus the number of discrete derivatives ∇ in its representation, where Δ is treated as two discrete derivatives. The classification of local monomials according to dimension is known as power counting in the renormalisation group literature. Relevant monomials are those with dimension strictly greater than d, marginal ones those with dimension equal to d, and irrelevant those with dimension strictly less than d. Concretely, we consider the following space of formal local field polynomials, consisting of the relevant and marginal monomials consistent with symmetry constraints.

Definition 3.5 Let $\mathcal{V}^\varnothing \cong \mathbb{C}^4$ be the linear space of formal local field polynomials of the form

$$V = y(\nabla\psi)(\nabla\bar\psi) + \frac{z}{2}((-\Delta\psi)\bar\psi + \psi(-\Delta\bar\psi)) + a\psi\bar\psi + b\psi\bar\psi(\nabla\psi)(\nabla\bar\psi). \quad (3.35)$$

We will identify elements $V \in \mathcal{V}^\varnothing$ with their coupling constants $(z, y, a, b) \in \mathbb{C}^4$. Sometimes we include a constant term u and write $u + V \in \mathbb{C} \oplus \mathcal{V}^\varnothing$ with $u + V \cong (u, z, y, a, b) \in \mathbb{C}^5$.

Given a set $X \subset \Lambda$, a formal local field polynomial P can be specialised to an element of $\mathcal{N}(\Lambda)$ by replacing formal monomials by evaluations. For example, if $P = \bar\psi\psi$, $P(X) = \sum_{x\in X} \bar\psi_x\psi_x$. We call polynomials arising in this way *local polynomials*. The most important case is $V \mapsto V(X)$, with

$$V(X) = \sum_{x\in X}\Big[y(\nabla\psi)_x(\nabla\bar\psi)_x + \frac{z}{2}((-\Delta\psi)_x\bar\psi_x + \psi_x(-\Delta\bar\psi)_x) + a\psi_x\bar\psi_x + b\psi_x\bar\psi_x(\nabla\psi)_x(\nabla\bar\psi)_x\Big], \quad (3.36)$$

where $\Delta = -\frac{1}{2}\sum_{e\in\mathcal{E}_d} \nabla_{-e}\nabla_e$ and $(\nabla\psi)_x(\nabla\bar\psi)_x = \frac{1}{2}\sum_{e\in\mathcal{E}_d} \nabla_e\psi_x\nabla_e\bar\psi_x$ are the lattice Laplacian and the square of the lattice gradient; recall that $\mathcal{E}_d = \{e_1, \dots, e_{2d}\}$. For a constant $u \in \mathbb{C}$ we write $u(X) = u|X|$, where $|X|$ is the number of points in $X \subset \Lambda$. Thus $(u + V)(X) = u(X) + V(X) = u|X| + V(X)$.

Definition 3.6 For $X \subset \Lambda$, define $\mathcal{V}^\varnothing(X) = \{V(X) : V \in \mathcal{V}^\varnothing\} \subset \mathcal{N}(\Lambda)$ and analogously $(\mathbb{C} \oplus \mathcal{V}^\varnothing)(X) = \{u|X| + V(X) : u \in \mathbb{C},\ V \in \mathcal{V}^\varnothing\} \subset \mathcal{N}(\Lambda)$.

The space $\mathcal{V}^\varnothing$ contains all formal local field polynomials whose constituent monomials have dimension at most d that are (i) $U(1)$ invariant, (ii) respect lattice symmetries (if $EX = X$ for an automorphism E, then $EV(X) = V(X)$), (iii) $V(X) \neq 0$, and (iv) have no constant terms. Note that $\mathbb{E}_C\theta$ preserves $\mathcal{V}^\varnothing(X)$ by the discussion in Sect. 3.3. We emphasise that there is no $(\bar\psi\psi)^2$ term, which would be consistent with having dimension as most d (if $d = 3, 4$) and symmetries, because it vanishes upon specialisation by anticommutativity of the fermionic variables.

Two further remarks are in order. First, the monomial $\psi\bar{\psi}(\nabla\psi)(\nabla\bar{\psi})$ has dimension $2d-2 > d$ for $d \geqslant 3$; we include it in $\mathcal{V}^{\varnothing}$ since it occurs in the initial potential. Second, the monomials multiplying z and y are equivalent upon specialisation when $X = \Lambda$ by summation by parts, and differ only by boundary terms for general $X \subset \Lambda$. This would allow us to keep only one of them, but it will be simpler to keep both.

3.6.2 Localisation

The localisation operators Loc_X and $\mathrm{Loc}_{X,Y}$ associate local field monomials to elements of $\mathcal{N}(X)$. In renormalisation group terminology, the image of Loc projects onto the space of all relevant and marginal local polynomials. The precise definitions of the localisation operators do not play a direct role in this paper. Rather, only their abstract properties, summarised in the following Proposition 3.8, will be required. Nonetheless, to give some intuition for the action of Loc_X and $\mathrm{Loc}_{X,Y}$, we include the following typical examples (see also [30, Sect. 1.5]). The examples indicate (as stated at the beginning of this section) that the localisation operators are generalised Taylor approximations.

Example 3.7 (i) Let F be a monomial in $\{\bar{\psi}_x, \psi_x\}$ of degree greater than four. Then $\mathrm{Loc}_X F = 0$.

(ii) Consider $F = \sum_{x\in X}\sum_{y\in\Lambda} q(x-y)\bar{\psi}_y\psi_y$ where the kernel $q\colon \mathbb{Z}^d \to \mathbb{R}$ has finite support and is invariant under lattice rotations. Then provided Λ is large enough,

$$\begin{aligned}\mathrm{Loc}_X F &= \sum_{x\in X}\left[q^{(1)}\bar{\psi}_x\psi_x + q^{(**)}\left(\frac{1}{2}\bar{\psi}_x(\Delta\psi)_x + \frac{1}{2}(\Delta\bar{\psi})_x\psi_x + (\nabla\bar{\psi})_x(\nabla\psi)_x\right)\right]\\ &= \sum_{x\in X} P_x,\end{aligned} \tag{3.37}$$

where $q^{(1)} = \sum_{y\in\mathbb{Z}^d} q(y)$ and $q^{(**)} = \sum_{y\in\mathbb{Z}^d} y_1^2 q(y)$ (and y_1 denotes the first component of $y \in \mathbb{Z}^d$), and with the same P_x as in (3.37),

$$\mathrm{Loc}_{X,Y} F = \sum_{y\in Y} P_y. \tag{3.38}$$

Thus $\sum_{i=1}^{n}\mathrm{Loc}_{X,X_i} F = \mathrm{Loc}_X F$ if X is the disjoint union of $X_1, \dots, X_n$.

For the definition of Loc_X and $\mathrm{Loc}_{X,Y}$, we use the general framework developed in [30]. In short, the definitions of Loc_X and $\mathrm{Loc}_{X,Y}$ are those of [30, Definition 1.6 and 1.15]. These definitions require a choice of field dimensions, which we choose as $[\psi] = [\bar{\psi}] = (d-2)/2$, a choice of maximal field dimension d_+, which we choose as $d_+ = d$, and a choice of a space $\hat{P}$ of test polynomials, which we define exactly as in [30, (1.19)] with the substitution $\nabla_e\nabla_e \to -\nabla_e\nabla_{-e}$ explained in [30, Example 1.3]. The following properties are then almost immediate from [30].

Proposition 3.8 *For $L = L(d)$ sufficiently large there is a universal $\bar{C} > 0$ such that: for $j < N$ and any small sets $Y \subset X \in \mathcal{S}_j$, the linear maps $\mathrm{Loc}_{X,Y} : \mathcal{N}(X^{\square}) \to \mathcal{N}(Y^{\square})$ have the following properties:*

(*i*) *They are bounded:*

$$\| \mathrm{Loc}_{X,Y} F \|_{T_j(\ell_j)} \leqslant \bar{C} \| F \|_{T_j(\ell_j)}. \tag{3.39}$$

(*ii*) *The maps $\mathrm{Loc}_X = \mathrm{Loc}_{X,X} : \mathcal{N}(X^{\square}) \to \mathcal{N}(X^{\square})$ satisfy the contraction bound*

$$\| (1 - \mathrm{Loc}_X) F \|_{T_{j+1}(2\ell_{j+1})} \leqslant \bar{C} L^{-d} L^{-(\frac{d-2}{2} \wedge 1)} \| F \|_{T_j(\ell_j)}. \tag{3.40}$$

(*iii*) *If X is the disjoint union of $X_1, \ldots, X_n$ then $\mathrm{Loc}_X = \sum_{i=1}^n \mathrm{Loc}_{X,X_i}$.*

(*iv*) *The maps are Euclidean invariant: if $E \in \mathrm{Aut}(\Lambda_N)$ then $E \,\mathrm{Loc}_{X,Y} F = \mathrm{Loc}_{EX,EY} EF$.*

(*v*) *For a block B, small polymers $X_1, \ldots, X_n$, and any $F_i \in \mathcal{N}_{\mathrm{sym}}(X_i^{\square})$ such that $\sum_{i=1}^n \mathrm{Loc}_{X_i,B} F_i$ is invariant under automorphisms of Λ_N that fix B,*

$$\sum_{i=1}^n \mathrm{Loc}_{X_i,B} F_i \in (\mathbb{C} \oplus \mathcal{V}^{\varnothing})(B). \tag{3.41}$$

We remark that the image of $\mathrm{Loc}_{X,Y}$ is in general a larger space of local field monomials than $\mathcal{V}^{\varnothing}(Y)$, often denoted $\mathcal{V}$ in [30] — for example first gradients of the field can arise which only need cancel upon the symmetrisation in (3.41). Since we will not use this larger space directly we have not assigned a symbol for it.

Proof of Proposition 3.8 The bound (i) is [30, Proposition 1.16], the contraction bound (ii) is [30, Proposition 1.12], the decomposition property (iii) holds by the definition of $\mathrm{Loc}_{X,Y}$ in [30, Definition 1.15], and the Euclidean invariance (iv) is [30, Proposition 1.9]. Note that the parameter A' in [30, Proposition 1.12] does not appear here as it applies to the boson field ϕ; our fermionic context corresponds to $\phi = 0$. For the application of [30, Proposition 1.12] we have used that p_Φ was fixed to be $2d$ in Definition 3.2, and that we have only considered the action of Loc on small sets.

Finally, property (v) follows from [30, Proposition 1.10] and the fact that the space $\mathcal{V}^{\varnothing}$ defined in Definition 3.5 contains all local polynomials of dimension at most d invariant under lattice automorphisms that fix a point. □

3.7 Definition of the renormalisation group map

The renormalisation group map $\Phi_{j+1} = \Phi_{j+1,N,m^2}$ is a map

$$\Phi_{j+1} : (V_j, K_j) \mapsto (u_{j+1}, V_{j+1}, K_{j+1}) \tag{3.42}$$

acting on

$$V_j \in \mathcal{V}^{\varnothing}, \qquad K_j \in \mathcal{K}_j^{\varnothing}(\Lambda_N), \tag{3.43}$$

with $u_{j+1} \in \mathbb{C}$, the space of coupling constants $\mathcal{V}^{\varnothing}$ as in Definition 3.5, and the space of polymer activities $\mathcal{K}_j^{\varnothing}(\Lambda_N)$ as in Definition 3.1. The map will have mild dependence on m^2 and Λ_N as a consequence of this dependence of the covariance matrices. As indicated above the u-coordinate does not influence the dynamics of the remaining coordinates. Thus we can always explicitly assume that the incoming u-component of Φ_{j+1} is 0 and separate it from V_{j+1} in the output. This means that we will often regard Φ_{j+1} as a map $(V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$ where $u_j = u_{j+1} = 0$.

The explicit definition of the map Φ_{j+1} is given in (3.48) and (3.49) below. The essential consequences of the definition are Proposition 3.11, which enables the iterative application of the renormalisation group maps, and the estimates of Theorem 3.13.

At first sight, the definition of Φ_{j+1} may appear somewhat complicated, but it follows from simple principles that are outlined in the proof of Proposition 3.11 below. Compared to other implementations of the fermionic renormalisation group, the finite range property of the covariances in our implementation means we do not require infinite expansions, nor do we require norms which control the spatial complexity of polymers beyond simple volume estimates. As such, establishing useful norm estimates becomes an essentially combinatorial problem. This feature is especially useful in models with bosonic fields, see [28, 36] and [18, Appendix A] for introductory discussions, but it also provides appealing features in the present fermionic context. For example, the flows on tori with two distinct side lengths $L^{N_1} < L^{N_2}$ *coincide* up to the final length scale L^{N_1-1} for polymers which do not wrap around either torus, making the definition of the infinite volume flow and its relation to the finite volume one particularly transparent (see Proposition 3.12 below).

For the definition of the renormalisation group map Φ_{j+1}, we identify $V_j \in \mathcal{V}^{\varnothing}$ with the tuple $(V_j(B))_{B\in\mathcal{B}_j(\Lambda_N)}$, i.e., the field monomials corresponding to the coupling constants V_j evaluated over a block B, and the tuple $(K_j(X))_{X\in\mathcal{C}_j(\Lambda_N)}$ with its extension $(K_j(X))_{X\in\mathcal{P}_j(\Lambda_N)}$ determined by the component factorisation property (3.19). We also introduce, assuming $j+1 < N$,

$$Q(B) = \sum_{X\in\mathcal{S}_j : X\supset B} \operatorname{Loc}_{X,B} K_j(X), \qquad (B \in \mathcal{B}_j), \tag{3.44}$$

$$J(B,B) = -\sum_{X\in\mathcal{S}_j\setminus\mathcal{B}_j : X\supset B} \operatorname{Loc}_{X,B} K_j(X), \qquad (B \in \mathcal{B}_j), \tag{3.45}$$

$$J(B,X) = \operatorname{Loc}_{X,B} K_j(X), \qquad (X \in \mathcal{S}_j \setminus \mathcal{B}_j,\ B \in \mathcal{B}_j(X)), \tag{3.46}$$

and $J(B,X) = 0$ otherwise. If $j+1 = N$ we simply set $Q = J = 0$.

As a consequence of the properties of Loc from Proposition 3.8 (c.f. in particular property (v)), $Q(B)$ arises from an element of $\mathcal{V}^{\varnothing}$ and represents the marginal and relevant contributions from K_j associated with the block B. These contributions (which are marginal or relevant in the sense of power counting) only come from $K(X)$ for small sets X, as large sets X will yield contracting contributions for entropic reasons (as opposed to power counting reasons) considered later. The $J(B,X)$

are a technical device for removing the Q contribution from K_j. An important property is that

$$\sum_X J(B, X) = 0, \qquad (B \in \mathcal{B}_j). \tag{3.47}$$

The application of this property occurs in (3.111). We indicate the motivation for the form of J below. For a fuller discussion we refer to [28, Lectures 4–5] and [19, Appendix A, 12.3.2].

We will specify the V- and K-components of the renormalisation group map separately.

V-component. The first definition defines the V-component of the renormalisation group map. This map is given by first-order perturbation theory, i.e., $\mathbb{E}_{C_{j+1}}[\theta V_j(B)]$, plus the higher-order contribution $\mathbb{E}_{C_{j+1}}[\theta Q(B)]$ representing the marginal and relevant contributions from K_j as discussed above. Here recall the definition of the doubling map θ from (3.8), i.e., θF is obtained from F by replacing ψ by $\psi + \zeta$ and $\bar\psi$ by $\bar\psi + \bar\zeta$, and that the expectation only acts on $(\zeta, \bar\zeta)$.

Definition 3.9 The map $(V_j, K_j) \mapsto (u_{j+1}, V_{j+1})$ is defined by

$$u_{j+1}|B| + V_{j+1}(B) = \mathbb{E}_{C_{j+1}}\Big[\theta(V_j(B) - Q(B))\Big], \qquad (B \in \mathcal{B}_j). \tag{3.48}$$

We emphasise that V_{j+1} is evaluated on $B \in \mathcal{B}_j$ here; V_{j+1} can then be extended to $\mathcal{B}_{j+1}$ by additivity. When K_j is automorphism invariant, which is the case if $K_j \in \mathcal{K}_j^{\varnothing}(\Lambda_N)$, the right-hand side of (3.48) is in $(\mathbb{C} \oplus \mathcal{V}^{\varnothing})(B)$ and can thus be identified with an element of $\mathbb{C} \oplus \mathcal{V}^{\varnothing} \cong \mathbb{C}^5$. This can be checked by using Proposition 3.8 (iv) and (v) and the properties of progressive integration discussed in Sect. 3.2. Recall that we sometimes write the left-hand side as $(u + V)_{j+1}(B)$. Since $V_{j+1}(B)$ has no constant term by definition, the constant u_{j+1} is unambiguously defined.

K-component. The following formula for the K-component of the renormalisation group map is more involved. It is engineered to achieve the desired factorisation and contraction properties of the renormalisation group map. The explicit formula will enable a relatively straightforward verification of the estimates which follow from it; the formula itself is the result of relatively simple manipulations explained in the proof Proposition 3.11 below.

Definition 3.10 For $U \in \mathcal{P}_{j+1}$, the map $(V_j, K_j) \mapsto K_{j+1}(U)$ is defined by

$$K_{j+1}(U) = e^{u_{j+1}|U|} \times \sum_{(\mathcal{X},\check{X}) \in \mathcal{G}(U)} e^{-(u+V)_{j+1}(U \setminus \check{X} \cup X_{\mathcal{X}})} \mathbb{E}_{C_{j+1}}\left[\check{K}_j(\check{X}) \prod_{(B,X)\in\mathcal{X}} \theta J(B, X)\right] \tag{3.49}$$

where

$$\check{K}_j(X) = \prod_{W \in \mathrm{Comp}(X)} \check{K}_j(W),$$
$$\check{K}_j(W) = \sum_{Y \in \mathcal{P}_j(W)} (\theta K_j(W \setminus Y))(\delta I)^Y - \sum_{B \in \mathcal{B}_j(W)} \theta J(B, W), \tag{3.50}$$

$$(\delta I)^X = \prod_{B \in \mathcal{B}_j(X)} \delta I(B), \qquad \delta I(B) = \theta e^{-V_j(B)} - e^{-(u+V)_{j+1}(B)}. \tag{3.51}$$

Following [28, Sect. 5.1.2], we define the set $\mathcal{G}(U)$ (and the corresponding notation $\mathcal{X}$ and $X_{\mathcal{X}}$) as follows: $\check{X} \in \mathcal{P}_j$ and $\mathcal{X}$ is a set of pairs (B, X) with $X \in \mathcal{S}_j$ and $B \in \mathcal{B}_j(X)$ with the following properties: each X appears in at most one pair $(B, X) \in \mathcal{X}$, the different X do not touch, $X_{\mathcal{X}} = \cup_{(B,X)\in\mathcal{X}} X$ does not touch $\check{X}$, and the closure of the union of $\check{X}$ with $\cup_{(B,X)\in\mathcal{X}} B^{\square}$ is U.

The following proposition is essentially [28, Proposition 5.1]. The only differences are that we have factored out the factor $e^{-u_{j+1}|\Lambda|}$ and that the doubling map θ is explicit (it is implicit in [28]). Explicitly, note that $K_{j+1}(U)$ and $V_{j+1}(X)$ are elements of the Grassmann algebra $\mathcal{N}(\Lambda)$, i.e., they depend on the fields $(\psi, \bar{\psi})$, but since the doubling map θ replaces $(\psi, \bar{\psi})$ by $(\psi + \zeta, \bar{\psi} + \bar{\zeta})$, the functions $\check{K}_j(X)$ and $(\delta I)^X$ above depend on all of $(\psi, \bar{\psi}, \zeta, \bar{\zeta})$.

For convenience and because it also demystifies the somewhat complicated formula for the renormalisation group map, we have included a proof along with some additional explanations. The proof of this proposition does not rely on the specific choice of Q and J in (3.44)–(3.46) or on the property (3.47). This choice only becomes important in the proof of Theorem 3.13.

Proposition 3.11 *Given (V_j, K_j) define Z_j by (3.18) with $u_j = 0$. Suppose K_j has the factorisation property (3.19) with respect to $\mathcal{P}_j$. Then with the above choice of $(u_{j+1}, V_{j+1}, K_{j+1})$ and Z_{j+1} given by (3.18) with $j+1$ in place of j, we have $Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j$, and K_{j+1} has the factorisation property (3.19) with respect to $\mathcal{P}_{j+1}$. Moreover, if K_j is automorphism invariant then so is K_{j+1}.*

Proof The proof essentially consists of algebraically manipulating the expression

$$Z_j = \sum_{X \in \mathcal{P}_j} I^{\Lambda \setminus X} K(X) \tag{3.52}$$

where $I(B) = e^{-V_j(B)}$ and $K(X) = K_j(X)$. These manipulations only rely on factorisation properties of I and K (and not on their precise definitions). Hence we will explicitly state the required factorisation properties, and later specialise to the context of Proposition 3.11. We will use that $I^Y = \prod_{B \in \mathcal{B}_j(Y)} I(B)$ factors over blocks and $K(X)$ factors over connected components of X.

Change of coupling constants. Given any $\tilde{I}(B) \in \mathcal{N}(B)$ for $B \in \mathcal{B}_j$, let $\delta I(B) = \theta I(B) - \tilde{I}(B)$ and $\tilde{I}^Y = \prod_{B \in \mathcal{B}_j(Y)} \tilde{I}(B)$, i.e., $\theta I(B)$ depends on $(\psi + \zeta, \bar{\psi} + \bar{\zeta})$ by definition of θ, $\tilde{I}(B)$ on $(\psi, \bar{\psi})$, and $\delta I(B)$ depends on $(\psi, \bar{\psi}, \zeta, \bar{\zeta})$. The binomial expansion identity

$$\theta I^{\Lambda \setminus X} = \sum_{Y \subset \Lambda \setminus X} \tilde{I}^Y (\delta I)^{\Lambda \setminus (X \cup Y)} \tag{3.53}$$

and (3.52) lead to, after changing the index of summation,

$$\theta Z_j = \sum_{X \in \mathcal{P}_j} \tilde{I}^{\Lambda \setminus X} \tilde{K}(X), \qquad \tilde{K}(X) = \sum_{Y \in \mathcal{P}_j(X)} (\delta I)^Y \theta K(X \setminus Y). \tag{3.54}$$

We will later make the particular choice of $\tilde{I}$ that corresponds to (3.51). Thus $\tilde{I}$ corresponds to the next-scale coupling constants V_{j+1}. This will be important for obtaining the desired contractive properties of the renormalisation group map in Theorem 3.13. Exhibiting that the K coordinate is contractive will be aided by the following re-arrangements.

Cancellation of small sets. Keeping in mind that $\tilde{K}$ factors over components (since K factors over components), we can then define $\check{K}(Y)$ to be $\tilde{K}(Y) - \sum_{B \in \mathcal{B}_j(Y)} \theta J(B, Y)$ for any connected polymer Y (and zero otherwise), where $J(B, Y) \in \mathcal{N}(B)$ are given. This yields the following formula for $\tilde{K}$:

$$\tilde{K}(X) = \prod_{Y \in \mathrm{Comp}(X)} \left(\check{K}(Y) + \sum_{B \in \mathcal{B}_j(Y)} \theta J(B, Y) \right). \tag{3.55}$$

Again we will later specialise to J as defined in (3.45) and (3.46), in particular $J(B, Y) = 0$ unless Y is a small set. The motivation for this step is that, for Y that are small sets but not blocks, the effect is that $\check{K}$ has the relevant and marginal contributions corresponding to $\sum_B J(B, Y)$ removed. This step does not remove the relevant and marginal contributions of blocks due to the choice (3.45). However, relevant and marginal contributions of blocks will be removed by appropriate choice of $\tilde{I}$. Both cancellations occur in the estimates in Sect. 3.8.3.

We next substitute (3.55) into (3.54) and re-arrange the resulting sum. Expanding the product, (3.55) can be written as

$$\tilde{K}(X) = \sum_{\check{X} \subset \mathrm{Comp}(X)} \check{K}(\check{X}) \prod_{Y \in \mathrm{Comp}(X \setminus \check{X})} \sum_{B \in \mathcal{B}_j(Y)} \theta J(B, Y). \tag{3.56}$$

Given the polymer $X \setminus \check{X}$, there is a $(X \setminus \check{X})$-dependent set of sets $\mathcal{X} \subset \{(B, Y) : B \in \mathcal{B}_j(Y), Y \in \mathcal{C}_j\}$ such that

$$\prod_{Y \in \mathrm{Comp}(X \setminus \check{X})} \sum_{B \in \mathcal{B}_j(Y)} J(B, Y) = \sum_{\mathcal{X}} \prod_{(B,Y) \in \mathcal{X}} J(B, Y), \tag{3.57}$$

where the sum on the right-hand is over the aforementioned sets of $\mathcal{X}$. Explicitly, the sets comprising $\mathcal{X}$ are sets of pairs (B, Y) where (i) B is a block in Y, (ii) each component Y occurs in exactly one pair, and (iii) Y is a component of $X \setminus \check{X}$. In particular, $X = \check{X} \cup X_{\mathcal{X}}$. Thus

$$\tilde{K}(X) = \sum_{(\mathcal{X},\check{X})} \check{K}(\check{X}) \prod_{(B,Y)\in\mathcal{X}} \theta J(B,Y). \tag{3.58}$$

Substituting this into (3.54), and using that $\tilde{I}^X \in \mathcal{N}(\Lambda)$ is a constant with respect to $\mathbb{E}_{C_{j+1}}$, i.e., the expectation acts on $(\zeta, \bar{\zeta})$ while $\tilde{I}^X$ depends on $(\psi, \bar{\psi})$,

$$\begin{aligned}\mathbb{E}_{C_{j+1}}\theta Z_j &= \sum_{X\in\mathcal{P}_j} \tilde{I}^{\Lambda\setminus X}\mathbb{E}_{C_{j+1}}\tilde{K}(X) \\ &= \sum_{X\in\mathcal{P}_j}\sum_{(\mathcal{X},\check{X})} \tilde{I}^{\Lambda\setminus(\check{X}\cup X_{\mathcal{X}})}\mathbb{E}_{C_{j+1}}\left[\check{K}(\check{X}) \prod_{(B,Y)\in\mathcal{X}} \theta J(B,Y)\right]\end{aligned} \tag{3.59}$$

where $X_{\mathcal{X}}$ is by definition $\bigcup_{(B,Y)\in\mathcal{X}} Y$, and hence $X = \check{X} \cup X_{\mathcal{X}}$ by the definition of the set $\mathcal{X}$.

Reblocking. Next we organise the last right-hand side of (3.59) as a sum over next-scale polymers $U \in \mathcal{P}_{j+1}$. We start by inserting the partition of unity

$$1 = \sum_{U\in\mathcal{P}_{j+1}} 1_{\overline{\check{X}\cup[\cup_{(B,Y)\in\mathcal{X}} B^{\square}]}=U} \qquad \text{for every } (\mathcal{X},\check{X}) \tag{3.60}$$

into the sum and changing the order of the sums. This gives

$$\mathbb{E}_{C_{j+1}}\theta Z_j = \sum_{U\in\mathcal{P}_{j+1}} \tilde{I}^{\Lambda\setminus U} K'(U) \tag{3.61}$$

with

$$K'(U) = \sum_{(\mathcal{X},\check{X})\in\mathcal{G}(U)} \tilde{I}^{U\setminus(\check{X}\cup X_{\mathcal{X}})}\mathbb{E}_{C_{j+1}}\left[\check{K}(\check{X}) \prod_{(B,Y)\in\mathcal{X}} \theta J(B,Y)\right], \tag{3.62}$$

where we make the definition that $\mathcal{G}(U)$ consists of $(\mathcal{X}, \check{X})$ such that $\check{X} \in \mathcal{P}_j$, $\mathcal{X}$ satisfies (i) and (ii) above, $X_{\mathcal{X}}$ does not touch $\check{X}$, and $\overline{\check{X} \cup [\cup_{(B,Y)\in\mathcal{X}} B^{\square}]} = U$. The sum over X in (3.59) has been incorporated into the sum over $(\check{X}, \mathcal{X})$.

Conclusion. We now specialise to the setting of Proposition 3.11. Thus we take J as in (3.45) and (3.46), and $\tilde{I}(B) = e^{-(u+V)_{j+1}(B)}$ with the exponent as defined in (3.48), and $K_{j+1}(U)$ as defined in (3.49). The arguments above show that

$$\mathbb{E}_{C_{j+1}}\theta Z_j = e^{-u_{j+1}|\Lambda|} \sum_{U\in\mathcal{P}_{j+1}} e^{-V_{j+1}(\Lambda\setminus U)} K_{j+1}(U). \tag{3.63}$$

What remains is to prove the claims regarding factorisation and automorphism invariance. For factorisation, note that $\sum_{\mathcal{G}(U_1\cup U_2)} = \sum_{\mathcal{G}(U_1)} \sum_{\mathcal{G}(U_2)}$ for $U_1, U_2 \in \mathcal{P}_{j+1}$

that do not touch. Moreover, since U_1 and U_2 are separated by a distance at least $L^{j+1} > \frac{1}{2}L^{j+1} + 2^{d+1}L^j$, the expectations in the definition of $K_{j+1}(U_1 \cup U_2)$ factor. Here we have used our standing assumption that $L > 2^{d+2}$, that $J(B, Y) = 0$ if $Y \notin \mathcal{S}_j$, and that the range of C_{j+1} is $\frac{1}{2}L^{j+1}$. Automorphism invariance follows from the formula for K_{j+1} and the properties of $\mathbb{E}_{C_{j+1}}\theta$ discussed in Sect. 3.3. □

Proposition 3.11 implies in particular that if K_j has the factorisation property (3.19), then we can identify $(K_{j+1}(X))_{X \in \mathcal{P}_{j+1}(\Lambda_N)}$ with its restriction to connected polymers $(K_{j+1}(X))_{X \in \mathcal{C}_{j+1}(\Lambda_N)}$. Moreover, if the K_j are also automorphism invariant, then $K_{j+1} \in \mathcal{K}^{\varnothing}_{j+1}(\Lambda_N)$.

By construction and the consistency of the covariances C_j with $j < N$ for different values of N, the maps defined for different Λ_N are also consistent in the following sense:

Proposition 3.12 *For $j + 1 < N$ and $U \in \mathcal{P}_{j+1}(\Lambda_N)$, $V_{j+1}(U)$ and $K_{j+1}(U)$ above depend on (V_j, K_j) only through $V_j(X)$, $K_j(X)$ with $X \in \mathcal{P}_j(U^{\square})$. Moreover, for $U \in \mathcal{P}_{j+1}(\Lambda_N) \cap \mathcal{P}_{j+1}(\Lambda_M)$ with the natural local identification of Λ_N and Λ_M, the map $(V_j, K_j) \mapsto (V_{j+1}(U), K_{j+1}(U))$ is independent of N and M.*

Temporarily indicating the N-dependence of $\Phi_{j+1} = \Phi_{j+1,N}$ explicitly, consistency implies the existence of an infinite volume limit $\Phi_{j+1,\infty} = \lim_{N\to\infty} \Phi_{j+1,N}$ defined for arguments $V_j \in \mathcal{V}^{\varnothing}$ and $K_j = (K_j(X))_{X \in \mathcal{C}_j(\mathbb{Z}^d)} \in \mathcal{K}^{\varnothing}_j(\mathbb{Z}^d)$. Explicitly, if we write $\Phi_{j+1,N}(V_j, K_j) = (V^N_{j+1}, K^N_{j+1})$ and omit the N for the infinite volume map, $K_{j+1}(U) = \lim_{N\to\infty} K^N_{j+1}(U)$, and similarly for V_{j+1}. The limits exist as the sequences are constant after finitely many terms. This infinite volume limit does not carry the full information from the $\Phi_{j+1,N}$ because terms indexed by polymers that wrap around the torus are lost, but it does carry complete information about small sets at all scales and thus about the flow of V_j. As for the finite-volume maps, the infinite volume limit carries a mild dependence on m^2. We typically omit this from the notation.

3.8 Estimates for the renormalisation group map

The renormalisation group map $\Phi_{j+1} = \Phi_{j+1,N}$ is a function of $(V, K) \in \mathcal{V}^{\varnothing} \oplus \mathcal{K}^{\varnothing}_j(\Lambda_N)$. The size of V and K will be measured in the norms

$$\|V\|_j = \sup_{B \in \mathcal{B}_j} \|V(B)\|_{T_j(\ell_j)} \tag{3.64}$$

$$\|K\|_j = \sup_{X \in \mathcal{C}_j} A^{(|\mathcal{B}_j(X)| - 2^d)_+} \|K(X)\|_{T_j(\ell_j)} \tag{3.65}$$

where $A > 1$ is a parameter that will be chosen sufficiently large. The space of bulk coupling constants $\mathcal{V}^{\varnothing} \cong \mathbb{C}^4$ has finite dimension. The space for polymer activities $\mathcal{K}^{\varnothing}_j(\Lambda_N)$ is also finite-dimensional for $N < \infty$ since an element $K_j \in \mathcal{K}^{\varnothing}_j(\Lambda_N)$ is a finite collection of elements $K_j(X)$ of the finite-dimensional Grassmann algebra

$\mathcal{N}(\Lambda_N)$. Thus $\mathcal{V}^{\varnothing} \oplus \mathcal{K}_j^{\varnothing}(\Lambda_N)$ is a finite-dimensional complex normed vector space with the above norms, and therefore a Banach space.

Theorem 3.13 *Let $d \geqslant 3$, $L \geqslant L_0(d)$, and $A \geqslant A_0(L, d)$. Assume that $u_j = 0$. There exists $\varepsilon = \varepsilon(L, A) > 0$ such that if $j + 1 < N$ and $\|V_j\|_j + \|K_j\|_j \leqslant \varepsilon$ then*

$$\|u_{j+1} + V_{j+1} - \mathbb{E}_{C_{j+1}}\theta V_j\|_{j+1} \leqslant O(L^d \|K_j\|_j) \tag{3.66}$$

$$\|K_{j+1}\|_{j+1} \leqslant O(L^{-(\frac{d-2}{2}\wedge 1)} + A^{-\eta})\|K_j\|_j + O(A^{\nu})(\|V_j\|_j^2 + \|K_j\|_j^2), \tag{3.67}$$

where $\eta = \eta(d)$ and $\nu = \nu(d)$ are positive geometric constants. The maps Φ_{j+1} are entire in (V_j, K_j) and hence all derivatives of any order are uniformly bounded on $\|V_j\|_j + \|K_j\|_j \leqslant \varepsilon$. Moreover, the maps Φ_{j+1} are continuous in $m^2 \geqslant 0$.

The last renormalisation group map Φ_N satisfies the same bound with $L^{-(\frac{d-2}{2}\wedge 1)}$ replaced by 1.

Theorem 3.13 is the analogue of [31, 32] for the four-dimensional weakly self-avoiding walk, but much simpler since (i) we are only working with fermionic variables, and (ii) we are above the lower critical dimension (two for our model). The factors L^d and A^{ν} in the error bounds are harmless. On the other hand, it is essential that $O(L^{-(\frac{d-2}{2}\wedge 1)} + A^{-\eta}) < 1$ for L and A large: this estimate establishes that K is *irrelevant* (contracting) in renormalisation group terminology.

The remainder of this subsection proves Theorem 3.13. Readers not familiar with the use of the renormalisation group might want to skip the somewhat technical proof on a first reading and proceed to Sects. 3.9 and 4 to get an idea of how these estimates are used.

Throughout the rest of Sect. 3.8 the hypotheses of Theorem 3.13 will be assumed to hold.

The substantive claims of Theorem 3.13 are the estimates (3.66) and (3.67): these quickly yield the claims regarding derivatives by a standard Cauchy estimate, as we now explain. Recall that given two Banach spaces X and Y and a domain $D \subset \mathbb{C}$ we say that a function $g\colon D \to X$ is analytic if it satisfies the Cauchy-Riemann equation $\partial_{\bar{z}} g = 0$. For an open set $O \subseteq X$, we then say that a function $F\colon O \to Y$ is analytic if $F \circ g$ is analytic for every analytic function $g\colon D \to X$. After (possibly) adding some additional coordinates to ensure all necessary monomials are in the domain, the maps $(V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$ are multivariate polynomials, and the norm estimates (3.66) and (3.67) extend to this larger space. Being multivariate polynomials, the Φ_{j+1} are analytic functions.

We use analyticity and the Cauchy integral formula to extract derivatives. Namely, if (V, K) and $(\dot{V}^{(i)}, \dot{K}^{(i)})_{i=1}^n$ are collections of polymer coordinates at scale j satisfying $\|V\|_j + \|K\|_j \leqslant \varepsilon/2$ and $\|\dot{V}^{(i)}\|_j + \|\dot{K}^{(i)}\|_j \leqslant 1$, then

$$D^n \Phi_{j+1}|_{(V,K)}(\dot{V}^{(i)}, \dot{K}^{(i)})_{i=1}^n = \oint \cdots \oint \prod_{i=1}^n \frac{\mathrm{d}w_i}{w_i^2} \Phi_{j+1}(V + \sum_{i=1}^n w_i \dot{V}^{(i)}, K + \sum_{i=1}^n w_i \dot{K}^{(i)}) \tag{3.68}$$

where the n-tuple of contours are circles around 0 with radius $\varepsilon/(2n)$. The statement of Theorem 3.13 regarding boundedness of derivatives follows.

The asserted continuity in $m^2 \geqslant 0$ follows from the explicit formulas for (V_{j+1}, K_{j+1}), that Loc is linear, and that the covariances C_j are continuous in $m^2 \geqslant 0$.

3.8.1 Coupling constants

We begin with the bound (3.66) for V_{j+1}. The first term on the right-hand side in the definition (3.48) of $u_{j+1} + V_{j+1}$ produces the expectation term in (3.66). For $B \in \mathcal{B}_j$, the remainder in (3.48) is bounded as follows:

$$\begin{aligned} \|Q(B)\|_{T_j(\ell_j)} &\leqslant \sum_{X \in \mathcal{S}_j : X \supset B} \|\operatorname{Loc}_{X,B} K_j(X)\|_{T_j(\ell_j)} \\ &\leqslant O(1) \sup_{B,X} \|\operatorname{Loc}_{X,B} K_j(X)\|_{T_j(\ell_j)} \leqslant O(1)\|K_j\|_j \end{aligned} \tag{3.69}$$

where we have used that the number of small sets containing a fixed block is $O(1)$ in the second step, and (3.39) in the third. Since each block in $\mathcal{B}_{j+1}$ contains L^d blocks in $\mathcal{B}_j$, by using (3.34) to bound the expectation and change of scale in the norm, the first claim (3.66) follows.

The remainder of Sect. 3.8 establishes the bound (3.67) for K_{j+1}. The following basic observations will be used repeatedly. Note that by (3.34) the main term contributing to $u_{j+1}|B| + V_{j+1}(B)$ is bounded by, for $B \in \mathcal{B}_j$,

$$\|\mathbb{E}_{C_{j+1}} \theta V_j(B)\|_{T_{j+1}(\ell_{j+1})} \leqslant \|V_j(B)\|_{T_j(\ell_j)}. \tag{3.70}$$

By combining this with (3.69) we have that, for $B \in \mathcal{B}_j$,

$$\begin{aligned} u_{j+1}|B| &\leqslant \|V_j\|_j + O(\|K_j\|_j), \\ \|V_{j+1}(B)\|_{T_{j+1}(\ell_{j+1})} &\leqslant \|V_j\|_j + O(\|K_j\|_j). \end{aligned} \tag{3.71}$$

3.8.2 Preparation for bound of the non-perturbative coordinate

To derive (3.67), we first separate from $K_{j+1}(U)$ a leading contribution. This contribution is:

$$\begin{aligned} \mathcal{L}_{j+1}(U) = &\sum_{X \in \mathcal{C}_j : \bar{X} = U} e^{-V_{j+1}(U \setminus X)} e^{u_{j+1}|X|} \mathbb{E}_{C_{j+1}} \left[\theta K_j(X) - \sum_{B \in \mathcal{B}_j} \theta J(B, X) \right] \\ &+ \sum_{X \in \mathcal{P}_j : \bar{X} = U} e^{-V_{j+1}(U \setminus X)} e^{u_{j+1}|X|} \mathbb{E}_{C_{j+1}} \Big[(\delta I)^X \Big]. \end{aligned} \tag{3.72}$$

Note that while the first sum on the right-hand side is over connected polymers, the second is over all polymers. This expression includes the contributions to K_{j+1} explicitly linear in K_j, and all other terms in the definition of K_{j+1} are higher order, see Sect. 3.8.5 below.

We may divide each of the sums on the right-hand side in (3.72) into the contributions from small sets $X \in \mathcal{S}_j$ and large sets $X \in \mathcal{P}_j \setminus \mathcal{S}_j$. We recall that small sets are, by definition, connected. These restricted sums will be denoted by $\mathcal{L}_{j+1,\mathcal{S}}(U)$ and $\mathcal{L}_{j+1,\mathcal{P}\setminus\mathcal{S}}(U)$ respectively:

$$\mathcal{L}_{j+1}(U) = \mathcal{L}_{j+1,\mathcal{S}}(U) + \mathcal{L}_{j+1,\mathcal{P}\setminus\mathcal{S}}(U). \tag{3.73}$$

Large sets are easier to handle because they lose combinatorial entropy under change of scale (reblocking), i.e., $|\mathcal{B}_j(X)|$ will be significantly larger than $|\mathcal{B}_{j+1}(\bar{X})|$. In renormalisation group terminology, they are *irrelevant*. Small sets, on the other hand, require careful treatment.

3.8.3 Small sets

The main estimate on small sets is summarised as follows.

Proposition 3.14 *The contribution $\mathcal{L}_{j+1,\mathcal{S}}$ to* (3.72) *satisfies*

$$\|\mathcal{L}_{j+1,\mathcal{S}}\|_{j+1} = O(L^{-(\frac{d-2}{2}\wedge 1)}\|K_j\|_j + L^d(\|V_j\|_j^2 + \|K_j\|_j^2)). \tag{3.74}$$

In order to bound

$$\mathcal{L}_{j+1,\mathcal{S}}(U) = \sum_{X\in\mathcal{S}_j:\bar{X}=U} e^{-V_{j+1}(U\setminus X)} e^{u_{j+1}|X|} \times \mathbb{E}_{C_{j+1}}\Big[\theta K_j(X) - \sum_{B\in\mathcal{B}_j} \theta J(B,X) + (\delta I)^X\Big], \tag{3.75}$$

we consider the terms $X \in \mathcal{S}_j \setminus \mathcal{B}_j$ and $X \in \mathcal{B}_j$ in the outer sum separately. By the definition of J in (3.46), for any $X \in \mathcal{S}_j \setminus \mathcal{B}_j$,

$$\sum_{B\in\mathcal{B}_j(X)} \mathbb{E}_{C_{j+1}}\Big[\theta J(B,X)\Big] = \sum_{B\in\mathcal{B}_j(X)} \mathbb{E}_{C_{j+1}}\Big[\theta \operatorname{Loc}_{X,B} K_j(X)\Big] = \mathbb{E}_{C_{j+1}}\Big[\theta \operatorname{Loc}_X K_j(X)\Big], \tag{3.76}$$

where the final equality follows from Proposition 3.8 (iii). Thus the contribution to $\mathcal{L}_{j+1,\mathcal{S}}$ from $X \in \mathcal{S}_j \setminus \mathcal{B}_j$ is

$$\mathbb{E}_{C_{j+1}}\Big[\theta(1-\operatorname{Loc}_X)K_j(X)\Big] + \mathbb{E}_{C_{j+1}}\Big[(\delta I)^X\Big]. \tag{3.77}$$

The contribution to $\mathcal{L}_{j+1,\mathcal{S}}$ from $X = B \in \mathcal{B}_j$ is

$$\mathbb{E}_{C_{j+1}}\Big[\theta K_j(B) + \delta I(B) - \theta J(B,B)\Big] = \mathbb{E}_{C_{j+1}}\Big[\theta(1-\operatorname{Loc}_B)K_j(B)\Big] + \mathbb{E}_{C_{j+1}}\Big[\delta I(B) + \theta Q(B)\Big]. \tag{3.78}$$

We now give a series of estimates, bounding the right-hand sides of (3.77) and (3.78) and the term outside the expectation in (3.75), and then assemble them into a proof for Proposition 3.14.

Lemma 3.15 *For any* $U \in \mathcal{C}_{j+1}$,

$$\sum_{X \in \mathcal{S}_j : \bar{X} = U} \left\| \mathbb{E}_{C_{j+1}} \Big[\theta (1 - \mathrm{Loc}_X) K_j(X) \Big] \right\|_{T_{j+1}(\ell_{j+1})} = O(L^{-(\frac{d-2}{2} \wedge 1)}) \|K_j\|_j. \quad (3.79)$$

Proof Note that $\bar{X} \in \mathcal{S}_{j+1}$ if $X \in \mathcal{S}_j$, so it suffices to prove the lemma for $U \in \mathcal{S}_{j+1}$. Now for any $U \in \mathcal{S}_{j+1}$, since there are $O(L^d)$ small sets $X \in \mathcal{S}_j$ such that $\bar{X} = U$ we get

$$\begin{aligned}
&\sum_{X \in \mathcal{S}_j : \bar{X} = U} \left\| \mathbb{E}_{C_{j+1}} \Big[\theta (1 - \mathrm{Loc}_X) K_j(X) \Big] \right\|_{T_{j+1}(\ell_{j+1})} \\
&\leqslant O(L^d) \sup_{X \in \mathcal{S}_j} \left\| \mathbb{E}_{C_{j+1}} \Big[\theta (1 - \mathrm{Loc}_X) K_j(X) \Big] \right\|_{T_{j+1}(\ell_{j+1})} \\
&\leqslant O(L^d) \sup_{X \in \mathcal{S}_j} \|(1 - \mathrm{Loc}_X) K_j(X)\|_{T_{j+1}(2\ell_{j+1})} \\
&\leqslant O(L^d) O(L^{-d}) (L^{-(\frac{d-2}{2} \wedge 1)}) \sup_{X \in \mathcal{S}_j} \|K_j(X)\|_{T_j(\ell_j)} \\
&\leqslant O(L^{-(\frac{d-2}{2} \wedge 1)}) \|K_j\|_j
\end{aligned} \quad (3.80)$$

where we have used the contraction estimate (3.33) for the expectation in the second step and the contraction estimate (3.40) for Loc_X in the third step. □

Lemma 3.16 *For* $B \in \mathcal{B}_j$,

$$\left\| \mathbb{E}_{C_{j+1}} \Big[\delta I(B) + \theta Q(B) \Big] \right\|_{T_{j+1}(\ell_{j+1})} = O(\|V_j\|_j^2 + \|K_j\|_j^2), \quad (3.81)$$

Proof By the definition of $(u+V)_{j+1}$ in (3.48) we have

$$\begin{aligned}
\mathbb{E}_{C_{j+1}} \Big[\delta I(B) + \theta Q(B) \Big] &= \mathbb{E}_{C_{j+1}} \Big[\theta e^{-V_j(B)} - 1 + \theta V_j(B) \Big] \\
&\quad - \Big[e^{-(u+V)_{j+1}(B)} - 1 + (u+V)_{j+1}(B) \Big].
\end{aligned} \quad (3.82)$$

By the product property (3.26), if for some V and some k we have $\|V(B)\|_{T_k(\ell_k)} \leqslant 1$, then

$$\|e^{-V(B)} - 1 + V(B)\|_{T_k(\ell_k)} \leqslant O(\|V(B)\|_{T_k(\ell_k)}^2). \quad (3.83)$$

Recall that $\mathbb{E}_{C_{j+1}} \theta$ is contractive as a map from $T_j(\ell_j)$ to $T_{j+1}(\ell_{j+1})$ by (3.34). Applying these estimates to the $T_{j+1}(\ell_{j+1})$ norm of (3.82) and using (3.71) gives the bound (3.81). □

Lemma 3.17 *For $X \in \mathcal{P}_j$,*

$$\left\| \mathbb{E}_{C_{j+1}} \left[(\delta I)^X \right] \right\|_{T_{j+1}(\ell_{j+1})} = (O(\|V_j\|_j + \|K_j\|_j))^{|\mathcal{B}_j(X)|}. \tag{3.84}$$

Proof Using that $\mathbb{E}_{C_{j+1}}$ satisfies the contraction estimate (3.32), it suffices to show

$$\|(\delta I)^X\|_{T_{j+1}(\ell_{j+1})} = (O(\|V_j\|_j + \|K_j\|_j))^{|\mathcal{B}_j(X)|}. \tag{3.85}$$

By the product property (3.26) it suffices to prove this estimate for a single block. In this case,

$$\begin{aligned} \|\delta I(B)\|_{T_{j+1}(\ell_{j+1})} &\leqslant \|\theta(e^{-V_j(B)} - 1)\|_{T_{j+1}(\ell_{j+1})} + \|e^{-(u+V)_{j+1}(B)} - 1\|_{T_{j+1}(\ell_{j+1})} \\ &\leqslant O(\|V_j(B)\|_{T_{j+1}(2\ell_{j+1})}) + O(\|(u+V)_{j+1}(B)\|_{T_{j+1}(\ell_{j+1})}) \end{aligned} \tag{3.86}$$

by the product property (3.26) of the norms and (3.31). Using $2\ell_{j+1} \leqslant \ell_j$ and (3.30) for the first term and (3.71) for the second term bounds the right-hand side by $O(\|V_j\|_j + \|K_j\|_j)$ as needed. □

We need one further general estimate.

Lemma 3.18 *If $\|K_j\|_j + \|V_j\|_j \leqslant \varepsilon = \varepsilon(d, L)$ is sufficiently small, then if $\bar{X} = U \in \mathcal{P}_{j+1}$,*

$$\|e^{-V_{j+1}(U \setminus X) + u_{j+1}|X|}\|_{T_{j+1}(\ell_{j+1})} \leqslant 2^{|\mathcal{B}_j(X)|}. \tag{3.87}$$

Proof By the product property (3.26) and (3.71) to bound V_{j+1} and u_{j+1},

$$\|e^{-V_{j+1}(U \setminus X) + u_{j+1}|X|}\|_{T_{j+1}(\ell_{j+1})} \leqslant (1 + O(\varepsilon))^{|\mathcal{B}_j(U)|}, \tag{3.88}$$

and $|\mathcal{B}_j(U)|$ is at most $L^d |\mathcal{B}_{j+1}(U)| \leqslant L^d |\mathcal{B}_j(X)|$. The claim follows provided $(1 + O(\varepsilon))^{L^d} \leqslant 2$. □

Proof of Proposition 3.14 To estimate the summands of $\mathcal{L}_{j+1,\mathcal{S}}(U)$, we use the product property of the $\| \cdot \|_{T_{j+1}(\ell)}$ norm to combine Lemma 3.18 with Lemma 3.15, Lemma 3.16 for $X \in \mathcal{B}_j$, and with Lemma 3.17 for $X \in \mathcal{S}_j \setminus \mathcal{B}_j$. For the sum of the terms $(\delta I)^X$ we use that $(1 + \|V_j\|_j^2 + \|K_j\|_j^2)^{L^d} \leqslant 2$ provided $\varepsilon = \varepsilon(L)$ is small enough. Altogether, we obtain

$$\|\mathcal{L}_{j+1,\mathcal{S}}(U)\|_{T_{j+1}(\ell_{j+1})} = O(L^{-(\frac{d-2}{2} \wedge 1)} \|K_j\|_j + L^d (\|V_j\|_j^2 + \|K_j\|_j^2)), \tag{3.89}$$

which proves the lemma. □

3.8.4 Large sets

Next we consider the contribution to (3.72) from the terms $X \notin \mathcal{S}_j$ in the sums, i.e., $\mathcal{L}_{j+1,\mathcal{P}\setminus\mathcal{S}}$. The main estimate of this section is summarised in the following proposition.

Proposition 3.19 *The contribution $\mathcal{L}_{j+1,\mathcal{P}\setminus\mathcal{S}}$ to (3.72) satisfies*

$$\|\mathcal{L}_{j+1,\mathcal{P}\setminus\mathcal{S}}\|_{j+1} = O(A^{-\eta}\|K_j\|_j + A^{\nu}[\|V_j\|_j + \|K_j\|_j]^2) \tag{3.90}$$

We begin by recording a combinatorial fact, see [28, Lemmas 6.15 and 6.16] or [32, Lemma C.3] for details on its proof. For the statement, recall that if $X \in \mathcal{P}_j$, then its closure $\bar{X} \in \mathcal{P}_{j+1}$ denotes the smallest next-scale polymer containing X.

Lemma 3.20 *Let $L \geqslant 2^d + 1$. There is a geometric constant $\eta = \eta(d) > 0$ depending only on d such that for all $X \in \mathcal{C}_j \setminus \mathcal{S}_j$,*

$$|\mathcal{B}_j(X)| \geqslant (1+2\eta)|\mathcal{B}_{j+1}(\bar{X})|. \tag{3.91}$$

Moreover, for all $X \in \mathcal{P}_j$, $|\mathcal{B}_j(X)| \geqslant |\mathcal{B}_{j+1}(X)|$ and

$$|\mathcal{B}_j(X)| \geqslant (1+\eta)|\mathcal{B}_{j+1}(\bar{X})| - (1+\eta)2^{d+1}|\mathrm{Comp}(X)|. \tag{3.92}$$

We also record an application of this estimate to sums indexed by large polymers which will be used in this and in the next section. By (3.91), if $A = A(L)$ is large enough,

$$A^{|\mathcal{B}_{j+1}(U)|} \sum_{X \in \mathcal{C}_j \setminus \mathcal{S}_j : \bar{X} = U} (A/2)^{-|\mathcal{B}_j(X)|} \leqslant (2^{L^d} 2^{1+2\eta} A^{-2\eta})^{|\mathcal{B}_{j+1}(U)|} \leqslant A^{-\eta|\mathcal{B}_{j+1}(U)|}, \tag{3.93}$$

as the set of $X \in \mathcal{P}_j$ with $\bar{X} = U$ has size at most $2^{L^d|\mathcal{B}_{j+1}(U)|}$. Similarly, by (3.92), if $\alpha \geqslant A^{(1+\eta)2^{d+1}}$,

$$A^{|\mathcal{B}_{j+1}(U)|} \sum_{X \in \mathcal{P}_j : \bar{X} = U} (A/2)^{-|\mathcal{B}_j(X)|} \alpha^{-|\mathrm{Comp}(X)|} \leqslant A^{-(\eta/2)|\mathcal{B}_{j+1}(U)|}. \tag{3.94}$$

Proof of Proposition 3.19 From (3.72) and (3.73), recall that (as $J(B,X) = 0$ for large X)

$$\mathcal{L}_{j+1,\mathcal{P}\setminus\mathcal{S}}(U) = \sum_{X \in \mathcal{C}_j \setminus \mathcal{S}_j : \bar{X} = U} e^{-V_{j+1}(U\setminus X) + u_{j+1}|X|} \mathbb{E}_{C_{j+1}}\Big[\theta K_j(X)\Big] + \sum_{X \in \mathcal{P}_j \setminus \mathcal{S}_j : \bar{X} = U} e^{-V_{j+1}(U\setminus X) + u_{j+1}|X|} \mathbb{E}_{C_{j+1}}\Big[(\delta I)^X\Big]. \tag{3.95}$$

We first consider the case $U \in \mathcal{C}_{j+1} \setminus \mathcal{S}_{j+1}$, and proceed as follows: for $\|K_j\|_j + \|V_j\|_j \leqslant \varepsilon$ with ε sufficiently small, by Lemma 3.18 the $j+1$ norm of the K contribution to (3.95) is bounded by

$$A^{|\mathcal{B}_{j+1}(U)|-2^d} \sum_{X \in \mathcal{C}_j \setminus \mathcal{S}_j : \bar{X}=U} 2^{|\mathcal{B}_j(X)|} \left\| \mathbb{E}_{C_{j+1}}\left[\theta K_j(X)\right] \right\|_{T_{j+1}(\ell_{j+1})}. \tag{3.96}$$

By the definition of $\|K_j\|_j$ and noting that $(|\mathcal{B}_j(X)| - 2^d)_+ = |\mathcal{B}_j(X)| - 2^d$ since $X \notin \mathcal{S}_j$,

$$\left\| \mathbb{E}_{C_{j+1}}\left[\theta K_j(X)\right] \right\|_{T_{j+1}(\ell_{j+1})} \leqslant A^{-(|\mathcal{B}_j(X)|-2^d)} \|K_j\|_j, \tag{3.97}$$

where we have also used the contraction estimates (3.33) and (3.30). Inserting this bound into (3.96) and using (3.93) gives that the K contribution to (3.95) is bounded by

$$A^{|\mathcal{B}_{j+1}(U)|} \sum_{X \in \mathcal{C}_j \setminus \mathcal{S}_j : \bar{X}=U} (A/2)^{-|\mathcal{B}_j(X)|} \|K_j\|_j \leqslant A^{-\eta} \|K_j\|_j. \tag{3.98}$$

This is the desired bound for the first term in (3.95).

To bound the $j+1$ norm of the δI contribution to (3.95), Lemmas 3.17 and 3.18 and the product property yield (provided ε is sufficiently small depending on L)

$$\begin{aligned} &A^{|\mathcal{B}_{j+1}(U)|-2^d} \left\| \sum_{X \in \mathcal{P}_j \setminus \mathcal{S}_j : \bar{X}=U} e^{-V_{j+1}(U \setminus X) + u_{j+1}|X|} \mathbb{E}_{C_{j+1}}\left[(\delta I)^X\right] \right\|_{T_{j+1}(\ell_{j+1})} \\ &\leqslant A^{|\mathcal{B}_{j+1}(U)|-2^d} \sum_{X \in \mathcal{P}_j \setminus \mathcal{S}_j : \bar{X}=U} \left[2O(\|V_j\|_j + \|K_j\|_j)\right]^{|\mathcal{B}_j(X)|}. \end{aligned} \tag{3.99}$$

Since $U \in \mathcal{C}_{j+1} \setminus \mathcal{S}_{j+1}$ and $\bar{X} = U$, each X in the last sum must have $|\mathcal{B}_j(X)| \geqslant 2^d + 1$. If $\|V_j\|_j + \|K_j\|_j < \varepsilon$ and ε is sufficiently small (depending on A), then the quantity in brackets is less than $1/A^{2+2(1+\eta)2^{d+1}}$. By the elementary inequality $(c^2)^{n-2} \leqslant c^n$ for $c \in (0,1)$, $n > 4$ and using that $|\mathcal{B}_j(X)| \geqslant 2^d + 1 > 4$ for each summand, we obtain the upper bound

$$[O(\|V_j\|_j + \|K_j\|_j)]^2 A^{|\mathcal{B}_{j+1}(U)|} \sum_{X \in \mathcal{P}_j \setminus \mathcal{S}_j : \bar{X}=U} A^{-|\mathcal{B}_j(X)|} A^{-(1+\eta)2^{d+1}|\mathcal{B}_j(X)|}. \tag{3.100}$$

Using (3.94), it follows that the δI contribution to (3.95) is bounded by

$$O(A^{-\eta/2}[\|V_j\|_j + \|K_j\|_j]^2) = O([\|V_j\|_j + \|K_j\|_j]^2), \tag{3.101}$$

for A sufficiently large. We have now completed the bound on (3.72) provided $U \in \mathcal{C}_{j+1} \setminus \mathcal{S}_{j+1}$.

The argument is similar if $U \in \mathcal{S}_{j+1}$. In this case the prefactor $A^{|\mathcal{B}_{j+1}(U)|-2^d}$ gets replaced by 1 in (3.96) and (3.99). For the K contribution, in place of (3.98) we obtain, since $1+2^d \leqslant |\mathcal{B}_j(X)| \leqslant L^d|\mathcal{B}_{j+1}(U)| \leqslant (2L)^d$ and the number of summands in this case is at most $2^{(2L)^d}$,

$$\left\| \sum_{X \in \mathcal{C}_j \setminus \mathcal{S}_j : \bar{X}=U} e^{-V_{j+1}(U \setminus X)+u_{j+1}|X|} \mathbb{E}_{C_{j+1}}\Big[\theta K_j(X)\Big] \right\|_{T_{j+1}(\ell_{j+1})}$$
$$\leqslant A^{-1} 2^{2(2L)^d} \|K_j\|_j = O(A^{-\eta}\|K_j\|_j) \tag{3.102}$$

for A large enough depending on L and d. For the δI contribution, in place of (3.99) we have

$$\left\| \sum_{X \in \mathcal{P}_j \setminus \mathcal{S}_j : \bar{X}=U} e^{-V_{j+1}(U \setminus X)+u_{j+1}|X|} \mathbb{E}_{C_{j+1}}\Big[(\delta I)^X\Big] \right\|_{T_{j+1}(\ell_{j+1})}$$
$$\leqslant O\left(2^{2(2L)^d}[\|V_j\|_j + \|K_j\|_j]^2\right) \tag{3.103}$$

since each summand on the left-hand side has $|\mathcal{B}_j(X)| \geqslant 2$.

Thus for $A = A(L,d)$ sufficiently large and $\varepsilon = \varepsilon(A, L)$ sufficiently small, the expression (3.95) is bounded in the $T_{j+1}(\ell_{j+1})$ norm by $O(A^{-\eta}\|K_j\|_j + A^{\nu}[\|V_j\|_j + \|K_j\|_j]^2)$ in all cases. □

3.8.5 Non-linear part

Finally, we consider the non-linear contribution $K_{j+1} - \mathcal{L}_{j+1}$. To conclude the proof of (3.67), and hence the proof of Theorem 3.13, we prove the following estimate.

Proposition 3.21

$$\|K_{j+1} - \mathcal{L}_{j+1}\|_{j+1} \leqslant A^{\nu} O(\|K_j\|(\|K_j\|_j + \|V_j\|_j)). \tag{3.104}$$

Before diving into the proof, let us review the terms which remain to be estimated. Recall the definition of $K_{j+1}(U)$ from (3.49) and the leading part $\mathcal{L}_{j+1}(U)$ from (3.72). Write $|\mathcal{X}|$ for the number of pairs in $\mathcal{X}$. With respect to the indexing of summands for $K_{j+1}(U)$, the leading part $\mathcal{L}_{j+1}(U)$ results from the terms with $|\mathcal{X}| = 0$ and $\check{X} = X$ by only keeping the terms in the formula for $\check{K}(X)$ with either a single factor $\theta K_j(X)$ when $X \in \mathcal{C}_j$, a single factor $(\delta I)^X$ when $X \in \mathcal{P}_j$, or a single factor $\sum_B \theta J(B, X)$. It follows that we can write

$$K_{j+1}(U) - \mathcal{L}_{j+1}(U) = \mathcal{R}^1(U) + \mathcal{R}^2(U) + \mathcal{R}^3(U), \tag{3.105}$$

where

$$\mathcal{R}^1(U) = e^{u_{j+1}|U|} \sum_{\mathcal{G}_1(U)} e^{-(u+V)_{j+1}(U \setminus X_{\mathcal{X}})} \mathbb{E}_{C_{j+1}}\Bigg[\prod_{(B,X)\in\mathcal{X}} \theta J(B,X)\Bigg], \tag{3.106}$$

$$\mathcal{R}^2(U) = e^{u_{j+1}|U|} \sum_{\mathcal{G}_2(U)} e^{-(u+V)_{j+1}(U\setminus \check{X}\cup X_{\mathcal{X}})} \times \mathbb{E}_{C_{j+1}}\Big[(\check{K}(\check{X}) - (\delta I)^{\check{X}} \mathbf{1}_{|\mathcal{X}|=0}) \prod_{(B,X)\in\mathcal{X}} \theta J(B,X)\Big], \quad (3.107)$$

and

$$\mathcal{R}^3(U) = e^{u_{j+1}|U|} \sum_{\mathcal{G}_3(U)} e^{-(u+V)_{j+1}(U\setminus \check{X})} \times \mathbb{E}_{C_{j+1}}\Big[\check{K}(\check{X}) - \theta K(\check{X}) - (\delta I)^{\check{X}} + \sum_{B\in\mathcal{B}_j} \theta J(B,\check{X})\Big], \quad (3.108)$$

when the subsets $\mathcal{G}_i(U) \subset \mathcal{G}(U)$ are defined as follows. The set $\mathcal{G}_1(U)$ is defined by imposing the conditions $|\mathcal{X}| = 1$ and $\check{X} = \varnothing$. The set $\mathcal{G}_2(U)$ is defined to consist of $(\mathcal{X}, \check{X})$ such that $X_{\mathcal{X}} \cup \check{X}$ has at least two components. In particular, if $\mathcal{X} = \varnothing$, $\check{X}$ has least two components and if $\check{X} = \varnothing$ then $|\mathcal{X}| \geqslant 2$. Finally, $\mathcal{G}_3(U)$ is defined by the conditions $|\mathcal{X}| = 0$ and $\check{X} \in \mathcal{C}_j$.

The next lemma clearly implies Proposition 3.21.

Lemma 3.22 *For $i \in \{1, 2, 3\}$,*

$$\|\mathcal{R}^i\|_{j+1} \leqslant A^\nu O(\|K_j\|(\|K_j\|_j + \|V_j\|_j)). \quad (3.109)$$

Proof of Lemma 3.22 for $i = 1$ We begin by bounding $\mathcal{R}^1$. This bound exploits that $\sum_X J(B,X) = 0$ for every $B \in \mathcal{B}_j$, see (3.45)–(3.46) and (3.47). As $\mathcal{X}$ is a single pair $\{(B,X)\}$, $X_{\mathcal{X}} = X$. Since $\check{X} = \varnothing$ we can write

$$\mathcal{R}^1(U) = e^{u_{j+1}|U|} \sum_{B\in\mathcal{B}_j} \sum_{X_{\mathcal{X}}\in\mathcal{S}_j} e^{-(u+V)_{j+1}(U\setminus X_{\mathcal{X}})} \mathbb{E}_{C_{j+1}}\Big[\theta J(B, X_{\mathcal{X}})\Big] \mathbf{1}_{\overline{B^\square}=U}. \quad (3.110)$$

Since $\sum_{X_{\mathcal{X}}} J(B, X_{\mathcal{X}}) = 0$ for $B \in \mathcal{B}_j$, we can rewrite

$$\mathcal{R}^1(U) = e^{u_{j+1}|U|} \sum_{B\in\mathcal{B}_j} \sum_{X_{\mathcal{X}}\in\mathcal{S}_j} e^{-(u+V)_{j+1}(U\setminus X_{\mathcal{X}})} (1 - e^{(u+V)_{j+1}(X_{\mathcal{X}})}) \times \mathbb{E}_{C_{j+1}}\Big[\theta J(B, X_{\mathcal{X}})\Big] \mathbf{1}_{\overline{B^\square}=U}. \quad (3.111)$$

Since $X_{\mathcal{X}} \in \mathcal{S}_j$ we have $\|1 - e^{(u+V)_{j+1}(X_{\mathcal{X}})}\|_{T_{j+1}(\ell_{j+1})} = O(L^d(\|V_j\|_j + \|K_j\|_j))$ by (3.71). Moreover, (3.39) implies $\|J(B,X)\|_{T_j(\ell_j)} = O(\|K_j\|_j)$, so $\|\mathbb{E}_{C_{j+1}}\theta J(B, X)\|_{T_{j+1}(\ell_{j+1})} = O(\|K_j\|_j)$ since $\mathbb{E}_{C_{j+1}}\theta$ is a contraction. Finally, exactly as in the proof of Lemma 3.18, $\|e^{u_{j+1}(U)} e^{-(u+V)_{j+1}(U\setminus X_{\mathcal{X}})}\|_{T_{j+1}(\ell_{j+1})} \leqslant (1 + O(\varepsilon))^{|\mathcal{B}_j(U)|} \leqslant 2$ for $\varepsilon = \varepsilon(L)$ small enough, since U is the closure of $B^\square$. As there are $O(L^d)$ summands in (3.111) we have shown

$$\|\mathcal{R}^1(U)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(L^{2d}(\|V_j\|_j + \|K_j\|_j)\|K_j\|_j)$$

$$\leqslant O(A^{\nu}(\|V_j\|_j + \|K_j\|_j)\|K_j\|_j). \tag{3.112}$$

Since $A^{(|\mathcal{B}_{j+1}(U)|-2^d)_+} = 1$ for any contributing U (as U is the closure of $B^{\square}$ for some block B), this concludes the desired bound on $\mathcal{R}^1(U)$. $\square$

Before proceeding to the proof of Lemma 3.22 for $i = 2$, we first provide estimates on the norms of $\check{K}(\check{X})$ and $\check{K}(\check{X}) - (\delta I)^{\check{X}}$.

Lemma 3.23 *If $\|V_j\|_j + \|K_j\|_j \leqslant \varepsilon$ and $\varepsilon = \varepsilon(A, L)$ is sufficiently small, then*

$$\|\check{K}(\check{X})\|_{T_{j+1}(\ell_{j+1})} \leqslant [A^{2^d} O(\|V_j\|_j + \|K_j\|_j)]^{|\mathrm{Comp}(\check{X})|} (\frac{A}{2})^{-|\mathcal{B}_j(\check{X})|}, \tag{3.113}$$

$$\|\check{K}(\check{X}) - (\delta I)^{\check{X}}\|_{T_{j+1}(\ell_{j+1})} \leqslant [A^{2^d} O(\|V_j\|_j + \|K_j\|_j)]^{|\mathrm{Comp}(\check{X})|-1} \times O(A^{2^d}\|K_j\|_j)(\frac{A}{2})^{-|\mathcal{B}_j(\check{X})|}. \tag{3.114}$$

The proof of these estimates is postponed until the conclusion of the main argument.

Proof of Lemma 3.22 for $i = 2$ If $\|V_j\|_j + \|K_j\|_j$ is small enough, arguing as in (3.87) implies $\|e^{u_{j+1}(U)} e^{-(u+V)_{j+1}(U \setminus \check{X} \cup X_{\mathcal{X}})}\|_{T_{j+1}(\ell_{j+1})} \leqslant 2^{|\mathcal{B}_j(U)|}$, and by (3.39), $\|J(B, X)\|_{T_j(\ell_j)} = O(\|K_j\|_j)$. Thus using that $\mathbb{E}_{C_{j+1}}$ contracts from $T_{j+1}(\ell_{j+1} \sqcup \ell_{j+1})$ into $T_{j+1}(\ell_{j+1})$ we obtain

$$\|\mathcal{R}^2(U)\|_{T_{j+1}(\ell_{j+1})} \leqslant 2^{|\mathcal{B}_j(U)|} \sum_{\mathcal{G}_2(U)} [O(\|K_j\|_j)]^{|\mathcal{X}|} \|\check{K}(\check{X}) - (\delta I)^{\check{X}} \mathbf{1}_{\mathcal{X}=\varnothing}\|_{T_{j+1}(\ell_{j+1})}. \tag{3.115}$$

For brevity let us write b for the factors $O(\|V_j\|_j + \|K_j\|_j)$ above. By (3.115) and Lemma 3.23, it suffices to show

$$A^{|\mathcal{B}_{j+1}(U)|} 2^{|\mathcal{B}_j(U)|} \sum_{\mathcal{G}_2(U)} (bA^{2^d})^{|\mathcal{X}|+|\mathrm{Comp}(\check{X})|} (\frac{A}{2})^{-|\mathcal{B}_j(\check{X})|} \leqslant A^{\nu} O(b^2). \tag{3.116}$$

Indeed, (3.115) is bounded by $O(A^{-|\mathcal{B}_{j+1}(U)|}\|K_j\|_j/b)$ times this quantity. The small $\|K_j\|_j/b$ is due to the fact that if $|\mathcal{X}| \geqslant 1$ there is a factor $\|K_j\|_j$ in (3.115) and if $|\mathcal{X}| = 0$ then (3.114) provides such a factor in place of b. Hence (3.116) gives

$$\|\mathcal{R}^2\|_{j+1} \leqslant O(A^{\nu})(\|V_j\|_j + \|K_j\|_j)\|K_j\|_j. \tag{3.117}$$

To verify (3.116), first note that since $|\mathcal{B}_j(U)| \leqslant L^d |\mathcal{B}_{j+1}(U)|$, for any $c > 0$ the prefactor can be bounded by

$$A^{|\mathcal{B}_{j+1}(U)|} 2^{|\mathcal{B}_j(U)|} \leqslant (\frac{A}{2})^{(1-c)|\mathcal{B}_{j+1}(U)|} 2^{(L^d+1)|\mathcal{B}_{j+1}(U)|} (\frac{A}{2})^{c|\mathcal{B}_{j+1}(U)|}. \tag{3.118}$$

Taking $c > 1$, the product of the first two terms on the last right-hand side is less than 1 for A sufficiently large. It thus suffices to prove that for some $c > 1$

$$(\frac{A}{2})^{c|\mathcal{B}_{j+1}(U)|} \sum_{\mathcal{G}_2(U)} (bA^{2^d})^{|\mathcal{X}|+|\mathrm{Comp}(\check{X})|} (\frac{A}{2})^{-|\mathcal{B}_j(\check{X})|} \leqslant A^{\nu} O(b^2). \tag{3.119}$$

At this point we appeal to [28, proof of Lemma 6.17]; this result estimates the same sum but over $\mathcal{G}(U)$ instead of $\mathcal{G}_2(U)$. However, following exactly the same proof as in [28] but using that the sum is over $\mathcal{G}_2(U)$, the supremum over $n \geqslant 1$ in [28, (6.85)] becomes a supremum over $n \geqslant 2$ since $|\mathcal{X}| + |\mathrm{Comp}(\check{X})| \geqslant 2$. This yields that there exists $c > 1$ such that if $A = A(L, d)$ is large enough, then there is an m such that for all $U \in \mathcal{P}_{j+1}$,

$$(\frac{A}{2})^{c|\mathcal{B}_{j+1}(U)|} \sum_{\mathcal{G}_2(U)} (bA^{2^d})^{|\mathcal{X}|+|\mathrm{Comp}(\check{X})|} (\frac{A}{2})^{-|\mathcal{B}_j(\check{X})|} = O((bA^m)^2), \tag{3.120}$$

which is $A^{\nu} O(b^2)$ as needed. □

Proof of Lemma 3.23 For notational convenience, for $Y \in \mathcal{C}_j$ let

$$\tilde{K}(Y) = \sum_{W \in \mathcal{P}_j(Y)} \theta K_j(Y \setminus W)(\delta I)^W. \tag{3.121}$$

We first establish that the claimed bounds follow from the definition of $\check{K}(X)$ in (3.50) if we show, for $Y \in \mathcal{C}_j$,

$$\left\| \tilde{K}(Y) - \sum_{B \in \mathcal{B}_j(Y)} \theta J(B, Y) \right\|_{T_{j+1}(\ell_{j+1})} \leqslant A^{2^d} O(\|V_j\|_j + \|K_j\|_j) \times (\frac{A}{2})^{-|\mathcal{B}_j(Y)|} \tag{3.122}$$

$$\left\| \tilde{K}(Y) - (\delta I)^Y - \sum_{B \in \mathcal{B}_j(Y)} \theta J(B, Y) \right\|_{T_{j+1}(\ell_{j+1})} \leqslant A^{2^d} O(\|K_j\|_j)(\frac{A}{2})^{-|\mathcal{B}_j(Y)|}. \tag{3.123}$$

Indeed, though $\check{K}(\check{X}) - (\delta I)^{\check{X}}$ does not factor over components X of $\check{X}$, it can be written as a sum of $|\mathrm{Comp}(\check{X})|$ terms, each of which is a product over the components X of $\check{X}$. That is, we use the formula $(a+b)^n - a^n = \sum_{k=0}^{n-1} a^k b(a+b)^{n-k-1}$ with $a = (\delta I)^X$ and $b = \check{K}(X) - (\delta I)^X$. Thus each summand contains one factor $\check{K}(X) - (\delta I)^X$ and the rest of the factors are either $\check{K}(X)$ or $(\delta I)^X$. The estimates (3.113)-(3.114) then follow by using (3.122)-(3.123) and Lemma 3.17.

To establish (3.122)–(3.123) we apply the triangle inequality. Since $J(B, Y) = 0$ if $Y \notin \mathcal{S}_j$,

$$\left\| \sum_{B \in \mathcal{B}_j(Y)} \theta J(B, Y) \right\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|K_j\|_j) \tag{3.124}$$

where we have used $\|J(B,X)\|_{T_j(\ell_j)} = O(\|K_j\|_j)$, that θ contracts from $T_j(\ell_j)$ into $T_{j+1}(\ell_{j+1})$ and that $|\mathcal{B}_j(Y)| \leqslant 2^d$. This shows the J contributions to (3.122) and (3.123) satisfy the requisite bounds. For the other contributions, note that by (3.85), component factorisation of K_j, and the contraction property of the norms and θ, for $B \in \mathcal{B}_j$ and $Z \in \mathcal{P}_j$ we have

$$\|\delta I(B)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|V_j\|_j + \|K_j\|_j), \tag{3.125}$$

$$\|\theta K_j(Z)\|_{T_{j+1}(\ell_{j+1})} \leqslant A^{-\sum_{W\in \mathrm{Comp}(Z)}(|\mathcal{B}_j(W)|-2^d)_+} \|K_j\|_j^{|\mathrm{Comp}(Z)|}. \tag{3.126}$$

We now impose the condition that $\varepsilon \leqslant A^{-2^d}$ and that $O(\varepsilon) \leqslant A^{-1}$ in the implicit bound below. Plugging the previous bounds into the expression for $\tilde{K}(Y)$ we have

$$\begin{aligned}
&\|\tilde{K}(Y) - (\delta I)^Y\|_{T_{j+1}(\ell_{j+1})}\\
&\leqslant \sum_{Z\in\mathcal{P}_j(Y):Y\neq Z} \|(\delta I)^Z \theta K_j(Y\setminus Z)\|_{T_{j+1}(\ell_{j+1})}\\
&\leqslant \sum_{Z\in\mathcal{P}_j(Y):Y\neq Z} (O(\|V_j\|_j + \|K_j\|_j))^{|\mathcal{B}_j(Z)|}\\
&\quad \times \|K_j\|_j^{|\mathrm{Comp}(Y\setminus Z)|} A^{-\sum_{W\in\mathrm{Comp}(Y\setminus Z)}(|\mathcal{B}_j(W)|-2^d)_+}\\
&\leqslant \sum_{Z\in\mathcal{P}_j(Y):Y\neq Z} \left(A^{2^d}\|K_j\|_j\right)^{|\mathrm{Comp}(Y\setminus Z)|} (O(\|V_j\|_j + \|K_j\|_j))^{|\mathcal{B}_j(Z)|} A^{-|\mathcal{B}_j(Y\setminus Z)|}\\
&\leqslant A^{2^d}\|K_j\|_j (O(\|V_j\|_j + \|K_j\|_j) + A^{-1})^{|\mathcal{B}_j(Y)|}\\
&\leqslant A^{2^d}\|K_j\|_j \left(\frac{A}{2}\right)^{-|\mathcal{B}_j(Y)|}.
\end{aligned} \tag{3.127}$$

Since $\|(\delta I)^Y\|_{T_{j+1}(\ell_{j+1})} \leqslant [O(\|V_j\|_j + \|K_j\|_j)]^{|\mathcal{B}_j(Y)|} \leqslant AO(\|V_j\|_j + \|K_j\|_j) \times A^{-|\mathcal{B}_j(Y)|}$ if $O(\varepsilon) \leqslant A^{-1}$, by the triangle inequality we also have

$$\|\tilde{K}(Y)\|_{T_{j+1}(\ell_{j+1})} \leqslant A^{2^d} O(\|V_j\|_j + \|K_j\|_j) \left(\frac{A}{2}\right)^{-|\mathcal{B}_j(Y)|}. \tag{3.128}$$

Together with (3.124) this proves the lemma. □

Proof of Lemma 3.22 for $i = 3$ The bound on $\mathcal{R}^3(U)$ is similar to that of $\mathcal{R}^2(U)$ but simpler since only connected $\check{X}$ are involved. In particular, the analogue of Lemma 3.23 only involves the reasoning leading to (3.127), with the key difference being that now also $Z \neq \emptyset$. We omit the details. □

3.9 Flow of the renormalisation group

Recall the infinite volume limit of the renormalisation group maps $\Phi_{j+1,\infty}$ discussed below Proposition 3.12. We equip $\mathcal{K}_j^{\varnothing}(\mathbb{Z}^d)$ with the norm $\|K\|_j$ defined by (3.65).

This space is now infinite dimensional, but it is clear that it is still complete as a normed space, i.e., a Banach space. Moreover, by the consistency of the finite volume renormalisation group maps (Proposition 3.12), the estimates given in Theorem 3.13 also hold for the infinite volume limit. Next we study the iteration of the renormalisation group maps as a dynamical system. In what follows $K_0 = 0$ means $K_0(X) = 1_{X=\varnothing}$ for $X \in \mathcal{P}_j$.

In the next theorem (and later in the paper) we write $O_L(\cdot)$ to indicate a bound with a constant that may depend on L, but where the constant is uniform in j, i.e., $f_j = O_L(L^{-j})$ if there is a $C = C(L)$ such that $f_j \leqslant CL^{-j}$ for all j.

Theorem 3.24 *Let $d \geqslant 3$, $L \geqslant L_0$, and $A \geqslant A_0(L)$. For $m^2 \geqslant 0$ arbitrary and b_0 small, there exist $V_0^c(m^2, b_0)$ and $\kappa > 0$ such that if $(V_0, K_0) = (V_0^c(m^2, b_0), 0)$ and $(V_{j+1}, K_{j+1}) = \Phi_{j+1,\infty,m^2}(V_j, K_j)$ is the flow of the infinite volume renormalisation group map then*

$$\|V_j\|_j = O_L(b_0 L^{-\kappa j}), \qquad \|K_j\|_j = O_L(b_0^2 L^{-\kappa j}). \tag{3.129}$$

The components of $V_0^c(m^2, b_0)$ are continuous and uniformly bounded in $m^2 \geqslant 0$ and differentiable in b_0 with uniformly bounded derivative.

Proof of Theorem 3.24 The proof is by a version of the stable manifold theorem for smooth dynamical systems. Specifically, we use [28, Theorem 2.16].

To start, we write down the dynamical system corresponding to the renormalisation group map. The definition of V_{j+1} is (3.48). We start with the contribution to V_{j+1} arising from the first term

$$\mathbb{E}_{C_{j+1}}\Big[\theta V_j(B)\Big] = \tilde{u}_{j+1}|B| + \tilde{V}_{j+1}(B), \tag{3.130}$$

where $\tilde{u}_{j+1}$ and $\tilde{V}_{j+1}$ are defined by the right-hand side. These can be computed by the Wick formula (3.10), which gives

$$\begin{aligned} \tilde{z}_{j+1} &= z_j + \kappa_j^{zb} b_j, & \tilde{y}_{j+1} &= y_j + \kappa_j^{yb} b_j, \\ \tilde{a}_{j+1} &= a_j + \kappa_j^{ab} b_j, & \tilde{b}_{j+1} &= b_j, \end{aligned} \tag{3.131}$$

with $\kappa_j^{yb} = -C_{j+1}(0)$, $\kappa_j^{ab} = \Delta C_{j+1}(0)$, and $\kappa_j^{zb} = \frac{1}{2d}\Delta C_{j+1}(0)$. Indeed, non-constant contributions only arise from quartic terms, and Wick's formula gives

$$\begin{aligned} &\mathbb{E}_C\Big[\theta(\psi_x \bar{\psi}_x (\nabla_e \psi)_x (\nabla_e \bar{\psi})_x)\Big] \\ &\quad = \psi_x \bar{\psi}_x (\nabla_e \psi)_x (\nabla_e \bar{\psi})_x - C(0)(\nabla_e \psi)_x (\nabla_e \bar{\psi})_x - \nabla_e \nabla_{-e} C(0) \psi_x \bar{\psi}_x \\ &\qquad + \nabla_{-e} C(0) \psi_x (\nabla_e \bar{\psi})_x - \nabla_e C(0) \bar{\psi}_x (\nabla_e \psi)_x + (\text{constant}). \end{aligned} \tag{3.132}$$

Since $\nabla_e C(0) = C(e) - C(0) = \frac{1}{2d}\sum_{e \in \mathcal{E}_d}(C(e) - C(0)) = \frac{1}{2d}\Delta C(0)$ for all $e \in \mathcal{E}_d$ by symmetry, and since the lattice Laplacian has the representations $\Delta =$

$-\frac{1}{2}\sum_{e\in\mathcal{E}_d}\nabla_e\nabla_{-e}=\sum_{e\in\mathcal{E}_d}\nabla_e$, therefore

$$\begin{aligned}
&\mathbb{E}_C\Big[\theta(\psi_x\bar\psi_x(\nabla\psi)_x(\nabla\bar\psi)_x)\Big]\\
&\quad=\frac{1}{2}\sum_{e\in\mathcal{E}_d}\mathbb{E}_C\Big[\theta(\psi_x\bar\psi_x(\nabla_e\psi)_x(\nabla_e\bar\psi)_x)\Big]\\
&\quad=\psi_x\bar\psi_x(\nabla\psi)_x(\nabla\bar\psi)_x-C(0)(\nabla\psi)_x(\nabla\bar\psi)_x+\Delta C(0)\psi_x\bar\psi_x\\
&\qquad+\frac{\Delta C(0)}{2d}\Big(\frac{1}{2}\psi_x(\Delta\bar\psi)_x+\frac{1}{2}(\Delta\psi)_x\bar\psi_x\Big)+(\text{constant}).
\end{aligned}\tag{3.133}$$

Since $\|V_j(B)\|_{T_j(\ell_j)}$ is comparable with $|z_j|+|y_j|+L^{2j}|a_j|+L^{-(d-2)j}|b_j|$, i.e., $\|V_j(B)\|_{T_j(\ell_j)}=O_L(|z_j|+|y_j|+L^{2j}|a_j|+L^{-(d-2)j}|b_j|)=O_L(\|V_j(B)\|_{T_j(\ell_j)})$, it is natural to define the rescaled variables and coefficients $\hat z_j=z_j$, $\hat y_j=y_j$, $\hat a_j=L^{2j}a_j$, $\hat b_j=L^{-(d-2)j}b_j$, $\hat\kappa_j^{ab}=L^{2+dj}\kappa_j^{ab}$, $\hat\kappa_j^{yb}=L^{(d-2)j}\kappa_j^{yb}$, and $\hat\kappa_j^{zb}=L^{(d-2)j}\kappa_j^{zb}$. The definition (3.48) of V_{j+1} then becomes

$$\hat z_{j+1}=\hat z_j+\hat\kappa_j^{zb}\hat b_j+\hat r_j^z,\qquad \hat y_{j+1}=\hat y_j+\hat\kappa_j^{yb}\hat b_j+\hat r_j^y,\tag{3.134}$$

$$\hat a_{j+1}=L^2\hat a_j+\hat\kappa_j^{ab}\hat b_j+\hat r_j^a,\qquad \hat b_{j+1}=L^{-(d-2)}\hat b_j+\hat r_j^b.\tag{3.135}$$

Here $\hat r_j$ is the 4-component vector of real numbers determined by the Loc step of the renormalisation group map. Each component is thus a linear function of K_j, and by (3.66) of Theorem 3.13 has size $O(L^d\|K_j\|_j)$. The $\hat\kappa_j$ are uniformly bounded in j by the covariance estimates (3.3).

We now reorganize variables. Set $v_j=(\hat y_j,\hat z_j,\hat a_j)$ and $w_j=(\hat b_j,K_j)$ and use $\|\cdot\|_j$ for the norm given by maximum of the (norm of the) respective components. The index j does not play a role for $\|v_j\|_j$, but it does for $\|w_j\|_j$. By the computation above and Theorem 3.13 the infinite volume renormalisation group map can be written in the block diagonal form

$$\begin{pmatrix}v_{j+1}\\ w_{j+1}\end{pmatrix}=\begin{pmatrix}E & B_j\\ 0 & D_j\end{pmatrix}\begin{pmatrix}v_j\\ w_j\end{pmatrix}+\begin{pmatrix}0\\ g_{j+1}(v_j,w_j)\end{pmatrix}.\tag{3.136}$$

In this formula, E comes from the first terms on the right-hand sides of the $\hat z$, $\hat y$, and $\hat a$ equations in (3.134) and (3.135), B_j represents the $\hat\kappa_j^{xb}$ and the $\hat r_j^x$ terms with $x=z,y,a$, and D_j represents the first term in the $\hat b$ equation in (3.135) and the linearisation of $(0,K_j)\mapsto K_{j+1}$. Finally, $g_{j+1}(v_j,w_j)$ is then the non-linear remainder of K_{j+1} after the linearisation is removed. It follows from these identifications that $g_j(0,0)=0$ and $Dg_j(0,0)=0$. Moreover g_j is analytic in its arguments, so that all structural hypotheses required to apply [28, Theorem 2.16] hold.

As for the requisite norm estimates, since it is a 3×3 triangular matrix with non-zero diagonal entries, E is invertible with bounded inverse E^{-1}. As indicated above, $\|\hat r_j\|_{j+1}=O(L^d\|K_j\|_j)$, so $\|B_j\|_{j\to j+1}$ is bounded. Finally, the derivative estimates on the renormalisation group map from Theorem 3.13 imply the following norm bounds on D_j:

$$\|D_j\|_{j \to j+1} \leqslant b_0 \max\{L^{-(d-2)}, O(L^{-3} + A^{-\eta})\} \leqslant O(b_0 L^{-\kappa}), \tag{3.137}$$

with the latter inequality holding provided A is large enough.

For every $m^2 \geqslant 0$, b_0 sufficiently small, and $K_0 = 0$, [28, Theorem 2.16] now implies that we can find an initial 3-tuple of coupling constants $v_0^c(m^2, b_0)$, or equivalently an initial local potential $V_0^c(m^2, b_0)$, so that for some $\kappa > 0$,

$$\|v_j\|_j + \|w_j\|_j = O_L(b_0 L^{-\kappa j}). \tag{3.138}$$

These bounds are not explicit in the statement of [28, Theorem 2.16], but are immediate from its proof (because the proof constructs a solution in a correspondingly weighted sequence space). In particular this bound implies that $\|K_j\|_j + \|V_j\|_j = O_L(\|v_j\|_j + \|w_j\|_j) = O_L(b_0 L^{-\kappa j})$.

Smoothness of the renormalisation group map implies that $V_0^c(m^2, b_0)$ is smooth in b_0. To see that $V_0^c(m^2, b_0)$ is also continuous in $m^2 \geqslant 0$, one can for example regard v_j and w_j as bounded continuous functions of m^2, i.e., consider $v_j \in C_b([0,\infty), \mathbb{R}^3)$ and $w_j \in C_b([0,\infty), \mathbb{R} \times \mathcal{K}_j^{\varnothing}(\mathbb{Z}^d))$. Since all the estimates above are uniform in $m^2 \geqslant 0$, the previous argument applies in these spaces and shows that the solution is continuous in m^2. □

Remark 3.25 Note that while Theorem 3.13 assumes that $u_j = 0$ and produces u_{j+1}, it is trivial to extend the statement to $u_j \neq 0$ by simply adding u_j to the u_{j+1} produced for $u_j = 0$.

By consistency, the finite volume renormalisation group flow for V_j agrees with the infinite volume renormalisation group flow up to scale $j < N$ provided both have the same initial condition. As a result we obtain the following corollary by iterating the recursion (3.67) for the K-coordinate using the *a priori* knowledge that $\|V_j\|_j = O_L(b_0 L^{-\kappa j})$ due to Theorem 3.24.

Corollary 3.26 *Under the same assumptions as in Theorem 3.24, the same estimates hold for the finite volume renormalisation group flow for all $j \leqslant N$, and the V_j and u_j produced by the finite volume renormalisation group flow are the same as those for the infinite volume flow when $j < N$.*

From this it follows that if $e^{-u_N|\Lambda_N|}$ denotes the total prefactor accumulated in the renormalisation group flow up to scale N, then u_N is uniformly bounded in N and m^2 as $m^2 \downarrow 0$ if we begin with V_0 as in Theorem 3.24. Indeed, up to scale $N-1$ this follows from the bounds (3.129) and (3.71). In passing from the scale $N-1$ to N, the renormalisation group step is Λ_N-dependent, but is nevertheless uniformly bounded by the last statement of Theorem 3.13.

4 Computation of the susceptibility

In the remainder of the paper, we will use the notation (with $\Lambda = \Lambda_N$)

$$\langle F \rangle = \langle F \rangle_{V_0} = \frac{\mathbb{E}_C\Big[e^{-V_0(\Lambda)}F\Big]}{\mathbb{E}_C\Big[e^{-V_0(\Lambda)}\Big]} \tag{4.1}$$

and assume that $(V_j, K_j)_{j=0,\dots,N}$ is a renormalisation group flow, i.e., $(V_{j+1}, K_{j+1}) = \Phi_{j+1}(V_j, K_j)$.

In this section we express the susceptibility in terms of the dynamical system generated by the (bulk) renormalisation group flow. First recall that $Z_0 = e^{-V_0(\Lambda)}$ and that

$$C = (-\Delta + m^2)^{-1} = C_1 + \cdots + C_{N-1} + C_{N,N}, \qquad C_{N,N} = C_N + t_N Q_N, \tag{4.2}$$

where Δ is the Laplacian on Λ_N, $t_N = m^{-2} - O(L^{2N})$ is a constant, and the matrix Q_N is the orthogonal projection onto constants (i.e., all entries equal $1/|\Lambda_N|$). Using (3.11) and (3.13), with u_N as in (3.18), we then set

$$Z_{N,N} = \mathbb{E}_{t_N Q_N}\Big[\theta Z_N\Big] = \mathbb{E}_C\Big[\theta Z_0\Big], \qquad \tilde{Z}_{N,N} = e^{u_N|\Lambda_N|} Z_{N,N} \tag{4.3}$$

where $\mathbb{E}_{t_N Q_N}\theta$ is the fermionic Gaussian convolution with covariance $t_N Q_N$ defined in Sect. 3.1. Thus $\tilde{Z}_{N,N}$ is still a function of ψ, $\bar\psi$, i.e., an element of $\mathcal{N}(\Lambda)$. Note that $\mathbb{E}_C[Z_0]$ is the constant term of $Z_{N,N}$, i.e., obtained from $Z_{N,N}$ by formally setting $\bar\psi$ and ψ to 0. The following technical device of restricting to constant fields ψ, $\bar\psi$ will be useful for extracting information. By restriction to constant ψ, $\bar\psi$ we mean applying the homomorphism from $\mathcal{N}(\Lambda)$ onto itself that acts on the generators by $\psi_x \mapsto \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\psi_x$ and likewise for the $\bar\psi_x$. The result is an element in the subalgebra of $\mathcal{N}(\Lambda)$ generated by $\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\psi_x$ and $\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\bar\psi_x$; we will simply denote these generators by ψ and $\bar\psi$ when no confusion can arise. To explain the notation in the next proposition, note that a general even element of this subalgebra can be written as $F^0 + F^2\psi\bar\psi$ for some constants F^0, F^2, c.f. (3.21).

Proposition 4.1 *Restricted to constant* ψ, $\bar\psi$,

$$\begin{aligned} \tilde{Z}_{N,N} &= 1 + \tilde{u}_{N,N} - |\Lambda_N|\tilde{a}_{N,N}\psi\bar\psi, \\ \tilde{u}_{N,N} &= k_N^0 + \tilde{a}_{N,N}t_N, \qquad \tilde{a}_{N,N} = a_N - \frac{k_N^2}{|\Lambda_N|} \end{aligned} \tag{4.4}$$

where

$$k_N^0 = O(\|K_N\|_N), \qquad k_N^2 = O(\ell_N^{-2}\|K_N\|_N). \tag{4.5}$$

If V_0, K_0 *are continuous in* $m^2 \geqslant 0$ *and* b_0 *small enough, then so are* k_N^0 *and* k_N^2.

Proof Since the set of N-polymers $\mathcal{P}_N(\Lambda_N)$ is $\{\varnothing, \Lambda_N\}$ and $e^{u_N|\Lambda_N|}$ is a constant, (3.18) and (4.3) simplify to

$$\tilde{Z}_{N,N} = \mathbb{E}_{t_N Q_N}\Big[\theta(e^{-V_N(\Lambda_N)} + K_N(\Lambda_N))\Big]. \tag{4.6}$$

We now evaluate the integral over the zero mode with covariance $t_N Q_N$. To this end, we restrict $V_N(\Lambda_N)$ and $K_N(\Lambda_N)$ to spatially constant ψ, $\bar{\psi}$ and denote these restrictions by $\widetilde{V}_N(\Lambda_N)$ and $\widetilde{K}_N(\Lambda_N)$. Since $\widetilde{V}_N$ and $\widetilde{K}_N$ are even, they are of the form

$$\widetilde{V}_N(\Lambda_N, \psi, \bar{\psi}) = |\Lambda_N| a_N \psi\bar{\psi} \tag{4.7}$$

$$\widetilde{K}_N(\Lambda_N, \psi, \bar{\psi}) = k_N^0 + k_N^2 \psi\bar{\psi}, \tag{4.8}$$

where the form of $\widetilde{V}_N$ follows from the representation (3.36). Thus restricted to constant ψ and $\bar{\psi}$ the integrand in (4.6) is

$$e^{-\widetilde{V}_N(\Lambda_N)} + \widetilde{K}_N(\Lambda_N) = 1 + k_N^0 - (|\Lambda_N| a_N - k_N^2)\psi\bar{\psi}. \tag{4.9}$$

Therefore applying the fermionic Wick formula $\mathbb{E}_{t_N Q_N}\theta\psi\bar{\psi} = -t_N|\Lambda_N|^{-1} + \psi\bar{\psi}$, we obtain (4.4). The continuity claims for k_N^0 and k_N^2 follow as (V_j, K_j) is a renormalisation group flow (see below (4.1)) and since the renormalisation group map has this continuity.

The bounds (4.5) follow from the definition of the $T_N(\ell_N)$ norm. Indeed, since k_0 is the constant coefficient of $K_N(\Lambda_N)$, clearly $k_N^0 = O(\|K_N\|_N)$. Similarly, setting $g_{(x,1),(y,-1)} = 1$ for all $x, y \in \Lambda_N$ and $g_z = 0$ for all other sequences, we have $\|g\|_{\Phi_N(\ell_N)} = \ell_N^{-2}$ and

$$|k_N^2| = |\langle K_N(\Lambda_N), g\rangle| \leqslant \|g\|_{\Phi_N(\ell_N)} \|K_N\|_N = \ell_N^{-2}\|K_N\|_N \tag{4.10}$$

where $\langle \cdot, \cdot \rangle$ is the pairing from Definition 3.3. □

Proposition 4.2 *Using the notation of Proposition* 4.1,

$$\sum_{x\in\Lambda_N} \langle \bar{\psi}_0 \psi_x \rangle = \frac{1}{m^2} - \frac{1}{m^4} \frac{\tilde{a}_{N,N}}{1 + \tilde{u}_{N,N}}. \tag{4.11}$$

Proof We amend the algebra $\mathcal{N}(\Lambda_N)$ by two Grassmann variables σ and $\bar{\sigma}$ which we view as constant fields $\sigma_x = \sigma$ and $\bar{\sigma}_x = \bar{\sigma}$. We then consider the fermionic cumulant generating function (an element of the Grassmann algebra generated by σ and $\bar{\sigma}$)

$$\Gamma(\sigma, \bar{\sigma}) = \log \mathbb{E}_C\Big[Z_0(\psi, \bar{\psi}) e^{(\sigma,\bar{\psi}) + (\psi,\bar{\sigma})}\Big], \tag{4.12}$$

where C is as in (4.2). By translation invariance

$$\sum_{x\in\Lambda_N} \langle \bar{\psi}_0 \psi_x \rangle = \frac{1}{|\Lambda_N|} \sum_{x,y\in\Lambda_N} \langle \bar{\psi}_x \psi_y \rangle = \frac{1}{|\Lambda_N|} \partial_{\bar{\sigma}} \partial_\sigma \Gamma(\sigma, \bar{\sigma}). \tag{4.13}$$

The linear change of generators $\psi \mapsto \psi + C\sigma$, $\bar\psi \mapsto \bar\psi + C\bar\sigma$ yields

$$\Gamma(\sigma,\bar\sigma) = (\sigma, C\bar\sigma) + \log \mathbb{E}_C\Big[\theta Z_0(C\sigma, C\bar\sigma)\Big], \tag{4.14}$$

where the right-hand side is to be interpreted as applying the doubling homomorphism and then restricting to constant σ, $\bar\sigma$. Since σ is constant in $x \in \Lambda_N$, we have $C\sigma = m^{-2}\sigma$. With (4.3) thus

$$\Gamma(\sigma,\bar\sigma) = m^{-2}|\Lambda_N|\sigma\bar\sigma + \log \tilde Z_{N,N}(m^{-2}\sigma, m^{-2}\bar\sigma) - u_N|\Lambda_N|. \tag{4.15}$$

As a result, by (4.4)–(4.5),

$$\frac{1}{|\Lambda_N|}\partial_{\bar\sigma}\partial_\sigma \Gamma(\sigma,\bar\sigma) = \frac{1}{m^2} - \frac{1}{m^4}\frac{\tilde a_{N,N}}{1+\tilde u_{N,N}}. \tag{4.16}$$

□

5 The observable renormalisation group flow

Recall that $\langle\cdot\rangle$ denotes the expectation (4.1), in which we will ultimately choose $V_0 = V_0^c(b_0, m^2)$ as in Theorem 3.24. This section sets up and analyses the renormalisation group flow associated to source fields. This will enable the computation of correlation functions (observables) like $\langle\bar\psi_a\psi_b\rangle$ in Sect. 6. Our strategy is inspired by that used in [14, 75], but with important differences arising due to the presence of a non-trivial zero mode in our setting.

5.1 Observable coupling constants

As in the proofs in Sect. 4, we amend the Grassmann algebra by two source fields. Now, however, the additional fields are not constant in space but rather are localised at two points $a, b \in \Lambda = \Lambda_N$. We distinguish between two cases:

Case (1). For the two point function $\langle\bar\psi_a\psi_b\rangle$ (which we call 'Case (1)'), the additional source fields σ_a and $\bar\sigma_b$ are two additional Grassmann variables that anticommute with each other and the ψ, $\bar\psi$.

Case (2). For the quartic correlation function $\langle\bar\psi_a\psi_a\bar\psi_b\psi_b\rangle$ (called 'Case (2)'), we introduce additional Grassmann variables $\bar\vartheta_x$, ϑ_x for $x \in \{a, b\}$ and the additional source fields σ_a and σ_b are the commuting variables $\sigma_x = \bar\vartheta_x\vartheta_x$ for $x \in \{a, b\}$. This explicit $U(1)$ invariant choice will be convenient when discussing symmetries.

In both cases we relabel the initial potential V_0 from Sect. 3 as $V_0^\varnothing$ and set $V_0 = V_0^\varnothing + V_0^\star$ where $V_0^\star$ is an observable part to be defined.

Case (1). In this case,

$$V_0^\star = -\lambda_{a,0}\sigma_a\bar\psi_a - \lambda_{b,0}\psi_b\bar\sigma_b. \tag{5.1}$$

The spatial index of $V_0^\star$ signals the local nature of the source fields. More precisely, the evaluation of $V_0^\star$ on a set X is defined to be spatially localised: $V_0^\star(X) =$

$-\lambda_{a,0}\sigma_a\bar{\psi}_a 1_{a\in X} - \lambda_{b,0}\psi_b\bar{\sigma}_b 1_{b\in X}$. Recalling that $C = (-\Delta + m^2)^{-1}$, see (3.1), it follows that

$$\langle\bar{\psi}_a\psi_b\rangle = \frac{1}{\lambda_{a,0}\lambda_{b,0}}\partial_{\bar{\sigma}_b}\partial_{\sigma_a}\log\mathbb{E}_C\Big[e^{-V_0(\Lambda)}\Big]. \tag{5.2}$$

Obtaining (5.2) is just a matter of expanding $e^{-V_0^\star(\Lambda)}$, using $\langle\bar{\psi}_a\rangle = \langle\psi_b\rangle = 0$, and applying the rules of Grassmann calculus. Note the order of $\partial_{\bar{\sigma}_b}$ and ∂_{σ_a}, which is important to obtain the correct sign. Although (5.2) holds for any constants $\lambda_{a,0}$, $\lambda_{b,0}$, it is convenient for us to leave these as variables to be tracked with respect to the renormalisation group flow.

Case (2). Similarly to the previous case, we choose

$$V_0^\star = -\lambda_{a,0}\sigma_a\bar{\psi}_a\psi_a - \lambda_{b,0}\sigma_b\bar{\psi}_b\psi_b, \tag{5.3}$$

so that

$$\langle\bar{\psi}_a\psi_a\rangle = \frac{1}{\lambda_{a,0}}\partial_{\sigma_a}\log\mathbb{E}_C\Big[e^{-V_0(\Lambda)}\Big]\Big|_{\lambda_{b,0}=0} \tag{5.4}$$

$$\langle\bar{\psi}_a\psi_a\bar{\psi}_b\psi_b\rangle - \langle\bar{\psi}_a\psi_a\rangle\langle\bar{\psi}_b\psi_b\rangle = \frac{1}{\lambda_{a,0}\lambda_{b,0}}\partial_{\sigma_b}\partial_{\sigma_a}\log\mathbb{E}_C\Big[e^{-V_0(\Lambda)}\Big]. \tag{5.5}$$

To distinguish the coupling constants in the two cases, we will sometimes write $\lambda_{a,0}^{(p)}$ with $p=1$ or $p=2$ instead of $\lambda_{a,0}$, and analogously for the other coupling constants.

5.2 The free observable flow

To orient the reader and motivate the discussion which follows, let us first consider the noninteracting case $V_0^\varnothing = 0$, in which the microscopic model is explicitly fermionic Gaussian. In this case, one may compute all correlations explicitly by applying the fermionic Wick rule. The same computation can be carried out inductively using the finite range decomposition of the covariance C, and we review this now as it will be the starting point for our analysis of the interacting case.

To begin the discussion, observe that all source fields square to zero, i.e., $\sigma_a^2 = \bar{\sigma}_b^2 = \sigma_b^2 = 0$. This implies that $V_0^\star(\Lambda)^3 = 0$ since $V_0^\star(\Lambda)$ has no constant term and has at least one least source field in each of its two summands. Given $V_0^\star$, we inductively define renormalised interaction potentials that share this property:

$$u_{j+1}^\star(\Lambda) + V_{j+1}^\star(\Lambda) = \mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda)\Big] - \frac{1}{2}\mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda);\theta V_j^\star(\Lambda)\Big] \tag{5.6}$$

where

$$\mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda);\theta V_j^\star(\Lambda)\Big] = \mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda)^2\Big] - \Big(\mathbb{E}_{C_{j+1}}\Big[V_j^\star(\Lambda)\Big]\Big)^2 \tag{5.7}$$

and $u^\star_{j+1}(\Lambda)$ collects the terms that do not contain ψ or $\bar\psi$. Consequently, one can check that

$$\mathbb{E}_{C_{j+1}}\Big[\theta e^{-V^\star_j(\Lambda)}\Big] = \mathbb{E}_{C_{j+1}}\Big[\theta(1 - V^\star_j(\Lambda) + \frac{1}{2}V^\star_j(\Lambda)^2)\Big] = e^{-u^\star_{j+1}(\Lambda)-V^\star_{j+1}(\Lambda)}. \tag{5.8}$$

For convenience, in the last step when $j = N$, we set $C_{N+1} = t_N Q_N$. This separation of the zero mode is not essential here but will be useful for our analysis in the interacting case.

For $j > 0$, the $V^\star_j$ have terms not present in $V^\star_0$, for example the terms involving q in the next definition. The nilpotency of the source fields σ_a, σ_b limits the possibilities.

Definition 5.1 Let $\mathcal{V}^\star$ be the space of formal field polynomials $u^\star + V^\star$ of the form:

$$\left.\begin{aligned} V^\star &= -\lambda_a\sigma_a\bar\psi_a - \lambda_b\psi_b\bar\sigma_b + \sigma_a\bar\sigma_b\frac{r}{2}(\bar\psi_a\psi_a + \bar\psi_b\psi_b),\\ u^\star &= -\sigma_a\bar\sigma_b q, \end{aligned}\right\} \quad \text{in Case (1)},$$

$$\left.\begin{aligned} V^\star &= -\sigma_a\lambda_a\bar\psi_a\psi_a - \sigma_b\lambda_b\bar\psi_b\psi_b - \sigma_a\sigma_b\frac{\eta}{2}(\bar\psi_a\psi_b + \bar\psi_b\psi_a)\\ &\qquad + \sigma_a\sigma_b\frac{r}{2}(\bar\psi_a\psi_a + \bar\psi_b\psi_b)),\\ u^\star &= -\sigma_a\gamma_a - \sigma_b\gamma_b - \sigma_a\sigma_b q, \end{aligned}\right\} \quad \text{in Case (2)},$$

for observable coupling constants $(\lambda_a, \lambda_b, q, r) \in \mathbb{C}^4$ respectively $(\lambda_a, \lambda_b, \gamma_a, \gamma_b, q, \eta, r) \in \mathbb{C}^7$. For $X \subset \Lambda$, we define $(u^\star + V^\star)(X) \in \mathcal{N}^\star(X \cap \{a, b\})$ by

$$\begin{aligned}(u^\star + V^\star)(X) &= -\lambda_a\sigma_a\bar\psi_a 1_{a\in X} - \lambda_b\psi_b\bar\sigma_b 1_{b\in X} - \sigma_a\bar\sigma_b q 1_{a\in X, b\in X}\\ &\qquad + \sigma_a\bar\sigma_b\frac{r}{2}(\bar\psi_a\psi_a + \bar\psi_b\psi_b)1_{a\in X, b\in X}\end{aligned} \tag{5.9}$$

in Case (1), and analogously in Case (2).

Remark 5.2 The terms corresponding to r do not appear at any step of the free observable flow (5.6) if we start with them equal to 0. We include them here in preparation for the interacting model.

The evolutions of $u^\star_j + V^\star_j \to u^\star_{j+1}$ and $V^\star_j \to V^\star_{j+1}$ are equivalent to the evolution of the coupling constants $(\lambda_a, \lambda_b, q, r)$ respectively $(\lambda_a, \lambda_b, \gamma_a, \gamma_b, q, \eta, r)$. By computation of the fermionic Gaussian moments in (5.6), the flow of the observable coupling constants according to (5.6) is then given as follows. Note that the evolution of coupling constants in $V^\star$ is independent of the coupling constants in $u^\star$.

Lemma 5.3 *Let $V^\varnothing_0 = 0$, and let $u^\star_j$ and $V^\star_j$ be of the form as in Definition 5.1. The map (5.6) is then given as follows. In Case (1), for $x \in \{a, b\}$,*

$$\lambda_{x,j+1} = \lambda_{x,j} \tag{5.10}$$

$$q_{j+1} = q_j + \lambda_{a,j}\lambda_{b,j}C_{j+1}(a,b) + r_j C_{j+1}(0,0) \tag{5.11}$$

$$r_{j+1} = r_j, \tag{5.12}$$

whereas in Case (2), *for* $x \in \{a, b\}$,

$$\lambda_{x,j+1} = \lambda_{x,j} \tag{5.13}$$

$$\gamma_{x,j+1} = \gamma_{x,j} + \lambda_{x,j}C_{j+1}(0,0) \tag{5.14}$$

$$q_{j+1} = q_j + \eta_j C_{j+1}(a,b) + r_j C_{j+1}(0,0) - \lambda_{a,j}\lambda_{b,j}C_{j+1}(a,b)^2 \tag{5.15}$$

$$\eta_{j+1} = \eta_j - 2\lambda_{a,j}\lambda_{b,j}C_{j+1}(a,b) \tag{5.16}$$

$$r_{j+1} = r_j. \tag{5.17}$$

Proof This follows from straightforward evaluation of (5.6) using (3.10). □

To continue the warm-up for the interacting case, we illustrate how these equations reproduce the direct computations of correlation functions (and explain the terminology of observable coupling constants). When $r_0 = 0$, by a computation using Definition 5.1, the formulas (5.2) and (5.4)–(5.5) imply the correlation functions in Cases (1) and (2) are given by

$$\langle \bar\psi_a \psi_b \rangle = \frac{q_{N+1}}{\lambda_{a,0}\lambda_{b,0}}, \qquad \langle \bar\psi_a \psi_a \rangle = \frac{\gamma_{a,N+1}}{\lambda_{a,0}}, \qquad \langle \bar\psi_a \psi_a; \bar\psi_b \psi_b \rangle = \frac{q_{N+1}}{\lambda_{a,0}\lambda_{b,0}}, \tag{5.18}$$

with $q = q^{(1)}$ and $\lambda = \lambda^{(1)}$ for the first equation and $q = q^{(2)}$, $\gamma = \gamma^{(2)}$, and $\lambda = \lambda^{(2)}$ for the last two. Recalling the convention $C_{N+1} = t_N Q_N$, for Case (2) with $r_0 = 0$ a computation using (5.13), (5.15) and (5.16) shows

$$\begin{aligned} q_{N+1} &= -\lambda_{0,a}\lambda_{0,b}\left(\sum_{k \leqslant N} C_k(a,b) + t_N Q_N(a,b)\right)^2 \\ &= -\lambda_{0,a}\lambda_{0,b}(-\Delta + m^2)^{-1}(a,b)^2, \end{aligned} \tag{5.19}$$

with the final equality by (3.1). Combined with (5.18), this gives

$$\langle \bar\psi_a \psi_a; \bar\psi_b \psi_b \rangle = -(-\Delta + m^2)^{-1}(a,b)^2, \tag{5.20}$$

as expected for free fermions.

Remark 5.4 In the preceding computation we kept the potential in the exponential for the entire computation, whereas in Sects. 4 and 6 the zero mode is integrated out directly without rewriting the integrand in this form (see, e.g., (4.3)). We distinguish these two approaches by using $N + 1$ subscripts for the former and (N, N) for the latter, and by putting tildes on quantities associated with the $(N, N)^{\text{th}}$ step as was done in Sect. 4.

Before moving to the interacting model, we introduce the *coalescence scale* j_{ab} as the largest integer j such that $C_{k+1}(a,b)=0$ for all $k<j$, i.e.,

$$j_{ab} = \lfloor \log_L(2|a-b|_\infty) \rfloor. \tag{5.21}$$

In the degenerate cases $\lambda_a = 0$ or $\lambda_b = 0$ when only one of the source fields is present we use the convention $j_{ab} = +\infty$. Note that the finite range property (3.2) implies that $q_j = \eta_j = r_j = 0$ for $j < j_{ab}$ provided they are all 0 when $j = 0$. This will also be true in the interacting case.

In connection with the coalescence scale, we also make a convenient choice of the block decomposition of Λ_N based on the relative positions of a and b. Namely, we center the block decomposition such that point a is in the center (up to rounding if L is even) of the blocks at all scales $1 \leqslant j \leqslant N$. This implies that if $|a-b|_\infty < \frac{1}{2}L^{j+1}$ the scale-j blocks containing a and b are contained in a common scale-$(j+1)$ block.

5.3 Norms with observables

To extend the above computation for $V^\varnothing = 0$ to the interacting case, we will extend the renormalisation group map to the Grassmann algebra amended by the source fields. In Case (2), recall that the source fields σ_a and σ_b are even elements rather than Grassmann generators themselves, i.e., they are commuting elements also satisfying $\sigma_a^2 = \sigma_b^2 = 0$. In both Cases (1) and (2), this algebra has the decomposition

$$\mathcal{N}(X) = \mathcal{N}^\varnothing(X) \oplus \mathcal{N}^a(X) \oplus \mathcal{N}^b(X) \oplus \mathcal{N}^{ab}(X) = \mathcal{N}^\varnothing(X) \oplus \mathcal{N}^\star(X) \tag{5.22}$$

where $\mathcal{N}^\varnothing(X)$ is spanned by monomials with no factors of σ, $\mathcal{N}^a(X)$ is spanned by monomials containing a factor σ_a but no factor $\bar\sigma_b$ (respectively σ_b), analogously for $\mathcal{N}^b(X)$, and $\mathcal{N}^{ab}(X)$ is spanned by monomials containing $\sigma_a\bar\sigma_b$ respectively $\sigma_a\sigma_b$. Thus any $F \in \mathcal{N}(X)$ can be written as

$$F = F^\varnothing + F^\star = \begin{cases} F_\varnothing + \sigma_a F_a + \bar\sigma_b F_b + \sigma_a\bar\sigma_b F_{ab}, & \text{Case (1)} \\ F_\varnothing + \sigma_a F_a + \sigma_b F_b + \sigma_a\sigma_b F_{ab}, & \text{Case (2),} \end{cases} \tag{5.23}$$

with $F_\varnothing, F_a, F_b, F_{ab} \in \mathcal{N}^\varnothing(X)$. We denote by $\pi_\varnothing$, π_a, π_b and π_{ab} the projections on the respective components, e.g., $\pi_a F = \sigma_a F_a$, and $\pi_\star = \pi_a + \pi_b + \pi_{ab}$. We will use superscripts instead of subscripts in the decomposition when the factors of σ are included, e.g., $F^a = \sigma_a F_a$ and $F^\varnothing = F_\varnothing$.

We say that F is $U(1)$ invariant if the number of generators with a bar is equal to the number without a bar. Explicitly, in Case (1) this means $F_\varnothing$ and F_{ab} are $U(1)$ invariant, F_a has one more factor with a bar than without, and similarly for F_b. In Case (2) this means all of $F_\varnothing$, F_a, F_b and F_{ab} are $U(1)$ invariant (recall that $\sigma_a = \bar\vartheta_a\vartheta_a$ and $\sigma_b = \bar\vartheta_b\vartheta_b$). Denote by $\mathcal{N}_{\mathrm{sym}}(X)$ the subalgebra of $U(1)$ invariant elements.

For F decomposed according to (5.23) we define

$$\|F\|_{T_j(\ell_j)} = \|F_\varnothing\|_{T_j(\ell_j)} + \ell_{a,j}\|F_a\|_{T_j(\ell_j)} + \ell_{b,j}\|F_b\|_{T_j(\ell_j)} + \ell_{ab,j}\|F_{ab}\|_{T_j(\ell_j)} \tag{5.24}$$

where

$$\ell_{a,j} = \ell_{b,j} = \begin{cases} \ell_j^{-1}, & \text{Case (1)} \\ \ell_j^{-2}, & \text{Case (2),} \end{cases} \qquad \ell_{ab,j} = \begin{cases} \ell_j^{-2}, & \text{Case (1)} \\ \ell_j^{-2}\ell_{j\wedge j_{ab}}^{-2}, & \text{Case (2).} \end{cases} \tag{5.25}$$

In particular, $\|\sigma_a\|_{T_j(\ell_j)} = \ell_{a,j}$ and $\|\sigma_a\sigma_b\|_{T_j(\ell_j)} = \ell_{ab,j}$ and, in Cases (1) and (2), respectively,

$$\|\sigma_a\bar{\psi}_a\|_{T_j(\ell_j)} = \ell_{a,j}\ell_j = 1, \qquad \|\sigma_a\bar{\psi}_a\psi_a\|_{T_j(\ell_j)} = \ell_{a,j}\ell_j^2 = 1 \tag{5.26}$$

and, again in the two cases respectively,

$$\|\sigma_a\bar{\sigma}_b\bar{\psi}_x\psi_x\|_{T_j(\ell_j)} = \ell_{ab,j}\ell_j^2 = 1, \qquad \|\sigma_a\sigma_b\bar{\psi}_x\psi_x\|_{T_j(\ell_j)} = \ell_{ab,j}\ell_j^2 = \ell_{j\wedge j_{ab}}^{-2}. \tag{5.27}$$

In both cases these terms do not change size under change of scale, provided that $j \geqslant j_{ab}$ for the last term. Thus they are *marginal*. As will be seen in Sect. 6, see the paragraph following Lemma 6.3, the choices of $\ell_{a,j}$ and $\ell_{ab,j}$ are appropriate to capture the leading behaviour of correlation functions.

The extended definition (5.24) of the $T_j(\ell_j)$ norm satisfies the properties discussed in Sect. 3.5, with the exception of the generalisation of the monotonicity estimate $\|F_\varnothing\|_{T_{j+1}(2\ell_{j+1})} \leqslant \|F_\varnothing\|_{T_j(\ell_j)}$. Checking these properties is straightforward by using the properties of the bulk norm, and, in the case of the product property, using that $\ell_{ab,j} \leqslant \ell_{a,j}\ell_{b,j}$ (recall (3.23)). Similar reasoning also yields a weaker monotonicity-type estimate: by (5.24), (3.23), and monotonicity in the bulk algebra,

$$\|F\|_{T_{j+1}(\ell_{j+1})} \leqslant \|F\|_{T_{j+1}(2\ell_{j+1})} \leqslant 16L^{2(d-2)}\|F\|_{T_j(\ell_j)}. \tag{5.28}$$

5.4 Localisation with observables

We combine the space $\mathcal{V}^\varnothing$ of bulk coupling constants from Definition 3.5 with the space $\mathcal{V}^\star$ of observable coupling constants from Definition 5.1 into

$$\mathcal{V} = \mathcal{V}^\varnothing \oplus \mathcal{V}^\star. \tag{5.29}$$

We extend the localisation operators $\mathrm{Loc}_{X,Y}$ from Sect. 3.6 to the amended Grassmann algebra (5.22) as follows. As in the bulk setting, we will focus on the key properties of the extended localisation operators. The extension of $\mathrm{Loc}_{X,Y}$ is linear and block diagonal with respect to the decomposition (5.22), and so can be defined separately on each summand. On $\mathcal{N}^\varnothing(X)$, the restriction $\mathrm{Loc}_{X,Y}$ is defined to coincide with the operators from Proposition 3.8. From now on we denote this restriction by $\mathrm{Loc}_X^\varnothing$ or $\mathrm{loc}_X^\varnothing$ if we want to distinguish it from the extended version. To define the restriction $\mathrm{Loc}_{X,Y}^\star$ of $\mathrm{Loc}_{X,Y}$ to $\mathcal{N}^\star(X)$, we continue to employ the systematic framework from [30, Sect. 1.7], as follows. Let loc^a, loc^b, and loc^{ab} be the localisation operators from [30, Definition 1.17] with maximal dimensions (for the Cases $(p=1)$ and $(p=2)$)

$$d_+^a = d_+^b = \frac{p}{2}(d-2), \qquad d_+^{ab} = d-2. \tag{5.30}$$

Case (1). For $\sigma_a F_a \in \mathcal{N}^a(X)$ we set $\mathrm{Loc}_{X,Y}(\sigma_a F_a) = \sigma_a \, \mathrm{loc}^a_{X\cap\{a\},Y\cap\{a\}} F_a$, and likewise for point b. For $\sigma_a\bar\sigma_b F_{ab} \in \mathcal{N}^{ab}(X)$ we set $\mathrm{Loc}_{X,Y}(\sigma_a\bar\sigma_b F_{ab}) = \sigma_a\bar\sigma_b \times \mathrm{loc}^{ab}_{X\cap\{a,b\},Y\cap\{a,b\}} F_{ab}$.

Case (2). The definitions in Case (2) are analogous, but with $\bar\sigma_b$ replaced by σ_b.

The superscripts $\varnothing$, a, b, ab are present to indicate that we have assigned different maximal dimensions to the summands in (5.22). We use the same choice of field dimensions $[\psi] = [\bar\psi] = (d-2)/2$ as in Sect. 3.6. We note that $\mathrm{loc}^a_{X,\varnothing} = \mathrm{loc}^b_{X,\varnothing} = \mathrm{loc}^{ab}_{X,\varnothing} = 0$. The main difference between these operators and $\mathrm{Loc}^\varnothing$ is that the expressions produced by loc^a, loc^b, loc^{ab} are local, i.e., supported near a and b. A second difference is that the maximal dimensions vary.

Before giving the general properties of the extended Loc we include an example in Case (1).

Example 5.5 Consider Case (1). Then:

(i) If $F \in \mathcal{N}(\Lambda)$ is a field monomial of degree greater than one or if F has gradients in it, then $\mathrm{Loc}^a_X F = \mathrm{Loc}^b_X F = 0$. Here (and in the rest of this example) we do not count factors of σ_a and σ_b in the degree. If F has degree greater than 2 or is degree two and has gradients $\mathrm{Loc}^{ab}_X F = 0$.

(ii) If $F = \sigma_a\bar\psi_x + \bar\sigma_b\psi_y + \sigma_a\bar\sigma_b\bar\psi_x\psi_y$, then

$$\begin{aligned}\mathrm{Loc}_X F &= \sigma_a\bar\psi_a 1_{a\in X} + \bar\sigma_b\psi_b 1_{b\in X}\\ &\quad + \sigma_a\bar\sigma_b\left(\bar\psi_a\psi_a 1_{a\in X,b\notin X} + \bar\psi_b\psi_b 1_{b\in X,a\notin X} + \frac{1}{2}[\bar\psi_a\psi_a + \bar\psi_b\psi_b]1_{a,b\in X}\right).\end{aligned} \tag{5.31}$$

The next proposition summarises the key properties of the operators $\mathrm{Loc}_{X,Y}$. As with Proposition 3.8, these properties follow from [30]. That the choice of maximal dimensions (5.30) produce contractive estimates can intuitively be understood by considering the marginal monomials. By (5.26) and (5.27), these are exactly the monomials with dimensions $d^a_+ = d^b_+$ respectively d^{ab}_+.

Proposition 5.6 *For $L = L(d)$ sufficiently large there is a universal $\bar C > 0$ such that: for $j < N$ and any small sets $Y \subset X \in \mathcal{S}_j$, the linear maps $\mathrm{Loc}^\star_{X,Y} : \mathcal{N}^\star(X^\square) \to \mathcal{N}^\star(Y^\square)$ have the following properties*:

(*i*) *They are bounded*:

$$\|\mathrm{Loc}^\star_{X,Y} F\|_{T_j(\ell_j)} \leqslant \bar C \|F\|_{T_j(\ell_j)}. \tag{5.32}$$

(*ii*) *For $j \geqslant j_{ab}$, the maps $\mathrm{Loc}^\star_X = \mathrm{Loc}^\star_{X,X} : \mathcal{N}^\star(X^\square) \to \mathcal{N}^\star(X^\square)$ satisfy the contraction bound*:

$$\|(1-\mathrm{Loc}^\star_X)F\|_{T_{j+1}(2\ell_{j+1})} \leqslant \bar C L^{-(\frac{d-2}{2}\wedge 1)} \|F\|_{T_j(\ell_j)}. \tag{5.33}$$

Moreover, the bound (5.33) holds also for $j < j_{ab}$ if $F^{ab} = 0$.

(*iii*) *If X is the disjoint union of $X_1, \ldots, X_n$ then $\mathrm{Loc}^\star_X = \sum_{i=1}^n \mathrm{Loc}^\star_{X,X_i}$.*

(*iv*) *For a block B and polymers $X \supset B$, $\mathrm{Loc}^\star_{X,B} F \in \mathcal{V}^\star(B)$ if $F \in \mathcal{N}^\star_{\mathrm{sym}}(X)$.*

Properties (i)–(iii) follow from [30] in the same way as the corresponding properties in Proposition 3.8 by making use of the observation that

$$\|\sigma_a\|_{T_{j+1}(2\ell_{j+1})} \leqslant 2L^{d_+^a}\|\sigma_a\|_{T_j(\ell_j)}, \qquad \|\sigma_b\|_{T_{j+1}(2\ell_{j+1})} \leqslant 2L^{d_+^b}\|\sigma_b\|_{T_j(\ell_j)}, \tag{5.34}$$

$$\|\sigma_a\sigma_b\|_{T_{j+1}(2\ell_{j+1})} \leqslant 4L^{d_+^{ab}}\|\sigma_a\sigma_b\|_{T_j(\ell_j)}, \qquad \text{if } j \geqslant j_{ab}, \tag{5.35}$$

in Case (2) and analogously in Case (1). These factors of L^{d_+} correspond to the missing L^{-d_+} factors in (5.33) as compared to Proposition 3.8. It only remains to verify (iv), i.e., to identify the image of $\mathrm{Loc}^\star_{X,B}$ when acting on $F \in \mathcal{N}^\star_{\mathrm{sym}}(X)$.

Case (1). By the choice of dimensions in its specification, the image of $\sigma_a\,\mathrm{loc}^a$ is spanned by the local monomials σ_a, $\sigma_a\bar\psi_a$, $\sigma_a\psi_a$. The condition of $U(1)$ invariance then implies that if $\sigma_a F_a \in \mathcal{N}^a_{\mathrm{sym}}(X)$ only the monomial $\sigma_a\bar\psi_a$ is admissible. The situation is analogous for loc^b. Similarly, $\sigma_a\bar\sigma_b\,\mathrm{loc}^{ab}$ has image spanned by $\sigma_a\bar\sigma_b$ and $\sigma_a\bar\sigma_b\bar\psi_x\psi_x$ for $x \in \{a,b\}$ as well as further first order monomials with at most $(d-2)/2$ gradients, e.g., $\sigma_a\bar\sigma_b\nabla_{e_1}\psi_x$. Only the even monomials $\sigma_a\bar\sigma_b$, $\sigma_a\bar\sigma_b\bar\psi_a\psi_a$, and $\sigma_a\bar\sigma_b\bar\psi_b\psi_b$ are compatible with $U(1)$ symmetry. In summary, $\mathrm{Loc}^\star_{X,Y}F$ is contained in $\mathcal{V}^\star$ if $F \in \mathcal{N}^\star_{\mathrm{sym}}(X)$.

Case (2). By the choice of dimensions, in this case $\sigma_a\,\mathrm{loc}^a$ has image spanned by the local monomials σ_a, $\sigma_a\bar\psi_a\psi_a$ as well as further first order monomials with at most $d-2$ gradients, and $U(1)$ symmetry implies that only the even terms σ_a and $\sigma_a\bar\psi_a\psi_a$ arise in the image if $F \in \mathcal{N}_{\mathrm{sym}}(X)$. The analysis for $\sigma_b\,\mathrm{loc}^b$ is analogous. Lastly, $\sigma_a\sigma_b\,\mathrm{loc}^{ab}$ has image spanned by $\sigma_a\sigma_b$ and the monomials $\sigma_a\sigma_b\bar\psi_x\psi_x$ for $x \in \{a,b\}$ and first order monomials with at most $(d-2)/2$ gradients. Again only the even monomials are compatible with $U(1)$ symmetry.

5.5 Definition of the renormalisation group map with observables

In this section the renormalisation group map $\Phi_{j+1} = \Phi_{j+1,N,m^2}$ is extended to include the observable components (as in Sect. 3, we omit the N and m^2-dependence when there is no risk of confusion). To this end, we now call the renormalisation group map from Sect. 3.8 the *bulk component* and denote it by $\Phi^\varnothing_{j+1}$, and $\Phi_{j+1} = (\Phi^\varnothing_{j+1}, \Phi^\star_{j+1})$ will now refer to the renormalisation group map extended to the algebra with observables. The map $\Phi^\star_{j+1}$ is the *observable component* of the renormalisation group map. This extension will be defined so that the bulk components of K_{j+1} and V_{j+1} only depend on the bulk components of K_j and V_j. In other words,

$$\pi_\varnothing\Phi_{j+1}(V_j, K_j) = \Phi^\varnothing_{j+1}(\pi_\varnothing V_j, \pi_\varnothing K_j). \tag{5.36}$$

On the other hand, the observable components $V^\star_{j+1}$ and $K^\star_{j+1}$ will depend on both the observable and the bulk components of (V_j, K_j). The observable component $\Phi^\star_{j+1}$ is upper-triangular in the sense that the a component $\pi_a\Phi^\star_{j+1}(V_j, K_j)$ of $\Phi^\star_{j+1}(V_j, K_j)$ only depends on $(V^\varnothing_j, K^\varnothing_j)$ and (V^a_j, K^a_j) but not on (V^b_j, K^b_j) or (V^{ab}_j, K^{ab}_j), and similarly for the b component. The ab component depends on all components from the previous scale. We will use an initial condition $V_0 \in \mathcal{V}$ and $K_0(X) = 1_{X=\varnothing}$.

We now give the precise definition of the observable component of the renormalisation group map $\Phi^\star_{j+1}\colon (V_j, K_j) \mapsto (u^\star_{j+1}, V^\star_{j+1}, K^\star_{j+1})$. For $j+1 < N$, given (V_j, K_j) and $B \in \mathcal{B}_j$, define $Q(B)$ and $J(B, X)$ as in (3.44)–(3.46) using the extended version of Loc from Sect. 5.4. If $j+1 = N$ set $Q = J = 0$. We let $Q^\star(B) = \pi_\star Q(B)$ and $J^\star(B, X) = \pi_\star J(B, X)$ denote the observable components. The new detail for the observable renormalisation group map is that, to define $V^\star_{j+1}$, we include the second order contribution from $V^\star_j$ in order to maintain better control on the renormalisation group flow. To this end, for $j+1 \leqslant N$ and $B, B' \in \mathcal{B}_j$, let

$$\begin{aligned} P^\star(B, B') &= \frac{1}{2}\mathbb{E}_{C_{j+1}}\Big[\theta(V^\star_j(B) - Q^\star(B)); \theta(V^\star_j(B') - Q^\star(B'))\Big], \\ P^\star(B) &= \sum_{B' \in \mathcal{B}_j} P^\star(B, B'). \end{aligned} \tag{5.37}$$

The following observations will be useful later. Since $V^\star(B), Q^\star(B) \in \mathcal{V}^\star(B)$, the sum over B' contains at most two non-zero terms, corresponding to the blocks containing a and b. Since the covariance matrix C_{j+1} has the finite range property (3.2), also $P^\star(B, B') = 0$ for $B \neq B'$ if $|a - b|_\infty \geqslant \frac{1}{2}L^{j+1}$. Finally, if a and b are not in the same block, then $P^\star(B, B) = 0$ since the source fields square to zero.

With these definitions in place, $u^\star_{j+1} + V^\star_{j+1}$ is defined in the same way as $u_{j+1} + V_{j+1}$ with the addition of the second order term $P^\star$, and $K^\star_{j+1}$ is then defined in the same way as K_{j+1}:

Definition 5.7 The map $(V_j, K_j) \mapsto (u^\star_{j+1}, V^\star_{j+1})$ is defined, for $B \in \mathcal{B}_j$, by

$$u^\star_{j+1}(B) + V^\star_{j+1}(B) = \mathbb{E}_{C_{j+1}}\Big[\theta(V^\star_j(B) - Q^\star(B))\Big] - P^\star(B) \tag{5.38}$$

where $u^\star_{j+1}$ consists of all monomials that do not contain factors of ψ or $\bar\psi$. Explicitly,

$$u^\star_{j+1} = \begin{cases} -\sigma_a \bar\sigma_b q_{j+1}, & \text{Case (1)}, \\ -\sigma_a \sigma_b q_{j+1} - \sigma_a \gamma_{a,j+1} - \sigma_b \gamma_{b,j+1}, & \text{Case (2)}. \end{cases} \tag{5.39}$$

The map $(V_j, K_j) \mapsto K^\varnothing_{j+1} + K^\star_{j+1}$ is defined by the same formula as in Definition 3.10 except that $V^\varnothing$ and $u^\varnothing$ are replaced by $V = V^\varnothing + V^\star$ and $u = u^\varnothing + u^\star$.

Propositions 3.11 and 3.12 also hold for this extended definition of the renormalisation group map. The proofs are the same as presented in Sect. 3.7.

5.6 Estimates for the renormalisation group map with observables

In this section, the O-notation refers to scale $j+1$ norms, i.e., for $F, G \in \mathcal{N}(\Lambda)$, we write $F = G + O(t)$ to denote that $\|F - G\|_{T_{j+1}(\ell_{j+1})} \leqslant O(t)$. We use $\|V_j\|_j$ and

$\|K_j\|_j$ defined in (3.64)–(3.65), with the understanding that the right-hand sides of the definitions are the norm (5.24) which accounts for source fields.

Theorem 5.8 *Under the assumptions of Theorem 3.13, if also $\|V_j^\star\|_j + \|K_j^\star\|_j \leqslant \varepsilon$ and $u_j^\star = 0$, then for $j+1 < N$ the observable components of the renormalisation group map $\Phi_{j+1}^\star$ satisfy*

$$u_{j+1}^\star(\Lambda) + V_{j+1}^\star(\Lambda) = \mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda)\Big] - \frac{1}{2}\mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda); \theta V_j^\star(\Lambda)\Big] + O(L^{2(d-2)}\|K_j^\star\|_j) \tag{5.40}$$

$$\|K_{j+1}^\star\|_{j+1} \leqslant O(L^{-(\frac{d-2}{2}\wedge 1)} + A^{-\eta})\|K_j^\star\|_j + O(A^\nu)(\|V_j^\varnothing\|_j + \|K_j\|_j)(\|V_j\|_j + \|K_j\|_j), \tag{5.41}$$

provided that $K_j^{ab}(X) = 0$ for $X \in \mathcal{S}_j$ if $j < j_{ab}$. Both $\eta = \eta(d)$ and $\nu = \nu(d)$ are positive geometric constants. For $j+1 = N$, $\Phi_N^\star$ is bounded.

The first estimate in the theorem expresses that the evolution of $u^\star + V^\star$ is given by second-order perturbation theory, plus a higher order remainder due to $K^\star$. The second estimate states that $K^\star$ is contracting (for L and A large), up to error terms at most as large as the bulk coupling constants $V^\varnothing$ and $K = K^\varnothing + K^\star$. The additional factor $\|V_j\|_j + \|K_j\|_j \geqslant \|V_j^\star\|_j$ will be small (but of order 1) while all other coordinates will be exponentially small in j. Indeed, as a consequence of the above theorem, Proposition 5.15 below states that if the bulk flow $(V^\varnothing, K^\varnothing)$ is as constructed in Sect. 3.9 then $V^\star$ remains bounded while $K^\star$ goes to 0 exponentially fast.

The proof of the theorem follows that of Theorem 3.13 closely, with improvements for the leading terms that allow for $V^\star$ to be tracked to second order. It is given in the remainder of this subsection. The reader may again wish to skip the details of this proof on a first reading and proceed to the application of these estimates Sect. 5.7.

5.6.1 Coupling constants

We first give a bound on $u_{j+1}^\star(\Lambda) + V_{j+1}^\star(\Lambda)$. By Proposition 5.6 (iii),

$$Q^\star(\Lambda) = \sum_{X \in \mathcal{S}_j} \mathrm{Loc}_X^\star K_j(X). \tag{5.42}$$

Since only small sets X that contain a or b contribute, Proposition 5.6 (i) implies

$$\|Q^\star(\Lambda)\|_{T_j(\ell_j)} \leqslant O(1)\|K_j^\star\|_j. \tag{5.43}$$

By algebraic manipulation, the product property, that $\mathbb{E}_{C_{j+1}}\theta$ is a contraction, (5.28), and (5.43),

$$P^\star(\Lambda) = \frac{1}{2}\mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda); \theta V_j^\star(\Lambda)\Big] + \mathbb{E}_{C_{j+1}}\Big[\theta Q_j^\star(\Lambda); \theta(V_j^\star(\Lambda) + \frac{1}{2}Q_j^\star(\Lambda))\Big]$$

$$= \frac{1}{2}\mathbb{E}_{C_{j+1}}\Big[\theta V_j^\star(\Lambda); \theta V_j^\star(\Lambda)\Big] + O(L^{4(d-2)}\|K_j^\star\|_j(\|V_j^\star\|_j + \|K_j^\star\|_j)). \tag{5.44}$$

Putting these pieces together establishes (5.40) as $L^{2(d-2)}(\|V_j\|_j + \|K_j\|_j) \leqslant 1$ if $\varepsilon = \varepsilon(L)$ is small enough. An immediate consequence is

$$\|u_{j+1}^\star(\Lambda)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|_j), \tag{5.45}$$

$$\|V_{j+1}^\star(\Lambda)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|_j). \tag{5.46}$$

The same bounds hold with Λ replaced by any $X \in \mathcal{P}_j$. These will be used in the following analysis.

5.6.2 Small sets

The most significant improvement in the analysis concerns small sets, which we now analyse to second order. To simplify notation, we write

$$\hat{V}_j^\star = V_j^\star - Q^\star, \qquad \tilde{V}_{j+1}^\star = u_{j+1}^\star + V_{j+1}^\star. \tag{5.47}$$

Lemma 5.9 *For any $B, B' \in \mathcal{B}_j$,*

$$P^\star(B, B') = \frac{1}{2}\mathbb{E}_{C_{j+1}}\Big[\theta\hat{V}_j^\star(B)\theta\hat{V}_j^\star(B')\Big] - \frac{1}{2}\tilde{V}_{j+1}^\star(B)\tilde{V}_{j+1}^\star(B'). \tag{5.48}$$

Proof Note that $P^\star(B, B') = \frac{1}{2}\mathbb{E}_{C_{j+1}}[\theta\hat{V}_j^\star(B); \theta\hat{V}_j^\star(B')]$. Since it is quadratic in $\hat{V}_j^\star \in \mathcal{V}^\star$, $P^\star(B, B')$ can only contain monomials with a factor of $\sigma_a\bar{\sigma}_b$ (Case (1)) or $\sigma_a\sigma_b$ (Case (2)) because $\sigma_a^2 = \sigma_b^2 = \bar{\sigma}_b^2 = 0$. Similarly, for any $W \in \mathcal{V}^\star$ and $B, B', B'' \in \mathcal{B}_j$, it follows that $P^\star(B, B')W(B'') = 0$. The claim follows as this implies that $(\mathbb{E}_{C_{j+1}}[\theta\hat{V}_j^\star(B)])(\mathbb{E}_{C_{j+1}}[\theta\hat{V}_j^\star(B')])$ is the same as

$$\Big(\mathbb{E}_{C_{j+1}}\Big[\theta\hat{V}_j^\star(B)\Big] - P^\star(B)\Big)\Big(\mathbb{E}_{C_{j+1}}\Big[\theta\hat{V}_j^\star(B')\Big] - P^\star(B')\Big) = \tilde{V}_{j+1}^\star(B)\tilde{V}_{j+1}^\star(B'). \tag{5.49}$$

□

The next lemmas are analogues of Lemmas 3.16–3.17 that apply to the observable components. We begin with the replacement for Lemma 3.17. For $B \in \mathcal{B}_j$, recall $\bar{B}$ denotes the scale $j+1$-block containing B.

Lemma 5.10 *Suppose that $\|V_j\|_j + \|K_j\|_j \leqslant 1$. Then for any $X \in \mathcal{P}_j$, denoting by $n \in \{0, 1, 2\}$ the number of $B \in \mathcal{B}_j(X)$ containing a or b,*

$$\Big\|\pi_\star\mathbb{E}_{C_{j+1}}\Big[(\delta I)^X\Big]\Big\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|V_j\|_j + L^{2(d-2)}\|K_j\|_j)^n(O(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j))^{|\mathcal{B}_j(X)|-n}. \tag{5.50}$$

For any $B \in \mathcal{B}_j$ such that $\bar{B}$ contains at most one of a and b,

$$\left\| \pi_\star \mathbb{E}_{C_{j+1}} \left[\delta I(B) \right] \right\|_{T_{j+1}(\ell_{j+1})} \leqslant O(L^{2(d-2)} \|K_j^\star\|_j) + O(\|V_j\|_j + L^{2(d-2)} \|K_j\|_j)(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j). \quad (5.51)$$

Moreover if $|a-b|_\infty \geqslant \frac{1}{2} L^{j+1}$ then for any $X \in \mathcal{P}_j$ with $|\mathcal{B}_j(X)| = 2$,

$$\left\| \pi_\star \mathbb{E}_{C_{j+1}} \left[(\delta I)^X \right] \right\|_{T_{j+1}(\ell_{j+1})} \leqslant O((\|V_j\|_j + L^{2(d-2)} \|K_j\|_j)(\|V_j^\varnothing\|_j + L^{2(d-2)} \|K_j\|_j)). \quad (5.52)$$

Proof Throughout the proof, we will use that for V representing either V_j or $u_{j+1} + V_{j+1}$ one has

$$\pi_\star e^{-V(B)} = \pi_\star (e^{-V^\varnothing(B) - V^\star(B)}) = -V^\star(B) + \frac{1}{2} V^\star(B)^2 + O(\|V^\star(B)\|_{T_{j+1}(\ell_{j+1})} \|V^\varnothing(B)\|_{T_{j+1}(\ell_{j+1})}), \quad (5.53)$$

where we recall that the O-notation refers to terms whose $T_{j+1}(\ell_{j+1})$-norms are bounded by the indicated numbers, up to multiplicative constants. For both of the choices for V, one has $\|V^\varnothing(B)\|_{T_{j+1}(\ell_{j+1})} \leqslant \|V_j^\varnothing\|_j + O(\|K_j^\varnothing\|_j) \leqslant O(1)$ by (3.71) and $\|V^\star(B)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|V_j^\star\|_j + L^{2(d-2)} \|K_j^\star\|_j)$ by using (5.45)–(5.46) (with B instead of Λ).

To show (5.50), for each $B \in \mathcal{B}_j$, write $\delta I(B) = \pi_\varnothing \delta I(B) + \pi_\star \delta I(B)$ and expand the product defining $(\delta I)^X$ using that there are n blocks B for which $\pi_\star \delta I(B) \neq 0$. The claim then follows since $\|\pi_\varnothing \delta I(B)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j)$ by Lemma 3.17 and $\|\pi_\star \delta I(B)\|_{T_{j+1}(\ell_{j+1})} = O(\|V_j^\star\|_j + L^{2(d-2)} \|K_j^\star\|_j)$ which follows from the previous paragraph (as the doubling map commutes with $\pi_\star$).

For the bound (5.51), using that B can contain only a or b by assumption and that source fields square to zero, one has $V^\star(B)^2 = 0$ for V either V_j or $u_{j+1} + V_{j+1}$. Thus (5.53) simplifies to

$$\pi_\star e^{-V(B)} = \pi_\star (e^{-V^\varnothing(B) - V^\star(B)}) = -V^\star(B) + O(\|V^\star(B)\|_{T_{j+1}(\ell_{j+1})} \|V^\varnothing(B)\|_{T_{j+1}(\ell_{j+1})}). \quad (5.54)$$

Observe that $P^\star(B) = 0$ since $\bar{B}$ contains only one of a and b, see the remark below (5.37). As a result, (5.38) and the above show that the term linear in $V_j^\star(B)$ in $\pi_\star \mathbb{E}_{C_{j+1}} \delta I(B)$ cancels in expectation. The claim (5.51) then follows from $\|\mathbb{E}_{C_{j+1}} \theta Q^\star(B)\|_{T_{j+1}(\ell_{j+1})} = O(L^{2(d-2)} \|K_j^\star\|_j)$ by (5.43) and (5.28), and bounding the quadratic terms using (3.71) and (5.45)–(5.46) as below (5.53).

For the final assertion (5.52), we first show that $\mathbb{E}_{C_{j+1}} (\pi_\star \delta I)^X = L^{4(d-2)} \times O(\|V_j^\varnothing\|_j + \|K_j\|_j)(\|V_j^\star\|_j + \|K_j^\star\|_j)$, where we emphasise that $\pi_\star$ is inside the product over X. To see this bound, let $X = B \cup B'$, and note that $V^\star(B)$ and $V^\star(B')$ are either 0 or polynomials in ψ_a, $\bar\psi_a$ and ψ_b, $\bar\psi_b$ respectively. Since by assumption $C_{j+1}(a,b) = 0$, $\mathbb{E}_{C_{j+1}}\theta V^\star(B)V^\star(B') = 0$. Hence a nonvanishing contribution to $\mathbb{E}_{C_{j+1}}(\pi_\star \delta I)^X$ involves at least one factor $V^\varnothing$ from the expansion of the δI by (5.53). The factor of $L^{4(d-2)}$ arises from applying (5.28). The estimate (5.52) now follows similarly to the previous cases:

$$\pi_\star \mathbb{E}_{C_{j+1}}(\delta I)^X = \pi_\star \mathbb{E}_{C_{j+1}}(\pi_\varnothing \delta I + \pi_\star \delta I)^X$$
$$= O((\|V_j\|_j + L^{2(d-2)}\|K_j\|_j)(\|V_j^\varnothing\|_j + L^{2(d-2)}\|K_j\|_j)) \quad (5.55)$$

as the cross terms with one factor $\pi_\star$ and one factor $\pi_\varnothing$ satisfy this bound as above. □

Next we replace Lemma 3.16. Unlike before we explicitly consider terms arising from two blocks, in order to obtain a cancellation up to a third order error in $V^\star$. Indeed, note that the right-hand side of (5.56) involves $\|V_j\|_j\|V_j^\varnothing\|_j \leqslant (\|V_j^\varnothing\|_j + \|V_j^\star\|_j)\|V_j^\varnothing\|_j$ but no term $\|V_j^\star\|_j^2$, and that $\|V_j^\varnothing\|_j$ is exponentially small in j along the flow while $\|V_j^\star\|_j$ is of order 1. The K-terms are higher order.

Lemma 5.11 *Suppose that* $\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j \leqslant \varepsilon$ *and* $\|V_j^\star\|_j + \|K_j^\star\|_j \leqslant \varepsilon$. *Then for* $B \in \mathcal{B}_j$,

$$\left\| \pi_\star \mathbb{E}_{C_{j+1}} \left[\delta I(B) + \frac{1}{2} \sum_{B' \neq B, B' \subset \bar B} \delta I(B)\delta I(B') + \theta Q(B) \right] \right\|_{T_{j+1}(\ell_{j+1})}$$
$$= O(L^{4d}(\|V_j\|_j + \|K_j\|_j)(\|V_j^\varnothing\|_j + \|K_j\|_j)). \quad (5.56)$$

Proof Recall $\tilde V_{j+1}^\star = u_{j+1}^\star + V_{j+1}^\star$. Using (5.38) to re-express $\mathbb{E}_{C_{j+1}}[\theta Q^\star(B)]$, the term inside the norm on the left-hand side of (5.56) equals

$$\pi_\star \mathbb{E}_{C_{j+1}} \left[\delta I(B) + \frac{1}{2} \sum_{B' \neq B, B' \subset \bar B} \delta I(B)\delta I(B') \right]$$
$$+ \mathbb{E}_{C_{j+1}}\left[\theta V_j^\star(B)\right] - \sum_{B'} P^\star(B,B') - \tilde V_{j+1}^\star(B). \quad (5.57)$$

We start with the one block terms $B' = B$ in (5.57). Using Lemma 5.9 to rewrite $P(B,B)$ and since $\delta I(B) = \theta e^{-V_j(B)} - e^{-(V_{j+1}+u_{j+1})(B)}$, these terms are

$$\pi_\star \mathbb{E}_{C_{j+1}}\theta \left[e^{-V_j(B)} - 1 + V_j^\star(B) - \frac{1}{2}\hat V_j^\star(B)^2 \right]$$
$$- \pi_\star \left[e^{-(V_{j+1}+u_{j+1})(B)} - 1 + \tilde V_{j+1}^\star(B) - \frac{1}{2}\tilde V_{j+1}^\star(B)^2 \right]. \quad (5.58)$$

To estimate these terms, first note that if $V = V_{j+1} + u_{j+1}$ then (5.40) and its consequences (5.45)–(5.46) imply $\|V^{\varnothing}\|_{T_{j+1}(\ell_{j+1})} \leqslant 1$, $\|V^{\star}\|_{T_{j+1}(\ell_{j+1})} \leqslant 1$. This bound also holds for $V = V_j$ provided ε is sufficiently small by (5.28), we then have for $V = V_j$ or $V = u_{j+1} + V_{j+1}$,

$$\begin{aligned} \pi_\star e^{-V(B)} &= \pi_\star (e^{-V^\star(B)} + (e^{-V^{\varnothing}(B)} - 1) e^{-V^\star(B)}) \\ &= -V^\star(B) + \frac{1}{2} V^\star(B)^2 \\ &\quad + O((\|V_j^\star\|_j + L^{2(d-2)} \|K_j^\star\|)(\|V_j^{\varnothing}\|_j + \|K_j^{\varnothing}\|_j)), \end{aligned} \tag{5.59}$$

where we have used $V^\star(B)^3 = 0$, and in the case $V = u_{j+1} + V_{j+1}$, (3.71) to control $\|u_{j+1}^{\varnothing} + V_{j+1}^{\varnothing}\|_{j+1}$ in terms of $\|V_j^{\varnothing}\|_j + \|K_j^{\varnothing}\|_j$ and (5.46) to control $\|u_{j+1}^\star + V_{j+1}^\star\|_{j+1}$ similarly. Using also

$$\begin{aligned} \hat{V}_j^\star(B)^2 &= (V_j^\star(B) - Q(B))^2 \\ &= V_j^\star(B)^2 + O(L^{4(d-2)} \|K_j^\star\|_j (\|V_j^\star\|_j + \|K_j^\star\|_j)), \end{aligned} \tag{5.60}$$

by the product property, (5.43), (5.28), and the assumed norm bounds, the estimate for the one block terms follow.

Recall that $P^\star(B, B') = 0$ unless a, b are each in one of the two blocks. Thus for $B' \neq B$ the two block terms in (5.57) are, by Lemma 5.9,

$$\frac{1}{2} \pi_\star \left(\mathbb{E}_{C_{j+1}} \Big[\delta I(B) \delta I(B') \Big] - \mathbb{E}_{C_{j+1}} \Big[\theta \hat{V}_j^\star(B) \theta \hat{V}_j^\star(B') \Big] + \tilde{V}_{j+1}^\star(B) \tilde{V}_{j+1}^\star(B') \right). \tag{5.61}$$

We start by rewriting this in a more convenient form. Let $\delta V_j^\star = \theta \hat{V}_j^\star - \tilde{V}_{j+1}^\star$. By (5.38), $\mathbb{E}_{C_{j+1}} \theta \hat{V}_j^\star = \tilde{V}_{j+1}^\star + P^\star = \tilde{V}_{j+1}^\star + O(\sigma_a \sigma_b)$, where $O(\sigma_a \sigma_b)$ denotes a monomial containing a factor $\sigma_a \bar{\sigma}_b$ in Case (1) or a factor $\sigma_a \sigma_b$ in Case (2). Since all terms in $\delta V_j^\star$ contain a source field (that is, a σ-factor) and source fields square to zero, we obtain

$$\begin{aligned} &\mathbb{E}_{C_{j+1}} \Big[\delta V_j^\star(B) \delta V_j^\star(B') \Big] \\ &\quad = \mathbb{E}_{C_{j+1}} \Big[\theta \hat{V}_j^\star(B) \theta \hat{V}_j^\star(B') \Big] + \tilde{V}_{j+1}^\star(B) \tilde{V}_{j+1}^\star(B') \\ &\qquad - \tilde{V}_{j+1}^\star(B) \mathbb{E}_{C_{j+1}} \Big[\theta \hat{V}_j^\star(B') \Big] - \tilde{V}_{j+1}^\star(B') \mathbb{E}_{C_{j+1}} \Big[\theta \hat{V}_j^\star(B) \Big] \\ &\quad = \mathbb{E}_{C_{j+1}} \Big[\theta \hat{V}_j^\star(B) \theta \hat{V}_j^\star(B') \Big] - \tilde{V}_{j+1}^\star(B) \tilde{V}_{j+1}^\star(B'). \end{aligned} \tag{5.62}$$

Therefore we need to estimate

$$\frac{1}{2} \pi_\star \mathbb{E}_{C_{j+1}} \Big[\delta I(B) \delta I(B') \Big] - \frac{1}{2} \mathbb{E}_{C_{j+1}} \Big[\delta V_j^\star(B) \delta V_j^\star(B') \Big]. \tag{5.63}$$

First write

$$\pi_\star [\delta I(B) \delta I(B')] = \pi_\star \delta I(B) \pi_\star \delta I(B') + \pi_\star \delta I(B) \pi_{\varnothing} \delta I(B')$$

$$+ \pi_\varnothing \delta I(B) \pi_\star \delta I(B'). \tag{5.64}$$

The second and third terms on the right-hand side are $O((\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|_j) \times (\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j))$ using Lemma 3.17 for $\pi_\varnothing \delta I$ and $\|\pi_\star \delta I(B)\|_{T_{j+1}(\ell_{j+1})} = O(\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|_j)$ by (5.46). Using (5.59), the term $\pi_\star \delta I(B) \pi_\star \delta I(B')$ can be estimated as

$$\begin{aligned} &\pi_\star(\delta V_j(B) - \frac{1}{2}(\theta V_j(B)^2 - \tilde{V}_{j+1}(B)^2))\pi_\star(\delta V_j(B') - \frac{1}{2}(\theta V_j(B')^2 - \tilde{V}_{j+1}(B')^2)) \\ &\qquad + O((\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|)(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j)) \\ &= \delta V_j^\star(B) \delta V_j^\star(B') + O((\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|)(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j)), \end{aligned} \tag{5.65}$$

since $\sigma_a^2 = \sigma_b^2 = \bar{\sigma}_b^2 = 0$. The factor L^{4d} is a convenient common bound. □

The next lemma replaces Lemma 3.15 on the observable components.

Lemma 5.12 *For any $U \in \mathcal{C}_{j+1}$, if $K_j^{ab}(Y) = 0$ for all $Y \in \mathcal{S}_j$ and all $j < j_{ab}$, then*

$$\sum_{X \in \mathcal{S}_j : \bar{X} = U} \left\| \mathbb{E}_{C_{j+1}} \left[\theta(1 - \mathrm{Loc}_X^\star) K_j^\star(X) \right] \right\|_{T_{j+1}(\ell_{j+1})} = O(L^{-(\frac{d-2}{2} \wedge 1)}) \|K^\star\|_j. \tag{5.66}$$

Proof The proof is the same as that of Lemma 3.15 except for the following observation. The sum over $X \in \mathcal{S}_j$ that contributes a factor $O(L^d)$ in the proof of Lemma 3.15 only contributes $O(1)$ on the observable components because for these only the small sets containing a or b contribute. Thus the bound for $\mathrm{Loc}^\star$ from Proposition 5.6, which lacks a factor L^{-d} compared to the bound for $\mathrm{Loc}^\varnothing$, produces the same final bound. □

Proof of Theorem 5.8 The proof is analogous to that of Theorem 3.13, and we proceed in a similar manner, by beginning with the coupling constants and then an estimate of $\pi_\star \mathcal{L}_{j+1}(U)$, where $\mathcal{L}_{j+1}(U)$ is defined by the formula (3.72) but with the extended coordinates introduced in Sect. 5.5.

For the coupling constants, i.e., the analogue of Sect. 3.8.1, the bound (3.69) gets replaced by (5.43) which gives $\|Q^\star(B)\|_{T_{j+1}(\ell_{j+1})} \leqslant O(L^{d-2}\|K_j^\star\|)$, and we also have $\|u_{j+1}^\star\|_{j+1} + \|V_{j+1}^\star\|_{j+1} \leqslant O(\|V_j^\star\| + L^{d-2}\|K_j^\star\|_j)$ by (5.45)–(5.46).

Note that the terms of $\mathcal{L}_{j+1}(U)$ are of the form $\sum_F e^{-V_{j+1}(U \setminus X) + u_j|X|} F$. We will use that

$$\begin{aligned} &\pi_\star(e^{-V_{j+1}(U \setminus X) + u_j|X|} F) \\ &\quad = (\pi_\star e^{-V_{j+1}(U \setminus X) + u_j|X|}) \pi_\varnothing F + \pi_\star(e^{-V_{j+1}(U \setminus X) + u_j|X|} \pi_\star F). \end{aligned} \tag{5.67}$$

We first explain how to estimate the sum arising from the first term, which only contributes to the second term in the estimate. The estimation of the terms $\pi_\varnothing F$ is exactly as in Sect. 3.8. Estimating $\pi_\star e^{-V_{j+1}(U \setminus X) + u_j|X|}$, and the resultant sum over F, requires a replacement of Lemma 3.18. For this it suffices to note that $\|\pi_\star e^{-V_{j+1}(U\setminus X)+u_{j+1}|X|}\|_{T_{j+1}(\ell_{j+1})} \leqslant (\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|_j)2^{|\mathcal{B}_j(X)|}$. This estimate follows by the product property, (5.45)–(5.46), and arguing as in the proof of Lemma 3.18. Hence this term is bounded by $O(\|V_j^\star\|_j + L^{2(d-2)}\|K_j^\star\|_j)(\|K_j^\varnothing\|_j + A^\nu(\|K_j^\varnothing\|_j^2 + \|V_j^\varnothing\|_j^2))$.

Next we explain how to estimate $e^{-V_{j+1}(U\setminus X)+u_j|X|}\pi_\star F$. The prefactor is at most $2^{|\mathcal{B}_j(X)|}$, i.e., the analogue of Lemma 3.18 applies when $V^\varnothing$, $u^\varnothing$ and $K^\varnothing$ are replaced by V, u and K if ε is small enough, and it suffices to estimate $\pi_\star F$.

Consider the small set contributions to $\mathcal{L}_{j+1}(U)$, i.e., the analogue of Sect. 3.8.3. As stated previously, Lemma 3.15 is replaced with Lemma 5.12 whereas Lemmas 5.11 and 5.10 replace Lemmas 3.16 and 3.17. In detail, in the analogue of (3.78) we now also include quadratic terms in δI, i.e., we replace (3.78) by

$$\begin{aligned}
&\pi_\star \mathbb{E}_{C_{j+1}}\left[\theta K_j(B) + \delta I(B) + \frac{1}{2}\sum_{B'\neq B, B'\subset \bar{B}} \delta I(B)\delta I(B') - \theta J(B,B)\right] \\
&\quad = \pi_\star \mathbb{E}_{C_{j+1}}\Big[\theta(1-\mathrm{Loc}_B)K_j(B)\Big] \\
&\qquad + \pi_\star \mathbb{E}_{C_{j+1}}\left[\delta I(B) + \frac{1}{2}\sum_{B'\neq B, B'\subset \bar{B}} \delta I(B)\delta I(B') + \theta Q(B)\right], \qquad (5.68)
\end{aligned}$$

with the corresponding analogue of (3.77) then being (for $X \in \mathcal{S}_j \setminus \mathcal{B}_j$)

$$\begin{aligned}
&\pi_\star \mathbb{E}_{C_{j+1}}\Big[\theta(1-\mathrm{Loc}_X)K_j(X)\Big] \\
&\qquad + \pi_\star \mathbb{E}_{C_{j+1}}\left[(\delta I)^X - \frac{1}{2}\sum_{B'\neq B, B'\subset \bar{B}} \delta I(B)\delta I(B')\mathbf{1}_{B\cup B'=X}\right]. \qquad (5.69)
\end{aligned}$$

Let us note that since $B \cup B'$ is not necessarily connected (so in that case not a small set), along with the third term in (5.69), there is a corresponding correction for polymers in the large set sum (3.95): the terms inside the sum are replaced by $e^{-V_{j+1}(U\setminus X)+u_{j+1}|X|}$ multiplied by

$$\begin{aligned}
&\pi_\star \mathbb{E}_{C_{j+1}}\Big[\theta K_j(X)\Big]\mathbf{1}_{X\in\mathcal{C}_j\setminus\mathcal{S}_j} \\
&\qquad + \pi_\star \mathbb{E}_{C_{j+1}}\left[(\delta I)^X - \frac{1}{2}\sum_{B'\neq B, B'\subset \bar{B}} \delta I(B)\delta I(B')\mathbf{1}_{B\cup B'=X}\right]\mathbf{1}_{X\in\mathcal{P}_j\setminus\mathcal{S}_j}. \qquad (5.70)
\end{aligned}$$

Now Lemma 5.12 bounds the sum over X of the $(1-\mathrm{Loc}_X)$ terms in (5.68) and (5.69). Lemma 5.11 bounds the second term on the right-hand side of (5.68). Finally, (5.50) of Lemma 5.10 bounds the second term in (5.69) by $O(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j)(\|V_j\|_j + L^{2(d-2)}\|K_j\|_j)$, after making use of the cancellation between

$(\delta I)^X$ and $\delta I(B)\delta I(B')$ when $X = B \cup B'$ and $B' \subset \bar{B}$. Indeed, note that for all other X at least one $B \in \mathcal{B}_j(X)$ does not contain a or b. Putting these bounds together (as in the proof of Theorem 3.13) then gives that the small set contribution to $\pi_\star \mathcal{L}_{j+1}(U)$ is $O(L^{-(\frac{d-2}{2}\wedge 1)}\|K_j^\star\|_j) + O(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j)(\|V_j\|_j + L^{2(d-2)}\|K_j\|_j)$.

To bound the large set term (5.70) and the non-linear contributions, we will use the principle that for $F_i \in \mathcal{V}$,

$$\pi_\star \prod_{i=1}^{k} F_i = \sum_i F_i^\star \prod_{l\neq i} F_l^\varnothing + \sum_{i\neq k} F_i^\star F_k^\star \prod_{l\neq i,k} F_l^\varnothing \tag{5.71}$$

as the product of any three elements of $\mathcal{V}^\star$ is zero. The bound on the sum over

$$\pi_\star \mathbb{E}_{C_{j+1}}\Big[\theta K_j(X)\Big]\mathbf{1}_{X\in\mathcal{C}_j\setminus\mathcal{S}_j} \tag{5.72}$$

proceeds exactly as in Sect. 3.8.4, bearing in mind (5.71) and (5.28). The resulting estimate is $O(A^{-\eta}\|K_j^\star\|_j)$. For the second term in (5.70), observe that if $|\mathcal{B}_j(X)| = 2$ and $\bar{X} \in \mathcal{B}_{j+1}$, the bound is identical to that of the same term in (5.69) above. The remaining possibilities are that either $|\mathcal{B}_j(X)| \geqslant 3$ or $|\mathcal{B}_j(X)| = 2$ but with constituent j-blocks which are in distinct $(j+1)$-blocks. In the former case, by applying (5.71), (5.50) of Lemma 5.10 and Lemma 3.17 and then proceeding as in Sect. 3.8.4, we obtain

$$\begin{aligned} &A^{(|\mathcal{B}_{j+1}(U)|-2^d)_+} \\ &\quad\times \Bigg\| \pi_\star \sum_{X\in\mathcal{P}_j\setminus\mathcal{S}_j : \bar{X}=U, |\mathcal{B}_j(X)|\geqslant 3} e^{-V_{j+1}(U\setminus X)+u_{j+1}|X|}\mathbb{E}_{C_{j+1}}\Big[(\delta I)^X\Big]\Bigg\|_{T_{j+1}(\ell_{j+1})} \\ &\leqslant O(\|V_j\|_j + \|K_j\|_j)(\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j). \end{aligned} \tag{5.73}$$

The remaining case is $|\mathcal{B}_{j+1}(U)| = 2$ and $|\mathcal{B}_j(X)| = 2$ where $U = \bar{X}$. Then the $\delta I(B)\delta I(B')\mathbf{1}_{B\cup B'=X}$ cancellation is absent, but $\pi_\star \mathbb{E}_{C_{j+1}}(\delta I)^X$ itself satisfies the desired bound by Lemma 5.10. Indeed, either a and b are in the same $(j+1)$ block or they are not. If they are, we use (5.50) with $n = 1$, and if not, this follows from (5.52) since a and b being in distinct $(j+1)$-blocks of U implies that $|a-b|_\infty \geqslant \frac{1}{2}L^{j+1}$ since a is positioned in the center of all of its blocks. The bound

$$\begin{aligned} &\Bigg\| \pi_\star \sum_{X\in\mathcal{P}_j\setminus\mathcal{S}_j : \bar{X}=U, |\mathcal{B}_j(X)|=2} e^{-V_{j+1}(U\setminus X)+u_{j+1}|X|}\mathbb{E}_{C_{j+1}}\Big[(\delta I)^X\Big]\Bigg\|_{T_{j+1}(\ell_{j+1})} \\ &\qquad\leqslant O(L^{6d}(\|V_j\|_j + \|K_j\|_j)(\|V_j^\varnothing\|_j + \|K_j\|_j)) \end{aligned} \tag{5.74}$$

follows as there are at most L^{2d} summands.

All together, after possibly increasing A, we obtain that the large set contribution to $\mathcal{L}_{j+1}(U)$ is

$$O(A^{-\eta}\|K_j^\star\|_j) + O(A^\nu(\|V_j^\star\|_j + \|K_j^\star\|_j)(\|V_j^\varnothing\|_j + \|K_j\|_j)). \tag{5.75}$$

The non-linear contribution does not require any changes as the bound from Sect. 3.8.5 already gives (after possibly increasing A) $A^\nu O(\|K_j\|_j(\|V_j\|_j + \|K_j\|_j))$. □

5.7 Flow of observable coupling constants

With Theorem 5.8 in place, the evolution of the observable coupling constants in $u^\star + V^\star$ is the same as the free one from Sect. 5.2 up to the addition of remainder terms from the K coordinate. To avoid carrying an unimportant factor of $L^{2(d-2)}$ through equations, recall that we write $O_L(\cdot)$ to indicate bounds with constants possibly depending on L (but we reemphasise that implicit constants are always independent of the scale j).

Lemma 5.13 *Suppose $j < N$, $x \in \{a, b\}$, and that* (5.40) *holds. If $j < j_{ab}$, further suppose that $K_j^{ab}(X) = 0$ for $X \in \mathcal{S}_j$. In Case* (1),

$$\lambda_{x,j+1} = \lambda_{x,j} + O_L(\ell_{x,j}^{-1}\ell_j^{-1}\|K_j^x\|_j), \tag{5.76}$$

$$q_{j+1} = q_j + \lambda_{a,j}\lambda_{b,j}C_{j+1}(a,b) + r_j C_{j+1}(0,0) + O_L(\ell_{ab,j}^{-1}\|K_j^{ab}\|_j 1_{j \geqslant j_{ab}}), \tag{5.77}$$

$$r_{j+1} = r_j + O_L(\ell_{ab,j}^{-1}\ell_j^{-2}\|K_j^{ab}\|_j 1_{j \geqslant j_{ab}}), \tag{5.78}$$

and in Case (2),

$$\lambda_{x,j+1} = \lambda_{x,j} + O_L(\ell_{x,j}^{-1}\ell_j^{-2}\|K_j^x\|_j), \tag{5.79}$$

$$\gamma_{x,j+1} = \gamma_{x,j} + \lambda_{x,j}C_{j+1}(x,x) + O_L(\ell_{x,j}^{-1}\|K_j^x\|_j), \tag{5.80}$$

$$\begin{aligned} q_{j+1} = q_j &+ \eta_j C_{j+1}(a,b) - \lambda_{a,j}\lambda_{b,j}C_{j+1}(a,b)^2 + r_j C_{j+1}(0,0) \\ &+ O_L(\ell_{ab,j}^{-1}\|K_j^{ab}\|_j 1_{j \geqslant j_{ab}}), \end{aligned} \tag{5.81}$$

$$\eta_{j+1} = \eta_j - 2\lambda_{a,j}\lambda_{b,j}C_{j+1}(a,b), \tag{5.82}$$

$$r_{j+1} = r_j + O_L(\ell_{ab,j}^{-1}\ell_j^{-2}\|K_j^{ab}\|_j 1_{j \geqslant j_{ab}}). \tag{5.83}$$

Moreover, for $j + 1 < N$, all coupling constants are independent of N.

Note that there is no error term in the equation for η, as the corresponding nonlocal field monomials $\bar\psi_a\psi_b$ and $\bar\psi_b\psi_a$ are not contained in the image of Loc.

Proof For $j < N$, the main contribution in (5.40) is identical to that in Lemma 5.3. The indicator functions $1_{j \geqslant j_{ab}}$ in the error terms are due to the assumption $K_j^{ab}(X) =$

0 for $j < j_{ab}$ and $X \in \mathcal{S}_j$. The bounds for the error terms follow from the definition of the norms as in obtaining (4.10). Finally, that the couplings are independent of N is a consequence of the consistency of the renormalisation group map, i.e., Proposition 3.12 (applied to the renormalisation group map extended by observables). □

The next lemma shows that if we maintain control of $\|K_k^\star\|_k$ up to scale j then we control the coupling constants in $V^\star$ on scale j.

Lemma 5.14 *Assume that $\|K_k^\star\|_k \leqslant M\lambda_0 b_0 L^{-\kappa k}$ for $k < j$ and that (5.40) holds for $k < j$. Then, in Case (1) if $q_0 = r_0 = 0$ and $\lambda_0 > 0$,*

$$\lambda_j = \lambda_0 + O_L(M\lambda_0 b_0) \tag{5.84}$$

$$r_j = O_L(M\lambda_0 b_0 |a-b|^{-\kappa}) 1_{j \geqslant j_{ab}} \tag{5.85}$$

and, in Case (2), if $q_0 = r_0 = \gamma_{x,0} = \eta_0 = 0$ and $\lambda_0 > 0$,

$$\lambda_j = \lambda_0 + O_L(M\lambda_0 b_0) \tag{5.86}$$

$$\eta_j = O_L(\lambda_0^2 |a-b|^{-(d-2)}) 1_{j \geqslant j_{ab}} \tag{5.87}$$

$$r_j = O_L(M\lambda_0 b_0 |a-b|^{-(d-2)-\kappa}) 1_{j \geqslant j_{ab}}, \tag{5.88}$$

where $\lambda_j = \lambda_{x,j}$ for either $x = a$ or $x = b$. In both Cases (1) and (2),

$$\|V_j^\star\|_j \leqslant \lambda_0 + O_L(\lambda_0^2) + O_L(M\lambda_0 b_0). \tag{5.89}$$

Proof The bounds on the coupling constants in (5.84)–(5.88) follow from Lemma 5.13; the hypothesis regarding $K_j(X) = 0$ for $j < j_{ab}$ and $X \in \mathcal{S}_j$ holds as Definition 5.7 implies that for an iteration (V_j, K_j) of the renormalisation group map, the $\mathcal{N}^{ab}$ components of $V_j(B)$ and $K_j(X)$ with $X \in \mathcal{S}_j$ can only be nonzero for $j > j_{ab}$ since we have started the flow with $r_0 = 0$ in Case (1), and $q_0 = \eta_0 = r_0 = 0$ in Case (2). What remains is to analyse the recurrences to establish (5.89).

For $\lambda_{x,j}$, since $\ell_{x,j}^{-1}\ell_j^{-p} = 1$ in Case (p), using (5.76), respectively (5.79),

$$\lambda_{x,j} = \lambda_0 + \sum_{k=0}^{j-1} O_L(\|K_k^\star\|_k) = \lambda_0 + \sum_{k=0}^{j-1} O_L(M\lambda_0 b_0 L^{-\kappa k}) = \lambda_0 + O_L(M\lambda_0 b_0). \tag{5.90}$$

The bounds on r_j follow from the fact that all contributions are 0 for scales $j < j_{ab}$ if $r_0 = 0$. For example, in Case (2), using (5.83),

$$\begin{aligned} |r_j| &= \lambda_0 b_0 O_L(M \sum_{k=j_{ab}}^{j-1} \ell_{ab,j}^{-1} \ell_j^{-2} L^{-\kappa j}) \\ &= \lambda_0 b_0 \ell_{j_{ab}}^2 O_L(M \sum_{k=j_{ab}}^{j-1} L^{-\kappa j}) = O_L(M\lambda_0 b_0 |a-b|^{-(d-2)-\kappa}). \end{aligned} \tag{5.91}$$

Case (1) is similar, except no factor $\ell_{j_{ab}}$ arises (see (5.25)). The bound on η_j in Case (2) follows from the preceding analysis of $\lambda_{x,j}$, the fact that $\eta_j = 0$ for $j < j_{ab}$ if $\eta_0 = 0$ since C_j has finite range ($C_j(a,b) = 0$ if $|a-b|_\infty \geqslant \frac{1}{2}L^j$), and that $C_{j+1}(a,b) \leqslant O_L(L^{-(d-2)j})$:

$$|\eta_j| = O_L(\lambda_0^2 \sum_{k=j_{ab}}^{j-1} L^{-(d-2)k}) = O_L(\lambda_0^2 |a-b|^{-(d-2)}). \tag{5.92}$$

For the bound on the norm of $\|V_j^\star\|_j$ recall that the q and γ terms have been taken out of $V^\star$. Thus in Case (1),

$$\|V_j^\star(B)\| \lesssim |\lambda_j| + |r_j|\ell_{ab,j}\ell_j^2 \lesssim |\lambda_j| + |r_j| = |\lambda_j| + O_L(M\lambda_0 b_0). \tag{5.93}$$

Similarly, in Case (2), using that $\ell_j^2 \ell_{ab,j} = \ell_{j\wedge j_{ab}}^{-2} = O_L(|a-b|^{d-2})$ for $j \geqslant j_{ab}$,

$$\begin{aligned}
\|V_j^\star(B)\| &\lesssim |\lambda_j| + |\eta_j|\ell_j^2\ell_{ab,j} 1_{j\geqslant j_{ab}} + |r_j|\ell_{ab,j}\ell_j^2 1_{j\geqslant j_{ab}} \\
&\lesssim |\lambda_j| + |\eta_j||a-b|^{d-2} 1_{j\geqslant j_{ab}} + |r_j||a-b|^{d-2} 1_{j\geqslant j_{ab}} \\
&\lesssim |\lambda_j| + O_L(\lambda_0^2) 1_{j\geqslant j_{ab}} + O_L(M\lambda_0 b_0 |a-b|^{-\kappa}) 1_{j\geqslant j_{ab}} \\
&= |\lambda_j| + O_L(\lambda_0^2) + O_L(M b_0 \lambda_0). \qquad \square
\end{aligned}$$

We now analyse the observable flow from initial conditions which extend the bulk initial conditions of Theorem 3.24, and verify the assumption on $K^\star$ made in Lemma 5.14 along this flow. As already remarked at the beginning of Sect. 5.5, the renormalisation group maps with observables are upper triangular, so that the observable components of the maps do not affect the bulk flow. Thus from now on, the bulk components $(V_j^\varnothing, K_j^\varnothing)_{j\leqslant N}$ are identified with the trajectory given by Theorem 3.24 and we may use the decay rates from that theorem as inputs in obtaining estimates on the remaining components. In particular, there is an $\alpha > 0$ such that

$$\|V_j^\varnothing\|_j = O_L(b_0 L^{-\alpha j}), \qquad \|K_j^\varnothing\|_j = O_L(b_0 L^{-\alpha j}). \tag{5.94}$$

Using this as input, we iterate the observable flow (5.40)–(5.41), with initial condition $\lambda_{a,0} = \lambda_{b,0} = \lambda_0$ small enough and all other observable coupling constants equal to 0.

Proposition 5.15 *Assume that the bulk renormalisation group flow* $(V_j^\varnothing, K_j^\varnothing)$ *obeys*

$$\|V_j^\varnothing\|_j + \|K_j^\varnothing\|_j = O_L(b_0 L^{-\alpha j}) \tag{5.95}$$

for some $\alpha > 0$. *Then there is* $\kappa > 0$ *such that for* $\lambda_{a,0} = \lambda_{b,0} = \lambda_0 > 0$ *and* $b_0 > 0$ *sufficiently small and all other observable coupling constants initially* 0,

$$\|V_j^\star\|_j \leqslant O_L(\lambda_0), \qquad \|K_j^\star\|_j \leqslant O_L(\lambda_0 b_0 L^{-\kappa j}). \tag{5.96}$$

Proof We may assume $\lambda_0 < 1$, and that κ is less than α and the exponents of L and A in (5.41). The proof is by induction. The inductive assumption is that $\|K_k^\star\|_k \leqslant Mb_0\lambda_0 L^{-\kappa k}$ for all $k \leqslant j$, for some $M = M(L)$ chosen large enough below. Clearly, this holds for $j = 0$. Lemma 5.14 then shows that $\|V_k^\star\|_k \leqslant O_L(1)\lambda_0 + O_L(1)M\lambda_0 b_0$ for all $k \leqslant j+1$. We now apply Theorem 5.8 to control $K_{j+1}^\star$. Since A is chosen as function of L, the second term on the right-hand side of (5.41) is

$$\begin{aligned} &O(A^\nu)(\|V_j^\varnothing\|_j + \|K_j\|_j)(\|V_j\|_j + \|K_j\|_j) \\ &\quad \leqslant O_L(1)(b_0 L^{-\alpha j} + \|K_j^\star\|_j)(b_0 L^{-\alpha j} + \lambda_0 + M\lambda_0 b_0) \\ &\quad \leqslant O_L(1) b_0\lambda_0 L^{-\alpha j} + \frac{1}{4} L^{-\kappa}\|K_j^\star\|_j, \end{aligned} \tag{5.97}$$

as long as $b_0 \leqslant \lambda_0$ and $Mb_0 + \lambda_0$ is sufficiently small (depending on L). As $A > L$, and using our second assumption on κ, we obtain from (5.41) that

$$\begin{aligned} \|K_{j+1}^\star\|_{j+1} &\leqslant \frac{1}{2} L^{-\kappa}\|K_j^\star\|_j + O_L(1) b_0\lambda_0 L^{-\alpha j} \\ &\leqslant (\frac{1}{2}M + O_L(1)) b_0\lambda_0 L^{-\kappa(j+1)} \leqslant M b_0\lambda_0 L^{-\kappa(j+1)} \end{aligned} \tag{5.98}$$

provided that M is sufficiently large and b_0 is sufficiently small (depending only on L). Hence if λ_0 and b_0 are small enough we have advanced the induction, completing the proof. □

6 Computation of pointwise correlation functions

In this section we use the results of Sect. 5 to prove the following estimates for the pointwise correlation functions $\langle\bar\psi_a\psi_b\rangle$, $\langle\bar\psi_a\psi_a\rangle$, and $\langle\bar\psi_a\psi_a\bar\psi_b\psi_b\rangle$. Recall (2.27) and (2.25), i.e.,

$$W_N(x) = W_{N,m^2}(x) = (-\Delta + m^2)^{-1}(0,x) - \frac{t_N}{|\Lambda_N|}, \qquad 0 < t_N = \frac{1}{m^2} - O(L^{2N}). \tag{6.1}$$

Thus $W_N(x)$ is essentially the torus Green function $(-\Delta + m^2)^{-1}(0,x)$ with the zero mode subtracted and t_N is essentially m^{-2} when $L^{2N} \ll m^{-2}$ and negligible otherwise.

Proposition 6.1 *For b_0 sufficiently small and $m^2 \geqslant 0$, there exists continuous functions*

$$\lambda = \lambda(b_0, m^2) = 1 + O_L(b_0), \qquad \gamma = \gamma(b_0, m^2) = (-\Delta^{\mathbb{Z}^d} + m^2)^{-1}(0,0) + O_L(b_0), \tag{6.2}$$

such that if $V_0^{\varnothing} = V_0^c(m^2, b_0)$ *is as in Theorem* 3.24, $V_0^\star$ *is as in Proposition* 5.15, *and* $\tilde{u}^c_{N,N} = \tilde{u}^c_{N,N}(b_0, m^2)$ *is as in Proposition* 4.1, *then*

$$\langle \bar\psi_a \psi_a \rangle = \gamma + \frac{\lambda t_N |\Lambda_N|^{-1} + O_L(b_0 L^{-(d-2+\kappa)N}) + O_L(b_0 L^{-\kappa N}(m^2|\Lambda_N|)^{-1})}{1+\tilde{u}_{N,N}}. \tag{6.3}$$

Proposition 6.2 *Under the same assumptions as in Proposition* 6.1,

$$\begin{aligned}\langle \bar\psi_a \psi_b \rangle = W_N(a-b) + \frac{t_N|\Lambda_N|^{-1}}{1+\tilde{u}_{N,N}} \\ + O_L(b_0|a-b|^{-(d-2+\kappa)}) + \frac{O_L(b_0|a-b|^{-\kappa}(m^2|\Lambda_N|)^{-1})}{1+\tilde{u}_{N,N}}\end{aligned} \tag{6.4}$$

$$\begin{aligned}\langle \bar\psi_a \psi_a \bar\psi_b \psi_b \rangle = -\lambda^2 W_N(a-b)^2 + \gamma^2 + \frac{-2\lambda^2 W_N(a-b) + 2\lambda\gamma}{1+\tilde{u}_{N,N}} t_N |\Lambda_N|^{-1} \\ + O_L(b_0|a-b|^{-2(d-2)-\kappa}) + O_L(b_0 L^{-(d-2+\kappa)N}) \\ + (O_L(b_0 L^{-\kappa N}) + O_L(b_0|a-b|^{-(d-2+\kappa)}))\frac{(m^2|\Lambda_N|)^{-1}}{1+\tilde{u}_{N,N}}.\end{aligned} \tag{6.5}$$

Throughout this section, we assume that the renormalisation group flow $(V_j, K_j)_{j\leqslant N}$ is given as in Corollary 3.26 (bulk) and Proposition 5.15 (observables).

6.1 Integration of the zero mode

As in the analysis of the susceptibility in Sect. 4, we treat the final integration over the zero mode explicitly. Again we will only require the restriction to constant ψ, $\bar\psi$ (as discussed below (4.3)) of

$$\mathbb{E}_C \theta Z_0 = \mathbb{E}_{t_N Q_N} \theta Z_N = e^{-u_N^{\varnothing}|\Lambda_N|} \tilde{Z}_{N,N}, \tag{6.6}$$

where the last equation defines $\tilde{Z}_{N,N}$. We write $\tilde{Z}_{N,N} = \tilde{Z}^{\varnothing}_{N,N} + \tilde{Z}^{\star}_{N,N}$ for its decomposition into bulk and observable parts (see (5.23)). The bulk term was already computed in Proposition 4.1. The observable term $\tilde{Z}^\star_{N,N}$ is computed by the next lemma. In the lemma we only give explicit formulas for the terms that will be used in the proofs of Propositions 6.1 and 6.2.

Lemma 6.3 *Restricted to constant* ψ, $\bar\psi$, *in Case* (1),

$$\tilde{Z}^\star_{N,N} = \sigma_a \bar\psi \tilde{Z}^{\sigma_a\bar\psi}_{N,N} + \bar\sigma_b \psi \tilde{Z}^{\bar\sigma_b\psi}_{N,N} + \sigma_a\bar\sigma_b \tilde{Z}^{\bar\sigma_b\sigma_a}_{N,N} + \sigma_a\bar\sigma_b \psi\bar\psi \tilde{Z}^{\bar\sigma_b\sigma_a\psi\bar\psi}_{N,N} \tag{6.7}$$

where

$$\tilde{Z}^{\sigma_a\bar\psi}_{N,N} = \lambda_{a,N} + O_L(\ell_{x,N}^{-1}\ell_N^{-1}\|K_N^\star\|_N) \tag{6.8}$$

$$\tilde{Z}^{\bar{\sigma}_b\psi}_{N,N} = \lambda_{b,N} + O_L(\ell_{x,N}^{-1}\ell_N^{-1}\|K_N^\star\|_N) \tag{6.9}$$

$$\tilde{Z}^{\bar{\sigma}_b\sigma_a}_{N,N} = q_N(1+\tilde{u}_{N,N}) + \lambda_{a,N}\lambda_{b,N}t_N|\Lambda_N|^{-1} - r_N t_N|\Lambda_N|^{-1} + O_L(m^{-2}|\Lambda_N|^{-1}\ell_N^{-2}\ell_{ab,N}^{-1}\|K_N^\star\|_N). \tag{6.10}$$

In Case (2),

$$\tilde{Z}^\star_{N,N} = \sigma_a\tilde{Z}^{\sigma_a}_{N,N} + \sigma_a\bar{\psi}\psi\tilde{Z}^{\sigma_a\bar{\psi}\psi}_{N,N} + \sigma_b\tilde{Z}^{\sigma_b}_{N,N} + \sigma_b\bar{\psi}\psi\tilde{Z}^{\sigma_b\bar{\psi}\psi}_{N,N} + \sigma_a\sigma_b\tilde{Z}^{\sigma_a\sigma_b}_{N,N} + \sigma_a\sigma_b\psi\bar{\psi}\tilde{Z}^{\sigma_a\sigma_b\psi\bar{\psi}}_{N,N} \tag{6.11}$$

where, setting $\tilde{\lambda}_{x,N,N} = \lambda_{x,N} + O_L(\ell_{x,N}^{-1}\ell_N^{-2}\|K_N^\star\|_N)$,

$$\tilde{Z}^{\sigma_x}_{N,N} = \gamma_{x,N}(1+\tilde{u}_{N,N}) + \tilde{\lambda}_{x,N,N}t_N|\Lambda_N|^{-1} + O_L(\ell_{x,N}^{-1}\|K_N^\star\|_N). \tag{6.12}$$

$$\begin{aligned}\tilde{Z}^{\sigma_a\sigma_b}_{N,N} = &(q_N + \gamma_{a,N}\gamma_{b,N})(1+\tilde{u}_{N,N}) + (\eta_N - r_N + \tilde{\lambda}_{a,N,N}\gamma_{b,N}\\ &+ \tilde{\lambda}_{b,N,N}\gamma_{a,N})t_N|\Lambda_N|^{-1}\\ &+ O_L((|\gamma_{a,N}| + |\gamma_{b,N}|)\ell_{x,N}^{-1} + \ell_{ab,N}^{-1}\\ &+ m^{-2}|\Lambda_N|^{-1}\ell_N^{-2}\ell_{ab,N}^{-1})\|K_N^\star\|_N.\end{aligned} \tag{6.13}$$

The error bounds above reveal the tension in the explicit choices of $\ell_{x,j}^{-1}$ and $\ell_{ab,j}^{-1}$. To obtain effective error estimates, we want $\ell_{x,N}^{-1}$ and $\ell_{ab,N}^{-1}$ to be as small as possible. On the other hand, to control the iterative estimates of Theorem 5.8 over the entire trajectory, i.e., to prove Proposition 5.15, we needed that $\ell_{x,j}$ and $\ell_{ab,j}$ were not too large. In particular, either of the more naive choices $\ell_{ab,j} = \ell_j^{-4}$ and $\ell_{ab,j} = \ell_{j_{ab}}^{-4}$ in Case (2) would have lead to difficulties, either in terms of forcing us to track additional terms in the flow and in terms of controlling norms inductively, or by leading to error bounds that are not strong enough to capture the zero mode sufficiently accurately.

Proof Throughout the proof, we restrict to constant ψ, $\bar{\psi}$. Since

$$\begin{aligned}\left(e^{+u_N^\varnothing(\Lambda)}Z_N\right)^\star &= \left(e^{-u_N^\star(\Lambda)}(e^{-V_N(\Lambda)} + K_N(\Lambda))\right)^\star\\ &= (e^{-u_N^\star(\Lambda)} - 1)(e^{-V_N^\varnothing(\Lambda)} + K_N^\varnothing(\Lambda))\\ &\quad + e^{-u_N^\star(\Lambda)}(e^{-V_N(\Lambda)} + K_N(\Lambda))^\star,\end{aligned} \tag{6.14}$$

by applying $\mathbb{E}_{t_N Q_N}\theta$ we obtain

$$\tilde{Z}^\star_{N,N} = \underbrace{(e^{-u_N^\star(\Lambda)} - 1)(1 + \tilde{u}_{N,N} - |\Lambda_N|\tilde{a}_{N,N}\psi\bar{\psi})}_{A} + \underbrace{e^{-u_N^\star(\Lambda)}\mathbb{E}_{t_N Q_N}\Big[\theta(e^{-V_N(\Lambda)} + K_N(\Lambda))^\star\Big]}_{B}. \tag{6.15}$$

In obtaining A we used (4.4) which gives $\mathbb{E}_{t_N Q_N}[\theta(e^{-V_N^{\varnothing}(\Lambda)} + K_N^{\varnothing}(\Lambda))] = 1 + \tilde{u}_{N,N} - |\Lambda_N| \tilde{a}_{N,N} \psi\bar{\psi}$. Since each term in $V_N^{\varnothing}(\Lambda)$ contains a factor $\psi\bar{\psi}$ and each term in $V_N^{\star}(\Lambda)$ either ψ or $\bar{\psi}$, we have $V_N^{\varnothing}(\Lambda) V_N^{\star}(\Lambda) = 0$. Thus

$$B = e^{-u_N^{\star}(\Lambda)} \mathbb{E}_{t_N Q_N}\Big[\theta(-V_N^{\star}(\Lambda) + \frac{1}{2} V_N^{\star}(\Lambda)^2 + K_N^{\star}(\Lambda))\Big]. \tag{6.16}$$

Case (1). Since $\sigma_a^2 = \bar{\sigma}_b^2 = 0$,

$$e^{-u_N^{\star}(\Lambda)} - 1 = -u_N^{\star}(\Lambda) = \sigma_a \bar{\sigma}_b q_N, \tag{6.17}$$

we get

$$A = \sigma_a \bar{\sigma}_b q_N (1 + \tilde{u}_{N,N} - |\Lambda_N| \tilde{a}_{N,N} \psi\bar{\psi}) \tag{6.18}$$

$$\begin{aligned} B &= \sigma_a \bar{\psi}(\lambda_{a,N} + k_N^{\sigma_a \bar{\psi}}) + \psi \bar{\sigma}_b (\lambda_{b,N} + k_N^{\bar{\sigma}_b \psi}) \\ &\quad + \sigma_a \bar{\sigma}_b \mathbb{E}_{t_N Q_N}\Big[\theta \bar{\psi}\psi(\lambda_{a,N}\lambda_{b,N} - r_N + k_N^{\sigma_a \bar{\sigma}_b \bar{\psi}\psi})\Big]. \end{aligned} \tag{6.19}$$

The constants $k_N^{\#}$ are given in terms of derivatives of $K_N(\Lambda)$ and bounded analogously as in (4.10). For example, $k_N^{\sigma_a \bar{\psi}} = O_L(\ell_{a,N}^{-1} \ell_N^{-1} \|K_N^{\star}\|_N)$, and similarly for the other $k_N^{\#}$ terms, the rule being that we have a factor $\ell_{x,N}^{-1}$ if there is a superscript σ_a or $\bar{\sigma}_b$ but not both, a factor $\ell_{ab,N}^{-1}$ for $\sigma_a \bar{\sigma}_b$ and a factor ℓ_N^{-1} for each superscript ψ or $\bar{\psi}$. These bounds follow from the definition of the $T_j(\ell_j)$ norm.

Since $\mathbb{E}_{t_N Q_N} \theta \psi\bar{\psi} = -t_N |\Lambda_N|^{-1} + \psi\bar{\psi}$ the claim follows by collecting terms and using (3.7).

Case (2). Using again that $\sigma_a^2 = \sigma_b^2 = 0$, but now taking in account that $u^{\star}(\Lambda)$ has additional terms compared to Case (1),

$$e^{-u_N^{\star}(\Lambda)} - 1 = -u_N^{\star}(\Lambda) + \frac{1}{2} u_N^{\star}(\Lambda)^2 = \sigma_a \sigma_b (q_N + \gamma_{a,N}\gamma_{b,N}) + \sigma_a \gamma_{a,N} + \sigma_b \gamma_{b,N}, \tag{6.20}$$

and therefore

$$A = (\sigma_a \sigma_b (q_N + \gamma_{a,N}\gamma_{b,N}) + \sigma_a \gamma_{a,N} + \sigma_b \gamma_{b,N})(1 + \tilde{u}_{N,N} - |\Lambda_N| \tilde{a}_{N,N} \psi\bar{\psi}). \tag{6.21}$$

Since in Case (2) each term in $V_N^{\star}(\Lambda)$ contains a factor of $\psi\bar{\psi}$, we have $V_N^{\star}(\Lambda)^2 = 0$ and thus

$$B = (1 - u_N^{\star}(\Lambda)) \mathbb{E}_{t_N Q_N}\Big[\theta(-V_N^{\star}(\Lambda) + K_N^{\star}(\Lambda))\Big]. \tag{6.22}$$

Therefore

$$\begin{aligned} B &= \sigma_a k_N^{\sigma_a} + \sigma_b k_N^{\sigma_b} + \sigma_a \sigma_b (\gamma_{a,N} k_N^{\sigma_b} + \gamma_{b,N} k_N^{\sigma_a} + k_N^{\sigma_a \sigma_b}) \\ &\quad + \mathbb{E}_{t_N Q_N}\Big[\theta(\bar{\psi}\psi(\sigma_a \tilde{\lambda}_{a,N} + \sigma_b \tilde{\lambda}_{b,N}) + \sigma_a \sigma_b \bar{\psi}\psi(\eta_N - r_N \end{aligned}$$

$$+\gamma_{a,N}\tilde{\lambda}_{b,N}+\gamma_{b,N}\tilde{\lambda}_{a,N}+k^{\sigma_a\sigma_b\bar{\psi}\psi}))\Big] \tag{6.23}$$

where we have set $\tilde{\lambda}_{x,N}=\lambda_{x,N}+k_N^{\sigma_x\bar{\psi}\psi}$. Taking the expectation and collecting all terms gives

$$\begin{aligned}\tilde{Z}^{\sigma_a\sigma_b}_{N,N} &= (q_N+\gamma_{a,N}\gamma_{b,N})(1+\tilde{u}_{N,N})\\ &\qquad + (\eta_N - r_N + \tilde{\lambda}_{a,N}\gamma_{b,N}+\tilde{\lambda}_{b,N}\gamma_{a,N}+k^{\sigma_a\sigma_b\bar{\psi}\psi})t_N|\Lambda_N|^{-1}\\ &\qquad + \gamma_{a,N}k_N^{\sigma_b}+\gamma_{b,N}k_N^{\sigma_a}+k_N^{\sigma_a\sigma_b} \end{aligned}\tag{6.24}$$

$$\tilde{Z}^{\sigma_a}_{N,N} = \gamma_{a,N}(1+\tilde{u}_{N,N})+\tilde{\lambda}_{a,N}t_N|\Lambda_N|^{-1}+k_N^{\sigma_a}. \tag{6.25}$$

The bounds on the constants $k_N^{\#}$ are analogous to those in Case (1). □

6.2 Analysis of one-point functions

We now analyse the observable flow given by Lemma 5.13 to derive the asymptotics of the correlation functions. Note that the coupling constants $\lambda_{x,j}$ and $\gamma_{x,j}$ can possibly depend on $x=a,b$ as the contributions from K can depend on the relative position of the points in the division of Λ_N into blocks. The following lemma shows that in the limit $j\to\infty$ they become independent of x; an analogous argument was used in [14, Lemma 4.6].

Lemma 6.4 *Under the hypotheses of Proposition 6.1 there are $\lambda_\infty^{(p)}=\lambda_0+O_L(\lambda_0 b_0)$ and $\gamma_\infty=O_L(\lambda_0)$, all continuous in $m^2\geqslant 0$ and b_0 small, such that for $x\in\{a,b\}$,*

$$\lambda^{(p)}_{x,j}=\lambda^{(p)}_\infty+O_L(\lambda_0 b_0 L^{-\kappa j}),\qquad \gamma_{x,j}=\gamma_\infty+O_L(\lambda_0 b_0 L^{-(d-2+\kappa)j}). \tag{6.26}$$

In Case (1), $\lambda^{(1)}_\infty=\lambda_0$. In Case (2), $\lambda^{(2)}_\infty=\lambda_0+O_L(\lambda_0 b_0)$ and $\gamma^{(2)}_\infty=\lambda^{(2)}_\infty(-\Delta^{\mathbb{Z}^d}+m^2)^{-1}(0,0)+O_L(\lambda_0 b_0)$, and with the abbreviations $\lambda=\lambda^{(2)}$ and $\gamma=\gamma^{(2)}$,

$$\langle\bar{\psi}_a\psi_a\rangle=\frac{\gamma_\infty}{\lambda_0}+\frac{\frac{\lambda_\infty}{\lambda_0}t_N|\Lambda_N|^{-1}+O_L(b_0L^{-(d-2+\kappa)N})+O_L(b_0L^{-\kappa N}(m^2|\Lambda_N|)^{-1})}{1+\tilde{u}_{N,N}}. \tag{6.27}$$

Proof We will typically drop the superscript (p). In both cases, we have already seen that

$$\lambda_{x,j}=\lambda_0+\sum_{k=0}^{j-1}O_L(\|K_k^\star\|_k)=\lambda_0+\sum_{k=0}^{j-1}O_L(\lambda_0 b_0 L^{-\kappa k}). \tag{6.28}$$

Since the $K_k^\star$ are independent of N for $k<N$ (by Proposition 3.12 for the extended renormalisation group map, see Sect. 5.5), the limit $\lambda_{x,\infty}$ makes sense, exists, and $|\lambda_{x,j}-\lambda_{x,\infty}|=O_L(\lambda_0 b_0 L^{-\kappa j})$. Similarly, in Case (2), by Lemma 5.13 and $\ell_{x,j}^{-1}=$

$\ell_j^2 = O_L(L^{-(d-2)j})$,

$$\gamma_{x,j} = \sum_{k=0}^{j-1} \Big[\lambda_{x,k} C_{k+1}(x,x) + O_L(L^{-(d-2)k} \|K_k^\star\|_k) \Big]. \tag{6.29}$$

In particular, by the above estimate for $|\lambda_{x,j} - \lambda_{x,\infty}|$, we have

$$\gamma_{x,\infty} = \lambda_{x,\infty} \sum_{k=0}^{\infty} C_{k+1}(0,0) + O_L(\lambda_0 b_0) = \lambda_{x,\infty}(-\Delta^{\mathbb{Z}^d} + m^2)^{-1}(0,0) + O_L(\lambda_0 b_0). \tag{6.30}$$

The continuity claims follow from the continuity of the covariances C_j in $m^2 \geqslant 0$, of the renormalisation group coordinates K_j, and that both λ_∞ and γ_∞ are uniformly convergent sums of terms continuous in b_0 and $m^2 \geqslant 0$.

To show that $\lambda_{x,\infty}^{(1)} = \lambda_0$ in Case (1), which is in particular independent of x, we argue as in the proof of [14, Lemma 4.6]. On the one hand, Lemma 6.3 implies as $N \to \infty$ with $m^2 > 0$ fixed,

$$\partial_{\bar\psi} \partial_{\sigma_a} \tilde Z_{N,N}|_0 = \lambda_{a,N} + O_L(\ell_N^{-1} \ell_{x,N}^{-1} \|K_N^\star\|_N) = \lambda_{a,N} + O_L(\|K_N^\star\|_N) \xrightarrow[N\to\infty]{} \lambda_{a,\infty}, \tag{6.31}$$

where $|_0$ denotes projection onto the degree 0 part, i.e., $\psi = \bar\psi = \sigma = \bar\sigma = 0$, and we have dropped the superscript (1) from $\lambda_{a,j}$. On the other hand, we claim

$$\partial_{\bar\psi} \partial_{\sigma_a} \tilde Z_{N,N}|_0 = \lambda_0 m^2 (1 + \tilde u_{N,N}) \sum_{x \in \Lambda_N} \langle \bar\psi_0 \psi_x \rangle = \lambda_0 \left(1 + \tilde u_{N,N} - \frac{\tilde a_{N,N}}{m^2} \right). \tag{6.32}$$

Indeed, the first equality in (6.32) follows analogously to [14, (4.51)–(4.53)]: let $\Gamma(\rho, \bar\rho)$ be as in (4.12), except that Z_0 now includes the observable terms σ_a and $\bar\sigma_b$ and we write ρ and $\bar\rho$ for the constant external field to distinguish them from σ_a and $\bar\sigma_b$. Then as in (4.15),

$$-\sum_{x \in \Lambda_N} \langle \psi_x \rangle_{\sigma_a, \bar\sigma_b} = \partial_{\bar\rho} \Gamma(\rho, \bar\rho)|_{\rho = \bar\rho = 0} = m^{-2} \frac{\partial_{\bar\psi} \tilde Z_{N,N}|_{\psi = \bar\psi = 0}}{\tilde Z_{N,N}|_{\psi = \bar\psi = 0}}, \tag{6.33}$$

and $\langle \cdot \rangle_{\sigma_a, \bar\sigma_b}$ denotes the expectation that still depends on the source fields σ_a and $\bar\sigma_b$. Differentiating with respect to σ_a and setting $\bar\sigma_b = 0$ gives

$$\lambda_0 \sum_{x \in \Lambda_N} \langle \bar\psi_a \psi_x \rangle = -m^{-2} \frac{\partial_{\sigma_a} \partial_{\bar\psi} \tilde Z_{N,N}|_0}{\tilde Z_{N,N}|_0} = m^{-2} \frac{\partial_{\bar\psi} \partial_{\sigma_a} \tilde Z_{N,N}|_0}{1 + \tilde u_{N,N}}, \tag{6.34}$$

which is the first equality of (6.32) upon rearranging. The second equality in (6.32) follows from Proposition 4.2.

The right-hand side of (6.32) converges to λ_0 in the limit $N \to \infty$ with $m^2 > 0$ fixed since $\tilde a_{N,N} = a_N - k_N^2/|\Lambda_N| = O_L(L^{-2N}\|V_N\|_N) + O_L(L^{-2N}\|K_N\|_N) \to 0$ and $\tilde u_{N,N} = k_N^0 + \tilde a_{N,N} t_N = O(\|K_N\|_N) + \tilde a_{N,N} t_N \to 0$ when $m^2 > 0$ is fixed. Since the left-hand sides of (6.31)–(6.32) are equal, we conclude that $\lambda_{a,\infty} = \lambda_0$ when $m^2 > 0$. By continuity this identity then extends to $m^2 = 0$.

In Case (2), to show (6.27), we use (6.12), that Proposition 5.15 implies $\|K_N^\star\|_N = O_L(\lambda_0 b_0 L^{-\kappa N})$, and $\ell_{x,N}^{-1}\ell_N^{-2} = 1$ and $\ell_{x,N}^{-1} = \ell_N^2 = O_L(L^{-(d-2)N})$ to obtain

$$\frac{\tilde{Z}^{\sigma_a}_{N,N}}{1+\tilde{u}_{N,N}} = \gamma_{a,N} + \frac{\lambda_{a,\infty} t_N|\Lambda_N|^{-1} + O_L(\lambda_0 b_0 L^{-(d-2+\kappa)N}) + O_L(\lambda_0 b_0 L^{-\kappa N}(m^2|\Lambda_N|)^{-1})}{1+\tilde{u}_{N,N}}. \tag{6.35}$$

Since

$$\lambda_0\langle\bar{\psi}_a\psi_a\rangle = \frac{\partial_{\sigma_a}\tilde{Z}^{N,N}|_0}{\tilde{Z}_{N,N}|_0} = \frac{\tilde{Z}^{\sigma_a}_{N,N}}{1+\tilde{u}_{N,N}}, \tag{6.36}$$

this gives (6.27). In particular, by the translation invariance of $\langle\bar{\psi}_a\psi_a\rangle$, taking $N \to \infty$ with $m^2 > 0$ fixed implies $\gamma_{a,\infty}$ is independent of a. Similarly, taking $m^2 \downarrow 0$ first and then $N \to \infty$ we see that $\lambda_{a,\infty}$ is independent of a. Indeed, using (4.4), as $N \to \infty$,

$$\lambda_0\langle\bar{\psi}_a\psi_a\rangle \sim \gamma_{a,\infty} + \lambda_{a,\infty}\lim_{m^2\downarrow 0}\frac{t_N|\Lambda_N|^{-1}}{\tilde{u}_{N,N}} \sim \gamma_{a,\infty} + \lambda_{a,\infty}\frac{1}{|\Lambda_N|\tilde{a}_{N,N}}, \tag{6.37}$$

where $\langle\bar{\psi}_a\psi_a\rangle$ and all scale-dependent coupling constants are evaluated at $m^2 = 0$. Thus $\lambda_{a,\infty} = \lim_{N\to\infty}|\Lambda_N|\tilde{a}_{N,N}(\lambda_0\langle\bar{\psi}_a\psi_a\rangle - \gamma_{a,\infty})$ and the right-hand side is independent of a. □

Proof of Proposition 6.1 Taking $\lambda_0 > 0$ small enough, the proposition follows immediately from Lemma 6.4 with $\lambda = \lambda_\infty^{(2)}/\lambda_0$ and $\gamma = \gamma_\infty^{(2)}/\lambda_0$. □

6.3 Analysis of two-point functions

Next we derive estimates for the two-point functions.

Lemma 6.5 *Under the hypotheses of Proposition 6.1,*

$$\langle\bar{\psi}_a\psi_b\rangle = W_N(a-b) + \frac{t_N|\Lambda_N|^{-1}}{1+\tilde{u}_{N,N}} + O_L(\frac{b_0}{\lambda_0}|a-b|^{-(d-2+\kappa)}) + O_L(\frac{b_0}{\lambda_0}|a-b|^{-\kappa})\frac{(m^2|\Lambda_N|)^{-1}}{1+\tilde{u}_{N,N}}, \tag{6.38}$$

and, setting $\lambda_\infty = \lambda_\infty^{(2)}$ and $\gamma_\infty = \gamma_\infty^{(2)}$ as in Lemma 6.4,

$$\langle\bar{\psi}_a\psi_a\bar{\psi}_b\psi_b\rangle = -\frac{\lambda_\infty^2}{\lambda_0^2}W_N(a-b)^2 + \frac{\gamma_\infty^2}{\lambda_0^2} + \frac{-2\lambda_\infty^2 W_N(a-b) + 2\lambda_\infty\gamma_\infty}{\lambda_0^2(1+\tilde{u}_{N,N})}t_N|\Lambda_N|^{-1}$$

$$+ O_L(\frac{b_0}{\lambda_0}|a-b|^{-2(d-2)-\kappa}) + O_L(\frac{b_0}{\lambda_0}L^{-(d-2+\kappa)N})$$
$$+ (O_L(\frac{b_0}{\lambda_0}|a-b|^{-(d-2+\kappa)}) + O_L(\frac{b_0}{\lambda_0}L^{-\kappa N}))\frac{(m^2|\Lambda_N|)^{-1}}{1+\tilde{u}_{N,N}}. \tag{6.39}$$

Proof The proofs of (6.38) and (6.39) corresponding to Cases (1) and (2) are again analogous.

Case (1). By Lemma 5.14 (whose hypotheses are verified by Proposition 5.15) and Lemma 6.4,

$$\begin{aligned} \lambda_{x,j} &= \lambda_\infty + O_L(\lambda_0 b_0 L^{-\kappa}) = \lambda_0 + O_L(\lambda_0 b_0 L^{-\kappa}), \\ r_j &= O_L(\lambda_0 b_0 |a-b|^{-\kappa}) 1_{j \geqslant j_{ab}}. \end{aligned} \tag{6.40}$$

Using that $\ell_{ab,j}^{-1}\|K_j^{ab}\|_j \leqslant O_L(\lambda_0 b_0 L^{-(d-2+\kappa)j}) 1_{j\geqslant j_{ab}}$ and $|C_{j+1}(a,b)| \leqslant C_{j+1}(0,0) \leqslant O_L(L^{-(d-2)j})$ it then follows from Lemma 5.13 that

$$\begin{aligned} q_N &= \sum_{j=j_{ab}-1}^{N-1} \Big[\lambda_{a,j}\lambda_{b,j} C_{j+1}(a,b) + r_j C_{j+1}(0,0) + O_L(\lambda_0 b_0 L^{-(d-2+\kappa)j})\Big] \\ &= \lambda_0^2 \sum_{j=1}^{N-1} C_j(a,b) + O_L(\lambda_0 b_0 |a-b|^{-(d-2)-\kappa}) \\ &= \lambda_0^2 W_N(a-b) + O_L(\lambda_0 b_0 |a-b|^{-(d-2)-\kappa}), \end{aligned} \tag{6.41}$$

where we have used (6.40), $|a-b| \leqslant L$, that $C_j(a,b) = 0$ for $j < j_{ab}$, and that $W_N(x-y) = C_1(x,y) + \cdots + C_N(x,y)$. By (6.10), using that $\ell_{ab,N}^{-1}\ell_N^{-2} = 1$ and again (6.40), therefore

$$\begin{aligned} \frac{\tilde{Z}_{N,N}^{\bar{\sigma}_b\sigma_a}}{1+\tilde{u}_{N,N}} &= \lambda_0^2 W_N(a-b) + O_L(\lambda_0 b_0 |a-b|^{-(d-2)-\kappa}) \\ &\quad + \frac{\lambda_0^2 t_N |\Lambda_N|^{-1} + O_L(\lambda_0 b_0 |a-b|^{-\kappa} m^{-2} |\Lambda_N|^{-1})}{1+\tilde{u}_{N,N}}. \end{aligned} \tag{6.42}$$

Since $\langle \bar{\psi}_a \psi_b \rangle = \tilde{Z}_{N,N}^{\bar{\sigma}_b\sigma_a}/(\lambda_0^2(1+\tilde{u}_{N,N}))$ and $|\lambda_0| \leqslant 1$, the claim for the two-point function follows.

Case (2). Again, the analogue of (6.40) holds:

$$\lambda_{x,j} = \lambda_\infty + O_L(b_0\lambda_0 L^{-\kappa j}), \qquad r_j = O_L(b_0\lambda_0 |a-b|^{-(d-2+\kappa)}) 1_{j\geqslant j_{ab}}. \tag{6.43}$$

The first estimate is by Lemma 6.4, the second by Lemma 5.14. Since $C_k(a,b) = 0$ for $k < j_{ab}$ and $|C_{k+1}(a,b)| \leqslant O_L(L^{-(d-2)k})$, then by Lemma 5.13 and as $|a-b| \leqslant$

L,

$$\eta_j = -2 \sum_{k=j_{ab}-1}^{j-1} \lambda_{a,k}\lambda_{b,k} C_{k+1}(a,b)$$
$$= -2\lambda_\infty^2 \sum_{k=1}^{j} C_k(a,b) + O_L(b_0\lambda_0|a-b|^{-(d-2)-\kappa}). \tag{6.44}$$

Note that

$$\sum_{k=j_{ab}-1}^{N-1} |r_k| C_{k+1}(0,0) \leqslant O_L(b_0\lambda_0|a-b|^{-(d-2+\kappa)}) \sum_{k\geqslant j_{ab}} L^{-(d-2)j}$$
$$\leqslant O_L(b_0\lambda_0|a-b|^{-2(d-2)-\kappa}). \tag{6.45}$$

As a result, again by Lemma 5.13, these bounds together then give

$$q_N = \sum_{k\leqslant N} [\eta_{k-1} C_k(a,b) - \lambda_\infty^2 C_k(a,b)^2] + O_L(b_0\lambda_0|a-b|^{-2(d-2)-\kappa})$$
$$= -\lambda_\infty^2 \sum_{k\leqslant N} [2\sum_{l<k} C_l(a,b)C_k(a,b) + C_k(a,b)^2] + O_L(b_0\lambda_0|a-b|^{-2(d-2)-\kappa})$$
$$= -\lambda_\infty^2 \left(\sum_{k\leqslant N} C_k(a,b) \right)^2 + O_L(b_0\lambda_0|a-b|^{-2(d-2)-\kappa}).$$
$$= -\lambda_\infty^2 W_N(a-b)^2 + O_L(b_0\lambda_0|a-b|^{-2(d-2)-\kappa}). \tag{6.46}$$

We finally substitute these estimates into (6.13). Using also that $\ell_{ab,N}^{-1}\ell_N^{-2} = \ell_{j_{ab}}^2 = O_L(|a-b|^{-(d-2)})$, that $\ell_{x,N}^{-1}\ell_N^{-2} = O_L(1)$, that $\gamma_{x,N} = \gamma_\infty + O_L(b_0\lambda_0 L^{-(d-2+\kappa)N})$ by (6.26), and $\|K_N^\star\|_N \leqslant O_L(b_0\lambda_0 L^{-\kappa})$, we obtain

$$\frac{\tilde{Z}_{N,N}^{\sigma_a\sigma_b}}{1+\tilde{u}_{N,N}}$$
$$= -\lambda_\infty^2 W_N(a-b)^2 + \gamma_\infty^2 + O_L(b_0\lambda_0|a-b|^{-2(d-2)-\kappa})$$
$$+ O_L(b_0\lambda_0 L^{-(d-2+\kappa)N})$$
$$+ \frac{-2\lambda_\infty^2 W_N(a-b) + 2\lambda_\infty\gamma_\infty}{1+\tilde{u}_{N,N}} t_N |\Lambda_N|^{-1}$$
$$+ \frac{O_L(b_0\lambda_0 L^{-\kappa N} m^{-2}|\Lambda_N|^{-1}) + O_L(b_0\lambda_0|a-b|^{-(d-2+\kappa)} m^{-2}|\Lambda_N|^{-1})}{1+\tilde{u}_{N,N}} \tag{6.47}$$

which gives (6.39) since $\langle \bar\psi_a\psi_a\bar\psi_b\psi_b\rangle = \tilde{Z}_{N,N}^{\sigma_a\sigma_b}/(\lambda_0^2(1+\tilde{u}_{N,N}))$. □

Proof of Proposition 6.2 The proposition follows immediately from Lemma 6.5 with the same λ and γ as in Proposition 6.1. □

7 Proof of Theorems 2.1 and 2.3

Proof of Theorems 2.1 and 2.3 By summation by parts on the whole torus Λ_N, we have

$$y_0(\nabla\psi, \nabla\bar{\psi}) + \frac{z_0}{2}\Big((-\Delta\psi, \bar{\psi}) + (\psi, -\Delta\bar{\psi})\Big) = (y_0 + z_0)(\nabla\psi, \nabla\bar{\psi}). \tag{7.1}$$

Given $m^2 \geqslant 0$ and b_0 small, we choose $V_0^c(b_0, m^2)$ as in Theorem 3.24. This defines the functions $s_0^c = y_0^c + z_0^c$ and a_0^c in (2.5) with the required regularity properties. The claims for the correlation functions and the partition function then follow from Propositions 4.1–4.2 and 6.1–6.2. The continuity of u_N^c follows from the continuity of V_0^c and the continuity of the renormalisation group maps.

For Theorem 2.1, note that the statements simplify by the assumption $m^2 \geqslant L^{-2N}$. Indeed, using that $(m^2|\Lambda_N|)^{-1} \leqslant L^{-(d-2)N}$ and $|a_N| \leqslant O_L(b_0 L^{-(2+\kappa)N})$, by Proposition 4.1, we have that $|\tilde{a}_{N,N}| \leqslant O_L(b_0 L^{-(2+\kappa)N})$ and $|\tilde{u}_{N,N}| \leqslant O_L(b_0 L^{-\kappa N})$. □

Appendix A: Random forests and the $\mathbb{H}^{0|2}$ model

A.1 Proof of Proposition 1.4

For any graph $G = (\Lambda, E)$ with edge weights (β_{xy}) and vertex weights (h_x), the partition function appearing in (1.1) can be generalised to

$$Z_{\beta,h} = \sum_{F \in \mathcal{F}} \prod_{xy \in F} \beta_{xy} \prod_{T \in F} (1 + \sum_{x \in T} h_x), \tag{A.1}$$

where $\mathcal{F}$ is the set of forest subgraphs of G. Recall from the discussion above (1.3) that expanding the product over T in (A.1) can be interpreted as choosing, for each T, either (i) a root vertex $x \in T$ with weight h_x or (ii) leaving T unrooted. This interpretation will be used in Lemma A.4.

By [20, Theorem 2.1] (which follows [37]),

$$Z_{\beta,h} = \int \prod_{x \in \Lambda} \partial_{\eta_x} \partial_{\xi_x} \frac{1}{z_x} e^{\sum_{xy} \beta_{xy}(u_x \cdot u_y + 1) - \sum_x h_x (z_x - 1)}. \tag{A.2}$$

Moreover, by [20, Corollary 2.2], if $h = 0$ then

$$\mathbb{P}_{\beta,0}[x \leftrightarrow y] = -\langle u_0 \cdot u_x \rangle_{\beta,0} = -\langle z_0 z_x \rangle_{\beta,0} = \langle \xi_x \eta_y \rangle_{\beta,0} = 1 - \langle \xi_x \eta_x \xi_y \eta_y \rangle_{\beta,0}. \tag{A.3}$$

Proposition 1.4 follows easily from this. For convenience, we restate the proposition as follows. In the statement and throughout this appendix, inequalities like $\beta \geqslant 0$ are to be interpreted pointwise, i.e., $\beta_{xy} \geqslant 0$ for all edges xy.

Proposition A.1 *For any finite graph G, any $\beta \geqslant 0$ and $h \geqslant 0$,*

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow \mathfrak{g}] = \langle z_0 \rangle_{\beta,h}, \tag{A.4}$$

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}] = \langle \xi_0 \eta_x \rangle_{\beta,h}, \tag{A.5}$$

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow x] + \mathbb{P}_{\beta,h}[0 \not\leftrightarrow x, 0 \leftrightarrow \mathfrak{g}, x \leftrightarrow \mathfrak{g}] = -\langle u_0 \cdot u_x \rangle_{\beta,h}, \tag{A.6}$$

and the normalising constants in (1.1) *and* (1.13) *are equal. In particular,*

$$\mathbb{P}_{\beta,0}[0 \leftrightarrow x] = -\langle u_0 \cdot u_x \rangle_{\beta,0} = -\langle z_0 z_x \rangle_{\beta,0} = \langle \xi_0 \eta_x \rangle_{\beta,0} = 1 - \langle \xi_0 \eta_0 \xi_x \eta_x \rangle_{\beta,0}. \tag{A.7}$$

Proof of Proposition A.1 For notational ease, we write the proof for constant h. The equality of the normalising constants is a special case of (A.2). To see (A.4), we use that $(z_0 - 1)^2 = 0$ so that $z_0 = 1 - (1 - z_0) = e^{-(1-z_0)}$. As a result $\langle z_0 \rangle_{\beta,h} = Z_{\beta,h-1_0}/Z_{\beta,h}$, and (A.1) gives

$$\langle z_0 \rangle_{\beta,h} = \mathbb{E}_{\beta,h} \frac{h|T_0|}{1 + h|T_0|} = \mathbb{P}_{\beta,h}[0 \leftrightarrow \mathfrak{g}]. \tag{A.8}$$

Similarly, $\langle z_0 z_x \rangle = Z_{\beta,h-1_0-1_x}/Z_{\beta,h}$ and thus (A.1) shows that

$$\begin{aligned} \langle z_0 z_x \rangle_{\beta,h} &= \mathbb{E}_{\beta,h} \frac{-1 + h|T_0|}{1 + h|T_0|} 1_{0 \leftrightarrow x} + \mathbb{E}_{\beta,h} \frac{h|T_0|}{1 + h|T_0|} \frac{h|T_x|}{1 + h|T_x|} 1_{0 \not\leftrightarrow x} \\ &= \mathbb{P}_{\beta,h}[0 \leftrightarrow x] - 2\mathbb{P}_{\beta,h}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}] + \mathbb{P}_{\beta,h}[0 \not\leftrightarrow x, 0 \leftrightarrow \mathfrak{g}, x \leftrightarrow \mathfrak{g}]. \end{aligned} \tag{A.9}$$

To see (A.6), we note that the left-hand side is the connection probability in the extended graph $G^{\mathfrak{g}}$. From (A.3) with $\beta_{xy} = \beta$ for $x, y \in \Lambda$ and $\beta_{x\mathfrak{g}} = h$ for $x \in \Lambda$ we thus obtain the claim:

$$-\langle u_0 \cdot u_x \rangle_{\beta,h} = \mathbb{P}_{\beta,h}[0 \leftrightarrow x] + \mathbb{P}_{\beta,h}[0 \not\leftrightarrow x, 0 \leftrightarrow \mathfrak{g}, x \leftrightarrow \mathfrak{g}]. \tag{A.10}$$

To see (A.5), we combine (A.9) and (A.10) to get

$$2\langle \xi_0 \eta_x \rangle_{\beta,h} = -\langle u_0 \cdot u_x \rangle_{\beta,h} - \langle z_0 z_x \rangle_{\beta,h} = 2\mathbb{P}_{\beta,h}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}]. \tag{A.11}$$

Finally, (A.7) is (A.3). This completes the proof. □

The extended graph $G^{\mathfrak{g}}$ allows z-observables to be interpreted in terms of edges connecting vertices in the base graph G to $\mathfrak{g}$. To state this, we denote by $\{x\mathfrak{g}\}$ the event the edge between x and $\mathfrak{g}$ is present. The next lemma will be used in Appendix A.3.

Proposition A.2

$$h_0 \langle z_0 - 1 \rangle_{\beta,h} = \mathbb{P}_{\beta,h}[0\mathfrak{g}] \tag{A.12}$$

$$h_0 h_x \langle z_0 - 1; z_x - 1 \rangle_{\beta,h} = \mathbb{P}_{\beta,h}[0\mathfrak{g}, x\mathfrak{g}] - \mathbb{P}_{\beta,h}[0\mathfrak{g}]\mathbb{P}_{\beta,h}[x\mathfrak{g}] \tag{A.13}$$

Proof As discussed above, after expanding the product in (A.1) the external fields h_x can be viewed as edge weights for edges from x to $\mathfrak{g}$. With this in mind the formulas follow by differentiating (A.2). □

A.2 High-temperature phase and positive external field

Proposition A.3 *If* $\beta < p_c(d)/(1-p_c(d))$, *then* $\theta_d(\beta) = 0$. *Moreover, there is a* $c = c(\beta) > 0$ *such that* $\mathbb{P}^{\Lambda_N}_{\beta,0}[0 \leftrightarrow x] \leqslant e^{-c|x|}$.

Proof In finite volume, Holley's inequality implies the stochastic domination $\mathbb{P}^{\Lambda_N}_{\beta,h} \preceq \mathbb{P}^{\Lambda_N}_{p,r}$, where the latter measure is Bernoulli bond percolation on the extended graph $G^{\mathfrak{g}}$ with $p = \beta/(1+\beta)$ and $r = h/(1+h)$, see [20, Appendix A]. In particular,

$$\mathbb{P}^{\Lambda_N}_{\beta,h}[0 \leftrightarrow \mathfrak{g}] \leqslant \mathbb{P}^{\Lambda_N}_{p,r}[0 \leftrightarrow \mathfrak{g}]. \tag{A.14}$$

Since each edge to the ghost is chosen independently with probability r, this latter quantity is

$$\mathbb{P}^{\Lambda_N}_{p,r}[0 \leftrightarrow \mathfrak{g}] = \sum_{n=1}^{|\Lambda_N|} \mathbb{P}^{\Lambda_N}_{p,r}[|C_0| = n](1-(1-r)^n) \leqslant r\mathbb{E}^{\Lambda_N}_{p,r}|C_0| \tag{A.15}$$

since $1-(1-r)^n \leqslant rn$ for $0 \leqslant r \leqslant 1$. Here C_0 is the cluster of the origin on the torus without the ghost site, so $\mathbb{E}^{\Lambda_N}_{p,r}|C_0| = \mathbb{E}^{\Lambda_N}_{p,0}|C_0|$. Now suppose β is such that $p < p_c(d)$. Then the right-hand side is finite and uniformly bounded in N. Hence

$$\theta_d(\beta) = \lim_{h\to 0}\lim_{N\to\infty} \mathbb{P}^{\Lambda_N}_{\beta,h}[0 \leftrightarrow \mathfrak{g}] \leqslant \lim_{r\to 0} r \sup_N \mathbb{E}^{\Lambda_N}_{p,0}|C_0| = 0. \tag{A.16}$$

The second claim follows from stochastic domination, as when $p < p_c(d)$ bond percolation has exponentially decaying connection probabilities [56]. □

Lemma A.4 *Let* $h > 0$ *and suppose that for all* x, $h_x = h$. *Then there are* $c, C > 0$ *depending on* d, β, h *such that*

$$\mathbb{P}^{\Lambda_N}_{\beta,h}[0 \leftrightarrow x, 0\mathfrak{g}] \leqslant Ce^{-c|x|}, \qquad \mathbb{P}^{\Lambda_N}_{\beta,h}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}] \leqslant Ce^{-c|x|}. \tag{A.17}$$

Proof We begin with the inequality on the left of (A.17). Define $\mathcal{F}(0 \leftrightarrow x)$ to be the set of forests in which both 0 is connected to x and T_0 is rooted at 0, and $\mathcal{F}$ the set of all forests. In this argument we treat $\mathcal{F}$ as being a set of (possibly) rooted forests, i.e., we identify edges to $\mathfrak{g}$ with roots. Without loss of generality we may assume $x \cdot e_1 \geqslant \alpha|x|$ for a fixed $\alpha > 0$. Note that if $F \in \mathcal{F}(0 \leftrightarrow x)$ there is a unique path γ_F from 0 to x in F, and there are at least $\alpha|x|$ edges of the form $\{u, u+e_1\}$ in γ_F.

We define a map $S\colon \mathcal{F}(0 \leftrightarrow x) \to 2^{\mathcal{F}}$ by, for $F \in \mathcal{F}(0 \leftrightarrow x)$,

1. choosing a subset $\{u_i, v_i\}$ of the edges $\{\{u,v\} \in \gamma_F \mid v = u + e_1\}$, and
2. removing each $\{u_i, v_i\}$ and rooting the tree containing v_i at v_i.

Thus $S(F)$ is the set of forests that results from all possible choices in the first step. The second step does yield an element of $2^{\mathcal{F}}$ since T_0 is rooted at 0, so it cannot be the case that the tree containing v_i is already rooted (connected to $\mathfrak{g}$).

The map S is injective, meaning that given $\bar{F} \in \bigcup_{F\in\mathcal{F}(0\leftrightarrow x)} S(F)$ there is a unique F such that $\bar{F} \in S(F)$. Indeed, given $\bar{F} \in S(F)$, F can be reconstructed as follows.

In $\bar{F}$, either the tree containing x contains 0, or else it is rooted at a unique vertex v' and it is not connected to $u' = v' - e_1$. Set $\bar{F}' = \bar{F} \cup \{u', v'\}$. The previous sentence applies to $\bar{F}'$ as well, and continuing until a connection to 0 is formed we recover F. This reconstruction was independent of F, and hence if $\bar{F}_1 = \bar{F}_2$, $\bar{F}_i \in S(F_i)$, we have $F_1 = F_2$.

Let $w(F) = h\beta^F \prod_{T \neq T_0}(1 + h|V(T)|)$. Then for $\bar{F} \in S(F)$, $w(\bar{F}) = w(F)(\frac{h}{\beta})^k$ if $\bar{F}$ had k edges removed. Hence if the connection from 0 to x in F has k edges of the form $\{u, v\}$, $v = u + e_1$,

$$\sum_{\bar{F} \in S(F)} w(\bar{F}) = (1 + \frac{h}{\beta})^k w(F). \tag{A.18}$$

Let $\mathcal{F}_k(x) \subset \mathcal{F}(0 \leftrightarrow x)$ be the set of forests where the connection from 0 to x contains k edges of the form $\{u, v\}$, $v = u + e_1$. We have the lower bound

$$Z_{\beta,h}^{\Lambda_N} = \sum_{F \in \mathcal{F}} \beta^F \prod_{T \in F}(1 + h|V(T)|) \geqslant \sum_{k \geqslant 0} \sum_{F \in \mathcal{F}_k(x)} \sum_{\bar{F} \in S(F)} w(\bar{F}) \tag{A.19}$$

since S is injective and all of the summands are non-negative. Hence we obtain, using (A.18),

$$\begin{aligned} \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x, 0\mathfrak{g}] &\leqslant \frac{\sum_{k \geqslant \alpha|x|} \sum_{F \in \mathcal{F}_k(x)} w(F)}{\sum_{k \geqslant 0} \sum_{F \in \mathcal{F}_k(x)} \sum_{\bar{F} \in S(F)} w(\bar{F})} \\ &= \frac{\sum_{k \geqslant \alpha|x|} \sum_{F \in \mathcal{F}_k(x)} (1 + \frac{h}{\beta})^{-k} \sum_{\bar{F} \in S(F)} w(\bar{F})}{\sum_{k \geqslant 0} \sum_{F \in \mathcal{F}_k(x)} \sum_{\bar{F} \in S(F)} w(\bar{F})} \\ &\leqslant (1 + \frac{h}{\beta})^{-\alpha|x|}. \end{aligned} \tag{A.20}$$

A similar argument applies when $0 \not\leftrightarrow \mathfrak{g}$; this condition is used in the second step defining S to ensure the trees containing the vertices v_i are not already connected to $\mathfrak{g}$. In this case the weight $w(F)$ does not have the factor h, but the remainder of the argument is identical. □

Proposition A.5 *Let $h > 0$ and suppose that for all x, $h_x = h$. Then there are $c, C > 0$ depending on d, β, h such that*

$$\mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x] \leqslant Ce^{-c|x|}. \tag{A.21}$$

Proof Since

$$\mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x] = \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x, 0 \leftrightarrow \mathfrak{g}] + \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}], \tag{A.22}$$

it is enough to estimate the first term, as the second is covered by Lemma A.4. Note

$$\mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x, 0 \leftrightarrow \mathfrak{g}] = \sum_y \mathbb{P}_{\beta,h}^{\Lambda_N}[1_{0 \leftrightarrow x} 1_{0 \leftrightarrow y} 1_{y\mathfrak{g}}] = \sum_y \mathbb{P}_{\beta,h}^{\Lambda_N}[1_{0 \leftrightarrow x} 1_{0 \leftrightarrow y} 1_{0\mathfrak{g}}]. \tag{A.23}$$

where the first equality follows from the fact that the only one vertex per component may connect to $\mathfrak{g}$, and the second follows from exchangeability of the choice of root. Examining the rightmost expression, there are most $c_d|x|^d$ summands in which $|y| \leqslant |x|$; for these terms we drop the condition $0 \leftrightarrow y$. For the rest we drop $0 \leftrightarrow x$. This gives, by Lemma A.4,

$$\mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow x, 0 \leftrightarrow \mathfrak{g}] \leqslant C|x|^d e^{-c|x|} + \sum_{|y|>|x|} Ce^{-c|y|} \leqslant Ce^{-c|x|}, \tag{A.24}$$

where c, C are changing from location to location but depend on d, β, h only. □

A.3 Infinite volume limit

We now discuss weak limits $\mathbb{P}_{\beta}^{\mathbb{Z}^d}$ obtained by (i) first taking a (possibly subsequential) infinite-volume weak limit $\mathbb{P}_{\beta,h}^{\mathbb{Z}^d} = \lim_N \mathbb{P}_{\beta,h}^{\Lambda_N}$ and (ii) subsequently taking a (possibly subsequential) limit $\mathbb{P}_{\beta}^{\mathbb{Z}^d} = \lim_{h\downarrow 0} \mathbb{P}_{\beta,h}^{\mathbb{Z}^d}$. We do not explicitly indicate the convergent subsequence chosen as what follows applies to any fixed choice. Define

$$\theta_{d,N}(\beta,h) = \mathbb{P}_{\beta,h}^{\Lambda_N}[0 \leftrightarrow \mathfrak{g}] = 1 - h^{-1}\mathbb{P}_{\beta,h}^{\Lambda_N}[0\mathfrak{g}] \tag{A.25}$$

where the second equality is due to (A.12). Since this last display only involves cylinder events,

$$\lim_{N\to\infty} \theta_{d,N}(\beta,h) = 1 - h^{-1}\mathbb{P}_{\beta,h}^{\mathbb{Z}^d}[0\mathfrak{g}] = \theta_d(\beta,h), \tag{A.26}$$

where the last equality defines $\theta_d(\beta,h)$.

Proposition A.6 *Assume* $\lim_{h\downarrow 0}\theta_d(\beta,h) = \theta_d(\beta)$ *exists. Then*

$$\mathbb{P}_{\beta}^{\mathbb{Z}^d}[|T_0| = \infty] = \theta_d(\beta). \tag{A.27}$$

Proof Write $\mathbb{P}_{\beta,h} = \mathbb{P}_{\beta,h}^{\mathbb{Z}^d}$. We claim that

$$\mathbb{P}_{\beta,h}[0\mathfrak{g}] = \sum_{n\geqslant 1} \mathbb{P}_{\beta,h}[|T_0| = n]\frac{h}{1+nh}, \tag{A.28}$$

and hence, since $\theta_d(\beta,h) = 1 - h^{-1}\mathbb{P}_{\beta,h}[0\mathfrak{g}]$,

$$\theta_d(\beta,h) = 1 - \sum_{n\geqslant 1} \mathbb{P}_{\beta,h}[|T_0| = n]\frac{1}{1+nh}. \tag{A.29}$$

Granting the claim, by dominated convergence we obtain

$$\mathbb{P}_{\beta,0}[|T_0| < \infty] = \sum_{n\geqslant 1} \mathbb{P}_{\beta,0}[|T_0| = n] = 1 - \theta_d(\beta), \tag{A.30}$$

as desired. To prove the claim, rewrite it as

$$\mathbb{P}_{\beta,h}[|T_0| = \infty, 0\mathfrak{g}] = \lim_{r\to\infty} \mathbb{P}_{\beta,h}[|T_0| \geqslant r, 0\mathfrak{g}] = 0. \tag{A.31}$$

The probabilities inside the limit are probabilities of cylinder events, and hence are limits of finite volume probabilities. For a fixed r the probability is at most $h/(1+rh)$ in finite volume, which vanishes as $r \to \infty$. □

Appendix B: Finite range decomposition

In this appendix, we give the precise references for the construction of the finite range decomposition (3.1). The general method we use was introduced in [12], and presented in the special case we use in [18, Chap. 3] and we will use this reference. For $t > 0$, first recall the polynomials P_t from [18, Chap. 3] (these polynomials are called W_t^* in [12]). These are polynomials of degree bounded by t satisfying

$$\frac{1}{\lambda} = \int_0^\infty t^2 P_t(\lambda)\,\frac{dt}{t}, \qquad 0 \leqslant P_t(u) \leqslant O_s(1+t^2u)^{-s} \tag{B.1}$$

for any $s > 0$ and $u \in [0, 2]$. Our decomposition (3.1) is defined by

$$C_1(x,y) = \frac{1}{(2d+m^2)|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(x-y)} \int_0^{\frac{1}{2}L} t^2 P_t\Big(\frac{\lambda(k)+m^2}{2d+m^2}\Big)\,\frac{dt}{t} \tag{B.2}$$

$$C_j(x,y) = \frac{1}{(2d+m^2)|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(x-y)} \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} t^2 P_t\Big(\frac{\lambda(k)+m^2}{2d+m^2}\Big)\,\frac{dt}{t} \tag{B.3}$$

$$C_{N,N}(x,y) = \frac{1}{(2d+m^2)|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(x-y)} \int_{\frac{1}{2}L^N}^{\infty} t^2 P_t\Big(\frac{\lambda(k)+m^2}{2d+m^2}\Big)\,\frac{dt}{t}, \tag{B.4}$$

where $\lambda(k) = 4\sum_{j=1}^d \sin^2(k_j/2)$ and $\Lambda_N^* \subset [-\pi,\pi)^d$ is the dual torus. The estimates for $C_1, \dots, C_{N-1}$ are straightforward from these Fourier representations and can be found in [18, Chap. 3]. We remark that in [18, Sect. 3.4], the torus covariances are defined by periodisation of the finite range covariances on $\mathbb{Z}^d$; by Poisson summation this is equivalent to the above definition.

The decomposition of $C_{N,N}$ in (3.6) is defined by removing the zero mode from $C_{N,N}$:

$$C_N(x,y) = \frac{1}{(2d+m^2)|\Lambda_N|} \sum_{k\in\Lambda_N^*:k\neq 0} e^{ik\cdot(x-y)} \int_{\frac{1}{2}L^N}^{\infty} t^2 P_t\Big(\frac{\lambda(k)+m^2}{2d+m^2}\Big)\,\frac{dt}{t} \tag{B.5}$$

$$t_N = \frac{1}{2d+m^2} \int_{\frac{1}{2}L^N}^{\infty} t^2 P_t\Big(\frac{m^2}{2d+m^2}\Big)\,\frac{dt}{t}, \tag{B.6}$$

from which (3.6) is immediate. For C_N estimates follows as in [18, Chap. 3]:

$$\begin{aligned} |C_N(x,y)| &\leqslant \frac{1}{|\Lambda_N|} \sum_{k \in \Lambda_N^*: k \neq 0} \left(\int_{\frac{1}{2}L^N}^{\infty} t^2 P_t(\frac{\lambda(k)+m^2}{2d+m^2}) \frac{dt}{t} \right) \\ &\lesssim \frac{1}{|\Lambda_N|} \sum_{k \in \Lambda_N^*: k \neq 0} \left(\int_{\frac{1}{2}L^N}^{\infty} t^2 t^{-2s} |k|^{-2s} \frac{dt}{t} \right) \\ &\lesssim \frac{L^{2N}}{|\Lambda_N|} \sum_{k \in \Lambda_N^*: k \neq 0} L^{-2sN} |k|^{-2s} \\ &\lesssim L^{-(d-2)N} \int_1^{\infty} r^{-2s+d-1} dr \lesssim L^{-(d-2)N} \end{aligned} \tag{B.7}$$

and analogously for the discrete gradients. Finally, by (B.1),

$$\begin{aligned} t_N &= \frac{1}{(2d+m^2)} \int_{\frac{1}{2}L^N}^{\infty} t^2 P_t(\frac{m^2}{2d+m^2}) \frac{dt}{t} \\ &= \frac{1}{m^2} - \frac{1}{(2d+m^2)} \int_0^{\frac{1}{2}L^N} t^2 P_t(\frac{m^2}{2d+m^2}) \frac{dt}{t} = \frac{1}{m^2} - O(L^{2N}). \end{aligned} \tag{B.8}$$

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