Stochastic differential games with controlled regime-switching

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Abstract

In this article, we consider a two-player zero-sum stochastic differential game with regime-switching. Different from the results in existing literature on stochastic differential games with regime-switching, we consider a game between a Markov chain and a state process which are two fully coupled stochastic processes. The payoff function is given by an integral with random terminal horizon. We first study the continuity of the lower and upper value functions under some additional conditions, based on which we establish the dynamic programming principle. We further prove that the lower and upper value functions are unique viscosity solutions of the associated lower and upper Hamilton-Jacobi-Bellman-Isaacs equations with regime-switching, respectively. These two value functions coincide under the Isaacs condition, which implies that the game admits a value. We finally apply our results to an example.

Keywords: Stochastic differential games; controlled regime-switching; dynamic programming principle; viscosity solutions; Hamilton-Jacobi-Bellman-Isaacs equations.

1 Introduction

The study of two-player zero-sum stochastic differential games has been rapidly developed since the pioneering work of Fleming and Souganidis [1]. In that article, the authors proved that the lower and upper value functions satisfy the dynamic programming principle and are the unique viscosity solutions of the associated Bellman-Isaacs partial differential equations, respectively. In the case that the Isaacs condition holds, the lower and upper value functions coincide, which implies that the game admits a value. Following the publication of [1], two-player zero-sum stochastic differential games in different situations were further considered in many works, for example, backward stochastic differential equations were introduced into the study of stochastic differential games in [2], stochastic differential games with jumps were considered in [3], stochastic differential games involving impulse controls were studied in [4], to name but a few.

Note, all the aforementioned works on two-player zero-sum stochastic differential games dealt with Itô diffusions. Due to the needs of modelling fluctuations of random environments arising in game problems and rich applications in real-world contexts, stochastic differential games with regime-switching have drawn considerable attentions in recent years, such as [5–11]. This kind of systems contains two components: the continuous diffusion component describing the evolution of the dynamics of the state process, and the discrete component describing the random switching of the environments. The dynamics of state process is modelled by a stochastic differential equation and the random switching is given by a Markov chain taking values in a finite state space. In these existing literature on stochastic differential games with regime-switching, Markov chains are normally assumed to be independent of the diffusion processes. In addition, both control processes act only on the coefficients of the stochastic differential equations.

In this article, we consider two-player zero-sum stochastic differential games with controlled regime-switching. In contrast to the classical stochastic differential games with regime-switching, two different kinds of controls are considered: one is on the dynamics of the state process, and the other is on the transition rates of the Markov chain. To the best of our knowledge, stochastic optimal control problems with such set-ups were firstly considered in [12], but this kind of stochastic differential games has never been studied in literature. Different from the classical stochastic models with regime-switching, the Markov chain we consider here also depends on the state process. In other words, the Markov chain and the state process are two fully coupled stochastic processes.

These two controls we consider represent two different kinds of control mechanisms. One controller, who is a relatively small player, can influence the state process by improving the mean and the volatility to maximize the payoff function. The other controller, on the contrary, can control the regime choice of the state processes to minimize the payoff function. The regime controller is relatively powerful, e.g., a policy maker can achieve their goal with more substantial impact to the game by controlling the regime choices.

Such kind of games can be found in many real-world contexts, such as in the field of finance and economics. The payoff function can be the tax paid by a group of companies, the Markov chain is the powerful tax policy controlled by the government who tries to maximize the payoff function, the state process is the revenue of the companies who try to minimize the payoff function. Note, higher tax rate may not necessarily imply higher tax collection as it can cause lower productivity. For this reason, in the process of controlling the tax policy, the government takes into the consideration of the revenue of the companies. The latter on the other hand, also depends on the tax regime. The stochastic differential games with controlled regime-switching considered in this article offers the right formulation for the tax-revenue model. The problem with players in unequal status was also considered in some literature on zero-sum stochastic differential games, e.g. in [13, 14], the game involves an agent as one player, who aims to find an optimal portfolio to maximize its own utility and the market as the other player, who controls the underlying probability measure to minimize the utility.

In addition, we consider the game before the state process leaves a bounded domain D. The bounded set could be regarded as the game domain that these two rival game players agree to play on in advance. When the state process is out of the region, the game should be stopped immediately. For example, the game mentioned in the above paragraph should stop as soon as the revenue of the company reaches zero or reaches a certain level.

The main difficulty of studying stochastic differential games with a random horizon is that the continuity of the associated value functions with respect to the time and state variables is not always satisfied due to that the domain D is bounded. But this is key in establishing the dynamic programming principle. In [15], the continuity of the value functions of stochastic optimal control problems was considered under some additional conditions, and more general results with weaker conditions can be found in [16] and [17]. In this article, we prove a continuity result of the value functions for the game problem. Consequently, we give their respective lower and upper dynamic programming equations. This further leads us to prove that they are unique viscosity solutions of the lower and upper regime-switching Hamilton-Jacobi-Bellman-Isaacs equations. These two value functions coincide under the Isaacs condition, which implies that the game admits a value.

The rest of this article is organized as follows. In the following section, we formulate the stochastic differential game problem and give some useful estimations. We prove the continuity of the lower and upper value functions and give the associated lower and upper dynamic programming equations in Section 3, and as the unique viscosity solutions of the lower and upper Hamilton-Jacobi-Bellman-Isaacs equations in Section 4. Finally, we give an example of these results in Section 5.

2 Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a d-dimensional stochastic differential equation (SDE) is considered over the finite time horizon $[0, T]$ for a fixed $T > 0$,

$$\begin{cases} dX_t = b(t, X_t, \theta_t, u_t)dt + \sigma(t, X_t, \theta_t, u_t)dB_t, & 0 \le s \le t \le T, \\ X_s = x \in \mathbb{R}^d, \ \theta_s = i \in S, \end{cases} \tag{1}$$

where $B.$ is a d-dimensional Brownian motion and $\theta.$ is a continuous-time Markov chain taking values in a finite state space S. Let $\{\mathcal{F}_t^s, s \le t \le T\}$ be the natural filtration generated by $B_\cdot$ and $\theta_\cdot$ and augmented by all P-null sets in $\mathcal{F}$. The $\mathcal{F}_t^s$-adapted process u in SDE (1), taking values in a compact subset U of $\mathbb{R}^k$, is called an *admissible* control of the state process X_t. We denote by $\mathcal{U}_s$ the set of all admissible controls u. The coefficients

$$b : [0,T] \times \mathbb{R}^d \times S \times U \to \mathbb{R}^d, \quad \sigma : [0,T] \times \mathbb{R}^d \times S \times U \to \mathbb{R}^{d\times d},$$

are assumed to satisfy the following conditions:
Assumption 1. *1) For any $(x, i) \in \mathbb{R}^d \times S$, the functions $b(\cdot, x, i, \cdot)$ and $\sigma(\cdot, x, i, \cdot)$ are continuous.*
2) There exists a constant $C > 0$ such that for all $0 \le s \le T$, $x, y \in \mathbb{R}^d$, $i \in S$ and $u \in U$,

$$|b(s, x, i, u) - b(s, y, i, u)| + \|\sigma(s, x, i, u) - \sigma(s, y, i, u)\| \le C|x - y|.$$

From Assumption 1, we can get the global linear growth conditions of b and σ with respect to x, *i.e.*, there exists a constant $C > 0$ such that for all $0 \le s \le T$, $x \in \mathbb{R}^d$, $i \in S$ and $u \in U$,

$$|b(s, x, i, u)| + \|\sigma(s, x, i, u)\| \le C(1 + |x|).$$

The Markov chain $\theta_\cdot$ is assumed to satisfy the regularity condition, *i.e.*

$$\lim_{s\downarrow t} P(\theta_s = j|\theta_t = i) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $j = i$ or 0 otherwise. We denote by $\mathcal{V}_s$ the set of all admissible controls v, taking values in another compact subset V of $\mathbb{R}^k$ and being adapted to $\{\mathcal{F}_t^s\}_{t\ge s}$, of the Markov chain $\theta_\cdot$. For any $v \in \mathcal{V}_s$, the infinitesimal transition probabilities of $\theta_\cdot$ are given by

$$P(\theta_{s+\delta} = j|\theta_s = i, X_s = x, v_s = v) = \begin{cases} q_{ij}(x, v)\delta + o(\delta), & \textit{if } j \ne i, \\ 1 + q_{ii}(x, v)\delta + o(\delta), & \textit{otherwise}, \end{cases} \tag{2}$$

where the state-control-dependent transition rates q are assumed to satisfy the following conditions:
Assumption 2. *1) For any $x \in \mathbb{R}^d$ and $v \in V$, $q_{ij}(x, v) \ge 0$, if $j \ne i$.*
2) For any $i, j \in S$, the transition rates $q_{ij}(\cdot, \cdot)$ are bounded and continuous.
3) For any $i \in S$, $x \in \mathbb{R}^d$ and $v \in V$,

$$\sum_{j\in S} q_{ij}(x, v) = 0.$$

In what follows, we denote X_t and θ_t by $X_{t,u,v}^{s,x,i}$ and $\theta_{t,u,v}^{s,x,i}$, respectively, if emphases on the initial conditions and controls are needed.

Consider a non-empty bounded open subset $D \subset \mathbb{R}^d$ with boundary ∂D and closure $\bar{D}$. Without loss of generality, we assume throughout this article that D is connected. Indeed, if D is disconnected, one can solve separately on each connected subset. We define $\tau_{u,v}^{s,x,i}$ as the first exit time of $X_{t,u,v}^{s,x,i}$ from the bounded domain D (or the first hitting time to the boundary ∂D), that is

$$\tau_{u,v}^{s,x,i} := \inf\{t \geq s, X_{t,u,v}^{s,x,i} \notin D\} \wedge T = \inf\{t \geq s, X_{t,u,v}^{s,x,i} \in \partial D\} \wedge T. \tag{3}$$

For notational simplicity, we only give the dependence of τ with respect to the control processes u and v, $\tau_{u,v}$, if no confusion arises.

We consider the following payoff function, for $(s,x,i) \in [0,T] \times \bar{D} \times S$

$$\begin{aligned} J(s,x,i,u,v) := E\bigg[\int_s^{\tau_{u,v}} & f(t, X_{t,u,v}^{s,x,i}, \theta_{t,u,v}^{s,x,i}, u_t, v_t)dt \\ & + g(\tau_{u,v}, X_{\tau_{u,v},u,v}^{s,x,i}, \theta_{\tau_{u,v},u,v}^{s,x,i})\bigg], \end{aligned} \tag{4}$$

where the mappings

$$f : [0,T] \times \mathbb{R}^d \times S \times U \times V \to \mathbb{R}, \quad g : [0,T] \times \mathbb{R}^d \times S \to \mathbb{R},$$

are assumed to satisfy the following conditions:

Assumption 3. *1) For any $(x,i) \in \mathbb{R}^d \times S$, the mappings $(t,u,v) \mapsto f(t,x,i,u,v)$ and $t \mapsto g(t,x,i)$ are continuous.*

2) There exists a constant $C > 0$ such that for any $s \in [0,T]$, $x, y \in \mathbb{R}^d$, $i \in S$, $u \in U$ and $v \in V$,

$$|f(s,x,i,u,v) - f(s,y,i,u,v)| + |g(s,x,i) - g(s,y,i)| \leq C|x-y|.$$

From Assumption 3, we can also get the linear growth conditions of f and g with respect to x. In fact, the functions b, σ, f and g are all bounded in $[0,T] \times \bar{D} \times S \times U \times V$.

Under Assumptions 1, 2 and 3, for each $u \in \mathcal{U}_s$ and $v \in \mathcal{V}_s$, SDE (1) has an unique strong solution, see [18]. Thus the payoff function (4) is well-defined. Moreover, we have the following estimates:

$$\begin{aligned} E \sup_{s \leq t \leq T} \left| X_{t,u,v}^{s,x,i} \right|^2 &\leq C(1 + |x|^2), \\ E \sup_{s \leq t \leq T} \left| X_{t,u,v}^{s,x,i} - X_{t,u,v}^{s,y,i} \right|^2 &\leq L|x-y|^2, \end{aligned} \tag{5}$$

where C and L are some positive constants. By the linear growth conditions of f, g and the estimates (5) about X_t, the payoff function (4) has at most linear growth in x as

$$|J(s,x,i,u,v)| \leq E\left[\int_s^T \left| f(t, X_{t,u,v}^{s,x,i}, \theta_{t,u,v}^{s,x,i}, u_t, v_t) \right| dt + \left| g(\tau_{u,v}, X_{\tau_{u,v},u,v}^{s,x,i}, \theta_{\tau_{u,v},u,v}^{s,x,i}) \right| \right]$$

$$
\begin{aligned}
&\leq C \int_s^T \left(1 + E\left|X_{t,u,v}^{s,x,i}\right|\right) dt + C\left(1 + E\left|X_{\tau_{u,v},u,v}^{s,x,i}\right|\right) \\
&\leq C(T+1)\left(1 + E \sup_{t\in[s,T]} \left|X_{t,u,v}^{s,x,i}\right|\right) \\
&\leq C(1+|x|),
\end{aligned}
$$

where the constant $C > 0$ may vary from line to line.

Besides the admissible controls, we also consider the admissible strategies.

Definition 1. *An admissible strategy for player I is a mapping $\alpha : \mathcal{V}_s \to \mathcal{U}_s$ satisfying that for any $\{\mathcal{F}_t^s\}$-stopping time τ and any $v_1, v_2 \in \mathcal{V}_s$ with $v_1 \equiv v_2$ on $[s,\tau]$, it holds that $\alpha(v_1) \equiv \alpha(v_2)$ on $[s,\tau]$. An admissible strategy β for player II is defined similarly. The set of all admissible strategies α and β are denoted as $\mathcal{A}_s$ and $\mathcal{B}_s$, respectively.*

Associated with the payoff function (4), the lower value function is defined as

$$
W^-(s,x,i) = \inf_{\beta\in\mathcal{B}_s} \sup_{u\in\mathcal{U}_s} J(s,x,i,u,\beta(u)), \tag{6}
$$

and the upper value function is defined as

$$
W^+(s,x,i) = \sup_{\alpha\in\mathcal{A}_s} \inf_{v\in\mathcal{V}_s} J(s,x,i,\alpha(v),v). \tag{7}
$$

Note, $W^+(s,x,i) \leq W^-(s,x,i)$ for all $(s,x,i) \in [0,T] \times \bar{D} \times S$ and both $W^-(s,x,i)$ and $W^+(s,x,i)$ are dominated by $C(1+|x|)$. In the case that $W^- = W^+$, we say that the game admits a value. The main objective in this article is to prove that the game admits a value.

3 Continuity of value functions

One of the most commonly used approaches to solve stochastic differential game problems is to establish the *dynamic programming principle* (DPP) based on the pioneering work of Bellman [19]. However, the continuity of the value functions with respect to the time and state variables, which is key to the establishment of DPP, is normally not satisfied when we consider the first exit time payoff function from a bounded domain D, see [20]. We need some more conditions to guarantee the continuity of the value functions. For the fixed finite horizon or infinite horizon stochastic differential game problems, the continuity holds naturally, see e.g. [6, 21–23].

Moreover, the interaction of the Markov chain and the state process can cause many difficulties. The Markov chain coupled in the evolution of the state process, may jumps from one state to another which leads the regime of the state process to switch to another one. The control-state-dependent Markov chain may have totally different distributions when the corresponding state diffusion processes have different initial positions. All these are difficulties of this problem. Yin and Zhu [18] provided some technical estimations about the dependence of the solution on the initial conditions. We use some of their ideas in the proof of the following lemma.

Lemma 1. *For any continuous function f which is Lipschitz continuous with respect to the state variable and any $T > 0$ there exists a constant $K > 0$ such that*

$$E \int_0^T \left| f(t, X^{x,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{y,i}_{t,u,v}) \right|^2 dt \le K|x-y|^2,$$

and

$$E \left| f(T, X^{x,i}_{T,u,v}, \theta^{x,i}_{T,u,v}) - f(T, X^{y,i}_{T,u,v}, \theta^{y,i}_{T,u,v}) \right| \le K|x-y|.$$

Proof. For any small $\eta > 0$,

$$\begin{aligned}
&E \int_s^{s+\eta} \left| f(t, X^{y,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{y,i}_{t,u,v}) \right|^2 dt \\
\le& KE \int_s^{s+\eta} \left| f(t, X^{y,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{s,u,v}, \theta^{x,i}_{t,u,v}) \right|^2 dt \\
&+ KE \int_s^{s+\eta} \left| f(t, X^{y,i}_{s,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{s,u,v}, \theta^{y,i}_{t,u,v}) \right|^2 dt \\
&+ KE \int_s^{s+\eta} \left| f(t, X^{y,i}_{s,u,v}, \theta^{y,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{y,i}_{t,u,v}) \right|^2 dt.
\end{aligned} \tag{8}$$

The constant K here and below is an universal one, its exact value may be different at different lines. We use this convention throughout the proof, as their exact values are not important.

By Lipschitz continuity, we obtain

$$\begin{aligned}
&E \int_s^{s+\eta} \left| f(t, X^{y,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{s,u,v}, \theta^{x,i}_{t,u,v}) \right|^2 dt \\
\le& KE \int_s^{s+\eta} \left| X^{y,i}_{t,u,v} - X^{y,i}_{s,u,v} \right|^2 dt \\
\le& K \int_s^{s+\eta} (t-s) dt \le K\eta^2.
\end{aligned} \tag{9}$$

Likewise, we also have

$$E \int_s^{s+\eta} \left| f(t, X^{y,i}_{s,u,v}, \theta^{y,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{y,i}_{t,u,v}) \right|^2 dt \le K\eta^2. \tag{10}$$

Note that

$$
\begin{aligned}
& E\int_s^{s+\eta}\left|f(t,X^{y,i}_{s,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{t,u,v})\right|^2 dt\\
\le & KE\int_s^{s+\eta}\left|f(t,X^{y,i}_{s,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{s,u,v})\right|^2 dt\\
& +KE\int_s^{s+\eta}\left|f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{s,u,v})-f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{t,u,v})\right|^2 dt.
\end{aligned}
\tag{11}
$$

The second term in (11) can be estimated as

$$
\begin{aligned}
& E\int_s^{s+\eta}|f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{s,u,v})-f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{t,u,v})|^2 dt\\
=& E\int_s^{s+\eta}|f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{s,u,v})-f(t,X^{y,i}_{s,u,v},\theta^{y,i}_{t,u,v})|^2 I_{\{\theta^{y,i}_{t,u,v}\neq\theta^{y,i}_{s,u,v}\}}dt\\
=& \sum_{m\in S}\sum_{j\neq m}E\int_s^{s+\eta}|f(t,X^{y,i}_{s,u,v},m)-f(t,X^{y,i}_{s,u,v},j)|^2 I_{\{\theta^{y,i}_{s,u,v}=m\}}I_{\{\theta^{y,i}_{t,u,v}=j\}}dt\\
\le & K\sum_{m\in S}\sum_{j\neq m}E\int_s^{s+\eta}[1+|X^{y,i}_{s,u,v}|^2]I_{\{\theta^{y,i}_{s,u,v}=m\}}E[I_{\{\theta^{y,i}_{t,u,v}=j\}}|X^{y,i}_{s,u,v},v_s,\theta^{y,i}_{s,u,v}=m]dt\\
\le & K\sum_{m\in S}E\int_s^{s+\eta}[1+|X^{y,i}_{s,u,v}|^2]I_{\{\theta^{y,i}_{s,u,v}=m\}}\\
& \qquad\qquad\times\left[\sum_{j\neq m}q_{mj}(X^{y,i}_{s,u,v},v_s)(t-s)+o(t-s)\right]dt\\
\le & K\int_s^{s+\eta}(t-s)dt\le K\eta^2.
\end{aligned}
\tag{12}
$$

To treat the first term in (11), a coupling method is used. For $x,\tilde{x}$ and $i,j\in S$, consider the measure $\Gamma((x,j),(\tilde{x},i))=|x-\tilde{x}|+d(j,i)$, where $d(j,i)=0$ if $j=i$ and $d(j,i)=1$ if $j\neq i$. That is, $\Gamma(\cdot,\cdot)$ is a measure obtained by piecing the usual Euclidean length of two vectors and the discrete measure together. Let $(\theta^{y,i}_s,\theta^{x,i}_s)$ be a discrete random process with a finite state space $S\times S$ such that

$$
\begin{aligned}
P[(\theta^{y,i}_{t+h},\theta^{x,i}_{t+h})=(m,n)|(\theta^{y,i}_t,\theta^{x,i}_t)=(k,l),(X^{y,i}_t,X^{x,i}_t)=(\tilde{y},\tilde{x}),v_t=v]\\
=\begin{cases} q_{(k,l)(m,n)}(\tilde{y},\tilde{x},v)h+o(h), & if\ (m,n)\neq(k,l),\\ 1+q_{(k,l)(k,l)}(\tilde{y},\tilde{x},v)h+o(h), & if\ (m,n)=(k,l),\end{cases}
\end{aligned}
$$

where the transition rates satisfy, for any function $\tilde{f}$ defined on $S\times S$

$$\begin{aligned}
&\sum_{(j,i)\in S\times S} q_{(k,l)(j,i)}(x,\tilde{x},v)(\tilde{f}(j,i)-\tilde{f}(k,l))\\
=&\sum_j (q_{kj}(x,v)-q_{lj}(\tilde{x},v))^+(\tilde{f}(j,l)-\tilde{f}(k,l))\\
&+\sum_j (q_{lj}(\tilde{x},v)-q_{kj}(x,v))^+(\tilde{f}(k,j)-\tilde{f}(k,l))\\
&+\sum_j (q_{kj}(x,v)\wedge q_{lj}(\tilde{x},v))(\tilde{f}(j,j)-\tilde{f}(k,l)).
\end{aligned}$$

Owing to the coupling defined above, for $t\in[s,s+\eta)$

$$\begin{aligned}
&E[I_{\{\theta_t^{x,i}=j\}}|\theta_s^{x,i}=i_1,\theta_s^{y,i}=i_2,X_s^{x,i}=\tilde{x},X_s^{y,i}=\tilde{y},v_s=v]\\
=&\sum_{l\in S} E[I_{\{\theta_t^{x,i}=j,\theta_t^{y,i}=l\}}|\theta_s^{x,i}=i_1,\theta_s^{y,i}=i_2,X_s^{x,i}=\tilde{x},X_s^{y,i}=\tilde{y},v_s=v]\\
=&\sum_{l\in S}\tilde{q}_{(i_1,i_2)(j,l)}(\tilde{x},\tilde{y},v)(t-s)+o(t-s)
\end{aligned}$$

So we have

$$\begin{aligned}
&E\int_s^{s+\eta}|f(t,X_{s,u,v}^{y,i},\theta_{t,u,v}^{x,i})-f(t,X_{s,u,v}^{y,i},\theta_{s,u,v}^{y,i})|^2dt\\
=&E\sum_{i_1\in S}\sum_{j\neq i_1}\int_s^{s+\eta}|f(t,X_{s,u,v}^{y,i},j)-f(t,X_{s,u,v}^{y,i},i_1)|^2 I_{\{\theta_{t,u,v}^{x,i}=j\}}I_{\{\theta_{s,u,v}^{y,i}=i_1\}}dt\\
\le&E\sum_{i_1,i_2\in S}\sum_{j\neq i_1}\int_s^{s+\eta}\left[1+|X_{s,u,v}^{y,i}|^2\right]I_{\{\theta_{s,u,v}^{y,i}=i_1,\theta_{s,u,v}^{x,i}=i_2\}}\\
&\qquad\times E\left[I_{\{\theta_{t,u,v}^{x,i}=j\}}|\theta_{s,u,v}^{x,i}=i_1,\theta_{s,u,v}^{y,i}=i_2,X_{s,u,v}^{x,i},X_{s,u,v}^{y,i},v_s\right]dt\\
\le&K\eta^2.
\end{aligned} \tag{13}$$

We thus now have proved from (9), (10), (12), (13) that (8) turns out to be

$$E\int_s^{s+\eta}|f(t,X_{t,u,v}^{y,i},\theta_{t,u,v}^{x,i})-f(t,X_{t,u,v}^{y,i},\theta_{t,u,v}^{y,i})|^2dt\le K\eta^2.$$

Then, we have

$$
\begin{aligned}
&E\int_0^T |f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2dt\\
\leq &\sum_{k=0}^{[T/\eta]+1} E\int_{k\eta}^{k\eta+\eta} |f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2dt \qquad (14)\\
\leq &K\eta.
\end{aligned}
$$

Similarly,

$$
E\int_T^{T+\eta} |f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2dt \leq K\eta^2.
$$

Here we noted that both processes X_s and θ_s are defined beyond T. For any $\delta > 0$,

$$
\frac{1}{\eta}E\int_T^{T+\eta} \frac{|f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2}{\eta+\delta}dt \leq K.
$$

In the limit of $\eta \to 0$, we have

$$
\lim_{\eta\to 0}\frac{1}{\eta}E\int_T^{T+\eta} \frac{|f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2}{\eta+\delta}dt \leq K,
$$

which implies that

$$
\frac{d}{d\eta}\int_T^{T+\eta} \frac{E|f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2}{\eta+\delta}dt\bigg|_{\eta=0} \leq K,
$$

i.e.

$$
\begin{aligned}
&-\int_T^{T+\eta} \frac{E|f(t,X^{y,i}_{t,u,v},\theta^{x,i}_{t,u,v})-f(t,X^{y,i}_{t,u,v},\theta^{y,i}_{t,u,v})|^2}{(\eta+\delta)^2}dt\bigg|_{\eta=0}\\
&+\frac{E|f(T,X^{y,i}_{T,u,v},\theta^{x,i}_{T,u,v})-f(T,X^{y,i}_{T,u,v},\theta^{y,i}_{T,u,v})|^2}{\eta+\delta}\bigg|_{\eta=0} \leq K.
\end{aligned}
$$

Here it is desirable to consider the interval $[T, T+\eta]$ rather than $[T-\eta, T]$ as the Markov chain $\theta_s, s \geq 0$ is only right-continuous. Hence, we have

$$
E|f(T,X^{y,i}_{T,u,v},\theta^{x,i}_{T,u,v})-f(T,X^{y,i}_{T,u,v},\theta^{y,i}_{T,u,v})|^2 \leq K\delta. \qquad (15)
$$

Since the η in (14) and the δ in (15) are arbitrary, let both η in (14) and δ in (15) be $|x-y|^{\gamma_0}$ with $\gamma_0 > 2$, then from estimates in (5) we have

$$
\begin{aligned}
& E\int_0^T |f(t, X^{x,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{y,i}_{t,u,v})|^2 dt \\
\leq & E\int_0^T |f(t, X^{x,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{x,i}_{t,u,v})|^2 dt \\
& + E\int_0^T |f(t, X^{y,i}_{t,u,v}, \theta^{x,i}_{t,u,v}) - f(t, X^{y,i}_{t,u,v}, \theta^{y,i}_{t,u,v})|^2 dt \\
\leq & KE\int_0^T |X^{x,i}_{t,u,v} - X^{y,i}_{t,u,v}|^2 dt + o(|x-y|^2) \\
\leq & K|x-y|^2.
\end{aligned}
\tag{16}
$$

Similarly, we have

$$
E|f(T, X^{x,i}_{T,u,v}, \theta^{x,i}_{T,u,v}) - f(T, X^{y,i}_{T,u,v}, \theta^{y,i}_{T,u,v})| \leq K|x-y|. \tag{17}
$$

□

In this section, we mainly give the proof of the continuity of the lower value function $W^-(s,x,i)$ with respect to the time and state variables, the proof for the upper value function $W^+(s,x,i)$ is analogous.

Let $\psi : \mathbb{R}^d \times S \to \mathbb{R}$ be a function such that for any $x, y \in \mathbb{R}^d$ and $i \in S$,

$$
|\psi^+(x,i) - \psi^+(y,i)| \leq C|x-y|, \tag{18}
$$

where $\psi^+ = \max\{\psi, 0\}$. For any $\varepsilon > 0$, define

$$
\Gamma^{s,x,i}_\varepsilon(t) := \exp\left\{ -\frac{1}{\varepsilon}\int_s^t \psi^+(X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}) dr \right\},\ t \in [s,T].
$$

Note, $0 < \Gamma^{s,x,i}_\varepsilon(t) \leq 1$. Consider an auxiliary payoff function

$$
\begin{aligned}
& J^\varepsilon(s,x,i,u,v) \\
=& E\left[\int_s^T \Gamma^{s,x,i}_\varepsilon(t) f(t, X^{s,x,i}_{t,u,v}, \theta^{s,x,i}_{t,u,v}, u_t, v_t) dt + \Gamma^{s,x,i}_\varepsilon(T) g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v}) \right],
\end{aligned}
\tag{19}
$$

and the corresponding value function

$$
W^\varepsilon(s,x,i) = \inf_{v\in\mathcal{V}_s} \sup_{u\in\mathcal{U}_s} J^\varepsilon(s,x,i,u,v). \tag{20}
$$

We will prove that for each fixed $i \in S$, the value function $W^\varepsilon(\cdot,\cdot,i)$ is continuous. Then by the approximation technique, we deduce that the lower value function $W^-(\cdot,\cdot,i)$ is also continuous.

Theorem 1. *For each fixed $\varepsilon > 0$ and $i \in S$, the value function $W^\varepsilon(s,x,i)$ is continuous with respect to s and x.*

Proof. We will divide the proof into two steps.

Step 1. In this part, we will prove that $W^\varepsilon(s,x,i)$ is continuous with respect to x. From the definition (20) of $W^\varepsilon(s,x,i)$, we have

$$\begin{aligned}
&|W^\varepsilon(s,x,i) - W^\varepsilon(s,y,i)| \\
=&\left|\inf_{v\in\mathcal{V}_s}\sup_{u\in\mathcal{U}_s} J^\varepsilon(s,x,i,u,v) - \inf_{v\in\mathcal{V}_s}\sup_{u\in\mathcal{U}_s} J^\varepsilon(s,y,i,u,v)\right| \\
\leq&\sup_{u,v}|J^\varepsilon(s,x,i,u,v) - J^\varepsilon(s,y,i,u,v)| \\
\leq&\sup_{u,v} E\int_s^T \left|\Gamma_\varepsilon^{s,x,i}(t) f(t, X^{s,x,i}_{t,u,v}, \theta^{s,x,i}_{t,u,v}, u_t, v_t) - \Gamma_\varepsilon^{s,y,i}(t) f(t, X^{s,y,i}_{t,u,v}, \theta^{s,y,i}_{t,u,v}, u_t, v_t)\right| dt \\
&+\sup_{u,v} E\left|\Gamma_\varepsilon^{s,x,i}(T) g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v}) - \Gamma_\varepsilon^{s,y,i}(T) g(T, X^{s,y,i}_{T,u,v}, \theta^{s,y,i}_{T,u,v})\right|.
\end{aligned}$$

By the Cauchy-Schwartz inequality and the fact that $0 < \Gamma_\varepsilon^{s,x,i}(t) \leq 1$

$$\begin{aligned}
&E\left|\Gamma_\varepsilon^{s,x,i}(T) g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v}) - \Gamma_\varepsilon^{s,y,i}(T) g(T, X^{s,y,i}_{T,u,v}, \theta^{s,y,i}_{T,u,v})\right| \\
\leq&E\left[\left|\Gamma_\varepsilon^{s,x,i}(T) - \Gamma_\varepsilon^{s,y,i}(T)\right| \times \left|g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v})\right|\right] \\
&+E\left[\left|\Gamma_\varepsilon^{s,y,i}(T)\right| \times \left|g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v}) - g(T, X^{s,y,i}_{T,u,v}, \theta^{s,y,i}_{T,u,v})\right|\right] \\
\leq&\left(E\left|\Gamma_\varepsilon^{s,x,i}(T) - \Gamma_\varepsilon^{s,y,i}(T)\right|^2\right)^{1/2} \times \left(E\left|g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v})\right|^2\right)^{1/2} \\
&+E\left|g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v}) - g(T, X^{s,y,i}_{T,u,v}, \theta^{s,y,i}_{T,u,v})\right|.
\end{aligned} \tag{21}$$

Moreover, noting that $|e^{-a} - e^{-b}| \leq |a-b|$ for any $a, b \geq 0$ and using the estimates (5) and (16), we can get that

$$\begin{aligned}
&E|\Gamma_\varepsilon^{s,x,i}(T) - \Gamma_\varepsilon^{s,y,i}(T)|^2 \\
\leq&\frac{K}{\varepsilon^2} E\int_s^T \left|\psi^+(X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}) - \psi^+(X^{s,y,i}_{r,u,v}, \theta^{s,y,i}_{r,u,v})\right|^2 dr \\
\leq&\frac{K}{\varepsilon^2}|x-y|^2,
\end{aligned}$$

and by (17)

$$E\left|g(T, X^{s,x,i}_{T,u,v}, \theta^{s,x,i}_{T,u,v}) - g(T, X^{s,y,i}_{T,u,v}, \theta^{s,y,i}_{T,u,v})\right| \leq K|x-y|.$$

Thus, we have

$$E\left|\Gamma_\varepsilon^{s,x,i}(T)g(T,X^{s,x,i}_{T,u,v},\theta^{s,x,i}_{T,u,v})-\Gamma_\varepsilon^{s,y,i}(T)g(T,X^{s,y,i}_{T,u,v},\theta^{s,y,i}_{T,u,v})\right|\le C'_\varepsilon|x-y|,$$

where the constant C'_ε does not depend on s,y and u,v. We similarly also have by (16) again that

$$\begin{aligned}&E\int_s^T|\Gamma_\varepsilon^{s,x,i}(t)f(t,X^{s,x,i}_{t,u,v},\theta^{s,x,i}_{t,u,v},u_t,v_t)-\Gamma_\varepsilon^{s,y,i}(t)f(t,X^{s,y,i}_{t,u,v},\theta^{s,y,i}_{t,u,v},u_t,v_t)|dt\\ \le& C''_\varepsilon|x-y|.\end{aligned}$$

Hence, we get there exists a constant $L_\varepsilon>0$ such that

$$|W^\varepsilon(s,x,i)-W^\varepsilon(s,y,i)|\le L_\varepsilon|x-y|. \tag{22}$$

That is, the value function $W^\varepsilon(s,x,i)$ is continuous with respect to x.

Step 2. In this step, we will prove that $W^\varepsilon(s,x,i)$ is continuous with respect to s. With the continuity of $W^\varepsilon(s,x,i)$ with respect to x, we have the following dynamic programming equation, for any small $\delta>0$

$$\begin{aligned}W^\varepsilon(s,x,i)=\inf_{v\in\mathcal{V}_s}\sup_{u\in\mathcal{U}_s}E\Bigg[&\int_s^{s+\delta}\Gamma_\varepsilon^{s,x,i}(r)f(r,X^{s,x,i}_{r,u,v},\theta^{s,x,i}_{r,u,v},u_r,v_r)dr\\ &+\Gamma_\varepsilon^{s,x,i}(s+\delta)W^\varepsilon(s+\delta,X^{s,x,i}_{s+\delta,u,v},\theta^{s,x,i}_{s+\delta,u,v})\Bigg].\end{aligned}$$

Then, we have

$$\begin{aligned}&|W^\varepsilon(s+\delta,x,i)-W^\varepsilon(s,x,i)|\\ =&\Bigg|\inf_{v\in\mathcal{V}_s}\sup_{u\in\mathcal{U}_s}E\Bigg[\int_s^{s+\delta}\Gamma_\varepsilon^{s,x,i}(r)f(r,X^{s,x,i}_{r,u,v},\theta^{s,x,i}_{r,u,v},u_r,v_r)dr\\ &\quad+\Gamma_\varepsilon^{s,x,i}(s+\delta)W^\varepsilon(s+\delta,X^{s,x,i}_{s+\delta,u,v},\theta^{s,x,i}_{s+\delta,u,v})-W^\varepsilon(s+\delta,x,i)\Bigg]\Bigg|\\ \le&\sup_{u,v}E\int_s^{s+\delta}|f(r,X^{s,x,i}_{r,u,v},\theta^{s,x,i}_{r,u,v},u_r,v_r)|dr\\ &+\sup_{u,v}E|\Gamma_\varepsilon^{s,x,i}(s+\delta)W^\varepsilon(s+\delta,X^{s,x,i}_{s+\delta,u,v},\theta^{s,x,i}_{s+\delta,u,v})-W^\varepsilon(s+\delta,x,i)|\\ \le&C\delta+\sup_{u,v}E|W^\varepsilon(s+\delta,X^{s,x,i}_{s+\delta,u,v},\theta^{s,x,i}_{s+\delta,u,v})-W^\varepsilon(s+\delta,x,\theta^{s,x,i}_{s+\delta,u,v})|\\ &+\sup_{u,v}E|W^\varepsilon(s+\delta,x,\theta^{s,x,i}_{s+\delta,u,v})-W^\varepsilon(s+\delta,x,i)|\\ &+\sup_{u,v}E|(\Gamma_\varepsilon^{s,x,i}(s+\delta)-1)W^\varepsilon(s+\delta,X^{s,x,i}_{s+\delta,u,v},\theta^{s,x,i}_{s+\delta,u,v})|.\end{aligned}$$

By (22),

$$
\begin{aligned}
&E|W^\varepsilon(s+\delta, X^{s,x,i}_{s+\delta,u,v}, \theta^{s,x,i}_{s+\delta,u,v}) - W^\varepsilon(s+\delta, x, \theta^{s,x,i}_{s+\delta,u,v})| \\
\leq &\sum_{j\in S} E\left[I_{\{\theta^{s,x,i}_{s+\delta,u,v}=j\}} \times |W^\varepsilon(s+\delta, X^{s,x,i}_{s+\delta,u,v}, j) - W^\varepsilon(s+\delta, x, j)| \right] \\
\leq &K L_\varepsilon E|X^{s,x,i}_{s+\delta,u,v} - x|.
\end{aligned}
$$

Moreover, it is easy to see that

$$
E|X^{s,x,i}_{s+\delta,u,v} - x| \leq C(\delta + \delta^{1/2}),
$$

and

$$
\begin{aligned}
&E\left| W^\varepsilon(s+\delta, x, \theta^{s,x,i}_{s+\delta,u,v}) - W^\varepsilon(s+\delta, x, i) \right| \\
= &\sum_{j\neq i} P\{\theta^{s,x,i}_{s+\delta,u,v} = j\} \times |W^\varepsilon(s+\delta, x, j) - W^\varepsilon(s+\delta, x, i)| \\
\leq &CP\{\theta^{s,x,i}_{s+\delta,u,v} \neq i\} \leq K\delta + o(\delta).
\end{aligned}
$$

The last inequality follows from (2). And

$$
\begin{aligned}
&E|(\Gamma^{s,x,i}_\varepsilon(s+\delta) - 1) W^\varepsilon(s+\delta, X^{s,x,i}_{s+\delta,u,v}, \theta^{s,x,i}_{s+\delta,u,v})| \\
\leq &(E|W^\varepsilon(s+\delta, X^{s,x,i}_{s+\delta,u,v}, \theta^{s,x,i}_{s+\delta,u,v})|^2)^{1/2} (E|\Gamma^{s,x,i}_\varepsilon(s+\delta) - 1|^2)^{1/2} \\
\leq &\frac{K}{\varepsilon}(E|\int_s^{s+\delta} \psi^+(X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}) dr|^2)^{1/2} \\
\leq &\frac{K\delta}{\varepsilon}.
\end{aligned}
$$

Thus, we obtain that

$$
|W^\varepsilon(s,x,i) - W^\varepsilon(s+\delta, x, i)| \to 0, \quad as \quad \delta \to 0.
$$

The continuity of $W^\varepsilon(s,x,i)$ with respect to s is proved. □

To see the continuity of the lower value function $W^-(s,x,i)$, we need to make more assumptions on ψ.

Assumption 4. *There exists a function $\psi : \mathbb{R}^d \times S \to \mathbb{R}$ satisfying (18) and that*

$$
\psi(x,i) \leq 0, \quad \forall (x,i) \in \bar{D} \times S,
$$

and for any $x \in \partial D$, there exists some $v \in \mathcal{V}_s$ such that

$$
\inf_{u\in\mathcal{U}_s} \int_s^t \psi^+(X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}) dr > 0, \quad a.s.\ for\ any\ t \in (s,T]. \tag{23}
$$

Remark 1. *For each u, v, the set of regular points is defined as*

$$\Gamma := \{x \in \partial D : P(T_{u,v}^{s,x,i} > s) = 0\},$$

where $T_{u,v}^{s,x,i} := \inf\{t > s, X_{t,u,v}^{s,x,i} \notin \bar{D}\}$. If $\partial D = \Gamma$, then the Assumption 4 with (23) without taking $\inf_{u \in \mathcal{U}_s}$ on (23) holds naturally, see Proposition 8 in [17]. If the domain D is convex with a continuous boundary and σ is nondegenerate, we can easily get the regular points set $\Gamma = \partial D$, see such as [24, 25]. But we need (23) uniformly for all control processes $u \in \mathcal{U}_s$. This is given by the following observation.

We say that σ has bounded inverse if

$$\|\sigma^{-1}\|_\infty < \infty.$$

We assume that the domain D is convex with a continuous boundary, σ is bounded with a bounded inverse.

Consider a SDE:

$$\begin{cases} d\tilde{X}_t = \sigma(t, \tilde{X}_t, \theta_t, u_t)dB_t, \quad 0 \le s \le t \le T, \\ \tilde{X}_s = x \in \mathbb{R}^d, \;\; \theta_s = i \in S, \end{cases}$$

Let $\tilde{P}(d\omega) = P(d\omega)M_t$, where

$$M_t = \exp\left\{-\frac{1}{2}\int_s^t |\alpha(r)|^2 dr - \int_s^t \alpha(r)dB_r\right\},$$

and $\alpha(r) = -\sigma^{-1}(r, \tilde{X}_r, \theta_r, u_r)b(r, \tilde{X}_r, \theta_r, u_r)$. The Girsanov theorem implies that

$$\tilde{B}_t = B_t + \int_s^t \alpha(r)dr$$

is a Brownian motion under the probability measure $\tilde{P}$. There exists a Brownian motion $(V_t, t \ge s)$ with initial value $V_s = x$ such that for all $t \ge s$,

$$\tilde{X}_t = V_{s+[\tilde{X}]_t}.$$

By (23), we let $\psi(x, i) = -dist(x, \partial D)$ if $x \in D$ or $dist(x, \partial D)$ if $x \notin D$. Now consider a fixed time $t > s$. We have for $i \in S$

$$\int_s^t \psi^+(V(r), i)dr > 0 \;\text{ almost surely.}$$

Since $V.$ is a Brownian motion which is independent of the control processes and there exist positive constants $\lambda_1 < \lambda_2$ independent of u, v and x such that $\frac{1}{r-s}[X]_r \in [\lambda_1, \lambda_2]$

for any $r > s$, so we have

$$\inf_{u\in\mathcal{U}_s}\int_s^t \psi^+(V(r),i)dr > 0 \ \text{ almost surely,}$$

i.e.,

$$\inf_{u\in\mathcal{U}_s}\int_s^t \psi^+(\tilde{X}^{s,x,i}_{r,u,v},i)dr > 0 \ \text{ almost surely.}$$

The Markov chain $\theta.$ becomes a new Markov chain with different transition rates under the probability measure $\tilde{P}$, so $\tilde{X}$ is not the solution of SDE (1) under the probability space $(\Omega, \mathcal{F}, \tilde{P})$. Consider the SDEs

$$\begin{cases} dX^1_t = b(t, X^1_t, \theta^1_t, u_t)dt + \sigma(t, X^1_t, \theta^1_t, u_t)dB_t \\ X^1_s = x \ \ \theta^1_s = i, \end{cases}$$

and

$$\begin{cases} dX^2_t = b(t, X^2_t, \theta^2_t, u_t)dt + \sigma(t, X^2_t, \theta^2_t, u_t)dB_t \\ X^2_s = x \ \ \theta^2_s = i. \end{cases}$$

For any small enough $\delta > 0$ and $t > s$ satisfying $t - s < \delta$, the solutions X^1_t and X^2_t have identical distribution because of the regularity of the Markov chains. So, we finally have

$$\inf_{u\in\mathcal{U}_s}\int_s^t \psi^+(X^{s,x,i}_{r,u,v},i)dr > 0 \ \text{ almost surely.}$$

Thus in this case, Assumption (4) holds.

Theorem 2. *In addition to Assumptions 1, 2, 3 and 4, we also suppose that the terminal function $g(\cdot,\cdot,i) \in C^{1,2}$ for each fixed $i \in S$, and that*

$$\partial_s g(s,x,i) + H(s,x,i,u,v,g,Dg,D^2g) \geq 0, \tag{24}$$

where the Hamiltonian function H is defined as

$$\begin{aligned} &H(s,x,i,u,v,g,Dg,D^2g) \\ =&f(s,x,i,u,v) + Dg(s,x,i)\cdot b(s,x,i,u) + \frac{1}{2}tr[\sigma\sigma'(s,x,i,u)D^2g(s,x,i)] \\ &+\sum_{j\neq i} q_{ij}(x,v)(g(s,x,j) - g(s,x,i)). \end{aligned}$$

Then $W^-(\cdot,\cdot,i)$ is continuous.

Proof. We divide the proof into two steps.

Step 1. Assume that $f \geq 0$ and $g \equiv 0$. Then by (23), for any $x \in \partial D$, $u \in \mathcal{U}_s$ and some $v \in \mathcal{V}_s$,

$$\lim_{\varepsilon \downarrow 0} \Gamma_\varepsilon^{s,x,i}(t) = \lim_{\varepsilon \downarrow 0} \exp\{-\frac{1}{\varepsilon}\int_s^t \psi^+(X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i})dr\} = 0, \quad a.s. \ for \ any \ t \in (s,T].$$

Hence, by the dominated convergence theorem, we obtain for any $x \in \partial D$ and $s \in [0,T]$

$$\lim_{\varepsilon \downarrow 0} W^\varepsilon(s,x,i) = 0. \tag{25}$$

Let

$$h(\varepsilon) := \sup\{W^\varepsilon(s,x,i) : (s,x,i) \in [0,T] \times \partial D \times S\},$$

we have $\lim_{\varepsilon \to 0} h(\varepsilon) = 0$ by the Dini's theorem. For any $(s,x,i) \in [0,T] \times \bar{D} \times S$, by the dynamic programming equation for W^ε, we have

$$\begin{aligned}
&W^\varepsilon(s,x,i) \\
&= \inf_{v \in \mathcal{V}_s} \sup_{u \in \mathcal{U}_s} E\left[\int_s^{\tau_{u,v}} f(r, X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i}, u_r, v_r)dr + W^\varepsilon(\tau_{u,v}, X_{\tau_{u,v},u,v}^{s,x,i}, \theta_{\tau_{u,v},u,v}^{s,x,i})\right] \\
&\leq \inf_{v \in \mathcal{V}_s} \sup_{u \in \mathcal{U}_s} E\int_s^{\tau_{u,v}} f(r, X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i}, u_r, v_r)dr + h(\varepsilon) \\
&= W^-(s,x,i) + h(\varepsilon).
\end{aligned} \tag{26}$$

Note that, for all $r \in [s, \tau_{u,v}]$, $X_r \in \bar{D}$. Thus, we have $\Gamma_\varepsilon^{s,x,i}(r) = 1$. By the definitions of J^ε and $\tau_{u,v}$ and the assumption that $g \equiv 0$, we have

$$\begin{aligned}
&J^\varepsilon(s,x,i,u,v) \\
&= E\int_s^T \Gamma_\varepsilon^{s,x,i}(r) f(r, X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i}, u_r, v_r)dr \\
&= E\Big[\int_s^{\tau_{u,v}} f(r, X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i}, u_r, v_r)dr \\
&\quad + I_{\{\tau_{u,v} < T\}} \int_{\tau_{u,v}}^T \Gamma_\varepsilon^{s,x,i}(r) f(r, X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i}, u_r, v_r)dr\Big] \\
&= J(s,x,i,u,v) + E\left[I_{\{\tau_{u,v} < T\}} \int_{\tau_{u,v}}^T \Gamma_\varepsilon^{s,x,i}(r) f(r, X_{r,u,v}^{s,x,i}, \theta_{r,u,v}^{s,x,i}, u_r, v_r)dr\right] \\
&\geq J(s,x,i,u,v).
\end{aligned}$$

That is,

$$W^\varepsilon(s,x,i) \geq W^-(s,x,i). \tag{27}$$

Combining (26) and (27), we have

$$W^-(s,x,i) \leq W^\varepsilon(s,x,i) \leq W^-(s,x,i) + h(\varepsilon).$$

This implies that $W^\varepsilon \to W^-$ uniformly on $[0,T] \times \bar{D} \times S$. Since W^ε is continuous with respect to s and x, so does W^-.

Step 2. For general f and g satisfying (24), let

$$\tilde{f}(s,x,i,u,v) = \partial_s g(s,x,i) + H(s,x,i,u,v,g,Dg,D^2 g),$$

and $\tilde{g} \equiv 0$. Then condition (24) implies that $\tilde{f} \geq 0$, and Step 1 implies that

$$\begin{aligned}\tilde{W}(s,x,i) &= \inf_{v\in\mathcal{V}_s} \sup_{u\in\mathcal{U}_s} \tilde{J}(s,x,i,u,v)\\ &= \inf_{v\in\mathcal{V}_s} \sup_{u\in\mathcal{U}_s} E\left[\int_s^{\tau_{u,v}} \tilde{f}(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}, u_r, v_r)dr\right],\end{aligned}$$

is continuous with respect to s and x. Now applying Itô's formula and Dynkin's formula to g from s to $\tau_{u,v}$, we have

$$\begin{aligned}&E[g(\tau_{u,v}, X^{s,x,i}_{\tau_{u,v},u,v}, \theta^{s,x,i}_{\tau_{u,v},u,v}) - g(s,x,i)]\\ =&E\int_s^{\tau_{u,v}} \partial_s g(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}) + Dg(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}) \cdot b(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}, u_r)\\ &+ \frac{1}{2} tr[\sigma\sigma'(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}, u_r) D^2 g(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v})]\\ &+ \sum_{j\neq\theta^{s,x,i}_{r,u,v}} q_{\theta^{s,x,i}_{r,u,v},j}(X^{s,x,i}_{r,u,v}, v_r)(g(r, X^{s,x,i}_{r,u,v}, j) - g(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}))dr.\end{aligned}$$

Then it follows that

$$\begin{aligned}\tilde{J}(s,x,i,u,v) =&E\int_s^{\tau_{u,v}} \tilde{f}(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}, u_r, v_r)dr\\ =&E\int_s^{\tau_{u,v}} f(r, X^{s,x,i}_{r,u,v}, \theta^{s,x,i}_{r,u,v}, u_r, v_r)dr\\ &+ E[g(\tau_{u,v}, X^{s,x,i}_{\tau_{u,v},u,v}, \theta^{s,x,i}_{\tau_{u,v},u,v}) - g(s,x,i)]\\ =&J(s,x,i,u,v) - g(s,x,i).\end{aligned}$$

Therefore, we conclude that $W^-(s,x,i) = \tilde{W}(s,x,i) + g(s,x,i)$ is continuous. □

With the continuity of W^- and W^+ at hand, we can establish the DPP satisfied by W^- and W^+, respectively. By virtue of the same procedure as in the proof of Theorem 4.1 of Buckdahn and Nie [24] employing the stochastic backward semigroup introduced by Peng [26], we give the following theorem without proof.

Theorem 3 (Dynamic programming principle)**.** *For any $\mathcal{F}_t^s$-stopping time $\Theta \geq s$, we have the following lower dynamic programming equation*

$$W^-(s,x,i) = \inf_{\beta\in\mathcal{B}_s} \sup_{u\in\mathcal{U}_s} E\left[\int_s^{\tau_{u,\beta(u)}\wedge\Theta} f(r,X_r,\theta_r,u_r,\beta(u)_r)dr \right. \\ \left. +W^-(\tau_{u,\beta(u)}\wedge\Theta, X_{\tau_{u,\beta(u)}\wedge\Theta}, \theta_{\tau_{u,\beta(u)}\wedge\Theta})\right], \tag{28}$$

and the upper dynamic programming equation

$$W^+(s,x,i) = \sup_{\alpha\in\mathcal{A}_s} \inf_{v\in\mathcal{V}_s} E\left[\int_s^{\tau_{\alpha(v),v}\wedge\Theta} f(r,X_r,\theta_r,\alpha(v)_r,v_r)dr \right. \\ \left. +W^+(\tau_{\alpha(v),v}\wedge\Theta, X_{\tau_{\alpha(v),v}\wedge\Theta}, \theta_{\tau_{\alpha(v),v}\wedge\Theta})\right]. \tag{29}$$

4 Viscosity solutions

In this section, we consider the following Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations with regime-switching: the lower case

$$\begin{cases} \partial_s W^-(s,x,i) + \inf_{v\in V}\sup_{u\in U} H(s,x,i,u,v,W^-,DW^-,D^2W^-) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (s,x,i)\in[0,T)\times D\times S, \\ W^-(s,x,i) = g(s,x,i), \quad (s,x,i)\in D_T\times S, \end{cases} \tag{30}$$

and the upper case

$$\begin{cases} \partial_s W^+(s,x,i) + \sup_{u\in U}\inf_{v\in V} H(s,x,i,u,v,W^+,DW^+,D^2W^+) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (s,x,i)\in[0,T)\times D\times S, \\ W^+(s,x,i) = g(s,x,i), \quad (s,x,i)\in D_T\times S, \end{cases} \tag{31}$$

where $D_T := \{T\}\times\bar{D} \ \cup \ [0,T]\times\partial D$ and the Hamiltonian function H is defined as in Theorem 2.

If for each $i\in S$, the lower and upper value functions $W^-(\cdot,\cdot,i)$ and $W^+(\cdot,\cdot,i)$ are both in $C^{1,2}([0,T]\times\bar{D})$, then they are the unique classical solutions of the lower and upper HJBI equations (30) and (31), respectively. By the continuities and the measurable selection theorem (see [27] and references therein), we can further prove that there exist optimal controls and strategies of this game, see such as [28–30]. However, the value functions are usually not smooth enough, to treat the non-smoothness case, viscosity solution as a kind of weak solution was introduced originally in [31] (another kind of weak solution: Sobolev weak solution, of such nonlinear partial differential equations (PDEs) is considered in [32]). In this section, we will prove that the lower and upper value functions $W^-(s,x,i)$ and $W^+(s,x,i)$ defined as in (6) and (7) are unique viscosity solutions of the lower and upper HJBI equations (30) and (31), respectively. Furthermore, the lower and upper value functions coincide under the Isaacs condition, which implies that the game admits a value, *i.e.*, $W^-(s,x,i) = W^+(s,x,i)$ for all $(s,x,i) \in [0,T] \times \bar{D} \times S$.

In this section, we mainly focus on the proof that the lower value function $W^-(s,x,i)$ is the unique viscosity solution of the lower HJBI equation (30). The proof of that the upper value function $W^+(s,x,i)$ is the unique viscosity solution of the upper HJBI equation (31) can be done similarly.

Definition 2. *A continuous function $w(s,x,i)$ is called a viscosity subsolution of the lower HJBI equation (30) if $w(s,x,i) \le g(s,x,i)$ for all $(s,x,i) \in D_T \times S$, and for any $i \in S$*

$$\partial_s \phi(s_0,x_0) + \inf_{v\in V} \sup_{u\in U} H(s_0,x_0,i,u,v,w,D\phi,D^2\phi) \ge 0.$$

whenever $\phi \in C^{1,2}([0,T] \times \bar{D})$ and $w(s,x,i) - \phi(s,x)$ attains a local maximum at $(s_0,x_0) \in [0,T) \times D$.

A continuous function $w(s,x,i)$ is called a viscosity supersolution of the lower HJBI equation (30) if $w(s,x,i) \ge g(s,x,i)$ for all $(s,x,i) \in D_T \times S$, and for any $i \in S$

$$\partial_s \phi(s_0,x_0) + \inf_{v\in V} \sup_{u\in U} H(s_0,x_0,i,u,v,w,D\phi,D^2\phi) \le 0.$$

whenever $\phi \in C^{1,2}([0,T] \times \bar{D})$ and $w(s,x,i) - \phi(s,x)$ attains a local minimum at $(s_0,x_0) \in [0,T) \times D$.

A continuous function $w(s,x,i)$ is called a viscosity solution of the lower HJBI equation (30), if it is both a viscosity subsolution and a viscosity supersolution.

Theorem 4. *The lower value function $W^-(s,x,i)$ defined as in (6) is a viscosity solution of the lower HJBI equation (30).*

Proof. Viscosity subsolution property. For a given $i \in S$ and $\phi \in C^{1,2}([0,T] \times \bar{D})$, suppose that $W^-(s,x,i) - \phi(s,x)$ attains its maximum at $(s_0,x_0) \in [0,T) \times D$ in a neighborhood $N(s_0,x_0) = [(s_0-\delta) \vee 0, s_0+\delta) \times B_\varepsilon(x_0)$, where $B_\varepsilon(x_0) = \{y \in \mathbb{R}^d, |y-x_0| < \varepsilon\}$ and $\delta, \varepsilon > 0$ are small enough so that $N(s_0,x_0) \subset [0,T) \times D$. Let τ_θ denote the first jump time of $\theta.$ after s_0 and $\tau_\varepsilon^{s_0,x_0,i} = \inf\{t \ge s_0, X_t^{s_0,x_0,i} \notin B_\varepsilon(x_0)\}$. Recall the lower dynamic programming equation (28) satisfied by $W^-(s,x,i)$, we know that for any $\beta \in \mathcal{B}_{s_0}$, there exists $\hat{u} \in \mathcal{U}_{s_0}$ such that

$$\begin{aligned} W^-(s_0,x_0,i) < E\bigg[\int_{s_0}^{\tau} & f(r, X_{r,\hat{u},\beta(\hat{u})}^{s_0,x_0,i}, \theta_{r,\hat{u},\beta(\hat{u})}^{s_0,x_0,i}, \hat{u}_r, \beta(\hat{u})_r)dr \\ & + W^-(\tau, X_{\tau,\hat{u},\beta(\hat{u})}^{s_0,x_0,i}, \theta_{\tau,\hat{u},\beta(\hat{u})}^{s_0,x_0,i})\bigg] + \delta^2, \end{aligned} \tag{32}$$

where $\tau = \tau_\theta \wedge \tau_\varepsilon^{s_0,x_0,i} \wedge (s_0+\delta)$.

Define $\Psi : [0,T] \times \mathbb{R}^d \times S \to \mathbb{R}$ as follows

$$\Psi(s,x,j) = \begin{cases} \phi(s,x) - \phi(s_0,x_0) + W^-(s_0,x_0,i), & j = i, \\ W^-(s,x,j), & j \neq i. \end{cases}$$

Applying Dynkin's formula between 0 and τ, we have

$$\begin{aligned} &E[\Psi(\tau, X^{s_0,x_0,i}_{\tau,\hat{u},\beta(\hat{u})}, \theta^{s_0,x_0,i}_{\tau,\hat{u},\beta(\hat{u})})] - \Psi(s_0,x_0,i) \\ = &E\left[\int_{s_0}^{\tau} \partial_s \phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}) + D\phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}) \cdot b(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i, \hat{u}_r) \right. \\ &+ \frac{1}{2} tr[\sigma\sigma'(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i, \hat{u}_r) D^2\phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})})] \\ &\left. + \sum_{j\neq i} q_{ij}(X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, \beta(\hat{u})_r) \times (\Psi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, j) - \Psi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i)) dr \right]. \end{aligned} \tag{33}$$

Moreover, for any $r \in (s_0, \tau]$,

$$W^-(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i) \leq W^-(s_0,x_0,i) - \phi(s_0,x_0) + \phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}).$$

Thus, (33) turns out to be

$$\begin{aligned} &E[W^-(\tau, X^{s_0,x_0,i}_{\tau,\hat{u},\beta(\hat{u})}, \theta^{s_0,x_0,i}_{\tau,\hat{u},\beta(\hat{u})})] - W^-(s_0,x_0,i) \\ \leq &E\left[\int_{s_0}^{\tau} \partial_s \phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}) + D\phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}) \cdot b(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i, \hat{u}_r) \right. \\ &+ \frac{1}{2} tr[\sigma\sigma'(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i, \hat{u}_r) D^2\phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})})] \\ &\left. + \sum_{j\neq i} q_{ij}(X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, \beta(\hat{u})_r) \times (W^-(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, j) - W^-(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i)) dr \right]. \end{aligned} \tag{34}$$

Combining (32) and (34), we have

$$\begin{aligned} -\delta^2 < &E\left[\int_{s_0}^{\tau} f(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, \theta^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, \hat{u}_r, \beta(\hat{u})_r) \right. \\ &+ \partial_s \phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}) + D\phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}) \cdot b(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i, \hat{u}_r) \\ &+ \frac{1}{2} tr[\sigma\sigma'(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i, \hat{u}_r) D^2\phi(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})})] \\ &\left. + \sum_{j\neq i} q_{ij}(X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, \beta(\hat{u})_r) \times (W^-(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, j) - W^-(r, X^{s_0,x_0,i}_{r,\hat{u},\beta(\hat{u})}, i)) dr \right]. \end{aligned}$$

Dividing both sides of the above equality by δ and letting $\delta \to 0$, we have

$$\partial_s \phi(s_0,x_0) + \inf_{v\in V} \sup_{u\in U} H(s_0,x_0,i,u,v,W^-,D\phi,D^2\phi) \geq 0.$$

Thus, we have proved that the lower value function W^- is a viscosity subsolution of the lower HJBI equation (30).

Viscosity supersolution property. For any fixed $i \in S$ and $\phi \in C^{1,2}([0,T] \times \bar{D})$, suppose that $W^-(s,x,i) - \phi(s,x)$ admits a minimum at $(s_0,x_0) \in [0,T) \times D$ in a neighborhood $N(s_0,x_0) = [(s_0-\delta) \vee 0, s_0+\delta) \times B_\varepsilon(x_0) \subset [0,T) \times D$. Let τ_θ denote the first jump time of θ. and $\tau_\varepsilon^{s_0,x_0,i} = \inf\{t \geq s_0, X_t^{s_0,x_0,i} \notin B_\varepsilon(x_0)\}$. Then by (28), for any $u \in \mathcal{U}_{s_0}$, there exists $\hat{\beta} \in \mathcal{B}_{s_0}$ such that

$$
\begin{aligned}
&W^-(s_0,x_0,i)\\
>&E\left[\int_{s_0}^{\tau} f(r, X^{s_0,x_0,i}_{r,u,\hat{\beta}(u)}, \theta^{s_0,x_0,i}_{r,u,\hat{\beta}(u)}, u_r, \hat{\beta}(u)_r)dr + W^-(\tau, X^{s_0,x_0,i}_{\tau,u,\hat{\beta}(u)}, \theta^{s_0,x_0,i}_{\tau,u,\hat{\beta}(u)})\right] - \delta^2,
\end{aligned}
$$

where $\tau = \tau_\theta \wedge \tau_\varepsilon^{s_0,x_0,i} \wedge (s_0+\delta)$.

With a similar argument as in the first part, we can prove that

$$
\partial_s \phi(s_0,x_0) + \inf_{v \in V} \sup_{u \in U} H(s_0,x_0,i,u,v,W^-,D\phi,D^2\phi) \leq 0.
$$

Thus, we have proved that the lower value function $W^-(s,x,i)$ is a viscosity supersolution of the lower HJBI equation (30).

Combining these two parts, we obtain that the lower value function $W^-(s,x,i)$ is a viscosity solution of the lower HJBI equation (30). □

There exist equivalent definitions of viscosity solutions of (30) and (31), respectively, which is useful for proving the uniqueness results, see [31]. Define the second order superdifferential of ϕ at $(t,x) \in [0,T) \times D$ as

$$
\begin{aligned}
&D_+^{1,2}\phi(t,x) := \{(q,p,P) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d :\\
&\limsup_{\substack{s \to t, s \in [0,T)\\ y \to x, y \in D}} \frac{\phi(s,y) - \phi(t,x) - q(s-t) - p \cdot (y-x) - \frac{1}{2}tr((y-x)(y-x)'P)}{|s-t| + |y-x|^2} \leq 0\},
\end{aligned}
$$

where $\mathcal{S}^d$ denotes the collection of $d \times d$ symmetric matrices. The second order subdifferential of ϕ at (t,x) is defined as $D_-^{1,2}\phi(t,x) = -D_+^{1,2}(-\phi)(t,x)$. We also denote by $\bar{D}_+^{1,2}\phi(t,x)$ and $\bar{D}_-^{1,2}\phi(t,x)$ the closures of $D_+^{1,2}\phi(t,x)$ and $D_-^{1,2}\phi(t,x)$, respectively.

Definition 3. *A continuous function $w(s,x,i)$ is said to be a viscosity subsolution of the lower HJBI equation (30) if $w(s,x,i) \leq g(s,x,i)$ for all $(s,x,i) \in D_T \times S$, and for any $(s,x,i) \in [0,T) \times D \times S$ and any $(q,p,P) \in D_+^{1,2}w(s,x,i)$,*

$$
q + \inf_{v \in V} \sup_{u \in U} H(s,x,i,u,v,w,p,P) \geq 0,
$$

and a continuous function $w(s,x,i)$ is said to be a viscosity supersolution of (30) if $w(s,x,i) \geq g(s,x,i)$ for all $(s,x,i) \in D_T \times S$, and for any $(s,x,i) \in [0,T) \times D \times S$

and any $(q,p,P) \in D_-^{1,2} w(s,x,i)$,

$$q + \inf_{v \in V} \sup_{u \in U} H(s,x,i,u,v,w,p,P) \leq 0,$$

a continuous function $w(s,x,i)$ *is called a viscosity solution of (30), if it is both a viscosity subsolution and a viscosity supersolution of (30).*

We refer to [23] for more details about the superdifferential and subdifferential and the connection of Definition 2 and Definition 3. To see the uniqueness properties of viscosity solutions of HJBI equations, we will give some lemmas first, this method is introduced in [33].

Lemma 2. *Let* $W_1^-(s,x,i)$ *be a viscosity subsolution and* $W_2^-(s,x,i)$ *be a viscosity supersolution of the lower HJBI equation (30). Then the function* $w(s,x) = \max_{i \in S}(W_1^-(s,x,i) - W_2^-(s,x,i))$ *is a viscosity subsolution of the following nonlinear PDE:*

$$\begin{cases} \partial_s w(s,x) + \sup_{u,i}\{\frac{1}{2} tr[\sigma\sigma'(s,x,i,u) D^2 w(s,x)] + Dw(s,x) \cdot b(s,x,i,u)\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (s,x) \in [0,T) \times D, \\ w(s,x) = 0, \quad (s,x) \in D_T. \end{cases} \tag{35}$$

Proof. Let $\phi \in C^{1,2}([0,T] \times \bar{D})$ and assume that $w(s,x) - \phi(s,x)$ attains a local maximum at $(s_0,x_0) \in [0,T) \times D$, let $\mathcal{O} \subset [0,T] \times \bar{D}$ be a closed neighborhood such that (s_0,x_0) is the global maximum in $\mathcal{O}$, and i_0 is such that $w(s_0,x_0) = W_1^-(s_0,x_0,i_0) - W_2^-(s_0,x_0,i_0)$. Define the function

$$\begin{aligned} \psi(s_1,x_1,s_2,x_2,i) =& W_1^-(s_1,x_1,i) - W_2^-(s_2,x_2,i) \\ & - \frac{|x_1 - x_2|^2}{\varepsilon^2} - \frac{|s_1 - s_2|^2}{\delta^2} - \phi(s_1,x_1), \end{aligned}$$

where $\varepsilon, \delta \in (0,1)$. Because of the continuity, there exists a global maximum point $(s_1^0, x_1^0, s_2^0, x_2^0, i)$ of ψ in $\mathcal{O} \times \mathcal{O} \times S$. In particular,

$$\psi(s_1^0,x_1^0,s_1^0,x_1^0,i) + \psi(s_2^0,x_2^0,s_2^0,x_2^0,i) \leq 2\psi(s_1^0,x_1^0,s_2^0,x_2^0,i),$$

which implies

$$\begin{aligned} & \frac{2|x_1^0 - x_2^0|^2}{\varepsilon^2} + \frac{2|s_1^0 - s_2^0|^2}{\delta^2} \\ \leq & W_1^-(s_1^0,x_1^0,i) - W_1^-(s_2^0,x_2^0,i) + W_2^-(s_1^0,x_1^0,i) \\ & \quad - W_2^-(s_2^0,x_2^0,i) - \phi(s_1^0,x_1^0) + \phi(s_2^0,x_2^0) \\ \leq & C. \end{aligned}$$

Thus it follows that (s_1^0,x_1^0,i), $(s_2^0,x_2^0,i) \to (s_0,x_0,i_0)$ and $\frac{|x_1^0 - x_2^0|^2}{\varepsilon^2}$, $\frac{|s_1^0 - s_2^0|^2}{\delta^2} \to 0$ when $\varepsilon, \delta \to 0$.

In addition, by [31], there exist X, Y satisfying

$$\left(\frac{2(s_1^0-s_2^0)}{\delta^2}+\partial_s\phi(s_1^0,x_1^0),\frac{2(x_1^0-x_2^0)}{\varepsilon^2}+D\phi(s_1^0,x_1^0),X\right)\in\bar{D}_+^{1,2}W_1^-(s_1^0,x_1^0,i),$$
$$\left(\frac{2(s_1^0-s_2^0)}{\delta^2},\frac{2(x_1^0-x_2^0)}{\varepsilon^2},Y\right)\in\bar{D}_-^{1,2}W_2^-(s_2^0,x_2^0,i),$$

and

$$\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\le\frac{4}{\varepsilon^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}+\begin{pmatrix}D^2\phi(s_1^0,x_1^0) & 0\\ 0 & 0\end{pmatrix}.$$

Since the definitions of viscosity solution of the lower HJBI equation (30), we have

$$\partial_s\phi(s_1^0,x_1^0)+\frac{2(s_1^0-s_2^0)}{\delta^2}+\inf_{v\in V}\sup_{u\in U}H(s_1^0,x_1^0,i,u,v,W_1^-,\frac{2(x_1^0-x_2^0)}{\varepsilon^2}+D\phi(s_1^0,x_1^0),X)\ge 0,$$
$$\frac{2(s_1^0-s_2^0)}{\delta^2}+\inf_{v\in V}\sup_{u\in U}H(s_2^0,x_2^0,i,u,v,W_2^-,\frac{2(x_1^0-x_2^0)}{\varepsilon^2},Y)\le 0.$$

They lead to that

$$\begin{aligned}-\partial_s\phi(s_1^0,x_1^0)\le\sup_{u,v}\Big\{&f(s_1^0,x_1^0,i,u,v)-f(s_2^0,x_2^0,i,u,v)\\&+\sum_{j\neq i}q_{ij}(x_1^0,v)(W_1^-(s_1^0,x_1^0,j)-W_1^-(s_1^0,x_1^0,i))\\&-\sum_{j\neq i}q_{ij}(x_2^0,v)(W_2^-(s_2^0,x_2^0,j)-W_2^-(s_2^0,x_2^0,i))\\&+\left(D\phi(s_1^0,x_1^0)+\frac{2(x_1^0-x_2^0)}{\varepsilon^2}\right)\cdot b(s_1^0,x_1^0,i,u)\\&-\frac{2(x_1^0-x_2^0)}{\varepsilon^2}\cdot b(s_2^0,x_2^0,i,u)\\&+\frac{1}{2}(tr[\sigma\sigma'(s_1^0,x_1^0,i,u)X]-tr[\sigma\sigma'(s_2^0,x_2^0,i,u)Y])\Big\}.\end{aligned}$$

In addition, (for $\xi=f,b,\sigma$)

$$|\xi(s_1^0,x_1^0,i,u,v)-\xi(s_2^0,x_2^0,i,u,v)|\le C|x_1^0-x_2^0|+\varpi(s_1^0-s_2^0),$$

where $\varpi(s) \to 0$ as $s \to 0$. In the limit of $\delta \to 0$ and then $\varepsilon \to 0$, we have

$$\begin{aligned}-\partial_s\phi(s_0,x_0) \le \sup_{u,v}\Bigg\{&\sum_{j\neq i_0} q_{i_0j}(x_0,v)(W_1^-(s_0,x_0,j)-W_1^-(s_0,x_0,i_0))\\&-\sum_{j\neq i_0} q_{i_0j}(x_0,v)(W_2^-(s_0,x_0,j)-W_2^-(s_0,x_0,i_0))\\&+D\phi(s_0,x_0)\cdot b(s_0,x_0,i_0,u)+\frac{1}{2}tr[\sigma\sigma'(s_0,x_0,i_0,u)D^2\phi(s_0,x_0)]\Bigg\}.\end{aligned} \tag{36}$$

Since the definition of $w(s,x)$, we have for any $j \in S$

$$W_1^-(s_0,x_0,j)-W_2^-(s_0,x_0,j) \le W_1^-(s_0,x_0,i_0)-W_2^-(s_0,x_0,i_0).$$

Hence, (36) turns out to be

$$\partial_s\phi(s_0,x_0)+\sup_u\{D\phi(s_0,x_0)\cdot b(s_0,x_0,i_0,u)+\frac{1}{2}tr[\sigma\sigma'(s_0,x_0,i_0,u)D^2\phi(s_0,x_0)]\}\ge 0,$$

i.e.,

$$\partial_s\phi(s_0,x_0)+\sup_{u,i}\{D\phi(s_0,x_0)\cdot b(s_0,x_0,i,u)+\frac{1}{2}tr[\sigma\sigma'(s_0,x_0,i,u)D^2\phi(s_0,x_0)]\}\ge 0.$$

Therefore, $w(s,x)$ is a viscosity subsolution of the PDE (35). □

Lemma 3. *Set*

$$f(x)=\Big[\log((|x|^2+1)^{1/2})+1\Big]^2,\ x\in\mathbb{R}^d.$$

For any $A>0$, there exists $C_1>0$ such that the function

$$\chi(s,x)=\exp\{(C_1(T-s)+A)f(x)\}$$

satisfies

$$\partial_s\chi(s,x)+\chi(s,x)+\sup_{u,i}\{D\chi(s,x)\cdot b(s,x,i,u)+\frac{1}{2}tr[\sigma\sigma'(s,x,i,u)D^2\chi(s,x)]\}<0,$$

in $[t_1,T]\times\mathbb{R}^d$, where $t_1=T-\frac{A}{C_1}$.

Proof. It is easy to verify this lemma directly, so we omit the proof. The reader is referred to [21] for detailed derivations. □

Theorem 5. *The lower HJBI equation (30) has at most one viscosity solution in* $[0,T]\times\bar{D}\times S$.

Proof. Let $W_1^-(s,x,i)$ and $W_2^-(s,x,i)$ be viscosity solutions of the lower HJBI equation (30) with same boundary condition. In fact, $W_1^-(s,x,i)$ is also a viscosity subsolution and $W_2^-(s,x,i)$ is also a viscosity supersolution of (30), we will prove that $W_1^-(s,x,i)\le W_2^-(s,x,i)$ for all $(s,x,i)\in[0,T]\times\bar{D}\times S$. Conversely, we can also prove that $W_1^-(s,x,i)\ge W_2^-(s,x,i)$ for all $(s,x,i)\in[0,T]\times\bar{D}\times S$ by the symmetry. Let $w(s,x)=\max_{i\in S}(W_1^-(s,x,i)-W_2^-(s,x,i))$. It is clear that for $t_1=T-\frac{A}{C_1}$ and $\kappa>0$,

$$M:=\max_{(s,x)\in[t_1,T]\times\bar{D}}(w(s,x)-\kappa\chi(s,x))\exp(s-T),$$

can be achieved at some point (s_0,x_0), where $\chi(s,x)>0$ is defined in Lemma 3. Without loss of generality, we assume that $w(s_0,x_0)>0$. Otherwise, $M\le 0$ and $w(s,x)\le\kappa\chi(s,x)$ in $[t_1,T]\times\bar{D}$ holds trivially for any $\kappa>0$. Consequently, letting $\kappa\to 0$, we obtain for all $(s,x,i)\in[t_1,T]\times\bar{D}\times S$

$$W_1^-(s,x,i)\le W_2^-(s,x,i).$$

Note, $w(s_0,x_0)>0$ implies $(s_0,x_0)\in[t_1,T)\times D$. From the definition of (s_0,x_0), we know that for all $(s,x)\in[t_1,T]\times\bar{D}$

$$w(s,x)-\kappa\chi(s,x)\le(w(s_0,x_0)-\kappa\chi(s_0,x_0))\exp(s_0-s).$$

Then (s_0,x_0) can be seen as a global maximum point for $w(s,x)-h(s,x)$ in $[t_1,T]\times\bar{D}$, where

$$h(s,x)=\kappa\chi(s,x)+(w(s_0,x_0)-\kappa\chi(s_0,x_0))\exp(s_0-s).$$

Since $w(s,x)$ is a viscosity subsolution of PDE (35), then we have

$$\partial_s h(s_0,x_0)+\sup_{u,i}\{Dh(s_0,x_0)\cdot b(s_0,x_0,i,u)+\frac{1}{2}tr[\sigma\sigma'(s_0,x_0,i,u)D^2h(s_0,x_0)]\}\ge 0.$$

By the definition of h, we deduce that

$$\begin{aligned}w(s_0,x_0)\le\kappa\sup_{u,i}\{&\partial_s\chi(s_0,x_0)+D\chi(s_0,x_0)\cdot b(s_0,x_0,i,u)\\&+\frac{1}{2}tr[\sigma\sigma'(s_0,x_0,i,u)D^2\chi(s_0,x_0)]+\chi(s_0,x_0)\}.\end{aligned}$$

By Lemma 3 and for any $\kappa>0$, we have

$$w(s_0,x_0)\le\kappa\sup_{u,i}\{\partial_s\chi(s_0,x_0)+D\chi(s_0,x_0)\cdot b(s_0,x_0,i,u)$$

$$+\frac{1}{2}tr[\sigma\sigma'(s_0,x_0,i,u)D^2\chi(s_0,x_0)]+\chi(s_0,x_0)\}<0,$$

which is a contradiction since $w(s_0,x_0)>0$. Finally, by applying successively the same argument on the interval $[t_2,t_1]$, with $t_2=(t_1-\frac{A}{C_1})$, and then, if $t_2>0$, on $[t_3,t_2]$, etc., we get $W_1^-(s,x,i)\leq W_2^-(s,x,i)$, for any $(s,x,i)\in[0,T]\times\bar{D}\times S$. In a similar way, we can prove $W_1^-(s,x,i)\geq W_2^-(s,x,i)$, for any $(s,x,i)\in[0,T]\times\bar{D}\times S$.

Thus, the proof is completed. □

In Theorem 4 and 5, we proved that the lower and upper value functions (6) and (7) are unique viscosity solutions of the lower and upper HJBI equations (30) and (31), respectively. Moreover, if the Isaacs condition

$$\inf_{v\in V}\sup_{u\in U}H(s,x,i,u,v,w,p,P)=\sup_{u\in U}\inf_{v\in V}H(s,x,i,u,v,w,p,P),$$

holds, then the lower and upper value functions coincide, which means that the game admits a value.

5 Example

In this section, we consider a one-dimensional stochastic differential equation

$$\begin{cases}dX_t=\theta_t^vX_tdt+\theta_t^vu_tdB_t,\\ X_s=x,\ \theta_s=i.\end{cases}\tag{37}$$

The state space of the controlled Markov chian θ_t^v is $S=\{1,2\}$. The transition rates of the Markov chain θ is

$$q_{11}(v)=-v,\quad q_{12}(v)=v,\quad q_{21}(v)=v,\quad q_{22}(v)=-v.$$

We denote by $\mathcal{U}$ and $\mathcal{V}$ the sets of feedback controls u_t and v_t taking values in $U=[a,b]$ and $V=[\lambda,\gamma]$, respectively.

We consider the following payoff function

$$J(s,x,i,u,v)=E\left[\int_s^\tau(v_t^2+u_t^2)dt\right],\tag{38}$$

where $\tau=\inf\{t\geq s,X_t\notin D\}\wedge T$, and D is an open bounded set. Then the lower value function is

$$W^-(s,x,i)=\inf_{v\in\mathcal{V}_s}\sup_{u\in\mathcal{U}_s}J(s,x,i,u,v),\tag{39}$$

and the upper value function is

$$W^+(s,x,i) = \sup_{u\in\mathcal{U}_s} \inf_{v\in\mathcal{V}_s} J(s,x,i,u,v). \tag{40}$$

It is worth pointing out that the continuity of the value function holds naturally by virtue of the assumptions in Section 3.

The lower value function (39) and the upper value function (40) respectively satisfy the lower dynamic programming equation

$$W^-(s,x,i) = \inf_{v\in\mathcal{V}_s} \sup_{u\in\mathcal{U}_s} E\left[\int_s^{t\wedge\tau} (v_r^2 + u_r^2)dr + W^-(t\wedge\tau, X_{t\wedge\tau}, \theta_{t\wedge\tau})\right],$$

and upper dynamic programming equation

$$W^+(s,x,i) = \sup_{u\in\mathcal{U}_s} \inf_{v\in\mathcal{V}_s} E\left[\int_s^{t\wedge\tau} (v_r^2 + u_r^2)dr + W^+(t\wedge\tau, X_{t\wedge\tau}, \theta_{t\wedge\tau})\right].$$

We also have the lower and upper HJBI equations

$$\begin{aligned}0 =& \partial_s W^-(s,x,i) + \partial_x W^-(s,x,i)ix + \sup_{u\in U}\{u^2 + \frac{1}{2}i^2u^2\partial_{xx}W^-(s,x,i)\}\\ &+ \inf_{v\in V}\{v^2 + v(W^-(s,x,j) - W^-(s,x,i))I_{\{j\neq i\}}\},\end{aligned}$$

and

$$\begin{aligned}0 =& \partial_s W^+(s,x,i) + \partial_x W^+(s,x,i)ix + \sup_{u\in U}\{u^2 + \frac{1}{2}i^2u^2\partial_{xx}W^+(s,x,i)\}\\ &+ \inf_{v\in V}\{v^2 + v(W^+(s,x,j) - W^+(s,x,i))I_{\{j\neq i\}}\}.\end{aligned}$$

This means the Isaacs condition holds, i.e., the game admits a value.

For a numerical experiment, let $D = (0, 50)$, $\Delta x = 0.1$, $U = \{1.0, 1.1, \cdots, 3.0\}$, $V = \{1.5, 1.6, \cdots, 2.5\}$, $T = 10$, $\Delta t = 0.01$ and $N = T/\Delta t$. The discrete-time dynamic programming principle is stated in the following steps.

Step 1: Let $W^-(n,x,i) = W^+(n,x,i) = 0$ for all $(n,x) \in \{N\} \times \{0.1, 0.2, \cdots, 49.9\} \cup \{0, 1, \cdots, N\} \times \{0, 50\}$ and $i \in \{1, 2\}$.

Step 2: For $x \notin \partial D$, $n \in \{N-1, \cdots, 1, 0\}$ and $j \neq i$

$$\begin{aligned}W^-(n,x,i) = \min_v \max_u \Bigg\{&(v^2+u^2)\Delta t + \sum_y W^-(n+1,y,i)P\{X_{n+1} = y|X_n = x\}\\ &+ \sum_y v\Delta t P\{X_{n+1} = y|X_n = x\} \times [W^-(n+1,y,j) - W^-(n+1,y,i)]\Bigg\},\end{aligned}$$

and

$$W^+(n,x,i) = \max_u \min_v \Big\{ (v^2+u^2)\Delta t + \sum_y W^+(n+1,y,i) P\{X_{n+1}=y|X_n=x\}$$
$$+ \sum_y v\Delta t P\{X_{n+1}=y|X_n=x\} \times [W^+(n+1,y,j) - W^+(n+1,y,i)] \Big\}.$$

Step 3: Record controls u and v.

In Step 2, we use the probability $P\{y-\Delta x/2 \le X_{n+1} \le y+\Delta x/2 | X_n = x\}$ as the approximation of the transition probability $P\{X_{n+1}=y|X_n=x\}$ by the 'pnorm' function in R language. The optimal control processes u and v are recorded as two matrixes, respectively, which can provide the optimal policies for each optimal control problems starting from any $(s,x,i) \in [0,T] \times \bar{D} \times S$.

We calculate $W^-(0,x,i)$ and $W^+(0,x,i)$ numerically using the above scheme for $x \in \{0, 0.1, 0.2, \cdots, 49.9, 50\}$ and $i \in \{1,2\}$, and plot them in Fig. 1.

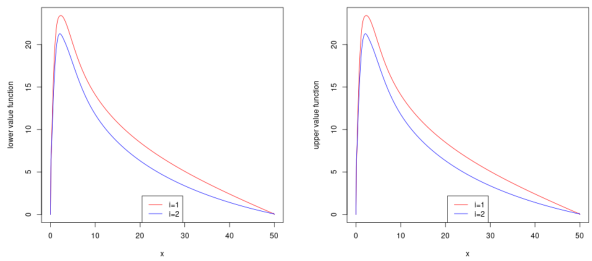

Fig. 1 Graphs of upper value function $W^-(0,x,i)$ and upper value function $W^+(0,x,i)$, respectively.

It is not difficult to see from Fig.1 that the lower and upper value functions coincide.

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Declarations

- Conflict of interest/Competing interests
  Author declare that there are no competing interests.

- Consent for publication
  All authors have read and approved the publication.

- Availability of data and materials
  All data generated or analysed during this study are available upon request.

- Code availability
  The codes generated are available upon request.

- Authors' contributions
  All authors have participated sufficiently in the work, including participation in the concept, format, analysis, writing, or revision of the manuscript. Furthermore, each author certifies that this manuscript has not been and will not be submitted to or published in any other publication.

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