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ON THE PROBABILITY OF POSITIVE FINITE-TIME LYAPUNOV EXPONENTS ON STRANGE NONCHAOTIC ATTRACTORS

Flavia Remo[1], Gabriel Fuhrmann[*2] and Tobias Jäger[1]

[1]Institute of Mathematics, Friedrich Schiller University Jena, Germany

[2]Department of Mathematical Sciences, Durham University, United Kingdom

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Abstract. We study strange non-chaotic attractors in a class of quasiperiodically forced monotone interval maps known as pinched skew products. We prove that the probability of positive time-N Lyapunov exponents—with respect to the unique physical measure on the attractor—decays exponentially as $N \to \infty$. The motivation for this work comes from the study of finite-time Lyapunov exponents as possible early-warning signals of critical transitions in the context of forced dynamics.

1. Introduction.

In this article, we study quasiperiodically forced interval maps of the form

$$F_\kappa : \mathbb{T}^D \times [0,1] \to \mathbb{T}^D \times [0,1], \quad F_\kappa(\theta, x) = (\theta + \rho, \tanh(\kappa x) \cdot g(\theta)) . \tag{1}$$

Here $\kappa > 0$ is a real parameter, $\rho \in \mathbb{T}^D$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is a totally irrational rotation vector (that is, there is no $n \in \mathbb{Z}^d \setminus \{0\}$ with $\langle \rho, n \rangle \in \mathbb{Z}$) and the multiplicative forcing term $g : \mathbb{T}^D \to [0,1]$ is given by

$$g(\theta) = \frac{1}{D} \cdot \sum_{i=1}^{D} \sin(\pi\theta_i), \tag{2}$$

where θ_i denotes the i-th component of the point $\theta \in \mathbb{T}^D$.

Systems of this kind are often called *pinched skew products*, where *pinched* refers to the fact that the forcing term g vanishes for some $\theta \in \mathbb{T}^D$ (here, at $\theta = 0$). Pinched skew-products received considerable attention due the occurrence of so-called strange non-chaotic attractors (SNAs) [1, 2, 3, 4, 5, 6, 7]. Due to their specific properties—in particular, the pinching in combination with the invariance of the zero line $\mathbb{T}^D \times \{0\}$—they are technically more accessible than other forced systems that exhibit SNAs so that they have been used on various occasions for case studies concerning the structural properties of such attractors. This led, for instance, to first results on the topological structure [6] and the dimensions [7] of SNAs, which have later been extended to the more difficult situation of additive quasiperiodic forcing [8, 9, 10].

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*Corresponding author: Gabriel Fuhrmann.

In a similar spirit, the aim of this article is to establish a quantitative result on the distribution of positive finite-time Lyapunov exponents on the SNA appearing in the system given by (1) and (2). Given $(\theta, x) \in \mathbb{T}^D \times [0,1]$ and $N \in \mathbb{N}$, we define the time-N-Lyapunov exponent as

$$\lambda_N(\theta, x) = \log\left(\partial_x\ \pi_2 \circ F_\kappa^N(\theta, x)\right)/N\ ,$$

where π_2 is the projection to the second coordinate. The (asymptotic) Lyapunov exponents are then given by

$$\lambda(\theta, x) = \lim_{N\to\infty} \lambda_N(\theta, x)\ .$$

As established in [3], for any $\kappa > \kappa_0 := e^{-\int_{\mathbb{T}^D} \log g(\theta) d\theta}$, there exists a unique physical measure $\mathbb{P}_\kappa$ of the system (1) that is ergodic and has a negative Lyapunov exponent. As a consequence, asymptotic Lyapunov exponents are $\mathbb{P}_\kappa$-almost surely negative. However, on the invariant zero line $\mathbb{T}^D \times \{0\}$, the pointwise Lyapunov exponents almost surely equal $\log\kappa - \log\kappa_0$ (see Remark 2.1 below). Hence, for $\kappa > \kappa_0$, positive asymptotic Lyapunov exponents are still present in the system and lead to a positive probability of positive finite-time exponents for all times $N \in \mathbb{N}$. Our main result provides information on the scaling behaviour of these probabilities if the rotation vector is Diophantine—see Section 3 for the specific Diophantine condition.

Theorem 1.1. *Denote by $\mathbb{P}_\kappa$ the unique physical measure of (1) with forcing function (2) and suppose ρ is Diophantine. Let $p_{\kappa,N} = \mathbb{P}_\kappa(\{(\theta, x) \mid \lambda_N(\theta, x) > 0\})$. Then there exists $\kappa_1 > \kappa_0$ such that for all $\kappa \geq \kappa_1$, there are constants $\gamma_+ \geq \gamma_- > 0$ (depending on κ) such that*

$$\exp(-\gamma_+ N) \leq p_{\kappa,N} \leq \exp(-\gamma_- N)$$

holds for all $N \in \mathbb{N}$.

Apart from its intrinsic interest, motivation for this result stems from the study of critical transitions. One major problem in this field is the identification of suitable (that is, observable and reliable) early warning signals [16, 17, 18, 19, 20, 21] for such transitions. A commonly proposed and utilized early warning signal for fold bifurcations—which are often cited as a paradigmatic example of critical transitions—are *slow recovery rates* (also referred to as a *critical slowing down*) [16, 17, 18, 20]. Since this notion has been coined in an interdisciplinary context and is used in a wide variety of situations, there is no comprehensive and rigorous mathematical definition of this term and we refrain from attempting to give one here. However, a more thorough discussion of this topic is given in the authors previous work [22], and we refer the interested reader to the latter for more details on the remarks in the remainder of this section.

In the classical case of an autonomous fold bifurcation, recovery rates can be identified with the Lyapunov exponents of the stable equilibria. Thus, in this situation, critical slowing down simply refers to the fact that when the stable and unstable equilibria involved in the bifurcation approach each other and eventually collide at the critical parameter, the resulting single fixed point is neutral, that is, it has exponent zero.

This picture changes significantly when a fold bifurcation takes place under the influence of external quasiperiodic forcing. First of all, the resulting *non-autonomous* systems generally do not allow for fixed points. Therefore, when carrying over ideas from an autonomous to a non-autonomous setting, one needs an appropriate replacement. In the present context, this part is played by so-called *invariant graphs* (see Section 2). Accordingly, non-autonomous fold bifurcations occur as invariant graphs approach each other and collide upon a change of system parameters. It is important to note that such collision does not necessarily take place everywhere. That is, at the bifurcation, the graphs may coincide in some points and differ in others, see [15] for the details. Specifically and in stark contrast to autonomous fold bifurcations, non-autonomous fold bifurcations do not necessarily yield neutral invariant graphs but may instead lead to a strange non-chaotic attractor-repeller-pair [14, 15] created at the bifucation point. This alternative pattern is referred to as a *non-smooth saddle-node bifurcation*. Moreover, just as for pinched systems, under suitable conditions, there exists a unique physical measure $\mathbb{P}$ which is supported on the attractor and has a negative Lyapunov exponent (see [3, 12]). However, this means that Lyapunov exponents remain $\mathbb{P}$-almost surely negative and bounded away from zero during a non-smooth saddle-node bifurcation (see Section 2 for more details).

While this seems to rule out the viability of slow-recovery rates as early warning signals for non-smooth fold bifurcations, one should bear in mind that experiments never measure the actual Lyapunov exponent but rather approximations of it—simply, because every experiment takes place over a finite period of time. In

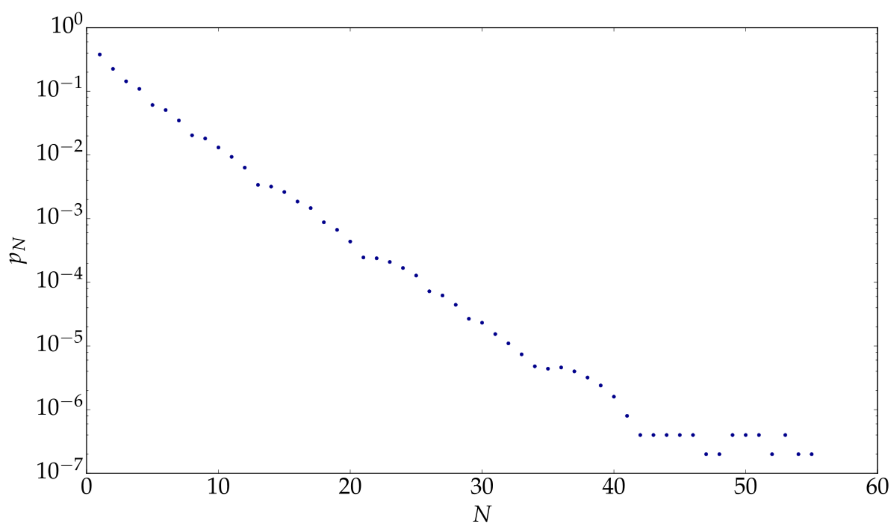

FIGURE 1. A logarithmic plot of the numerically obtained probability $p_N = p_{\kappa,N}$ over N for the system (1) with $D = 1$, $\kappa = 3$ and ρ the golden mean. More specifically, the graph shows the relative frequency of non-negative finite-time Lyapunov exponents among a grid of $5 \cdot 10^6$ initial conditions on the SNA (see also Figure 2). Consistent with the statement of Theorem 1.1, the plot indicates an exponential decay. We note that for $N > 40$, the sample size becomes too small to obtain a reliable estimate on the probabilities. This explains why the constant slope observed before does not extend all the way to the right.

other words, one rather measures finite-time Lyapunov exponents instead of asymptotic ones. Now, it is known that the presence of an SNA implies that positive finite-time exponents occur with positive probability for any time $N \in \mathbb{N}$ [2, 22]. Accordingly, one may wonder whether the observation of non-negative finite-time Lyapunov exponents can help to detect an SNA in practice. However, if N is chosen too small, then positive time-N-exponents can be observed already far from a bifurcation. Conversely, for large N, the probability of observing positive exponents on this time-scale converges to zero since the unique physical measure has a negative exponent—see Section 2 for the relation between pointwise Lyapunov exponents and the Lyapunov exponent of the physical measure. It is in this context that the scaling behaviour of the probabilities of time-N-exponents with $N \to \infty$ becomes important. Numerical studies for the quasiperiodically forced Allee model performed in [22] remained somewhat inconclusive, which is partly explained by the fact that the simulation of continuous-time systems is considerably more time-consuming than that of discrete-time systems. The exponential decay obtained in Theorem 1.1 is an indication that very large data sets may be required to detect positive finite-time Lyapunov exponents as early-warning signals in practice. As mentioned before, this interpretation relies on the hypothesis that quasiperiodically forced systems undergoing a saddle-node bifurcation—as studied in [22]—show a behaviour comparable to that of pinched systems treated here. We expect that using techniques from [8, 12, 10], similar statements can be established for non-pinched systems but this would require a considerably more involved analysis due to the inherent technical difficulties.

This article is organised as follows. In the next section, we introduce some technical background on forced monotone interval maps and their invariant graphs. There, we also describe the physical measure $\mathbb{P}$ from above in more detail. In Section 3, we specify the class of pinched skew-products for which we prove (a more general version of) the above theorem. This proof and the full statement—Theorem 4.4 and Theorem 4.8 (which gives the upper bound and is the harder part)—are given in the final section, Section 4.

2. **Forced monotone interval maps and invariant graphs—the general setting.** Throughout this article, we deal with *quasiperiodically forced (qpf) monotone interval maps*, that is, skew products of the form

$$F : \mathbb{T}^D \times [0,1] \to \mathbb{T}^D \times [0,1], \quad (\theta, x) \mapsto (\rho(\theta),\ F_\theta(x)), \tag{3}$$

where $\mathbb{T}^D = \mathbb{R}^D/\mathbb{Z}^D$ is the D-dimensional torus (for some $D \geq 1$),

$$\rho \colon \mathbb{T}^D \to \mathbb{T}^D, \qquad \theta \mapsto \theta + \rho$$

is a minimal rotation with a *rotation vector* ρ (the slight abuse of notation should not cause any confusion in the following) and for each $\theta \in \mathbb{T}^D$, F_θ is a continuously differentiable non-decreasing map on $[0,1]$ such that $(\theta, x) \mapsto F'_\theta(x)$ is continuous. It is customary to refer to $(\mathbb{T}^D, \rho)$ as the *forcing system* (defined on the *base* $\mathbb{T}^D$); the maps F_θ $(\theta \in \mathbb{T}^D)$ are also referred to as *fibre maps* (defined on the *fibres* $\{\theta\} \times [0,1]$).

An *invariant graph* of (3) is a measurable function $\phi : \mathbb{T}^D \to [0,1]$ which satisfies

$$F_\theta(\phi(\theta)) = \phi(\theta + \rho) \qquad \text{for all } \theta \in \mathbb{T}^D.$$

From an intuitive perspective, invariant graphs are to be seen as non-autonomous fixed points of (3)—observe that due to the minimality of ρ, (3) does not allow for actual fixed points. This idea is the basis for a bifurcation theory of invariant graphs, see [14, 15]. Independent of this analogy, invariant graphs of qpf monotone interval maps are key to understanding the dynamics of (3) due to their intimate relationship with the invariant sets and ergodic measures of the system (e.g. [5]).

Every invariant graph ϕ comes with an ergodic measure μ_ϕ where $\mu_\phi(A) = \mathrm{Leb}_{\mathbb{T}^D}(\phi^{-1}A)$ for each measurable $A \subseteq \mathbb{T}^D \times [0,1]$. Likewise, to each ergodic measure μ of (3) there is an invariant graph ϕ with $\mu = \mu_\phi$ [25, 26]. Moreover, given a compact invariant set $A \subseteq \mathbb{T}^D \times [0,1]$ (that is, A is compact and $F(A) = A$), let

$$\phi_A^+(\theta) = \sup\{x \in [0,1]\colon (\theta,x) \in A\} \qquad \text{and} \qquad \phi_A^-(\theta) = \inf\{x \in [0,1]\colon (\theta,x) \in A\}$$

for each $\theta \in \mathbb{T}^D$. Then ϕ_A^+ and ϕ_A^- define the so-called *upper* and *lower boundary graphs* of A which are invariant and—due to the compactness of A—upper and lower semi-continuous, respectively [5]. Of particular relevance for us will be the upper boundary graph of the *global attractor*

$$\bigcap_{n\in\mathbb{N}} F^n(\mathbb{T}^D \times [0,1]),$$

which we simply denote by ϕ^+.

The long-term behaviour of orbits near an invariant graph ϕ is largely characterized by its *Lyapunov exponent*

$$\lambda(\phi) = \int_{\mathbb{T}^D} \log F_\theta'(\phi(\theta)) d\theta,$$

provided this integral exists, see also [3].

If $\lambda(\phi) > 0$, then ϕ is repelling; if $\lambda(\phi) < 0$, then ϕ is attracting and μ_ϕ is a physical measure, see [23] for the details. Here, by *physical measure*, we refer to an F-invariant ergodic measure $\mathbb{P}$ for which there is a positive Lebesgue measure set $V \subseteq \mathbb{T}^D \times [0,1]$ such that for every continuous observable $f : \mathbb{T}^D \times [0,1] \to \mathbb{R}$ and all $(\theta, x) \in V$

$$1/n \cdot \sum_{\ell=0}^{n-1} f(F^\ell(\theta,x)) = \int f \, d\mathbb{P},$$

see also [27]. It is noteworthy that under suitable concavity assumptions on F_θ (which are verified by (1)), (3) has a unique physical measure given by the measure μ_{ϕ^+} on the upper boundary graph ϕ^+ of the global attractor, see [3, 15].

Observe that due to Birkhoff's Ergodic Theorem, $\lambda(\phi)$ equals the Lyapunov exponent of the point $(\theta, \phi(\theta))$ for $\mathrm{Leb}_{\mathbb{T}^D}$-almost every θ (equivalently: for μ_ϕ-almost every (θ, x)) since

$$\lambda(\theta,\phi(\theta)) = \lim_{n\to\infty} \frac{1}{n} \log(F_\theta^n)'(\phi(\theta)) = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \log F'_{\rho^\ell(\theta)}(\phi(\rho^\ell(\theta)))$$

for $\mathrm{Leb}_{\mathbb{T}^D}$-almost every $\theta \in \mathbb{T}^D$. Note that on the left-hand side of the above equation, we made use of the customary notation

$$F_\theta^n(x) = \pi_2 \circ F^n(\theta,x) = F_{\theta+(n-1)\rho} \circ \ldots \circ F_{\theta+\rho} \circ F_\theta(x), \tag{4}$$

where $\pi_2 \colon \mathbb{T}^D \times [0,1] \to [0,1]$ denotes the projection to the second coordinate.

Remark 2.1. Note that for the model (1) and the *a priori* invariant graph $\psi = 0$ given by zero line, we have that $F'_\theta(\psi(\theta)) = F'_\theta(0) = \kappa \cdot g(\theta)$, so that

$$\lambda(\psi) = \log\kappa + \int_{\mathbb{T}^D} \log g(\theta)\ d\theta = \log\kappa - \log\kappa_0$$

in this case. Hence, the zero line is repelling for all $\kappa > \kappa_0$, and pointwise Lyapunov exponents on this line are positive almost surely (with respect to the Lebesgue measure on $\mathbb{T}^D \times \{0\}$).

As we will discuss below, the unique physical measure of (1) is given by μ_{ϕ^+}, where ϕ^+ is the upper boundary graph of the global attractor. Therefore, a big part of the proof of Theorem 1.1 boils down to analysing ϕ^+ in considerable detail. In that context, we will utilize the obvious fact that ϕ^+ is the pointwise limit of the sequence of *iterated upper boundary lines* $(\phi_n)_{n\in\mathbb{N}_{\geq 0}}$, where

$$\phi_n\colon \mathbb{T}^D \to [0,1], \qquad \theta \mapsto F^n_{\theta-n\rho}(1). \tag{5}$$

Note that the graph of ϕ_n coincides with $F^n(\mathbb{T}^D \times \{1\})$ (recall the notation from (4)). It is further easy to see (and important to note) that the monotonicity of the fibre maps F_θ implies $\phi_{n+1} \leq \phi_n$ for all $n \in \mathbb{N}$.

For the convenience of the reader, we close this section with a brief description of the invariant graphs of (1). While this description will help to develop an intuition for the dynamics of (1) and, more broadly, for the results discussed in this work, it is strictly speaking not a prerequisite for the discussion in Section 3 and Section 4. For simplicity, we may assume that $D = 1$ in the remainder of this section.

It is immediate that independently of the value of κ, one invariant graph of (1) is given by the 0-line (which just happens to be the lower boundary graph of the global attractor). Let us denote this graph by ϕ^-. By direct computation, one can obtain that $\lambda(\phi^-) = \log\kappa - \log 2$.

Clearly, if ϕ^- equals the upper boundary graph ϕ^+ of the global attractor, then ϕ^- is the only invariant graph of F_κ. However, with help of the iterated upper boundary lines, one can show that $\lambda(\phi^+) \leq 0$, see [23]. Accordingly, if $\kappa > 2$, the 0-line ϕ^- is $\mathrm{Leb}_{\mathbb{T}^1}$-almost surely distinct from ϕ^+.

In other words, F_κ has at least two invariant graphs if $\kappa > 2$. Moreover, just as concavity of interval maps implies the existence of at most two fixed points (one of which is attracting and one of which is repelling), one can show that the concavity of the fibre maps of F_κ implies that ϕ^- and ϕ^+ are the only invariant graphs (and further, $\lambda(\phi^-) > 0 > \lambda(\phi^+)$), see [3, 15]. Note that accordingly, the physical measure $\mathbb{P}$ in the introduction has to coincide with μ_{ϕ^+}.

Now, since $F(0,x) = 0$, we have that ϕ^+ necessarily intersects the 0-line along the orbit of $(\rho, 0)$ which is, by minimality of ρ, dense in $\mathbb{T}^D \times \{0\}$. Therefore, while ϕ^+ is upper-semicontinuous (as the upper boundary graph of the global attractor) it clearly is not continuous and ϕ^+ is referred to as a *strange non-chaotic attractor*, see Figure 2 for a plot of ϕ^+.

3. **Pinched skew-product systems.** In this section, we specify the class of skew products within which we derive asymptotic estimates on the probability of positive finite-time Lyapunov exponents. For later reference, we repeat some of the assumptions from the previous section. By $\mathcal{F}$, we refer to the class of quasiperiodically forced monotone interval maps of the form

$$F: \mathbb{T}^D \times [0,1] \to \mathbb{T}^D \times [0,1], \quad (\theta, x) \mapsto (\rho(\theta),\ F_\theta(x)),$$

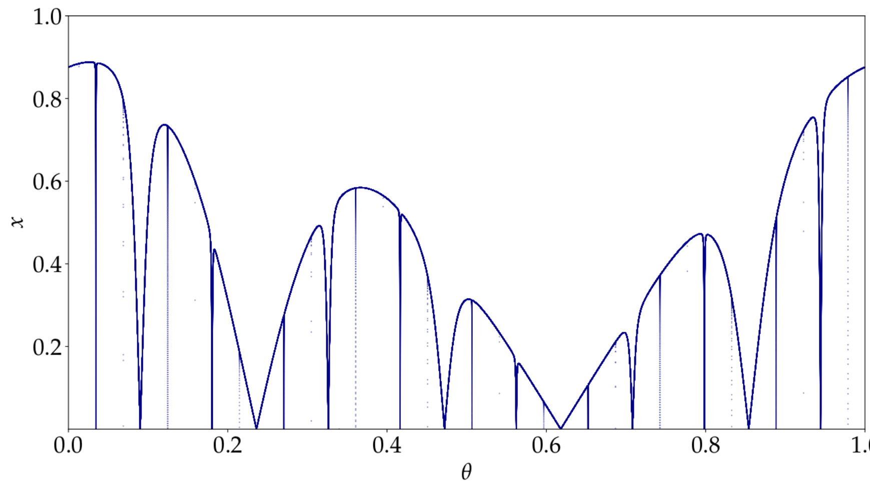

FIGURE 2. The SNA ϕ^+ of the parameter family $(\theta, x) \mapsto (\theta + \rho, \tanh(\kappa x) \cdot \sin(\pi\theta))$ with $\kappa = 3$ and ρ the golden mean. The points in the above plot are exactly the initial conditions used to estimate $p_{\kappa,N}$ in Figure 1.

which satisfy

($\mathcal{F}_1$): the fibre maps F_θ are non-decreasing;
($\mathcal{F}_2$): the fibre maps F_θ are differentiable and $(\theta, x) \mapsto F'_\theta(x)$ is continuous on $\mathbb{T}^D \times [0,1]$;
($\mathcal{F}_3$): F is *pinched*, that is, there is $\theta_* \in \mathbb{T}^D$ with $F_{\theta_*}(x) = 0$ for all $x \in [0,1]$;
($\mathcal{F}_4$): $F_\theta(0) = 0$ for all $\theta \in \mathbb{T}^D$ (invariance of the 0-line).

Besides the qualitative assumptions ($\mathcal{F}_1$)–($\mathcal{F}_4$), we need a number of quantitative assumptions. Let $F \in \mathcal{F}$ and assume that there exist parameters $\alpha > 2,\ \beta > 0,\ \gamma > 0$ and $L_0 \in (0,1)$ such that for all $\theta \in \mathbb{T}^D$, the following holds.

$$|F_\theta(x) - F_\theta(y)| \le \alpha\,|x-y| \qquad \text{for all } x, y \in [0,1], \tag{F1}$$

$$|F_\theta(x) - F_\theta(y)| \le \alpha^{-\gamma}\,|x-y| \qquad \text{for all } x, y \in [L_0,1], \tag{F2}$$

$$|F_\theta(x) - F_{\theta'}(x)| \le \beta\, d(\theta,\theta') \qquad \text{for all } x \in [0,1]. \tag{F3}$$

In particular, (F2) implies that the fibre maps F_θ are contracting in $[L_0, 1]$.

While the above assumptions specify the shape of F in the fibres, we need some additional control over the forcing on $\mathbb{T}^D$. To that end, we assume that the rotation vector $\rho \in \mathbb{T}^D$ is *Diophantine*. More specifically, setting $\tau_n = \rho^n(\theta_*) = \theta_* + n\rho$ (the n^{th}-iterate of the *pinched point* θ_*), we assume that there are constants $c > 0$ and $d > 1$ such that

$$d(\tau_n, \theta_*) \ge c \cdot n^{-d} \qquad \text{for all } n \in \mathbb{N}. \tag{F4}$$

Finally, bringing the behaviour along the fibres and the dynamics on the base $\mathbb{T}^D$ together, we assume that there are constants $m \in \mathbb{N},\ a > 1$ and $0 < b < 1$ with

$$m > 22\left(1 + \frac{1}{\gamma}\right), \tag{F5}$$

$$a \geq (m+1)^d, \tag{F6}$$

$$b \leq c, \tag{F7}$$

$$d(\tau_n, \theta_*) > b \qquad \text{for all } n \in \{1, \cdots, m-1\} \tag{F8}$$

such that

$$F_\theta(x) \geq \min\{2L_0, ax\} \cdot \min\left\{1, \frac{2}{b} d(\theta, \theta_*)\right\} \qquad \text{for all } (\theta, x) \in \mathbb{T}^D \times [0,1]. \tag{F9}$$

Our analysis of positive finite-time Lyapunov exponents will take place within the class

$$\mathcal{F}^* = \{F \in \mathcal{F} \colon F \text{ satisfies (F1)–(F9)}\}.$$

Instead of the abstract description of $\mathcal{F}^*$ given above, readers may simply think of the system given in (1) (for large κ) in all of the following. This is justified by the next statement.

Lemma 3.1 (see [7, Lemma 4.2])**.** *Consider F_κ as in (1) and let ρ satisfy the Diophantine condition (F4) for some $c > 0$ and $d > 1$. There exists a constant $\kappa_0 = \kappa_0(c, d, D)$ such that for all $\kappa \geq \kappa_0$, the map F_κ satisfies (F1)–(F9) (with appropriately chosen constants $\alpha, \gamma, \beta, L_0, m, a, b, c$).*

Note that as $[0,1] \ni x \mapsto F'_\theta(x)$ is continuous for each $\theta \in \mathbb{T}^D$ (due to $(\mathcal{F}_2)$), the mean value theorem and (F9) imply

$$F'_\theta(0) \geq a \cdot \min\left\{1, \frac{2}{b} d(\theta, \theta_*)\right\} \qquad \text{for all } \theta \in \mathbb{T}^D. \tag{F10}$$

While (F10) yields the existence of positive finite-time Lyapunov exponents on the zero line (see Lemma 4.3 below), in order to ensure big enough lower bounds on the probability of positive finite-time Lyapunov exponents outside the zero line, we additionally assume that for all $\delta > 0$ there is $x_\delta > 0$ with

$$F'_\theta(x) \geq (1-\delta) \cdot F'_\theta(0) \qquad \text{for all } x \in [0, x_\delta] \text{ and all } \theta \in \mathbb{T}^D. \tag{F11}$$

Clearly, this additional assumption is satisfied by (1) (for all κ, D and ρ).

4. **Rigorous bounds on the probability of positive finite-time Lyapunov exponents in pinched skew-products.** In this section, we show that within the class of pinched skew-products, the μ_{ϕ^+}-measure of points (θ, x) with $\lambda_N(\theta, x) \geq 0$ decays exponentially as $N \to \infty$—recall that ϕ^+ refers to the upper boundary graph of the global attractor, see Section 2.

We start by deriving a lower bound for this probability. To that end, we first need to study the occurrence of positive finite-time Lyapunov exponents on the zero line. Due to (F10), this essentially amounts to analysing the frequency of visits of points $\theta \in \mathbb{T}^D$ to the vicinity of θ_*.

For $j \in \mathbb{N}$, set

$$r_j = \frac{b}{2} a^{-(j-1)} \qquad \text{and} \qquad R_j = \frac{b}{2} a^{\frac{-(j-1)}{m}}. \tag{6}$$

Proposition 4.1. *Suppose (F4)–(F8) are satisfied. Then, for $n \in \{1, \ldots, m-1\}$, we have*

$$B_{r_1}(\theta_*) \cap (B_{r_1}(\theta_*) + n\rho) = \emptyset.$$

Similarly, for $j \geq 2$ and $n \in \{1, \ldots, (m+1)^{(j-1)}\}$, we have

$$B_{r_j}(\theta_*) \cap (B_{r_j}(\theta_*) + n\rho) = \emptyset.$$

Proof. We only discuss $j \geq 2$. With (F8), the other case is obvious.

Suppose $B_{r_j}(\theta_*) \cap (B_{r_j}(\theta_*) + n\rho) \neq \emptyset$ for some n, that is, $d(\theta_*, \tau_n) < 2r_j = ba^{-(j-1)}$. Note that (F4) gives $d(\theta_*, \tau_n) \geq c \cdot n^{-d}$. Therefore, $ba^{-(j-1)} > c \cdot n^{-d}$ and thus, $n > (c/b)^{1/d} \cdot a^{(j-1)/d} \geq (m+1)^{j-1}$, where we used (F6) and (F7) in the last step. □

This immediately gives

Corollary 4.2. *Assume (F4)–(F8) and let $\theta \in \mathbb{T}^D$. Suppose $n_1 < n_2 \in \mathbb{N}$ are such that $\theta + n_1\rho \in B_{r_j}(\theta_*)$ and $\theta + n_2\rho \in B_{r_j}(\theta_*)$. If $j = 1$, then $n_2 - n_1 \geq m$ and if $j \geq 2$, then $n_2 - n_1 \geq (m+1)^{j-1}$.*

In the following, recall that we set $\mathcal{F}^* = \{F \in \mathcal{F} \colon F \text{ satisfies (F1)–(F9)}\}$.

Lemma 4.3. *Suppose $F \in \mathcal{F}^*$ and $N \in \mathbb{N}$. For each $\theta \in B_{r_N}(\theta_*)$, we have*

$$\lambda_N(\theta + \rho, 0) \geq 1/2 \cdot \log a.$$

Proof. Set $\Delta_j = (m+1)^{j-1}$. By the previous corollary, given θ as in the assumptions and $2 \leq j \leq N$, we have

$$\#\{\ell \in \{1, \dots, N\} \colon \theta + \ell\rho \in B_{r_j}(\theta_*)\} \leq \lfloor N/\Delta_j \rfloor$$

and

$$\#\{\ell \in \{1, \dots, N\} \colon \theta + \ell\rho \in B_{r_1}(\theta_*)\} \leq \lfloor N/m \rfloor.$$

Note further that for $j \geq 0$, (F10) gives

$$F'_{\theta+\ell\rho}(0) \geq a \cdot \frac{2}{b} r_{j+1} = a^{-(j-1)} \qquad \text{whenever } \theta + \ell\rho \in B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*),$$

where—for notational convenience—$r_0 = \sqrt{D}$ and hence $B_{r_0}(\theta_*) = \mathbb{T}^D$. We therefore have

$$\begin{aligned}
&\lambda_N(\theta + \rho, 0) \\
&= 1/N \cdot \sum_{\ell=1}^{N} \log F'_{\theta+\ell\rho}(0) \geq 1/N \cdot \sum_{\ell=1}^{N} \sum_{j \geq 0} \log a^{-(j-1)} \cdot \mathbf{1}_{B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*)}(\theta + \ell\rho) \\
&= 1/N \cdot \sum_{\ell=1}^{N} \sum_{j \geq 0} (1-j) \log a \cdot \mathbf{1}_{B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*)}(\theta + \ell\rho) \\
&= \log a - \log a \cdot 1/N \cdot \sum_{j \geq 1} j \cdot \sum_{\ell=1}^{N} \mathbf{1}_{B_{r_j}(\theta_*) \setminus B_{r_{j+1}}(\theta_*)}(\theta + \ell\rho) \\
&\geq \log a - \log a \cdot 1/N \cdot (\lfloor N/m \rfloor + \sum_{j \geq 2} j \cdot \lfloor N/\Delta_j \rfloor) \geq \log a - \log a \cdot (1/m + \sum_{j \geq 2} j/\Delta_j) \\
&= \log a - \log a \cdot (1/m + \sum_{j \geq 2} j/(m+1)^{j-1}) \geq 1/2 \cdot \log a,
\end{aligned}$$

where we used (F5) in the last step. □

In order to prove the lower bound in Theorem 1.1, it remains to show that the positive finite-time Lyapunov exponents on the zero line are observable not only *on* but already *close to* the zero line. This is what the proof of the next statement is about.

Theorem 4.4. *Suppose $F \in \mathcal{F}^*$ satisfies (F11). Then there is $\gamma_+ > 0$ such that for all $N \in \mathbb{N}$*

$$\mu_{\phi^+}\{(\theta, x) \in \mathbb{T}^D \times [0,1] \colon \lambda_N(\theta, x) \geq 0\} \geq e^{-\gamma_+ N}.$$

Proof. Choose $\delta > 0$ small enough such that

$$\log(1-\delta) > -(\log a)/4 \tag{7}$$

and let x_δ be such that (F11) holds true. Without loss of generality, we may assume that $x_\delta \leq \beta b/2$ (with β from (F3)). For $N \in \mathbb{N}$, set $\tilde{r}_N = x_\delta/\beta \cdot \alpha^{-(N-1)}$. Observe that $\alpha \geq a$ (because of (F1) and (F9)) so that $\tilde{r}_N \leq r_N$ for all N. We first show that for $\theta \in B_{\tilde{r}_N}(\theta_*)$ and $j = 1, \ldots, N$, we have $\phi^+(\theta + j\rho) \leq x_\delta$.

To that end, observe that the monotonicity of the sequence of the iterated upper boundary lines $(\phi_n)_{n\in\mathbb{N}}$ (recall (5)) and (F3) yield

$$\phi^+(\theta+\rho) = \lim_{n\to\infty} \phi_n(\theta+\rho) \leq F_\theta(1) \leq \beta \cdot d(\theta, \theta_*).$$

Therefore, given $\theta \in B_{\tilde{r}_N}(\theta_*)$ and $j = 1, \ldots, N$, we have—due to $\mathcal{F}_1$ and (F1)—that

$$\phi^+(\theta + j\rho) = F^{j-1}_{\theta+\rho}\left(\phi^+(\theta+\rho)\right) \leq F^{j-1}_{\theta+\rho}(\beta \cdot d(\theta,\theta_*)) \leq \alpha^{j-1}\beta \cdot d(\theta, \theta_*) \leq x_\delta.$$

As a consequence, Lemma 4.3 and (7) in conjunction with (F11) give that $\lambda_N(\theta + \rho, x) \geq (\log a)/4$ for all $(\theta, \phi^+(\theta))$ with $\theta \in B_{\tilde{r}_N}(\theta_*)$. The statement follows. □

Having thus seen how within $\mathcal{F}^*$ (under the additional assumption of (F11)) the probability of positive finite-time Lyapunov exponents decays at most exponentially, we next come to show that this decay is, in fact, not slower than exponential.

Before we turn to the rigorous analysis, we briefly explain its idea on an intuitive level. First, note that (F2) implies that above L_0, fibres are contracted—we emphasize this fact by calling $\mathbb{T}^D \times [L_0, 1]$ the *contracting region*. In other words, visits to $\mathbb{T}^D \times [L_0, 1]$ contribute negatively to the (finite-time) Lyapunov exponent of an orbit. Second, (F9) enables us to control the number of times an orbit spends outside of the contracting region. Finally, since (F1) gives an upper bound for the possible fibre-wise expansion, the control obtained through (F9) enables us to ensure an overall contraction, that is, a negative (finite-time) Lyapunov exponent, along *most* finite orbits.

Let us specify this control in quantitative terms by collecting two auxiliary statements from [7]. Given $\theta \in \mathbb{T}^D$ and $n \in \mathbb{N}$, let $\theta_k := \rho^{k-n}(\theta)$ and $x_k := \phi_k(\theta_k)$ for $0 \leq k \leq n$. Note that $\phi_k(\theta_k) = F^k_{\theta_0}(1)$ and $\phi_n(\theta) = F^{n-k}_{\theta_k}(x_k)$. Let

$$s^n_k := \#\{k \leq j < n \colon x_j < 2L_0\}$$

and set $s^n_n(\theta) = 0$. Recall the definition of R_j in (6).

Lemma 4.5 ([7, Lemma 4.6])**.** *Let $F \in \mathcal{F}^*$ and $q,\ n \in \mathbb{N}$ with $n \geq mq+1$. Suppose that $\theta \notin \bigcup_{j=q}^{n} B_{R_j}(\tau_j)$ and consider $t \geq mq$. Then*

$$s^n_{n-t}(\theta) \leq \frac{11t}{m}.$$

As discussed in Section 2, the iterated upper boundary lines ϕ_n approximate the graph ϕ^+ whose measure μ_{ϕ^+} we are interested in. The next statement effectively provides numerical bounds for this approximation.

Proposition 4.6 ([7, Proposition 4.4])**.** *Let $F \in \mathcal{F}^*$, $q \in \mathbb{N}$ and $\eta = \gamma - \frac{11}{m}(1+\gamma) > 0$. Then, for $n \geq mq+1$ and $\theta \notin \bigcup_{j=q}^{n} B_{R_j}(\tau_j)$, we have that $|\phi_n(\theta) - \phi_{n-1}(\theta)| \leq \alpha^{-\eta(n-1)}$.*

Remark 4.7. Observe that $\eta > 0$ due to (F5) and note that η is independent of q.

Clearly, Proposition 4.6 gives that if $k, n \in \mathbb{N}$ satisfy $mq \leq k < n$ and $\theta \notin \bigcup_{j=q}^{n} B_{R_j}(\tau_j)$, then

$$|\phi_n(\theta) - \phi_k(\theta)| \leq \sum_{i=k+1}^{n} |\phi_i(\theta) - \phi_{i-1}(\theta)| \leq \sum_{i=k+1}^{n} \alpha^{-\eta(i-1)} \leq \frac{\alpha^{-\eta k}}{1-\alpha^{-\eta}}. \tag{8}$$

With the above statements, we are now in a position to prove the upper bound in Theorem 1.1.

Theorem 4.8. *Suppose $F \in \mathcal{F}^*$. Then there is $\gamma_- > 0$ such that for all $N \in \mathbb{N}$*

$$\mu_{\phi^+}\{(\theta, x) \in \mathbb{T}^D \times [0,1] \colon \lambda_N(\theta, x) \geq 0\} \leq e^{-\gamma_- N}.$$

Proof. Note that it suffices to show the statement for sufficiently large N.

Let $N \in \mathbb{N}$ be given. As ϕ^+ is the pointwise limit of the non-increasing sequence ϕ_n and due to the continuous differentiability of the fibre maps (see ($\mathcal{F}_2$)), we have that for each θ

$$\lambda_N(\theta_0, \phi^+(\theta_0)) = \frac{1}{N}\sum_{k=0}^{N-1} \log\left|F'_{\theta_k}(\phi^+(\theta_k))\right| = \lim_{n\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \log\left|F'_{\theta_k}(\phi_n(\theta_k))\right|,$$

where—as above—$\theta_k = \rho^{k-N}(\theta)$.

In a first instance, our goal is to derive assumptions on θ which ensure that the expression $\frac{1}{N}\sum_{k=0}^{N-1}\log\left|F'_{\theta_k}(\phi_n(\theta_k))\right|$ is negative and bounded away from zero for $n \geq N$ and large enough N. The statement then follows by showing that these assumptions are only violated in a set of exponentially small measure as $N \to \infty$.

We start by collecting a number of estimates which we will then combine to obtain an upper bound for $\frac{1}{N}\sum_{k=0}^{N-1}\log\left|F'_{\theta_k}(\phi_n(\theta_k))\right|$. First, let $\kappa \in \mathbb{N}$ be large enough such that $\kappa \geq m$ and $\frac{\alpha^{-\eta\cdot\kappa}}{1-\alpha^{-\eta}} < L_0$. Consider $k_0 \in \mathbb{N}$ with $k_0 \geq \kappa q$ (for some $q \in \mathbb{N}$ which we may consider fixed for now). Then (8) gives that for every $n > k \geq k_0$ and $\theta \notin \bigcup_{j=q}^{n} B_{R_j}(\tau_j)$, we obtain $|\phi_n(\theta) - \phi_k(\theta)| < L_0$. In particular, if $\phi_k(\theta) \geq 2L_0$, then $\phi_n(\theta) \geq L_0$. Therefore, if $n \geq k$ and $\theta_k \notin \bigcup_{j=q}^{n} B_{R_j}(\tau_j)$ for some $k \geq k_0$ with $\phi_k(\theta_k) > 2L_0$, (F2) yields

$$|F'_{\theta_k}(\phi_n(\theta_k))| \leq \alpha^{-\gamma}. \tag{9}$$

Second, observe that due to (F1), we have

$$\sum_{k=0}^{N-1}\log\left|F'_{\theta_k}(\phi_n(\theta_k))\right| \quad\leq\quad \sum_{k=0}^{k_0+1}\log\alpha + \sum_{k=k_0}^{N-1}\log\left|F'_{\theta_k}(\phi_n(\theta_k))\right|. \tag{10}$$

Third, let $N_0 = N_0(k_0) \in \mathbb{N}$ be the smallest integer such that $N_0 - k_0 \geq mk_0/\kappa \geq mq$. Then, Lemma 4.5 allows us to estimate the number of times for which $\phi_k(\theta_k) <$

$2L_0$. Specifically, if $N \geq N_0$ and $k \in \{k_0, \ldots, N-1\}$, we obtain for all $\theta \notin \bigcup_{j=q}^{N} B_{R_j}(\tau_j)$

$$s^N_{k_0}(\theta) = s^N_{N-(N-k_0)}(\theta) \leq \frac{11}{m}(N-k_0). \tag{11}$$

Observe that with (F1), (11) and (9), we get

$$\begin{aligned}
&\sum_{k=k_0}^{N-1} \log \left|F'_{\theta_k}(\phi_n(\theta_k))\right| \leq s^N_{k_0}(\theta)\log\alpha - \left(N - k_0 - s^N_{k_0}(\theta)\right)\gamma\log\alpha \\
&\leq \frac{11}{m}(N-k_0)\log\alpha - \gamma\cdot\left(N - k_0 - \frac{11}{m}(N-k_0)\right)\log\alpha \\
&= \left(\left(\gamma - \frac{11}{m}(1+\gamma)\right)k_0 + \left(\frac{11}{m}(1+\gamma) - \gamma\right)N\right)\log\alpha = (\eta k_0 - \eta N)\log\alpha
\end{aligned} \tag{12}$$

whenever $n \geq N \geq N_0$ and $\theta_k \notin \bigcup_{j=q}^{n} B_{R_j}(\tau_j)$ for all $k = k_0, \ldots, N$ and where η is as in Proposition 4.6. Plugging (12) into (10), we obtain (under the same assumptions as above)

$$\sum_{k=0}^{N-1} \log\left|F'_{\theta_k}(\phi_n(\theta_k))\right| \leq (2 + (\eta+1)k_0 - \eta N)\log\alpha. \tag{13}$$

Now, as $\eta > 0$, there is $\nu = \nu(\eta) > 0$ such that for all $N \geq k_0/\nu$, the right-hand side in (13) is negative (so that for all $n \geq N$, the left-hand side is negative and bounded away from 0). Note that we may assume without loss of generality that ν is small enough to ensure $k_0/\nu \geq N_0(k_0)$.

Set

$$B_{q,k_0,N} = \left\{\theta \in \mathbb{T}^D \colon \theta_k \in \bigcup_{j=q}^{\infty} B_{R_j}(\tau_j) \text{ for some } k \in \{k_0, \ldots, N\}\right\}.$$

Observe that (13) holds whenever θ is in the complement of the set $B_{q,k_0,N}$ (given $k_0 \geq \kappa q$ and $n \geq N \geq N_0(k_0)$). Note that

$$\begin{aligned}
&\mathrm{Leb}_{\mathbb{T}^D}(B_{q,k_0,N}) \leq (N-k_0+1)\cdot \mathrm{Leb}_{\mathbb{T}^D}\left(\bigcup_{j=q}^{\infty} B_{R_j}(\tau_j)\right) \\
&\leq (N-k_0+1)\cdot\zeta_D\cdot\left(\frac{b}{2}\right)^D \sum_{j=q}^{\infty} a^{\frac{-(j-1)D}{m}} \\
&= (N-k_0+1)\cdot\zeta_D\cdot\left(\frac{b}{2}\right)^D a^{-\frac{(q-1)D}{m}} \sum_{j=0}^{\infty}(a^{-\frac{D}{m}})^j = (N-k_0+1)\cdot a^{-\frac{(q-1)D}{m}} c(D),
\end{aligned}$$

where ζ_D denotes the $\mathrm{Leb}_{\mathbb{T}^D}$-measure of the D-dimensional unit ball and $c(D)$ simply collects all the terms in the above estimate which are independent of q, k_0 and N, that is, $c(D) = \zeta_D \cdot \left(\frac{b}{2}\right)^D \sum_{j=0}^{\infty}(a^{-\frac{D}{m}})^j$.

Now, set $k_0(N) = \lfloor \delta N \rfloor$ for some $\delta \in (0,\nu)$ and $q(N) = \lfloor N_\varepsilon \rfloor$ for some $\in_\varepsilon (0, \delta/\kappa)$. Note that for large enough N, we have $k_0(N) \geq \kappa q(N)$ and $N \geq k_0(N)/\nu \geq N_0[= N_0(k_0(N))]$. Hence, for sufficiently large N, the above gives

$$\mathrm{Leb}_{\mathbb{T}^D}(\{\theta \in \mathbb{T}^D \colon (\lambda_N(\theta, \phi^+(\theta)) \geq 0\}) \leq \mathrm{Leb}_{\mathbb{T}^D}(B_{q(N),k_0(N),N})$$

$$\leq N \cdot a^{-\frac{(\lfloor N_\varepsilon \rfloor - 1)D}{m}} c(D) \leq a^{-\frac{N_\varepsilon D}{2m}}.$$

The statement follows with $\gamma_- = (\log a) \cdot D_\varepsilon / 2m$. □

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