Published for SISSA by Springer

Received: *December 4, 2023*
Accepted: *March 1, 2024*
Published: *March 29, 2024*

# Noninvertible symmetries and anomalies from gauging 1-form electric centers

**Mohamed M. Anber and Samson Y.L. Chan**

*Centre for Particle Theory, Department of Mathematical Sciences, Durham University, South Road, Durham DH1 3LE, U.K.*

*E-mail:* mohamed.anber@durham.ac.uk, samson.y.chan@durham.ac.uk

Abstract: We devise a general method for obtaining 0-form noninvertible discrete chiral symmetries in 4-dimensional $\mathrm{SU}(N)/\mathbb{Z}_p$ and $\mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_p$ gauge theories with matter in arbitrary representations, where $\mathbb{Z}_p$ is a subgroup of the electric 1-form center symmetry. Our approach involves placing the theory on a three-torus and utilizing the Hamiltonian formalism to construct noninvertible operators by introducing twists compatible with the gauging of $\mathbb{Z}_p$. These theories exhibit electric 1-form and magnetic 1-form global symmetries, and their generators play a crucial role in constructing the corresponding Hilbert space. The noninvertible operators are demonstrated to project onto specific Hilbert space sectors characterized by particular magnetic fluxes. Furthermore, when subjected to twists by the electric 1-form global symmetry, these surviving sectors reveal an anomaly between the noninvertible and the 1-form symmetries. We argue that an anomaly implies that certain sectors, characterized by the eigenvalues of the electric symmetry generators, exhibit multi-fold degeneracies. When we couple these theories to axions, infrared axionic noninvertible operators inherit the ultraviolet structure of the theory, including the projective nature of the operators and their anomalies. We discuss various examples of vector and chiral gauge theories that showcase the versatility of our approach.

Keywords: Anomalies in Field and String Theories, Discrete Symmetries, Global Symmetries

ArXiv ePrint: 2311.07662



https://doi.org/10.1007/JHEP03(2024)169

## Contents



## 1 Introduction

A seismic shift has occurred in our understanding of symmetries in the past decade, transcending their conventional application to mere point-like particles. In the contemporary paradigm, a $p$-form symmetry in 4 dimensions is linked to operators residing on $(3-p)$-dimensional topological manifolds that act on $p$-dimensional objects charged under the symmetry. Moreover, over the last couple of years, symmetries have expanded their domain to encompass operators that defy the conventional notion of inversion. These are known as noninvertible symmetries. While noninvertible symmetries initially found their roots and applications in the realm of 2-dimensional QFT, see, e.g., [1, 2], their significance in the context of 4-dimensional QFT sparked a deluge of research endeavors in this area (a non-comprehensive list is [3–33]. Also, see [34, 35] for reviews.)

It is well known that quantum electrodynamics has a classical $\mathrm{U}(1)_\chi$ axial symmetry that breaks down because of the Adler-Bell-Jackiw (ABJ) anomaly. However, it was realized in [10, 11] that the axial symmetry does not completely disappear. Instead, it resurfaces as a noninvertible symmetry for each fractional element of the classical $\mathrm{U}(1)_\chi$. This new reinterpretation of symmetries triggered an interest in finding analogous structures in QFT. In [36], one of the authors established a technique for unveiling noninvertible 0-form symmetries within $\mathrm{SU}(N) \times \mathrm{U}(1)$ gauge theories in the presence of matter in representation $\mathcal{R}$. This approach employed the Hamiltonian formalism, where the theory was put on a three-dimensional torus $\mathbb{T}^3$, subjecting it to $\mathbb{Z}_N$ magnetic twists along all three spatial directions. Taking the matter



JHEP03(2024)169

to be a single Dirac fermion, this theory is endowed with invertible $\mathbb{Z}^{\chi}_{2\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$ 0-form chiral symmetry, where $T_{\mathcal{R}}$ and $d_{\mathcal{R}}$ are the Dynkin index and dimension of $\mathcal{R}$, respectively. Yet, it was shown that the theory also possesses a noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ 0-form chiral symmetry.[1] Such symmetry acts on the Hilbert space projectively by selecting special sectors characterized by certain magnetic numbers. New noninvertible symmetries were also revealed in [37] in theories with mixed anomalies between $\mathbb{Z}^{(1)}_2$ 1-form and 0-form discrete chiral symmetries.

The topological essence of symmetries, encompassing the noninvertible variants, underscores their sensitivity to the global structure of the gauge group. Consequently, the inquiry arises: how do we identify these noninvertible symmetries within a general gauge group, characterized as either $\mathrm{SU}(N)/\mathbb{Z}_p$ or $\mathrm{SU}(N)\times\mathrm{U}(1)/\mathbb{Z}_p$ where $\mathbb{Z}_p$ is a subgroup of the center symmetry? In this work, we answer this question by devising a general method that applies to any theory with a direct multiplication of abelian and semi-simple nonabelian gauge groups quotiented by a discrete center, whether the theory is vector-like or chiral. This is achieved by putting the theory on $\mathbb{T}^3$ and turning on magnetic fluxes in a refined subgroup of $\mathbb{Z}_N$, depending on the matter content as well as the global structure of the gauge group.

In the context of $\mathrm{SU}(N)$ gauge theory, the introduction of matter characterized by an $N$-ality $n$ has the effect of breaking the $\mathbb{Z}_N$ center of the group down to a subgroup $\mathbb{Z}_q$, where $q$ is the greatest common divisor (gcd) of $N$ and $n$. Our focus is on understanding the noninvertible 0-form symmetries present in the $\mathrm{SU}(N)/\mathbb{Z}_p$ gauge theories, where $\mathbb{Z}_p$ is a subgroup of the remaining center $\mathbb{Z}_q$. These theories exhibit both electric $\mathbb{Z}^{(1)}_{q/p}$ and magnetic $\mathbb{Z}^{(1)}_p$ 1-form global symmetries.[2] To identify the noninvertible symmetries, we initiate the process starting from $\mathrm{SU}(N)$ theory endowed with a single Dirac fermion in representation $\mathcal{R}$, which possesses an invertible $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry. We then subject this theory to electric and magnetic twists characterized by elements of $\mathbb{Z}_p$. If the theory exhibits a mixed anomaly between its chiral and electric $\mathbb{Z}^{(1)}_p$ 1-form symmetries, the act of gauging $\mathbb{Z}_p$ effectively reveals the chiral symmetry as noninvertible. The construction of a gauge-invariant operator corresponding to the noninvertible symmetry $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ involves several steps. First, we create a topological operator by integrating the anomalous current conservation law over $\mathbb{T}^3$. The resulting operator is not invariant under $\mathbb{Z}_p$ gauge transformations. Yet, we can restore gauge invariance by summing over all possible $\mathbb{Z}_p$ gauge-transformed operators. This process results in a noninvertible chiral symmetry operator that projects onto specific sectors in the Hilbert space, each characterized by certain 't Hooft lines charged under the magnetic $\mathbb{Z}^{(1)}_p$ 1-form symmetry. $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ can exhibit further anomalies when subjected to twists by the electric $\mathbb{Z}^{(1)}_{q/p}$ 1-form symmetry, implying that states within the Hilbert space of the $\mathrm{SU}(N)/\mathbb{Z}_p$ gauge theory will display multiple degeneracies.

We employ a similar approach to identify noninvertible symmetries in $\mathrm{SU}(N)\times\mathrm{U}(1)/\mathbb{Z}_p$ gauge theories, where $\mathbb{Z}_p$ is a subgroup of the electric $\mathbb{Z}^{(1)}_N$ 1-form center symmetry. Unlike in $\mathrm{SU}(N)$ theories, the introduction of matter does not reduce the $\mathbb{Z}_N$ center. This is due to the presence of an abelian U(1) sector, which ensures that all matter representations adhere to

[1] In this paper, a tilde is used to indicate that a given symmetry or operator is noninvertible.

[2] There are $p$ distinct theories $\left(\mathrm{SU}(N)/\mathbb{Z}_p\right)_n$, where $n=0,1,\ldots,p-1$ are the discrete $\theta$-like parameters [38]. These theories differ by the set of compatible line operators (Wilson, 't Hooft, and dyonic operator). Here, we restrict our analysis to $n=0$.



JHEP03(2024)169

the cocycle condition. In addition to the 1-form electric center symmetry, this theory is also endowed with a magnetic $U_m^{(1)}(1)$ 1-form symmetry. $\mathrm{SU}(N)$ gauge theory with matter exhibits an anomaly between its chiral and U(1) baryon-number symmetries. Gauging the latter transforms the theory into an $\mathrm{SU}(N) \times \mathrm{U}(1)$ gauge theory and reveals the chiral symmetry $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ as noninvertible. Placing the theory on $\mathbb{T}^3$ enables us to construct the corresponding noninvertible chiral operator by summing over large U(1) gauge transformations with distinct winding numbers. Furthermore, since the theory exhibits a 1-form electric center symmetry, we can decorate the noninvertible operator with $\mathbb{Z}_N$ magnetic twists. If we choose to further gauge a $\mathbb{Z}_p^{(1)}$ subgroup of the electric $\mathbb{Z}_N^{(1)}$ symmetry, thereby resulting in the $\mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_p$ theory, we must ensure that the noninvertible operator remains invariant under $\mathbb{Z}_p$ gauge transformation. This is accomplished by summing over all $\mathbb{Z}_p$ gauge-transformed chiral operators. Once again, we discover that the resultant operator projects onto specific sectors within the Hilbert space, distinguished by the presence of 't Hooft lines charged under $U_m^{(1)}(1)$. The noninvertible symmetry also exhibits a mixed anomaly with the remaining electric $\mathbb{Z}^{(1)}_{N/p}$ global symmetry. The anomaly implies that certain sectors of the theory, designated by certain $\mathbb{Z}^{(1)}_{N/p}$ electric fluxes, exhibit multi-fold degeneracy.

Placing the theory on $\mathbb{T}^3$ offers a distinct advantage: it presents a systematic approach for computing the 't Hooft anomalies inherent to a given theory. Simultaneously, it provides a means to construct the Hilbert space explicitly. In our work, we put a significant emphasis on this Hilbert space construction, shedding light on the intricate relationship between Wilson's lines, 't Hooft lines, and the noninvertible operator. Specifically, through several illustrative examples, we showcase how the noninvertible chiral operator, within the framework of the Hilbert space and Hamiltonian formalism, acts to annihilate the minimal 't Hooft lines.

We also introduce couplings of gauge theories to axions. The underlying renormalization group invariance of the noninvertible symmetries, along with their associated anomalies, guarantees that the infrared (IR) axion physics faithfully inherits all the characteristics of the theory at the ultraviolet (UV) level. We substantiate this by explicitly constructing noninvertible chiral operators, commencing from the IR anomalous axion current conservation law. In our exploration, we offer concrete illustrations of various UV theories and their corresponding IR axion physics manifestations.

This paper is organized as follows. In section 2, we provide a concise overview of the essential elements required for the development of noninvertible symmetries. This section encompasses the introduction of our notation, a review of the path-integral formalism on the 4-torus ($\mathbb{T}^4$), 't Hooft twists, and the Hamiltonian formalism on $\mathbb{T}^3$. Moving on to section 3, we proceed to construct noninvertible symmetries within the context of $\mathrm{SU}(N)/\mathbb{Z}_p$ theories while also identifying their associated anomalies. This section concludes with the presentation of specific examples of noninvertible symmetries in both vector and chiral gauge theories. In section 4, we replicate the same analysis, this time focusing on $\mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_p$. Two examples are discussed, including the Standard Model (SM), and we demonstrate that the SM lacks noninvertible symmetries within its non-gravitational sector. Finally, our paper culminates in section 5, where we explore the coupling of gauge theories to axions. We show that noninvertible symmetry operators can also be constructed using the axion anomalous current.



JHEP03(2024)169

## 2 Preliminaries

In this section, we review the path integral and the Hamiltonian formalisms of gauge theories put on a compact manifold with possible 't Hooft twists, both in space and time directions. Additionally, we examine the global symmetries and anomalies in both formalisms, providing an exploration of these key aspects. We base our formalism and notation on [36, 39–41], and set the stage for constructing the noninvertible operators we carry out in the subsequent sections. While some results in this section are new, many are a mere review of previous results. Moreover, some details are avoided, referring the reader to the literature for an in-depth discussion. Yet, the information encapsulated here is necessary to make this paper self-contained.

### 2.1 Twisting in the path integral

**Pure SU($N$) theory.** We begin by reviewing 't Hooft twists on a compact 4-dimensional Euclidean manifold with nontrivial 2-cycles. We consider SU($N$) pure Yang-Mills (YM) theory on $\mathbb{T}^4$, where $\mathbb{T}^4$ is a 4-torus with periods of length $L_\mu$, $\mu = 1, 2, 3, 4$.[3] The SU($N$) gauge fields $A_\mu$ are taken to obey the boundary conditions

$$A_\nu(x + L_\mu \hat{e}_\mu) = \Omega_\mu \circ A_\nu(x) \equiv \Omega_\mu(x) A_\nu(x) \Omega_\mu^{-1}(x) - i\Omega_\mu(x)\partial_\nu \Omega_\mu^{-1}(x)\,, \tag{2.1}$$

upon traversing $\mathbb{T}^4$ in each direction. The transition functions $\Omega_\mu$ are $N \times N$ unitary matrices in the defining representation of SU($N$), and $\hat{e}_\nu$ are unit vectors in the $x_\nu$ direction. The subscript $\mu$ in $\Omega_\mu$ means that the function $\Omega_\mu$ does not depend on the coordinate $x_\mu$. Then, the compatibility of (2.1) at the corners of the $x_\mu - x_\nu$ plane of $\mathbb{T}^4$ gives the cocycle condition

$$\Omega_\mu(x + \hat{e}_\nu L_\nu)\, \Omega_\nu(x) = e^{i\frac{2\pi n_{\mu\nu}}{N}} \Omega_\nu(x + \hat{e}_\mu L_\mu)\, \Omega_\mu(x)\,. \tag{2.2}$$

The exponent $e^{i\frac{2\pi n_{\mu\nu}}{N}}$, with anti-symmetric integers $n_{\mu\nu} = -n_{\nu\mu}$, is the $\mathbb{Z}_N$ center of SU($N$). The freedom to twist by elements of the center stems from the fact that both the transition function and its inverse appear in (2.1). This is also equivalent to the fact that the Wilson lines in pure SU($N$) gauge theory are charged under the electric $\mathbb{Z}_N^{(1)}$ 1-form center symmetry. The fundamental (defining representation) Wilson lines wind around the 4 cycles and are given by

$$W_\mu = \mathrm{tr}_\Box \left[ P e^{i\int_{x_\mu=0}^{x_\mu=L_\mu} A_\mu} \Omega_\mu \right]\,, \tag{2.3}$$

where $\Box$ denotes the defining representation of SU($N$) and the insertion of the transition function $\Omega_\mu$ ensures the gauge invariance of the lines. It will be useful to break $n_{\mu\nu}$ into spatial (magnetic) $m_i$ and temporal (electric) $k_i$ twists:

$$k_i \equiv n_{i4}\,, \qquad n_{ij} \equiv \epsilon_{ijk} m_k\,, \tag{2.4}$$

[3] YM theory on $\mathbb{T}^4$ with $\mathbb{Z}_N$ 't Hooft twists dates back to the original work by 't Hooft [39]. The groundbreaking paper [42] unveiled a novel mixed anomaly, specifically involving the electric $\mathbb{Z}_N^{(1)}$ 1-form symmetry. The background of the 1-form symmetry is a 2-form field that can be implemented via a 't Hooft twist. This fact led to a wave of enthusiasm to understand the semi-classical limit of gauge theories on $\mathbb{T}^4$ or $\mathbb{T}^2 \times \mathbb{R}^2$ [43–46].

and $i,j=1,2,3$ or $x,y,z$. We also use bold-face letters, e.g., $\boldsymbol{k}\equiv(k_1,k_2,k_3)$, to denote 3-dimensional vectors. When applied to the gauge fields on $\mathbb{T}^4$, the twists induce a background with fractional topological charge[4] [39]:

$$Q=\frac{1}{8\pi^2}\int_{\mathbb{T}^4}\text{tr}[F\wedge F]=-\frac{1}{8N}\epsilon_{\mu\nu\alpha\beta}n_{\mu\nu}n_{\alpha\beta}+\mathbb{Z}=\frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N}+\mathbb{Z}\,, \tag{2.5}$$

where $F$ is the field strength of $A$. Notice that the twists $(\boldsymbol{m},\boldsymbol{k})\in(\mathbb{Z}\,\text{Mod}\,N)^6$. Adding multiples of $N$ to $\boldsymbol{m}$ or $\boldsymbol{k}$ leaves the cocycle condition intact. However, this has the effect of changing the topological charges by integers. Hence, from here on, we shall take the twists $m_i,k_i\in\mathbb{Z}$, not $\text{Mod}\,N$. The partition function of the SU($N$) gauge theory with given twists $(\boldsymbol{m},\boldsymbol{k})$ is

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\text{SU}(N)}=\sum_{\nu\in\mathbb{Z}}\int[DA_\mu]_{(\boldsymbol{m},\boldsymbol{k})}\,e^{-S_{YM}-i\left(\frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N}+\nu\right)\theta}\,. \tag{2.6}$$

Here, $S_{YM}$ is the YM action, and the subscript $(\boldsymbol{m},\boldsymbol{k})$ indicates that the path integral is to be performed with a given set of twisted boundary conditions. Summation over the integer-valued topological sectors, $\nu\in\mathbb{Z}$, is necessary so that the theory satisfies locality (cluster decomposition).

**SU($N$) theory with matter.** Next, we add matter fields in a representation $\mathcal{R}$ under SU($N$). The matter representation has $N$-ality $n$. Then, the full $\mathbb{Z}_N$ center breaks down to $\mathbb{Z}_q$, $q=\gcd(N,n)$, i.e., the Wilson lines are charged under $\mathbb{Z}_q^{(1)}$ 1-from center symmetry.[5] Putting the matter, which, from now on, will be assumed to be fermions, on $\mathbb{T}^4$ modifies the cocycle conditions. Let $\psi$ be a left-handed Weyl fermion transforming under $\mathcal{R}$ of SU($N$). Then, the fermion obeys the boundary conditions

$$\psi(x+\hat{e}_\mu L_\mu)=\mathcal{R}(\Omega_\mu(x))\psi(x)\,. \tag{2.7}$$

The matrix $\mathcal{R}(\Omega_\mu(x))$ is built from $\Omega_\mu$, transforming in the defining representation of SU($N$), with suitable symmetrization or anti-symmetrization over $n$ indices (the $N$-ality of the representation) according to the specific representation $\mathcal{R}$. Thus, schematically (ignoring symmetrization over indices)

$$\mathcal{R}(\Omega_\mu)\sim\underbrace{\Omega_\mu\ldots\Omega_\mu}_{n}\,. \tag{2.8}$$

$\mathcal{R}(\Omega_\mu)$ must satisfy the cocycle condition

$$\mathcal{R}(\Omega_\mu(x+\hat{e}_\nu L_\nu))\,\mathcal{R}(\Omega_\nu(x))=\mathcal{R}(\Omega_\nu(x+\hat{e}_\mu L_\mu))\,\mathcal{R}(\Omega_\mu(x))\,, \tag{2.9}$$

[4]The simplest way to find the topological charge is by activating the electric and magnetic 't Hooft fluxes along the Cartan generators of SU($N$); see, e.g., [47]. We set $F_{12}=-\frac{2\pi m_3}{L_1L_2}\nu_aH_a$ and $F_{34}=\frac{2\pi k_3}{L_3L_4}\nu_aH_a$ along the 1-2 and 3-4 planes (and similar expressions in the rest of the planes), where $H_a$ are the Cartan generators, $\nu_a$ are the weights of the defining representation, $a=1,\ldots,N-1$, with summation over repeated indices. Plugging into $Q=\frac{1}{8\pi^2}\int_{\mathbb{T}^4}\text{tr}[F\wedge F]$, and using $\text{tr}[H_aH_b]=\delta_{ab}$ and $\nu_a\nu_a=1-1/N$, we find $Q=\frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N}+\mathbb{Z}$.

[5]For example, SU($2M$) gauge theory with matter in the 2-index (anti)symmetric representation has a $\mathbb{Z}_2^{(1)}$ center symmetry that acts on Wilson lines.

which, via eq. (2.2), reveals that the allowed values of the twists $\boldsymbol{m}$ and $\boldsymbol{k}$ are $\frac{N}{q}, \frac{2N}{q}, \ldots$. Twisting by the center subgroup $\mathbb{Z}_q$ induces a background field with fractional topological charge

$$Q = \frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N} + \mathbb{Z}\,, \quad \boldsymbol{m}, \boldsymbol{k} \in \frac{N}{q}\mathbb{Z}\,, \tag{2.10}$$

and the partition function in the presence of matter reads

$$\begin{aligned} \mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}} &= \sum_{\nu\in\mathbb{Z}} \int \{[DA_\mu]\,[D\text{matter}]\}_{(\boldsymbol{m},\boldsymbol{k})} e^{-S_{YM}-S_{\text{matter}}-i\left(\frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N}+\nu\right)\theta}\,, \\ m_i, k_i &\in \frac{N}{q}\mathbb{Z}\,, \quad i=1,2,3\,. \end{aligned} \tag{2.11}$$

In the presence of matter, the theory is endowed with classical nonabelian and abelian flavor symmetries. The U(1) baryon-number symmetry survives the quantum corrections. In contrast, the chiral part of the abelian symmetry, denoted by $\mathrm{U}(1)_\chi$, will generally break down to a discrete symmetry because of the Adler-Bell-Jackiw (ABJ) anomaly of $\mathrm{U}(1)_\chi$ in the background of color instantons (which have integer topological charges). To fix ideas, we consider a single flavor of a Dirac fermion with classical U(1) baryon number and $\mathrm{U}(1)_\chi$ chiral symmetries. We take the U(1) baryon charge of the Dirac fermion to be $+1$. The ABJ anomaly breaks $\mathrm{U}(1)_\chi$ down to invertible $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ chiral symmetry, where $T_\mathcal{R}$ is the Dynkin index of the representation. Generalizing the theory to include many flavors is straightforward, and we shall work out examples of this sort later in the paper. In the presence of the twists $(\boldsymbol{m},\boldsymbol{k})$, there can be an anomaly of $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ in the background of $\mathbb{Z}^{(1)}_q$. The anomaly is a non-trivial phase acquired by $\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}}$ as we apply a transformation by an element of $\mathbb{Z}_{2T_\mathcal{R}}$:

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}}|_{k_i,m_i\in N\mathbb{Z}/q} \longrightarrow e^{i2\pi\ell\frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N}}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}}\,, \tag{2.12}$$

and $\ell = 0,1,2,\ldots,T_\mathcal{R}-1$ are the elements of $\mathbb{Z}^\chi_{2T_\mathcal{R}}$. For the smallest twists $m_j = k_j = \frac{N}{q}$ in the $j$-th direction, we obtain [48]

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}}|_{m_3=k_3=\frac{N}{q}} \longrightarrow e^{i2\pi\ell\frac{N}{q^2}}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}}\,. \tag{2.13}$$

Bearing in mind that $N/q\in\mathbb{Z}$, we can generally absorb the integral part of $N/q^2$ by adding an integer topological charge, which cannot change the anomaly. Nevertheless, we will retain the phase as indicated in eq. (2.13). The phase is nontrivial, and hence there is an anomaly, if and only if $\ell\frac{N}{q^2}\notin\mathbb{Z}$. In the next section, we show how to obtain the same anomaly using the Hamiltonian formalism.

We can do more regarding turning on fractional fluxes in SU($N$) with matter. Instead of limiting ourselves to $\mathbb{Z}_q$ twists, we can twist with the full $\mathbb{Z}_N$ center symmetry or any subgroup of it provided we also turn on backgrounds of U(1) baryon number symmetry [47, 48]. Let $\omega_\mu$ denote the U(1) transition functions such that for the U(1) gauge field $a_\mu$, we have $a_\nu(x+\hat{e}_\mu) = \omega_\mu \circ a_\nu(x) \equiv a_\nu(x) - i\omega_\mu^{-1}\partial_\nu\omega_\mu$. Then, $\Omega_\mu$ and $\omega_\mu$ obey the cocycle conditions:

$$\begin{aligned} \Omega_\mu(x+\hat{e}_\nu L_\nu)\,\Omega_\nu(x) &= e^{i\frac{2\pi n_{\mu\nu}}{N}}\Omega_\nu(x+\hat{e}_\mu L_\mu)\,\Omega_\mu(x)\,, \\ \omega_\mu(x+\hat{e}_\nu L_\nu)\,\omega_\nu(x) &= e^{-i\frac{2\pi n n_{\mu\nu}}{N}}\omega_\nu(x+\hat{e}_\mu L_\mu)\,\omega_\mu(x)\,, \end{aligned} \tag{2.14}$$

where the $N$-ality of the matter representation is incorporated in the abelian transition functions. The topological charges of both the nonabelian center and abelian backgrounds read[6]

$$Q_{\mathrm{SU}(N)} = \frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N} + \mathbb{Z}\,, \quad Q_u = \left(\frac{n}{N}\boldsymbol{m} + \boldsymbol{A}\right)\cdot\left(\frac{n}{N}\boldsymbol{k} + \boldsymbol{B}\right)\,, \quad \boldsymbol{m},\boldsymbol{k},\boldsymbol{A},\boldsymbol{B} \in \mathbb{Z}^3\,. \tag{2.15}$$

Here, $\boldsymbol{A}, \boldsymbol{B}$ are arbitrary integral magnetic and electric quantum numbers that we can always turn on since they leave the cocycle condition intact.

**SU($\boldsymbol{N}$) × U(1) theory with matter.** We may also choose to make U(1) dynamical, which entails summing over small and large gauge transformations of U(1), with the latter implementing integer winding. This results in SU($N$) × U(1) gauge theory with a Dirac fermion in representation $\mathcal{R}$, with $N$-ality $N$ and baryon-charge +1. In this case, the U(1) instantons reduce $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ down to the genuine (invertible) symmetry $\mathbb{Z}^{\chi}_{2\mathrm{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$, and $d_{\mathcal{R}}$ is the dimension of $\mathcal{R}$. The easiest way to see that is by recalling the partition function under a $\mathrm{U}(1)_\chi$ transformation acquires a phase:

$$\exp\left[i2\alpha T_{\mathcal{R}}\int_{\mathbb{T}^4}\frac{\mathrm{tr}\,(F\wedge F)}{8\pi^2} + i2\alpha d_{\mathcal{R}}\int_{\mathbb{T}^4}\frac{f\wedge f}{8\pi^2}\right]\,, \tag{2.16}$$

where $f$ is the field strength of the U(1) field. Recalling that for the dynamical SU($N$) and U(1) fields we have $\int_{\mathbb{T}^4}\frac{\mathrm{tr}(F\wedge F)}{8\pi^2}\in\mathbb{Z}$, $\int_{\mathbb{T}^4}\frac{f\wedge f}{8\pi^2}\in\mathbb{Z}$, we conclude that only $\mathbb{Z}^{\chi}_{2\mathrm{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$ survises the chiral transformation. The theory admits Wilson's lines:

$$W_{\mu,\mathrm{SU}(N)} = \mathrm{tr}_\square\left[P e^{i\int_{x_\mu=0}^{x_\mu=L_\mu}A_\mu}\Omega_\mu\right]\,, \quad W_{\mu,\mathrm{U}(1)} = e^{-i\int_{x_\mu=0}^{x_\mu=L_\mu}a_\mu}\omega_\mu\,, \tag{2.17}$$

which are charged under an electric $\mathbb{Z}_N^{(1)}$ 1-form center symmetry. In addition, the theory is endowed with a magnetic $U_m^{(1)}(1)$ 1-form symmetry because of the absence of magnetic monopoles. For the sake of completeness, we also give the partition function of SU($N$) × U(1) theory with matter in the background of given $(\boldsymbol{m},\boldsymbol{k})$ fluxes:

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)\times\mathrm{U}(1)+\mathrm{matter}} = \sum_{\nu,\nu_{\mathrm{U}(1)}\in\mathbb{Z}}\int\{[DA_\mu]\,[Da_\mu]\,[D\,\mathrm{matter}]\}_{(\boldsymbol{m},\boldsymbol{k})}e^{-S_{YM}-S_{\mathrm{U}(1)}-S_{\mathrm{matter}}}\,,$$
$$m_i, k_i \in \mathbb{Z}\,, \quad i = 1,2,3\,, \tag{2.18}$$

and in addition to the SU($N$) integer topological charges $\nu$, we included a sum over integer topological charges $\nu_{\mathrm{U}(1)}$ of the U(1) sector.

## 2.2 Twisting in the Hamiltonian formalism

**Pure SU($\boldsymbol{N}$) theory.** Let us repeat the above discussion using the Hamiltonian formalism, starting with pure SU($N$) YM theory (we use a hat to distinguish an operator in this section.) To this end, we put the gauge theory on a spatial 3-torus $\mathbb{T}^3$ and apply the magnetic $\boldsymbol{m}$ twists along the 3-spatial directions. The transition functions in the defining representation

[6]Here, we use footnote 4 along with abelian field strengths $f_{12} = \frac{2\pi}{L_1L_2}(\frac{n}{N}m_3 + A_3)$ and $f_{34} = \frac{2\pi}{L_3L_4}(\frac{n}{N}k_3 + B_3)$ in the 1-2 and 3-4 planes, and similar expressions in the rest of the planes. Substituting into $Q_u = \int_{\mathbb{T}^4}\frac{f\wedge f}{8\pi^2}$, we obtain the fractional U(1) topological charge.

along the spatial directions, denoted by $\Gamma_i$, can be chosen to be constant $N \times N$ matrices obeying the cocycle condition

$$\Gamma_i \, \Gamma_j = e^{i\frac{2\pi\epsilon_{ijk}m_k}{N}} \Gamma_j \, \Gamma_i \,. \tag{2.19}$$

Then, one can construct the states of the physical Hilbert space using the temporal gauge condition $A_0 = 0$. The states can be written using the "position" eigenstates of the gauge fields $A_j$, $j = 1, 2, 3$ (or $i = x, y, z$) as follows:

$$|\psi\rangle_{\boldsymbol{m}} \equiv |A_1, A_2, A_3\rangle_{\boldsymbol{m}} \,, \quad \hat{A}_j |A_1, A_2, A_3\rangle_{\boldsymbol{m}} = A_j |A_1, A_2, A_3\rangle_{\boldsymbol{m}} \,, \tag{2.20}$$

and the subscript $\boldsymbol{m}$ emphasizes that the Hilbert space is constructed in the background of the magnetic twists. In writing eq. (2.20), we have put many details under the rug, and the reader is referred to [39–41] for details. For example, notice that the gauge fields $A_i$ need to respect the twisted boundary conditions (2.19), i.e., they transform according to (2.1) as we traverse any spatial direction on $\mathbb{T}^3$. The theory admits 3 fundamental Wilson lines wrapping the three cycles of $\mathbb{T}^3$; these are given by (2.3) by restricting $\mu$ to the spatial directions. The Wilson lines are charged under the $\mathbb{Z}_N^{(1)}$ 1-form symmetry generated by three symmetry generators $\hat{T}_j$, the Gukov-Witten operators, supported on co-dimension 2 surfaces. Thus, we have

$$\hat{T}_j \hat{W}_j = e^{i\frac{2\pi}{N}} \hat{W}_j \hat{T}_j \,, \tag{2.21}$$

and there are $\hat{W}_j^{e_j}$ distinct Wilson's lines with $N$ distinct $N$-alities $e_j = 0, 1, \ldots, N-1$. The center-symmetry generators $\hat{T}_i$ are hard to construct explicitly. However, their explicit form is not important to us. What is important is that they commute with the YM Hamiltonian $\hat{H}$, and thus, $\hat{H}$ and $\hat{T}_i$ can be simultaneously diagonalized. The physical states of the theory $|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$ are designated by the eigenvalues of $\hat{T}_i$. It can be shown that the action of $\hat{T}_i$ on $|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$ is given by

$$\hat{T}_j |\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = e^{i\frac{2\pi}{N}e_j - i\theta\frac{m_j}{N}} |\psi\rangle_{\text{phy}\,,\boldsymbol{m}} \,, \tag{2.22}$$

where $e_j, m_j \in \mathbb{Z}_N$ and the $\theta$ term ensures that $\hat{T}_i^N |\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = e^{-i\theta m_j} |\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$, and hence, $\hat{T}_i^N$ works as a large gauge transformation. The combination $e_j - \frac{\theta}{2\pi} m_j$ is the $\mathbb{Z}_N$ electric flux in the $j$-th direction. This is justified as follows. Consider the state $\hat{W}_j |\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$, obtained from $|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$ by the action of $\hat{W}_j$. Using eqs. (2.21), (2.22), we find $\hat{T}_j \hat{W}_j |\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = e^{i\frac{2\pi}{N}(e_j+1) - i\theta\frac{m_j}{N}} \hat{W}_j |\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$. Therefore, acting with $\hat{W}_j$ on the state $|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}$ increases $e_j$ by one unit in the $j$-th direction. Since $\hat{W}_j$ inserts an electric flux tube winding in the $j$-th direction, the interpretation of $e_j$ as electric flux follows. Notice also that because $\hat{T}_j$ and $\hat{H}$ can be simultaneously diagonalized, we may label the states by the energy and the electric flux:

$$|\psi\rangle_{\text{phy}\,,\boldsymbol{m}} \equiv |E, \boldsymbol{e}\rangle_{\boldsymbol{m}} \,, \quad \boldsymbol{e} \in \mathbb{Z}_N^3 \,. \tag{2.23}$$

It is worth spending some time to explain our notation in eq. (2.23), as we shall use this notation extensively in our paper. The physical state is labeled by the eigenvalues of a set of commuting operators, here the energy and the electric flux. The SU($N$) theory does not admit a 1-form magnetic symmetry, and thus, we cannot label the states by magnetic fluxes.



JHEP03(2024)169

Yet, we can turn on a background magnetic flux $\boldsymbol{m}$, indicated as a subscript; all physical quantities are calculated in this magnetic background. Also, we use the letter $\boldsymbol{m}$ to denote the set of magnetic fluxes we can consistently turn on. Here, we have $\boldsymbol{m} \in \mathbb{Z}^3$.

How can we make sense of the fractional topological charge (2.10) on $\mathbb{T}^3$? We consider the product of $\mathbb{T}^3$ and the time interval $[0, L_4]$ and consider the boundary conditions $\hat{A}_i(t = L_4) = C[\boldsymbol{k}] \circ \hat{A}_i(t = 0)$, where $C[\boldsymbol{k}]$ is an "improper gauge" transformation implementing a twist $\boldsymbol{k} \in \mathbb{Z}^3$ on the gauge fields by an element of the center.[7] In the presence of the magnetic twists $\boldsymbol{m}$, it can be shown that an application of $C[\boldsymbol{k}]$ results in the topological charge (Pontryagin square) [39–41]:

$$Q[C[\boldsymbol{k}]] = \int_{\mathbb{T}^3} K(C \circ \hat{A}) - K(\hat{A}) = \frac{1}{24\pi^2} \int_{\mathbb{T}^3} \mathrm{tr}\left[C dC^{-1}\right]^3 = \frac{\boldsymbol{m} \cdot \boldsymbol{k}}{N} + \mathbb{Z}\,, \tag{2.24}$$

where $K(\hat{A})$ is the topological current density operator $K(\hat{A}) = \frac{1}{8\pi^2}\mathrm{tr}\left[\hat{A} \wedge \hat{F} - \frac{i}{3}\hat{A} \wedge \hat{A} \wedge \hat{A}\right]$, or in terms of the components: $\hat{K}^\mu(A) = \frac{1}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}\left(\hat{A}^a_\nu \partial_\lambda \hat{A}^a_\sigma - \frac{1}{3} f^{abc} \hat{A}^a_\nu \hat{A}^b_\lambda \hat{A}^c_\sigma\right)$.

**SU($N$) theory with matter.** Adding fermions of $N$-ality $n$ changes the center from $\mathbb{Z}_N$ to $\mathbb{Z}_q$, $q = \gcd(N, n)$, and the twists $(\boldsymbol{m}, \boldsymbol{k})$ are now in $(N\mathbb{Z}/q)^6$. Otherwise, all the steps used to put the theory on $\mathbb{T}^3$ and construct the Hilbert space carry over. In particular, $\hat{T}_i$ now are the generators of the $\mathbb{Z}_q^{(1)}$ 1-form symmetry, and their action on the physical states in the Hilbert space is given by[8] (now we turn off the $\theta$ angle as we can rotate it away via a chiral transformation acting on the fermion)

$$\hat{T}_j |\psi\rangle_{\mathrm{phy}\,,\boldsymbol{m}} = e^{i\frac{2\pi}{q} e_j} |\psi\rangle_{\mathrm{phy}\,,\boldsymbol{m}}\,, \tag{2.25}$$

and the theory has $e_j = 0, 1, 2, \ldots, q-1$ electric flux sectors in each direction $j = 1, 2, 3$. The operators $\hat{T}_j$ act on the spatial Wilson lines in the defining representation of SU($N$) as $\hat{T}_j \hat{W}_j = e^{i\frac{2\pi}{q}} \hat{W}_j \hat{T}_j$, and there are $q$ distinct Wilson's lines $W_j^{e_j}$. The physical states $|\psi\rangle_{\mathrm{phy}\,,\boldsymbol{m}}$ are simultaneous eigenstates of the Hamiltonian and $\hat{T}_j$ since both operators commute. Thus, we can write the physical states in the magnetic flux background $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$ as

$$|\psi\rangle_{\mathrm{phy}\,,\boldsymbol{m}} = |E, \boldsymbol{e}N/q\rangle_{\boldsymbol{m}}\,, \quad \boldsymbol{e} = (e_1, e_2, e_3) \in \mathbb{Z}_q^3\,, \tag{2.26}$$

and $Ne_j/q$ is the amount of electric flux carried by the state in direction $j$. We may also say that $e_j$ is the number of electric fluxes in units of $N/q$. For matter with $N$-ality $n = 0$, e.g., in the adjoint representation, $q = N$ and we recover what we have said about pure SU($N$) gauge theory.

[7] In fact, $C$ should be designated by both $\boldsymbol{k}$ and the integral instanton number $\nu$; see [39]. However, $\nu$ does not play a role in this work.

[8] It is conceivable to introduce an additional label to signify the distinct symmetries generated by different operators $\hat{T}_j$. For instance, we could designate $\hat{T}_{N,j}$ as the generator of $\mathbb{Z}_N^{(1)}$ and $\hat{T}_{q,j}$ as the generator of $\mathbb{Z}_q^{(1)}$. Nonetheless, this approach may lead to increased complexity in our expressions, and we opt not to pursue it. Instead, we will explicitly specify the symmetry in question when discussing these distinct operators.

The partition function (2.11) can be written in the Hamiltonian formalism as a trace over states in Hilbert space:

$$\begin{aligned}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{\mathrm{SU}(N)+\text{matter}} &= \mathrm{tr}_{\boldsymbol{m}}\left[e^{-L_4\hat{H}}(\hat{T}_x)^{k_x}(\hat{T}_y)^{k_y}(\hat{T}_z)^{k_z}\right] \\ &= \sum_{\boldsymbol{e}\in\{0,1,\ldots,q-1\}^3} e^{i\frac{2\pi\boldsymbol{e}\cdot\boldsymbol{k}}{q}}\,{}_{\boldsymbol{m}}\langle E,\boldsymbol{e}N/q|e^{-L_4\hat{H}}|E,\boldsymbol{e}N/q\rangle_{\boldsymbol{m}}\,,\end{aligned} \tag{2.27}$$

where the subscript $\boldsymbol{m}$ in the trace means that we are considering the states in the background of the magnetic flux $\boldsymbol{m}\in(N\mathbb{Z}/q)^3$. We also used eqs. (2.25), (2.26), the fact that the states are eigenstates of both the energy and the 1-form center operators.

To detect the anomaly between $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ and $\mathbb{Z}^{(1)}_q$ in the Hamiltonian formalism, we first define the operator that implements the discrete chiral symmetry. To this end, we recall that under a chiral $\mathrm{U}(1)_A$ rotation, the presence of the ABJ anomaly indicates non-conservation of the corresponding symmetry rotation:

$$\partial_\mu \hat{j}^\mu_A = 2T_{\mathcal{R}}\partial_\mu \hat{K}^\mu(A)\,. \tag{2.28}$$

Yet, we can define a conserved current:

$$\hat{j}^\mu_5 \equiv \hat{j}^\mu_A - 2T_{\mathcal{R}}\hat{K}^\mu\,, \tag{2.29}$$

and correspondingly a conserved charge:

$$\hat{Q}_5 = \int_{\mathbb{T}^3}\hat{j}^0_5\,. \tag{2.30}$$

Therefore, it is natural to define the operator

$$\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \equiv \exp\left[i\frac{2\pi\ell}{2T_{\mathcal{R}}}\hat{Q}_5\right] = \exp\left[i\frac{2\pi\ell}{2T_{\mathcal{R}}}\int_{\mathbb{T}^3}(\hat{j}^0_A - 2T_{\mathcal{R}}\hat{K}^0(\hat{A}))\right]\,, \tag{2.31}$$

for $\ell = 0,1,\ldots,T_{\mathcal{R}}-1$, which implements the action of the $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry. $\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}}}$ is invariant under both small and large $\mathrm{SU}(N)$ gauge transformations (with integer winding). To find the mixed anomaly between $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ and $\mathbb{Z}^{(1)}_q$, we compute the commutation between $\hat{T}_j$, which implements the action of the electric center symmetry in the $j$-th direction, and $\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}}}$:

$$\hat{T}_j\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\hat{T}_j^{-1}\,, \tag{2.32}$$

remembering that the theory is in the background of a magnetic twist $m_j \in \frac{N\mathbb{Z}}{q}$ in the $j$-th direction.[9] First, $\hat{T}_j$ commutes with the current $j^0_A$ since the latter is a color singlet operator. However, $\hat{K}^0$ fails to commute with $\hat{T}_j$; the commutation between the two operators is found

[9]Similar to the footnote 8, we could use a label that denotes the specific magnetic flux background when we are dealing with the operator $\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$. This background can be taken in sets such as $\frac{N\mathbb{Z}}{q}$ or $\frac{N\mathbb{Z}}{p}$, among others. However, adopting this approach may introduce unnecessary complexity to our notation. As a result, we have chosen to adopt a more transparent approach: we will explicitly mention the magnetic flux background whenever we discuss this operator.

by recalling that the action of $\hat{T}_j$ is implemented on the gauge fields $\hat{A}_j$ as $\hat{A}_j = C[k_j] \circ \hat{A}_j$. Thus, we find, after making use of (2.24),

$$\begin{aligned}
&\hat{T}_j \exp\left[i2\pi\ell \int_{\mathbb{T}^3} \hat{K}^0(\hat{A})\right] \hat{T}_j^{-1} \\
&= \exp\left[i2\pi\ell \int_{\mathbb{T}^3} \hat{K}^0\left(C[k_j]\circ \hat{A}\right) - \hat{K}^0(\hat{A})\right] \exp\left[i2\pi\ell \int_{\mathbb{T}^3} \hat{K}^0(\hat{A})\right] \\
&= \exp\left[i2\pi\ell \frac{m_j k_j}{N}\right]_{m_j,k_j\in\left(\frac{N\mathbb{Z}}{q}\right)^2} \exp\left[i2\pi\ell \int_{\mathbb{T}^3} \hat{K}^0(\hat{A})\right],
\end{aligned} \tag{2.33}$$

noting the restriction $m_j, k_j \in \left(\frac{N\mathbb{Z}}{q}\right)^2$ due to the presence of matter; otherwise, we would not satisfy the cocycle condition. Collecting everything and using the minimal twists $m_j = k_j = \frac{N}{q}$ we conclude

$$\hat{T}_j \hat{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \hat{T}_j^{-1} = e^{i2\pi\ell\frac{N}{q^2}} \hat{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\,, \tag{2.34}$$

which is exactly the mixed anomaly between the $\mathbb{Z}^{\chi}_{2T_\mathcal{R}}$ chiral and the $\mathbb{Z}^{(1)}_q$ 1-form center symmetries found in (2.13) from the path integral formalism. The anomaly along with the commutation relations (remember that both $\mathbb{Z}^{(1)}_q$ and $\mathbb{Z}^{\chi}_{2T_\mathcal{R}}$ are good symmetries of the theory, and hence, the corresponding operators commute with the Hamiltonian)

$$[\hat{H}, \hat{T}_j] = 0\,, \quad [\hat{H}, \hat{U}_{\mathbb{Z}_{2T_\mathcal{R}}}] = 0\,, \tag{2.35}$$

furnishes a finite-dimensional space with a minimum dimension of $q^2/\gcd(q^2, N)$. This means that sectors in Hilbert space exhibit a $q^2/\gcd(q^2, N)$-fold degeneracy.

**SU($N$) × U(1) theory with matter.** Next, we discuss the Hamiltonian quantization of SU($N$) × U(1) gauge theory with matter fields on $\mathbb{T}^3$ in the background of twists. In this case, we may twist with the full $\mathbb{Z}_N$ center symmetry provided we also turn on a background of U(1). Thus, we replace the cocycle conditions (2.19) with

$$\begin{aligned}
\Gamma_i\, \Gamma_j &= e^{i\frac{2\pi\epsilon_{ijk}m_k}{N}} \Gamma_j\, \Gamma_i\,, \\
\omega_i(x+\hat{e}_j L_j)\, \omega_j(x) &= e^{-i\frac{2\pi n\epsilon_{ijk}m_k}{N}} \omega_j(x+\hat{e}_i L_i)\, \omega_i(x)\,,
\end{aligned} \tag{2.36}$$

and we included the $N$-ality of the matter representation $n$ in the cocycle condition of the abelian field. This guarantees that the combined transition functions satisfy the correct cocycle conditions in the presence of matter. Here, we can allow background center fluxes with $(\boldsymbol{m}, \boldsymbol{k}) \in \mathbb{Z}^6$ for all matter representations, thanks to the U(1) gauge group. We also introduce the operators $\hat{T}_j$ for SU($N$) and $\hat{t}_j$ for U(1), $j = 1, 2, 3$. The combinations $\hat{T}_j \hat{t}_j$ are the generators of the electric $\mathbb{Z}^{(1)}_N$ 1-form global symmetry and act on the spatial Wilson lines in (2.17) as: $\hat{T}_j W_{j,\mathrm{SU}(N)} = e^{i\frac{2\pi}{N}} W_{j,\mathrm{SU}(N)} \hat{T}_j$ and $\hat{t}_j W_{j,\mathrm{U}(1)} = e^{-i\frac{2\pi}{N}} W_{j,\mathrm{U}(1))} \hat{t}_j$. The action of $\hat{t}_j$ is implemented on the gauge fields, as usual, by improper gauge transformations of $\hat{a}_j$ as $\hat{a}_j = c[k_j] \circ \hat{a}_j$, and amounts to applying $nk_j$ (Mod $N$) electric twists (notice the appearance of the $N$-ality). Unlike $\hat{T}_j$, the explicit form of $\hat{t}_j$ is simple:

$$\hat{t}_j \equiv e^{i\lambda_j(x)}\,, \quad \lambda_j(x) = \frac{-2\pi n}{N} \frac{x_j k_j}{L_j}\,. \tag{2.37}$$

Since $\mathbb{Z}_N^{(1)}$ is a good global symmetry, we can choose the states in Hilbert space to be eigenstates of the $\mathbb{Z}_N^{(1)}$ generators $\hat{T}_j \hat{t}_j$:

$$\hat{T}_j \hat{t}_j |\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = e^{i\frac{2\pi e_j}{N}} |\psi\rangle_{\text{phy}\,,\boldsymbol{m}}\,, \tag{2.38}$$

where $e_j = 0, 1, \ldots, N-1$. Notice that the states are constructed in the "fractional" background magnetic flux $\boldsymbol{m} \in \mathbb{Z}^3$ (remember that in principle $m_i \in \mathbb{Z}$ Mod $N$, and thus, it implements the fractional magnetic twist. However, we can always add multiples of $N$ to $m_i$ without affecting the cocycle conditions, and hence, we drop the Mod $N$ restriction.) In addition, the theory has a magnetic $\mathrm{U}(1)_m^{(1)}$ 1-form global symmetry, which can be used to characterize the physical states by an "integer" value of the magnetic flux. Therefore, a state in the physical Hilbert space can be labeled as

$$|\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = |E, \boldsymbol{e}, \boldsymbol{N}\rangle_{\boldsymbol{m}}\,, \quad \boldsymbol{e} \in \mathbb{Z}_N^3\,, \tag{2.39}$$

and $\boldsymbol{N} = (N_x, N_y, N_z) \in \mathbb{Z}^3$ (not Mod $N$) label the integral magnetic fluxes of the U(1) gauge group. The partition function (2.18) can be written as a trace over states in Hilbert space in $(\boldsymbol{m}, \boldsymbol{k})$ backgrounds as follows:

$$\begin{aligned}\mathcal{Z}[\boldsymbol{m}, \boldsymbol{k}]_{\mathrm{SU}(N)\times\mathrm{U}(1)+\text{matter}} &= \operatorname{tr}_{\boldsymbol{m}}\left[e^{-L_4 \hat{H}} (\hat{T}_x \hat{t}_x)^{k_x} (\hat{T}_y \hat{t}_y)^{k_y} (\hat{T}_z \hat{t}_z)^{k_z}\right] \\ &= \sum_{\boldsymbol{e}\in\{0,1,\ldots,N-1\}^3, \boldsymbol{N}\in\mathbb{Z}^3} e^{i\frac{2\pi \boldsymbol{e}\cdot\boldsymbol{k}}{N}}\,{}_{\boldsymbol{m}}\langle E, \boldsymbol{e}, \boldsymbol{N}| e^{-L_4 \hat{H}} |E, \boldsymbol{e}, \boldsymbol{N}\rangle_{\boldsymbol{m}}\,.\end{aligned} \tag{2.40}$$

We also build the operator that corresponds to the chiral transformation. This construction was detailed in [36], and we do not repeat it here. Instead, we only give a synopsis of the derivation, which is needed in this work. The anomaly equation of the chiral current is

$$\partial_\mu \hat{j}_A^\mu - 2T_{\mathcal{R}} \partial_\mu \hat{K}^\mu(\hat{A}) - \frac{2d_{\mathcal{R}}}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma} \partial^\mu \hat{a}^\nu \partial^\lambda \hat{a}^\sigma = 0\,. \tag{2.41}$$

Then, the chiral symmetry operator in the background of the $m_j$ magnetic flux is given by

$$\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} = \exp\left[i\frac{2\pi\ell}{2T_{\mathcal{R}}} \hat{Q}_5\right]\,, \tag{2.42}$$

where the conserved charge $\hat{Q}_5$ is given by

$$\begin{aligned}\hat{Q}_5 = &\int_{\mathbb{T}^3} d^3x \left[\hat{j}_\chi^0 - 2T_{\mathcal{R}} K^0(\hat{A}) - 2\frac{d_{\mathcal{R}}}{8\pi^2} \epsilon^{ijk} \hat{a}_i \partial_j \hat{a}_k\right] \\ &+ \frac{d_R}{4\pi}\left(N_z + \frac{n}{N} n_z\right)\left[\int_0^{L_y} \frac{dy}{L_y} \int_0^{L_z} dz \hat{a}_z(x=0, y, z) + \int_0^{L_x} \frac{dx}{L_x} \int_0^{L_z} dz \hat{a}_z(x, y=0, z)\right] \\ &+ \sum_{\text{cyclic}} (x \to y \to z \to x)\,.\end{aligned} \tag{2.43}$$

The last term comes from carefully treating the boundary term implied from the transition functions $\omega_j(x)$, since, unlike $\Gamma_j$, they depend explicitly on $x_j$, see [36] for details. In addition to the background flux $n_j$, which introduces the fractional winding number, we also allow integer magnetic winding $N_j$. Under a transformation with $\hat{t}_j$, the integral of the abelian Chern-Simons term $\hat{K}^0(\hat{a}) = \epsilon^{ijk}\hat{a}_i\partial_j\hat{a}_k$ in the background of the integral $M_j$ and fractional $m_j$ magnetic fluxes transforms as (recall (2.37))

$$\begin{aligned}\hat{t}_j \exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]t_j^{-1} &= \exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(c\circ\hat{a}) - i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]\exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]\\ &= \left(N_j + \frac{nn_j}{N}\right)\left(\frac{nk_j}{N}\right)\exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]. \end{aligned} \tag{2.44}$$

The reader will notice that we switched from the letter $\boldsymbol{m}$, which we use to signify the set of fractional fluxes we can activate, e.g., here we have $\boldsymbol{m} \in \mathbb{Z}^3$, to the letter $\boldsymbol{n}$, which is the actual number of fractional magnetic fluxes we turn on. We shall use the same labeling throughout the paper.

In the next sections, we use these constructions to argue that $\mathrm{SU}(N)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_q$ as well as $\mathrm{SU}(N)\times \mathrm{U}(1)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_N$ enjoy a noninvertible 0-form chiral symmetry, with a possible mixed anomaly with the 1-form center symmetry.

# 3 $\mathrm{SU}(N)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_q$ theories, noninvertible symmetries, and their anomalies

In this section, we direct our attention to YM theories featuring matter fields residing in a particular representation $\mathcal{R}$ and characterized by an $N$-ality $n$. Building upon the discussion in the preceding section, it is established that $\mathrm{SU}(N)$ gauge theories, when coupled to matter, exhibit an electric $\mathbb{Z}_q^{(1)}$ 1-form center symmetry (recall $q = \gcd(N, n)$). A notable maneuver within this framework involves the gauging of $\mathbb{Z}_q^{(1)}$ or a subgroup of it, leading to $\mathrm{SU}(N)/\mathbb{Z}_p$ theory, $\mathbb{Z}_p \subseteq \mathbb{Z}_q$, whose partition function is obtained by summing over integer and fractional topological charge sectors. Thus, gauge transformations with fractional winding numbers are part of the gauge structure, and well-defined operators should be invariant under such gauge transformations. Here, we would like to emphasize that there are $p$ distinct theories: $(\mathrm{SU}(N)/\mathbb{Z}_p)_n$, $n = 0, 1, \ldots, p$, which differ by the admissible genuine (electric, magnetic, or dyonic) line operators. In this paper, we limit our treatment to $(\mathrm{SU}(N)/\mathbb{Z}_p)_{n=0}$, and whenever we mention $\mathrm{SU}(N)/\mathbb{Z}_p$, we particularly mean $(\mathrm{SU}(N)/\mathbb{Z}_p)_0$. What happens to the invertible $\mathbb{Z}^{\chi}_{2T_\mathcal{R}}$ discrete chiral symmetry of this theory? As we shall discuss, this symmetry can stay invertible or become noninvertible, depending on whether it exhibits a mixed anomaly with $\mathbb{Z}_p^{(1)}$ symmetry in the original $\mathrm{SU}(N)$ theory.

## 3.1 $\mathrm{SU}(N)/\mathbb{Z}_q$

We start by discussing noninvertible 0-form chiral symmetries in $\mathrm{SU}(N)/\mathbb{Z}_q$ theories, i.e., theories obtained by gauging the full electric $\mathbb{Z}_q^{(1)}$ 1-form center symmetry. Such theories do not possess global electric 1-form symmetry; hence, there are no genuine Wilson's lines. This can be understood as follows. We start with pure $\mathrm{SU}(N)$ gauge theory, which has an electric $\mathbb{Z}_N^{(1)}$ 1-form symmetry and admits the full spectrum of Wilson lines, i.e., it admits Wilson's lines with all $N$-alities $n = 0, 1, 2, \ldots, N-1$. Gauging a $\mathbb{Z}_q$ subgroup of $\mathbb{Z}_N$, we obtain $\mathrm{SU}(N)/\mathbb{Z}_q$ gauge theory. Now, the spectrum of allowed Wilson lines must be invariant under

$\mathbb{Z}_q$, forcing us to remove those lines with $N$-alities that are not multiples of $q$. The remaining lines in pure $SU(N)/\mathbb{Z}_q$ theory are charged under an electric $\mathbb{Z}^{(1)}_{N/q}$ 1-form symmetry; these are $W_j^{qe_j}$, with $e_j = 0, 1, \ldots, N/q - 1$ and $W_j$ is Wilson's line in the defining representation of $SU(N)$. Finally, introducing matter with $N$-ality $q$ means that those remaining lines can end on the matter and must also be removed from the spectrum. This deprives $SU(N)/\mathbb{Z}_q$ gauge theory with matter from all genuine Wilson's lines.

Despite that $SU(N)/\mathbb{Z}_q$ theory with matter does not possess an electric 1-form symmetry, it is endowed with a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form global symmetry. This can be understood, again, starting from the pure $SU(N)/\mathbb{Z}_q$ theory. As we discussed above, the pure theory has an electric $\mathbb{Z}^{(1)}_{N/q}$ 1-form symmetry. The magnetic dual of $SU(N)/\mathbb{Z}_q$ is $SU(N)/\mathbb{Z}_{N/q}$, which admits a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form symmetry. The pure $SU(N)/\mathbb{Z}_q$ theory has $q$ distinct magnetic fluxes ('t Hooft lines) in its spectrum. Let $\mathcal{T}_j$ be the 't Hooft line winding around direction $j$ in the defining representation of $SU(N)$, i.e., it has $N$-ality 1. Then, the pure $SU(N)/\mathbb{Z}_q$ theory possesses the following set of 't Hooft lines $\mathcal{T}_j^{n_j N/q}$, $n_j = 0, 1, \ldots, q-1$ for $j = 1, 2, 3$, which are mutually local with the set of Wilson's lines $W_j^{qe_j}$, $e_j = 0, 1, \ldots, N/q-1$.[10] Introducing electric matter removes all Wilson's lines (as stated above) but does not alter the magnetic symmetry. Thus, we conclude that $SU(N)/\mathbb{Z}_q$ theory with matter possesses a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form global symmetry acting on a set of 't Hooft lines $\mathcal{T}_j^{n_j N/q}$, $n_j = 0, 1, \ldots, q-1$ for $j = 1, 2, 3$.

We can label the states in the physical Hilbert space of $SU(N)/\mathbb{Z}_q$ theory with matter by both energy and magnetic fluxes since the Hamiltonian commutes with the generators of the magnetic $\mathbb{Z}_q^{m(1)}$ 1-form symmetry:[11]

$$|\psi\rangle_{\text{phy}} = |E, \boldsymbol{n}N/q\rangle\,, \quad \boldsymbol{n} = (n_x, n_y, n_z) \in (\mathbb{Z}_q)^3\,. \tag{3.1}$$

The partition function of these theories involves summing over sectors with fractional topological charges $N\mathbb{Z}/q^2$ (use eq. (2.10) and set $k_i = m_i = N/q$), which can be written in the path-integral formalism as (we set the vacuum angle $\theta = 0$)

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_q+\text{matter}} = \sum_{\nu\in\mathbb{Z},(\boldsymbol{m},\boldsymbol{k})\in(N\mathbb{Z}/q)^6} \int \{[DA_\mu]\, D\,[\text{matter}]\}_{(\boldsymbol{m},\boldsymbol{k})} e^{-S_{YM}-S_{\text{matter}}}\,, \tag{3.2}$$

or in the Hamiltonian formalism as

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_q+\text{matter}} = \text{tr}\left[e^{-L_4\hat{H}}\right] = \sum_{\text{physical states}} {}_{\text{phy}}\langle\psi|e^{-L_4\hat{H}}|\psi\rangle_{\text{phy}}\,. \tag{3.3}$$

Our main task is to build a gauge invariant operator that implements the $\mathbb{Z}^{\chi}_{2T_\mathcal{R}}$ chiral transformation in $SU(N)/\mathbb{Z}_q$ theory with matter. To this end, we use the Hamiltonian formalism of section 2.2, dropping the hats from all operators to reduce clutter. We also use $x, y, z$ to label the three spatial directions. For $\ell \in \mathbb{Z}^{\chi}_{2T_\mathcal{R}}$, the chiral symmetry operator is given by:

$$U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} = e^{2\pi i \frac{\ell}{2T_\mathcal{R}} \int_{\mathbb{T}^3}\left(j_A^0 - 2T_\mathcal{R} K^0(A)\right)}\,. \tag{3.4}$$

[10]This can be easily seen since $\mathcal{T}_j^{n_j N/q}$ and $W_j^{qe_j}$ satisfy the Dirac quantization condition.

[11]Recall that the allowed magnetic twists in the $SU(N)$ theory with matter are $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$.

This operator is invariant under large gauge transformations with integer winding numbers. We will now gauge the $\mathbb{Z}_q^{(1)}$ one-form symmetry. In $\mathrm{SU}(N)/\mathbb{Z}_q$ gauge theory with matter, we sum over arbitrary $\mathbb{Z}_q$ twists with fractional topological charges $N\mathbb{Z}/q^2$. We consider the operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ in the presence of magnetic fluxes $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$ (these are the magnetic fluxes that label the physical states in eq. (3.1).) Let $T_x$ be the generator of an electric $\mathbb{Z}_q$ center twist along the $x$ direction (i.e., a $\mathbb{Z}_q$ gauge transformation), and we take it to have the minimal twist of $N/q$. It acts on $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ via (recall the discussion around eq. (2.33))

$$T_x U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} T_x^{-1} = e^{-2\pi i \ell Q} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} = e^{-2\pi i \ell \frac{n_x N}{q^2}} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\,, \quad n_x \in \mathbb{Z}\,. \tag{3.5}$$

$n_x$ counts the magnetic fluxes inserted in the $y$-$z$ plane in units of $N/q$. Identical relations to (3.5) hold in the $y$ and $z$ directions. As we saw in the previous section, if $\ell\frac{N}{q^2} \notin \mathbb{Z}$, there is a mixed 't Hooft anomaly between the electric $\mathbb{Z}_q^{(1)}$ 1-form center and the discrete chiral symmetries of $\mathrm{SU}(N)$ theory with matter. Eq. (3.5) implies that the operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ is not gauge invariant under a $\mathbb{Z}_q$ gauge transformation as we attempt to gauge $\mathbb{Z}_q^{(1)}$. We can remedy this problem and reconstruct a gauge-invariant operator, denoted by $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}}}$, by summing over all $\mathbb{Z}_q$ gauge transformations generated by $T_x$, $T_y$ and $T_z$:

$$\begin{aligned}
\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} &\equiv \sum_{p_x,p_y,p_z\in\mathbb{Z}} (T_x)^{p_x}(T_y)^{p_y}(T_z)^{p_z} U_{\mathbb{Z}_{2T_\mathcal{R}}} (T_x)^{-p_x}(T_y)^{-p_y}(T_z)^{-p_z} \\
&= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{p_x,p_y,p_z\in\mathbb{Z}} e^{-2\pi i \frac{\ell N}{q^2}(p_x n_x + p_y n_y + p_z n_z)} \equiv U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{\boldsymbol{p}\in\mathbb{Z}^3} e^{-2\pi i \frac{\ell N}{q^2}\boldsymbol{p}\cdot\boldsymbol{n}} \\
&= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{n_x \ell N}{q^2} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{n_y \ell N}{q^2} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{n_z \ell N}{q^2} - l_z\right)\,.
\end{aligned} \tag{3.6}$$

In the first line, we included a sum over arbitrary powers of $T_x, T_y, T_z$ to enforce the gauge invariance. Then, we used eq. (3.5) in going from the first to the second line and the Poisson resummation formula in going from the second to the third line. Even though $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ is gauge invariant, it has no inverse; it is, in general, a noninvertible operator that implements the action of $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$, and we use a tilde to denote the noninvertible nature of symmetries and their operators. The noninvertibility stems from the fact that $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}}}$ works as a projector: the insertion of this operator in the path integral of $\mathrm{SU}(N)/\mathbb{Z}_q$ theory with matter projects onto specific topological charge sectors of $\mathrm{SU}(N)/\mathbb{Z}_q$, depending on $\ell$. This can be seen from the second line in (3.6), which is a sum over Fourier modes that projects in and out sectors, depending on their topological charge, upon acting on them. One can see the projective nature of $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ by inserting it into the partition function (3.3):

$$\langle \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \rangle = \sum_{\text{physical states}} {}_{\text{phy}}\langle \psi | e^{-L_4 \hat{H}} \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} | \psi \rangle_{\text{phy}}\,, \tag{3.7}$$

and then using the physical states defined in eq. (3.1). We find that $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ annihilates sectors with $\frac{n_x \ell N}{q^2} \notin \mathbb{Z}$, etc. We remind that $\frac{n_x N}{q^2}$ is the topological charge (see eq. (2.10)), which we can write as

$$\frac{n_x N}{q^2} = \underbrace{\frac{n_x N}{q}}_{m_x} \underbrace{\frac{N}{q}}_{k_x} \frac{1}{N}\,, \tag{3.8}$$

and, as we mentioned earlier and emphasize now, $n_x$ is the number of magnetic fluxes in units of $N/q$. The same applies to the magnetic sectors in the $y$ and $z$ directions. We conclude that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ selects sectors in Hilbert space with certain magnetic fluxes.

We can make the following observations about $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$:

1. If $\ell \in q\mathbb{Z}$, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ is invertible since in this case $\frac{n_{x,y,z}\ell N}{q^2} \in \mathbb{Z}$ for all values of $n_x, n_y, n_z \in \mathbb{Z}$. The invertible subgroup of $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ is $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}/q}$.

2. If $\gcd(\ell N/q, q) = 1$, then we must have $n_x, n_y, n_z \in q\mathbb{Z}$. In other words, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ projects onto untwisted flux sectors. In particular, in the sector given by $n_x, n_y, n_z \in q\mathbb{Z}$, the symmetry operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ act invertibly for all elements of the chiral symmetry $\ell = 1, 2, \ldots, T_{\mathcal{R}}$.

3. If $\gcd(\ell N/q, q) = a \neq 1$ and $\ell < q$, then let $q = aq'$, and we must have $n_{x,y,z} \in q'\mathbb{Z}$. $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ projects onto background fluxes with topological charge $Q \in \mathbb{Z}/q'$, i.e. sectors that have $\mathbb{Z}_{q'}$ twists.

4. The noninvertibility of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ can be seen by multiplying the operator by its "potential inverse" $\overline{\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}}$ to find

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \times \overline{\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}} \sim \sum_{\boldsymbol{p}\in\mathbb{Z}^3} e^{-2\pi i \frac{\ell N}{q^2} \boldsymbol{p}\cdot\boldsymbol{n}} \equiv \mathcal{C}\,. \tag{3.9}$$

   $\mathcal{C}$ is known as the condensation operator, which can be thought of as a sum over topological surface operators $\exp[-i \oint_{\mathbb{T}^2\subset\mathbb{T}^3} B^{(2)}] = \exp[-i2\pi\mathbb{Z}/q]$ wrapping the three 2-cycles of $\mathbb{T}^3$, and $B^{(2)}$ is the 2-form field of the $\mathbb{Z}_q^{(1)}$ 1-form symmetry.

We use the fact that $\mathrm{SU}(N)/\mathbb{Z}_q$ theory possesses a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form global symmetry to make one more observation. Let $\mathcal{T}_j$ be 't Hooft line of $N$-ality 1 in direction $j$. Then, the minimal 't Hooft line in $\mathrm{SU}(N)/\mathbb{Z}_q$ theory is $\mathcal{T}_j^{N/q}$, i.e., it has $N$-ality $N/q$. The minimal line acts on a physical state by increasing its magnetic flux by one in units of $N/q$.[12] Now, let us take a theory with $\gcd(N/q, q) = 1$ so that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ acts projectively on certain states. Then, $|E, (n_x = q, n_y = q, n_z = q)N/q\rangle$ is one of the physical states that survive under the action of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$. We have $\mathcal{T}_x^{N/q}|E,(q,q,q)N/q\rangle = |E,(q+1,q,q)N/q\rangle$. Thus, we immediately see from eq. (3.6) that

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}\mathcal{T}_x^{N/q}|E,(q,q,q)N/q\rangle = \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}|E,(q+1,q,q)N/q\rangle\rangle = 0\,. \tag{3.10}$$

We write this result as

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}\mathcal{T}_j^{N/q} = 0\,, \quad j = x, y, z\,. \tag{3.11}$$

In other words, the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ annihilates the minimal 't Hooft lines in this theory. It also annihilates all 't Hooft lines $\mathcal{T}_j^{n_j N/q}$, $n_j \neq 0$ Mod $q$. This is an alternative way to see the projective nature of this operator.

[12] Similar to the discussion we had after eq. (2.22), we can also consider the generators of the magnetic 1-form symmetry and argue that $\mathcal{T}_j^{N/q}$ inserts a magnetic flux $N/q$, as measured by the action of the magnetic 1-form symmetry on the state $\mathcal{T}_j^{N/q}|\psi\rangle_{\text{phy}}$.

### 3.2 $\mathrm{SU}(N)/\mathbb{Z}_p$

Next, we discuss $\mathrm{SU}(N)/\mathbb{Z}_p$ theory with matter with $N$-ality $n$, and $\mathbb{Z}_p \subseteq \mathbb{Z}_q = \mathbb{Z}_{\gcd(N,n)}$. The partition function of this theory is given by the path integral in eq. (3.2), now restricting the sum over the electric and magnetic twists $(\boldsymbol{m}, \boldsymbol{k}) \in (N\mathbb{Z}/p)^6$. The theory possess an electric $\mathbb{Z}^{(1)}_{q/p}$ 1-form global symmetry. As before, $T_x$ is taken to be the generator of the electric $\mathbb{Z}^{(1)}_q$ symmetry. Then, the electric $\mathbb{Z}^{(1)}_{q/p}$ 1-form global symmetry is generated by $T^p_x$ (as well as $T^p_y$ and $T^p_z$). The theory has $q/p$ distinct Wilson's lines $W^{e_j p}_j$, with $e_j = 0, 1, 2, \ldots, q/p-1$. These lines are invariant under $\mathbb{Z}_p$, as they should be since $\mathbb{Z}_p$ is gauged. The minimal admissible Wilson's line $W^p_j$ carries one electric flux in units of $pN/q$. In the limiting case $p = q$, the line $W^{p=q}_j$ coincides with the matter content and must be removed from the spectrum of line operators. Therefore, in this case, the theory does not possess a 1-form electric symmetry, as discussed in the previous section.

In addition, the theory has a magnetic $\mathbb{Z}^{m(1)}_p$ 1-form symmetry. If $\mathcal{T}_j$ is the 't Hooft line with $N$-ality 1, then the minimal admissible 't Hooft line in the theory is $\mathcal{T}^{N/p}_j$, which carries one magnetic flux in units of $N/p$. There are $p$ distinct 't Hooft lines in the theory $\mathcal{T}^{n_j N/p}_j$, $n_j = 0, 1, \ldots, p-1$, which are mutually local with Wilson's lines $W^{e_j p}_j$. The Hamiltonian, Wilson's lines generators, and the 't Hooft lines generators of this theory can be simultaneously diagonalized. Therefore, the energies and eigenvalues of the set of Wilson and 't Hooft operators can be used to label the physical states of Hilbert space:

$$|\psi\rangle_{\text{phy}} = |E, \boldsymbol{e}pN/q, \boldsymbol{n}N/p\rangle\,, \quad \boldsymbol{e} \in (\mathbb{Z}_{q/p})^3\,, \boldsymbol{n} \in (\mathbb{Z}_p)^3\,. \tag{3.12}$$

Next, we need to build a gauge invariant chiral symmetry operator. Our starting point, as usual, is the operator

$$U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} = e^{2\pi i \frac{\ell}{2T_{\mathcal{R}}} \int_{\mathbb{T}^3}\left(j^0_A - 2T_{\mathcal{R}} K^0(A)\right)} \tag{3.13}$$

taken in the presence of the fractional magnetic fluxes $\boldsymbol{m} \in (N\mathbb{Z}/p)^3$, which label the Hilbert space in eq. (3.12). The operator $T^{q/p}_x$ generates the electric $\mathbb{Z}^{(1)}_p$ 1-form symmetry, which is gauged. In other words, $T^{q/p}_x$ implements the twists $\boldsymbol{k} \in (N\mathbb{Z}/p)^3$. In analogy with $\mathrm{SU}(N)/\mathbb{Z}_q$ theories, we need to build gauge invariants of the chiral symmetry operator using the building block $T^{q/p}_x U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} T^{-q/p}_x$. To compute this block, we use the discussion around eq. (2.33), taking the minimal twist $N/p$ generated by $T^{q/p}_x$, to obtain

$$T^{q/p}_x U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} T^{-q/p}_x = e^{-2\pi i \ell \frac{n_x N}{p^2}} U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,, \quad n_x \in \mathbb{Z}\,, \tag{3.14}$$

and $n_x$ counts the magnetic fluxes in units of $N/p$. If $\ell \frac{N}{p^2} \notin \mathbb{Z}$, there is a mixed anomaly between $\mathbb{Z}^\chi_{2T_R}$ and the electric $\mathbb{Z}^{(1)}_p$ symmetries in $\mathrm{SU}(N)$ theory with matter, and we expect the chiral symmetry becomes noninvertible upon gauging $\mathbb{Z}^{(1)}_p$. The corresponding gauge invariant operator of the $\tilde{\mathbb{Z}}^\chi_{2T_{\mathcal{R}}}$ symmetry is then given by the summations

$$\begin{aligned}
\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} &= \sum_{p_x,p_y,p_z\in\mathbb{Z}} (T_x)^{qp_x/p}(T_y)^{qp_y/p}(T_z)^{qp_z/p} U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} (T_x)^{-qp_x/p}(T_y)^{-qp_y/p}(T_z)^{-qp_z/p} \\
&= U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \sum_{p_x,p_y,p_z\in\mathbb{Z}} e^{-2\pi i \frac{\ell N}{p^2}(p_x n_x + p_y n_y + p_z n_z)} \\
&= U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{n_x \ell N}{p^2} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{n_y \ell N}{p^2} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{n_z \ell N}{p^2} - l_z\right)\,.
\end{aligned} \tag{3.15}$$

This noninvertible operator generalizes (3.6) to any $\mathbb{Z}_p \subseteq \mathbb{Z}_q$, and it projects onto sectors with finer topological charges than the sectors admissible by (3.6). This means there exist sectors where $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ act invertibly for all $\ell = 1, 2, \ldots, T_{\mathcal{R}}$ if and only if

$$l_x = \frac{n_x N}{p^2} \in \mathbb{Z}\,, \tag{3.16}$$

with similar conditions in the $y$ and $z$ directions. As special cases, we may first set $p = q$ to readily cover (3.6). Also, setting $p = 1$, the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ becomes invertible, as can be easily seen from the second line in (3.15). Notice that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ does not act on Wilson's lines in this theory, as the noninvertible operator is built from $(T_j)^{qp_j/p}$ and its inverse; thus, one can push a Wilson line through $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ without being affected.[13] We can write this observation as

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} W_j^{e_j p} = W_j^{e_j p} \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,, \quad e_j = 0, 1, 2, \ldots, q/p - 1\,, \quad j = x, y, z\,. \tag{3.17}$$

This is very different from the action of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ on 't Hooft lines, as we discussed before.

The procedure employed to construct the noninvertible operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ contains an additional layer of underlying physics. It is essential to keep in mind that this operator is constructed in $\mathrm{SU}(N)/\mathbb{Z}_p$ theory, where its creation involved a sum over magnetic $\boldsymbol{m} \in (N\mathbb{Z}/p)^3$ and electric $\boldsymbol{k} \in (N\mathbb{Z}/p)^3$ twists. These twists do not encompass the entire range of permissible twists that can be applied. Recall that the theory encompasses a global $\mathbb{Z}^{(1)}_{q/p}$ symmetry, which affords us the opportunity to introduce the electric twists $\boldsymbol{k} \in (pN\mathbb{Z}/q)^3$. Moreover, we can turn on magnetic twists $\boldsymbol{m} \in (pN\mathbb{Z}/q)^3$, compatible with the cocycle condition.[14] This broader scope of twists provides a richer set of possibilities within the theory. We recall that $T_x^p$ is the generator of $\mathbb{Z}^{(1)}_{q/p}$ symmetry that implements the twists $k_x \in pN\mathbb{Z}/q$. Then, one can write the partition function of $\mathrm{SU}(N)/\mathbb{Z}_p$ theory in these background twists as

$$\begin{aligned}\mathcal{Z}_{\mathrm{SU}(N)/\mathbb{Z}_p+\text{matter}}[\boldsymbol{m},\boldsymbol{k}] &= \mathrm{tr}_{\boldsymbol{m}\in(pN\mathbb{Z}/q)^3}\left[e^{-L_4 H} T_x^{k_x p} T_y^{k_y p} T_z^{k_z p}\right] \\ &= \sum_{\boldsymbol{e}\in(\mathbb{Z}_{q/p})^3} e^{-i2\pi\frac{p\boldsymbol{k}\cdot\boldsymbol{e}}{q}} {}_{\text{phy}}\langle\psi|e^{-L_4 H}|\psi\rangle_{\text{phy}}\big|_{\boldsymbol{m}\in(pN\mathbb{Z}/q)^3}\,,\end{aligned} \tag{3.18}$$

and we used eq. (3.12) along with $T_j^{k_j p}|\psi\rangle_{\text{phy}} = e^{-i2\pi\frac{pk_j e_j}{q}}|\psi\rangle_{\text{phy}}$; see the discussion around eqs. (3.21), (3.22) below.

Next, consider the commutation relation between $T_x^p$ and $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$, the latter operator is being in the background of the magnetic twist $\boldsymbol{m} \in (pN\mathbb{Z}/q)^3$. Using the discussion and procedure around eq. (2.33), we obtain

$$T_x^p \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} T_x^{-p} = e^{-2\pi i \ell n_x \frac{p^2 N}{q^2}} \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,. \tag{3.19}$$

[13] Although we do not give the explicit form of $T_j$, it can be thought of as an exponential of an integral of the chromoelectric field over a 2-dimensional submanifold; see [49]. A Wilson line would acquire a phase as we push it past $T_j^{q/p}$ (we use $[A_j^a(\boldsymbol{x},t), E_k^b(\boldsymbol{y},t)] = i\delta_{jk}\delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ab}$, where $a, b$ are the color indices, along with the Baker-Campbell-Hausdorff formula). It also acquires the negative of the same phase as it is pushed past $T_j^{-q/p}$. Therefore, the phases cancel out, and hence, the result in eq. (3.17).

[14] Recall that twists in $N\mathbb{Z}/q$ are compatible with the cocycle conditions. Therefore, twists in $pN\mathbb{Z}/q$ are a subset of the allowed twists. Notice that the twists $\boldsymbol{m} \in (pN\mathbb{Z}/q)^3$ provide background magnetic fluxes and do not label the physical states in Hilbert space, eq. (3.12).

The failure of the commutation between $T_x^p$ and $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ by the phase $e^{-2\pi i \ell n_x \frac{p^2 N}{q^2}}$, assuming $\ell n_x \frac{p^2 N}{q^2} \notin \mathbb{Z}$, signals a mixed anomaly between the noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry and the electric $\mathbb{Z}^{(1)}_{q/p}$ 1-form global symmetry. This anomaly means that certain sectors in Hilbert space exhibit degeneracy. Let us analyze this situation more closely. We assume there exists a sector with $n_x, n_y, n_z$ that satisfies eq. (3.16), and thus, in this sector, the symmetry operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ acts invertibly for all elements $\ell = 1, 2, \ldots, T_{\mathcal{R}}$. Now, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$, being a global symmetry operator, commutes with the Hamiltonian:

$$[\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}, H] = 0\,. \tag{3.20}$$

Likewise, since $\mathbb{Z}^{(1)}_{q/p}$ is a global symmetry, its generators $T_j^p$ commute with the Hamiltonian:

$$[T_j^p, H] = 0\,. \tag{3.21}$$

This commutation relation, along with eq. (2.25), implies that $T_j^p$ acts on physical states in Hilbert space as (the label $\boldsymbol{l} = (l_x, l_y, l_z)$ emphasizes that such states satisfy condition (3.16), such that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ acts invertibly on such states. Also, we suppressed the detailed dependence on the different quantum numbers to reduce clutter)

$$T_j^p |E, e_j\rangle_{\boldsymbol{l}} = e^{i\frac{2\pi p}{q} e_j} |E, e_j\rangle_{\boldsymbol{l}}\,, \tag{3.22}$$

and that the states are labeled by their energies as well as $e_j = 1, 2, \ldots, q/p$ distinct labels; these are the eigenvalues (fluxes) of the $\mathbb{Z}^{(1)}_{q/p}$ symmetry operator. The algebra defined by the commutation relations eqs. (3.20), (3.21), along with the mixed anomaly represented as eq. (3.19), under the assumption of a nontrivial phase, furnishes a finite-dimensional space with a minimum dimension of $q^2/\gcd(n_x p^2 N, q^2)$ (we take $n_x = n_y = n_z$). The Hilbert space of physical states, which are labeled by $q/p$ distinct fluxes, sit in $q^2/\gcd(n_x p^2 N, q^2)$ orbits, and a rotation by $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ links a state with a flux $e_j$ to a state with a flux $e_{j+\gcd(n_j p^2 N, q^2)/(qp)}$ as:

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1} |E, e_j\rangle_{\boldsymbol{l}} = |E, e_j + \gcd(n_j p^2 N, q^2)/(qp)\rangle_{\boldsymbol{l}}\,. \tag{3.23}$$

Using the commutation relation (3.20), one immediately sees that the states $|E, e_j\rangle_{\boldsymbol{l}}$ and $|E, e_j + \gcd(n_j p^2 N, q^2)/(qp)\rangle_{\boldsymbol{l}}$ have the same energy.[15]

In the following subsections, we apply our formalism to examples of theories with fermions in specific representations.

## 3.3 Examples

### 3.3.1 SU$(4n+2)/\mathbb{Z}_2$ and SU$(4n)/\mathbb{Z}_2$ with a Dirac fermion in the 2-index anti-symmetric representation

The SU$(4n+2)/\mathbb{Z}_2$ gauge theory with a 2-index anti-symmetric Dirac fermion ($N$-ality 2) has a $\mathbb{Z}^{\chi}_{8n}$ chiral symmetry. The SU$(4n+2)$ theory possesses an electric $\mathbb{Z}^{(1)}_2$ one-form

[15] It is helpful to give a numerical example. Take $N = 1000$, $q = 500$, and $p = 20$. Such numbers are contrived and do not necessarily correspond to any realistic theory. Condition (3.16) is satisfied if we take $n_x = 2$. Then, the phase in the anomaly eq. (3.19) is $e^{-i2\pi/5}$, implying a 5-fold degeneracy. The theory has an electric $\mathbb{Z}^{(1)}_{25}$ 1-form symmetry, and thus, 25 distinct flux states. These states set in 5 different orbits such that the states labeled with $e_1$, $e_6$, $e_{11}$, $e_{16}$, $e_{21}$ have the same energy, and the states $e_2$,$e_7$,…,$e_{22}$, have the same energy, etc.

symmetry. In [37], the authors argued that upon gauging $\mathbb{Z}_2^{(1)}$, the odd rotations of $\mathbb{Z}_{8n}^\chi$ become non-invertible. We can show this is the case on $\mathbb{T}^3$ using our construction. Setting $N = 4n+2$ in (3.6), we obtain

$$\tilde{U}_{\mathbb{Z}_{8n},\ell} = U_{\mathbb{Z}_{8n},\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{n_x\ell}{2} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{n_y\ell}{2} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{n_z\ell}{2} - l_z\right) . \tag{3.24}$$

For $\ell$ odd, $\tilde{U}_{\mathbb{Z}_{8n},\ell}$ projects onto untwisted gauge sectors and becomes non-invertible.

The $SU(4n)/\mathbb{Z}_2$ theory with a 2-index anti-symmetric Dirac fermion has a $\mathbb{Z}_{8n-4}^\chi$ chiral symmetry. The cocycle conditions, say in the $x$-direction, must satisfy (see eq. (2.9))

$$e^{2\pi i \frac{2n_{yz}}{4n}} = 1 . \tag{3.25}$$

Therefore we must have $n_{yz} \in 2n\mathbb{Z}$. There is no mixed anomaly between $\mathbb{Z}_{8n-4}^\chi$ and the electric $\mathbb{Z}_2^{(1)}$ symmetries in the $SU(4n)$ theory since the anomaly phase $\frac{n_{yz}}{2} \in n\mathbb{Z}$ is trivial. Thus, the full chiral symmetry $\mathbb{Z}_{8n-4}^\chi$ is invertible. This is also in agreement with [37].

#### 3.3.2 $SU(6)/\mathbb{Z}_3$ with a Dirac fermion in the 3-index anti-symmetric representation

This theory has a $\mathbb{Z}_6^\chi$ chiral symmetry. What is special about this theory is that its bilinear fermion operator vanishes identically because of Fermi statistics. Moreover, the $SU(6)$ theory exhibits a mixed anomaly between its electric $\mathbb{Z}_3^{(1)}$ 1-form center and chiral symmetries [50, 51]. Assuming confinement, then the chiral symmetry must be broken in the infrared. Yet, this breaking has to be accomplished via higher-order condensate. Using (3.6), we find that the operator corresponding to a chiral transformation in $SU(6)/\mathbb{Z}_3$ theory is

$$\tilde{U}_{\mathbb{Z}_6,\ell} = U_{\mathbb{Z}_6,\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{2n_x\ell}{3} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{2n_y\ell}{3} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{2n_z\ell}{3} - l_z\right) . \tag{3.26}$$

Hence, for $\ell \in \{1,2,4,5\}$, the operator $\tilde{U}_{\mathbb{Z}_6,\ell}$ projects onto untwisted gauge sectors, and the chiral symmetry operator becomes noninvertible.

#### 3.3.3 2-index $SU(6)$ chiral gauge theory

Our next example is a chiral gauge theory. This is $SU(6)$ YM theory with a single left-handed Weyl fermion $\psi$ in the 2-index symmetric representation and 5 flavors of left-handed Weyl fermions $\chi$ in the complex conjugate 2-index anti-symmetric representation. The fermion budget ensures the theory is free from gauge anomalies. The theory encompasses continuous global symmetry $SU(5)_\chi \times U(1)_A$, where $SU(5)_\chi$ acts on $\chi$. The charges of $\psi$ and $\chi$ under $U(1)_A$ are $q_\psi = -5\,, q_\chi = 2$. The theory is also endowed with a $\mathbb{Z}_4^\chi$ chiral symmetry, which is taken to act on $\chi$ with a unit charge. It can be checked that this is a genuine symmetry since neither $\mathbb{Z}_4$ nor a subgroup of it can be absorbed in rotations in the centers of $SU(6)\times SU(5)_\chi$. It turns out, see [52] for details (also see [53]), that we must divide the global symmetry by $\mathbb{Z}_3 \times \mathbb{Z}_5$ to remove redundancies. Putting everything together and remembering that the theory possesses an electric $\mathbb{Z}_2^{(1)}$ 1-form center symmetry (since all fermions have $N$-ality $n=2$), we write the faithful global group as:

$$G^{\mathrm{g}} = \frac{SU(5)_\chi \times U(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_5} \times \mathbb{Z}_4^\chi \times \mathbb{Z}_2^{(1)} . \tag{3.27}$$

This theory has an anomaly between its $\mathbb{Z}_2^{(1)}$ center symmetry and $\mathbb{Z}_4^\chi$ chiral symmetry. To see the anomaly, we recall that we can turn on the magnetic and electric twists $(\boldsymbol{m}, \boldsymbol{k}) \in (3\mathbb{Z})^6$. This gives the topological charge $Q \in \mathbb{Z}/2$. Thus, under a chiral transformation, the partition function acquires a phase

$$\mathcal{Z}[\boldsymbol{m}, \boldsymbol{k}] \longrightarrow \exp\left[i\frac{2\pi\ell N_\chi T_\chi Q}{4}\right] \mathcal{Z}[\boldsymbol{m}, \boldsymbol{k}] = \exp\left[i2\pi\ell/2\right] \mathcal{Z}[\boldsymbol{m}, \boldsymbol{k}]\,, \tag{3.28}$$

where $N_\chi = 5$ is the number of the $\chi$ flavors and $T_\chi = 4$ is the Dynkin index of $\chi$. Therefore, we expect that $\mathbb{Z}_4^\chi$ becomes noninvertible in the $SU(6)/\mathbb{Z}_2$ chiral theory. Using (3.6), the noninvertible operator corresponding to a chiral transformation in $SU(6)/\mathbb{Z}_2$ theory is

$$\tilde{U}_{\mathbb{Z}_4,\ell} = U_{\mathbb{Z}_4,\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{n_x \ell}{2} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{n_y \ell}{2} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{n_z \ell}{2} - l_z\right). \tag{3.29}$$

Hence, for $\ell \in \{1, 3\}$, the operator $\tilde{U}_{\mathbb{Z}_4,\ell}$ projects onto untwisted gauge sectors, and the chiral symmetry operator becomes noninvertible.

## 4 $SU(N) \times U(1)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_N$ theories, noninvertible symmetries, and their anomalies

In this section, we also gauge the $U(1)$ baryon number symmetry. Thus, we are discussing $SU(N) \times U(1)$ gauge theory with a Dirac fermion in a representation $\mathcal{R}$, $N$-ality $n$, and $U(1)$ charge $+1$. This theory, as we discussed in section 2, is endowed with an invertible $\mathbb{Z}^\chi_{2\gcd(T_\mathcal{R}, d_\mathcal{R})}$ chiral symmetry as well as an electric $\mathbb{Z}_N^{(1)}$ center symmetry acting on its Wilson's lines; see eqs. (2.17). However, in [36], it was shown that $SU(N) \times U(1)$ theories also have noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ chiral symmetry. In the following, we first review the construction of the noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ operator in $SU(N) \times U(1)$ theories, and next, we discuss this operator in $SU(N) \times U(1)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_N$, theories.

### 4.1 $SU(N) \times U(1)$

Our starting point is the $SU(N)\times U(1)$ theory and its $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} = e^{i\frac{2\pi\ell}{2T_\mathcal{R}}Q_5}$, where $Q_5$ is the conserved chiral charge defined in eq. (2.43) in the background of the fractional $n_{x,y,z}$ and integer $N_{x,y,z}$ magnetic fluxes in the $x, y, z$ directions. We remind that we can turn on fractional fluxes in $\mathbb{Z}_N$ irrespective of the $N$-ality of the matter content since we use $U(1)$ transition functions to impose the cocycle condition; see eq. (2.36). No nontrivial electric twists are applied at this stage, i.e., we take $\boldsymbol{k} \in (N\mathbb{Z})^3$, since our nonabelian gauge group is $SU(N)$ rather than $SU(N)/\mathbb{Z}_p$. The operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ is invariant under $SU(N)$. To see that, we apply a large $SU(N)$ gauge transformation, recalling eq. (2.24) and setting $\boldsymbol{k} \in (N\mathbb{Z})^3$, which immediately gives the change in the nonabelian winding number by $Q \in \mathbb{Z}$. In addition, $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ must be invariant under $U(1)$ gauge symmetry. The photon gauge field $a_i$ transforms under $U(1)$ gauge symmetry as $a_j(x + \hat{e}_k L_k) = a_j - \partial_k \xi(x)$, and $\xi(x)$ is a periodic gauge function: $\xi(x + \hat{e}_k L_k) = \xi(x) + 2\pi p$, $p \in \mathbb{Z}$. Applying a large $U(1)$ gauge transformation to $Q_5$, we find (see [36] for the derivation)

$$U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \longrightarrow U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\, e^{-2\pi i\ell\left(p_x \frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x + \frac{n n_x}{N}\right) + p_y \frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_y + \frac{n n_y}{N}\right) + p_z \frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_z + \frac{n n_z}{N}\right)\right)}\,, \tag{4.1}$$

where $p_{x,y,z}$ are arbitrary integers corresponding to the U(1) gauge transformation. Eq. (4.1) shows that the operator $U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ fails to be gauge invariant under U(1) gauge symmetry. To remedy this problem, we follow the procedure of the previous section and define a new operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ by summing over all gauge-transformations of $U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$:

$$\begin{aligned}\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} &= U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \sum_{p_x,p_y,p_z\in\mathbb{Z}} e^{-2\pi i\ell\left(p_x\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x+\frac{nn_x}{N}\right)+p_y\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_y+\frac{nn_y}{N}\right)+p_z\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_z+\frac{nn_z}{N}\right)\right)} \\ &= U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\ell\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x+\frac{nn_x}{N}\right)-l_x\right)\left(\sum_{l_y\in\mathbb{Z}}\dots\right)\left(\sum_{l_z\in\mathbb{Z}}\dots\right). \end{aligned} \tag{4.2}$$

The operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ implements the chiral transformation of the now-noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ symmetry, as it acts projectively by selecting certain nonvanishing sectors in Hilbert space labeled by the integers $l_{x,y,z}$, such that for $\ell=1$ we must have

$$l_x = \frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x+\frac{nn_x}{N}\right)\in\mathbb{Z}\,, \tag{4.3}$$

with identical expressions for $l_y$ and $l_z$. Condition (4.3) ensures that all the symmetry elements $\ell = 1, 2, \dots, T_{\mathcal{R}}$ act invertibly on the same admissible sector. To explicitly see the projective nature of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ on states in Hilbert space, we use the partition function of the $SU(N)\times U(1)$ theory given by eq. (2.40) (we set the electric flux background $\boldsymbol{k}$=0 and, as usual, we use $\boldsymbol{n}$ to label a specific fractional magnetic flux background: $\boldsymbol{n}=(n_x,n_y,n_z)$) to compute $\langle\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\rangle$:[16]

$$\langle\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\rangle = \sum_{\boldsymbol{e}\in\mathbb{Z}^3_N,\,\boldsymbol{N}\in\mathbb{Z}^3} {}_{\boldsymbol{n}}\langle E,\boldsymbol{e},\boldsymbol{N}|e^{-L_4\hat{H}}\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}|E,\boldsymbol{e},\boldsymbol{N}\rangle_{\boldsymbol{n}}\,. \tag{4.4}$$

We immediately see from the Kronecker deltas in eq. (4.2) that only those sectors with $\boldsymbol{N}$ satisfying eq. (4.3) are selected.

Turning off the fractional magnetic flux background (i.e., setting $\boldsymbol{n}=0$), the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ becomes invertible for $\ell\in T_{\mathcal{R}}\mathbb{Z}/\gcd(T_{\mathcal{R}},d_{\mathcal{R}})$. We recognize that we have just recovered the invertible $\mathbb{Z}^{\chi}_{2\gcd(T_{\mathcal{R}},d_{\mathcal{R}})}$ subgroup of $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$. Furthermore, setting $\boldsymbol{n}=0$, the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ annihilates all Hilbert space sectors characterized with integral magnetic fluxes $\boldsymbol{N}\notin T_{\mathcal{R}}\mathbb{Z}^3/\gcd(T_{\mathcal{R}},d_{\mathcal{R}})$. This noninvertible nature of the chiral operator should have been anticipated. When we start with the SU(N) theory with matter, we find an 't Hooft anomaly between its invertible $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry and U(1) baryon symmetry. This anomaly is valued in $\mathbb{Z}_{T_{\mathcal{R}}/\gcd(T_{\mathcal{R}},d_{\mathcal{R}})}$. Upon gauging U(1), this anomaly becomes of the ABJ type, and the chiral symmetry becomes noninvertible. Now, If we take the Euclidean version of our theory in the infinite volume limit and apply a $\pi/2$ rotation to $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$, the operator becomes a defect. Alternatively, we may also use the half-gauging procedure to construct this defect, which was done in [36]. Inserting this defect at some position will generally create a domain wall (since it enforces a chiral transformation) dressed with a TQFT that accounts for the noninvertible nature of the defect. It will be interesting to analyze what happens to the domain walls when we turn on an external magnetic field with flux $\boldsymbol{N}\notin T_{\mathcal{R}}\mathbb{Z}^3/\gcd(T_{\mathcal{R}},d_{\mathcal{R}})$.

[16] Recall from our earlier analysis that the theory is endowed with electric $\mathbb{Z}^{(1)}_N$ and magnetic $U^{(1)}_m(1)$ symmetries, and the states of the theory are labeled by the eigenstates of these symmetries, $\boldsymbol{e}$ and $\boldsymbol{N}$, respectively.

SU($N$) $\times$ U(1) gauge theory has an electric $\mathbb{Z}_N^{(1)}$ 1-form global center symmetry, and the immediate exercise would be checking whether there is a mixed anomaly between the center and the noninvertible chiral symmetries. To this end, we turn on both electric and magnetic twists[17] $(\boldsymbol{m}, \boldsymbol{k}) \in \mathbb{Z}^6$, giving rise to nonabelian fractional topological charge $Q_{\rm SU(N)} \in \mathbb{Z}/N$ as well as abelian topological charge $Q_u = \left(\frac{n}{N}\right)^2$; see eq. (2.15). Using eqs. (2.24), (2.44), setting $\boldsymbol{k} = (1,0,0)$, we find

$$\begin{aligned} T_x t_x \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} (T_x t_x)^{-1} &= \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} e^{-2\pi i \ell\left(\frac{n_x}{N} - \frac{n}{N}\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x + \frac{n n_x}{N}\right)\right)} \\ &= \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} e^{-i2\pi\ell\left(\frac{n_x - n l_x}{N}\right)}\,, \end{aligned} \tag{4.5}$$

and we used Condition (4.3) to go from the first to the second line. If the phase is nontrivial, then there is a mixed anomaly between the electric $\mathbb{Z}_N^{(1)}$ 1-form center and the 0-form noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ symmetries, leading to spectral degeneracy of states (those that already selected by the operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$). The algebra defined by the commutation relations $[H, T_j t_j] = [H, \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}] = 0$ along with the mixed anomaly (4.5), under the assumption of a nontrivial phase, furnishes a finite-dimensional space with dimension $N/\gcd(N, n_x - n l_x)$ (we take $n_x = n_y = n_z$). The Hilbert space of physical states, which are labeled by $N$ different electric fluxes $\boldsymbol{e}$, sit in $N/\gcd(N, n_x - n l_x)$ orbits, and a rotation by $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell=1}$ links a state with a flux $e_j$ to a state with a flux $e_j + \gcd(N, n_j - n l_j)$, i.e., they have the same energy.

### 4.2 $\mathbf{SU(N) \times U(1)/\mathbb{Z}_p}$, $\mathbb{Z}_p \subseteq \mathbb{Z}_N$

Next, we study the noninvertible operators in SU($N$) $\times$ U(1)/$\mathbb{Z}_p$ gauge theory, where $\mathbb{Z}_p$ is a subgroup of the $\mathbb{Z}_N$ center symmetry. This theory has an electric $\mathbb{Z}_{N/p}^{(1)}$ 1-form global symmetry acting on the $p$-th power of the spatial components of the abelian and nonabelian Wilson's lines defined in eq. (2.17):

$$W^{e_j p}_{j,\,\rm SU(N)}\,, W^{e_j p}_{j,\,\rm U(1)}\,, \quad e_j = 1, 2, \ldots, N/p, \quad j = 1, 2, 3\,. \tag{4.6}$$

These Wilson's lines are invariant under $\mathbb{Z}_p$, as they should be, as this symmetry is gauged. Notice that the allowed abelian probe charges $q$ need to satisfy $q = z_e$, where $z_e = e_j p$ is the $N$-ality of the nonabelian line. Thus, we can represent the lines in eq. (4.6) by the pair $(z_e, q = z_e)$. The theory also possesses a magnetic $U_m^{(1)}(1)$ 1-form symmetry acting on 't Hooft lines. Let $z_m = 0, 1, \ldots, p-1$, and $g$ be the $N$-ality of the nonabelian 't Hooft line and the abelian magnetic charge, respectively. Then, the pairs $(z_e, q = z_e)$ and $(z_m, g)$ must satisfy the Dirac quantization condition $e^{i2\pi(-qg + z_e z_m/p)} = 1$ or $z_e z_m - pqg \in p\mathbb{Z}$, which gives a constraint on the magnetic charges: $g = \frac{z_m}{p} + \mathbb{Z}$, i.e., the abelian magnetic charges can be fractional [54]. Another way of putting it is that the presence of the Abelian Wilson's lines $W^{e_j p}_{j,\,\rm U(1)}$ demand that the Abelian 't Hooft lines are $\mathcal{T}^{N_j + n_j/p}_{j,\,\rm U(1)}$, $n_j \in \mathbb{Z}_p$, $N_j \in \mathbb{Z}$, such that the electric and magnetic lines are mutually local. The physical states in Hilbert space are taken to be eigenstates of the commuting set of the Hamiltonian, the generators of electric

[17] Notice that these electric twists $\boldsymbol{k} \in \mathbb{Z}^3$ are $\boldsymbol{e}$ that label the physical states in Hilbert space: $|E, \boldsymbol{e}, \boldsymbol{N}\rangle_{\boldsymbol{n}}$. In principle, $k_j$ should be in $\mathbb{Z}$ Mod $N$, but, as usual, we drop the modding as this does not affect the cocycle conditions.

symmetry, and the generators of magnetic symmetry:

$$|\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = |E, p\boldsymbol{e}, \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}\,, \quad e_j = 0,1,\ldots,N/p-1\,, \quad N_j \in \mathbb{Z} \quad n_j = 0,1,\ldots,p-1\,, \\ j = 1,2,3\,, \tag{4.7}$$

and $\boldsymbol{m} \in \mathbb{Z}^3$ is the fractional magnetic flux background (or background magnetic twist). Remember that, in principle, $\boldsymbol{m} \in (\mathbb{Z}\,\mathrm{Mod}\,N)^3$; however, we drop the modding by $N$ since this cannot affect the cocycle condition. Notice that we can always activate a $\mathbb{Z}_N$ magnetic twist since, as emphasized several times, we use a combination of nonabelian and abelian transition functions. Also, in the special case $p = N$, we should remove the subscript $\boldsymbol{m}$ since, in this case, the Hilbert space is spanned by eigenstates of the full magnetic $\mathbb{Z}_N$ fluxes, i.e., $n_j = 0,1,\ldots,N-1$.

The operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ defined in eq. (4.2) is invariant under both SU$(N)$ and U(1) gauge transformations. However, because we are now gauging $\mathbb{Z}_p$, the operator must also be invariant under $\mathbb{Z}_p$ gauge transformations. Let us recall that $T_j t_j$ is the generator of the electric $\mathbb{Z}_N^{(1)}$ symmetry, and therefore, $(T_j t_j)^{N/p}$ generates the $\mathbb{Z}_p$ symmetry, which must be gauged. The action of $(T_j t_j)^{N/p}$ on $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ can be read from the first line in eq. (4.5) by applying the operation $N/p$ times:

$$(T_j t_j)^{N/p}\,\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,(T_j t_j)^{-N/p} = \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,e^{-2\pi i\ell\left(\frac{n_x}{p} - \frac{n}{p}\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x + \frac{n n_x}{N}\right)\right)}\,. \tag{4.8}$$

This relation shows that for a general $\ell$, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ fails to be gauge invariant under a $\mathbb{Z}_p$ gauge transformation.[18] Being acquainted with the remedy of this problem, we use $(T_j t_j)^{N/p}\,\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,(T_j t_j)^{-N/p}$ as a building block of a gauge invariant operator by summing over gauge transformations of the block. The noninvertible operator is then given by

$$\begin{aligned}
\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} &= \sum_{p_x,p_y,p_z\in\mathbb{Z}} (T_x t_x)^{\frac{Np_x}{p}} (T_y t_y)^{\frac{Np_y}{p}} (T_z t_z)^{\frac{Np_z}{p}}\, U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} (T_x t_x)^{-\frac{Np_x}{p}} (T_y t_y)^{-\frac{Np_y}{p}} (T_z t_z)^{-\frac{Np_z}{p}} \\
&= U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \sum_{p_x,p_y,p_z\in\mathbb{Z}} e^{-2\pi i\ell\left(\frac{n_x}{p} - \frac{n}{p}\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x + \frac{n n_x}{N}\right)\right) + (x\to y) + (x\to z)} \\
&= U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{\ell n_x}{p} - \frac{\ell n}{p}\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x + \frac{n n_x}{N}\right) - l_x\right)\left(\sum_{l_y\in\mathbb{Z}}\cdots\right)\left(\sum_{l_z\in\mathbb{Z}}\cdots\right).
\end{aligned} \tag{4.9}$$

The operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ acts invertibly on sectors in Hilbert space that, for $\ell = 1$, satisy the condition

$$l_x = \frac{n_x}{p} - \frac{n}{p}\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x + \frac{n n_x}{N}\right) \in \mathbb{Z}\,, \tag{4.10}$$

with identical expressions in the $y$ and $z$ directions. The operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$, as introduced in eq. (4.9), within the context of SU$(N)\times$U$(1)/\mathbb{Z}_p$ gauge theory, is a generalization of the operator defined in eq. (4.2) for the conventional SU$(N)\times$U(1) theory. Furthermore,

[18] In the special case $p = 1$, the phase becomes $e^{2\pi i\ell n\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x + \frac{n n_x}{N}\right)}$, and using Condition (4.3), the phase trivializes. This shows that this operator is gauge invariant in SU$(N)\times$U(1) theory, as expected.

Condition (4.10) represents a broader generalization of Condition (4.3). In the specific scenario where $p = N$ holds, corresponding to the $\mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_N$ theory, Condition (4.10) precisely mirrors the criterion for the absence of a mixed anomaly between the electric $\mathbb{Z}_N^{(1)}$ 1-form global symmetry and the noninvertible chiral symmetry inherent to the $\mathrm{SU}(N) \times \mathrm{U}(1)$ theory. This correspondence is clear from the first line of eq. (4.5).

The $\mathrm{SU}(N) \times \mathrm{U}(1)/\mathbb{Z}_p$ theory exhibits an electric $\mathbb{Z}_{N/p}^{(1)}$ one-form global symmetry, which is generated by the operators $(T_j t_j)^p$. When introducing a background for this symmetry, we uncover a mixed anomaly between the noninvertible chiral symmetry and the $\mathbb{Z}_{N/p}^{(1)}$ symmetry. Sandwiching $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$, defined in eq. (4.9), between $(T_x t_x)^p$ and $(T_x t_x)^{-p}$ and using eqs. (2.24), (2.44), we find

$$
\begin{aligned}
(T_x t_x)^p \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} (T_x t_x)^{-p} &= \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} e^{-2\pi i \ell \left( \frac{p n_x}{N} - \frac{pn}{N} \frac{d_{\mathcal{R}}}{T_{\mathcal{R}}} \left(N_x + \frac{n n_x}{N}\right)\right)} \\
&= \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} e^{-i 2\pi l_x \ell \frac{p^2}{N}} \,,
\end{aligned} \tag{4.11}
$$

where we used $l_x$ defined in eq. (4.10) in going from the first to the second line. When the phase $e^{-i2\pi l_x \ell \frac{p^2}{N}}$ is nontrivial, it signifies the presence of a degeneracy within the spectrum. Notice that the anomaly phase coincides with the phase in eq. (3.19) if we set $q = N$ in the latter. This should not surprise us since, in this section, we employ the full $\mathbb{Z}_N$ center symmetry, thanks to gauging $\mathrm{U}(1)$. The anomaly in (4.11) is valued in $\mathbb{Z}_{N/\gcd(N,p^2 l_x)}$ (we take $n_x = n_y = n_z$) indicating a $N/\gcd(N, p^2 l_x)$-fold degeneracy. The Hilbert space of physical states, which are labeled by $N/p$ distinct electric fluxes, sit in $N/\gcd(N, p^2 l_x)$ orbits, and a rotation by $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ maps a state with an electric flux $pe_j$ to a state with a flux $p(e_j + \gcd(N, p^2 l_j)/p)$, i.e., they have the same energy.

### 4.3 Examples

#### 4.3.1 $\mathrm{SU}(4k + 2) \times \mathrm{U}(1)/\mathbb{Z}_p$ with 2-index antisymmetric fermions

$\mathrm{SU}(4k + 2) \times \mathrm{U}(1)$ theory with a single 2-index anti-symmetric Dirac fermion was considered in [36]. Here, we study this theory when we gauge a $\mathbb{Z}_p \subseteq \mathbb{Z}_N$ subgroup of the center. Numerical scans reveal that condition (4.10) is always satisfied for specific values of $n_x$ and $N_x$. Also, the anomaly (4.11) is trivial unless both $p$ and $l_x$ are odd; then, the anomaly is valued in $\mathbb{Z}_2$. The Hilbert space is spanned by the physical states

$$
\begin{aligned}
&|\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = |E, p\boldsymbol{e}, \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}} \,, \\
&\qquad e_j = 0, 1, \ldots, (4k+2)/p - 1 \,, \quad N_j \in \mathbb{Z} \,, \quad n_j = 0, 1, \ldots, p - 1 \,, \quad j = 1, 2, 3 \,,
\end{aligned} \tag{4.12}
$$

and the anomaly means that the states live in two orbits such that $|E, p\boldsymbol{e}, \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}$, $|E, p(\boldsymbol{e} + \gcd(N, p^2 \boldsymbol{l})/p), \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}$, $|E, p(\boldsymbol{e} + 2\gcd(N, p^2 \boldsymbol{l})/p), \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}$, etc. have the same energy (we take $n_x = n_y = n_z$).

#### 4.3.2 The Standard Model

The methods presented in this paper provide a systematic means to find noninvertible symmetries in any given gauge theory. As an important application, we employ our approach

| field | SU(3) | SU(2) | U(1) | $\mathrm{U}(1)_B$ | $\mathrm{U}(1)_L$ |
|---|---|---|---|---|---|
| $q_L$ | $\square$ | $\square$ | 1 | $\frac{1}{3}$ | 0 |
| $l_L$ | $\mathbf{1}$ | $\square$ | $-3$ | 0 | 1 |
| $\tilde{e}_R$ | $\mathbf{1}$ | $\mathbf{1}$ | 6 | 0 | $-1$ |
| $\tilde{u}_R$ | $\overline{\square}$ | $\mathbf{1}$ | $-4$ | $-\frac{1}{3}$ | 0 |
| $\tilde{d}_R$ | $\overline{\square}$ | $\mathbf{1}$ | 2 | $-\frac{1}{3}$ | 0 |
| $h$ | $\mathbf{1}$ | $\square$ | 3 | 0 | 0 |

**Table 1.** Matter content and charges of SM: $q_L$ and $l_L$ are the quark and lepton doublets, $\tilde{e}_R, \tilde{u}_R, \tilde{d}_R$ are the electron and up and down quarks singlets, while $h$ is the Higgs doublet. Notice that we take the hyper U(1) charges to be integers, while the matter content has the standard charges under the baryon number $\mathrm{U}(1)_B$ and lepton number $\mathrm{U}(1)_L$ symmetries.

to search for noninvertible symmetries in the nongravitational sector of the Standard Model (SM). SM is based on $su(3) \times su(2) \times u(1)$ Lie algebra. Yet, the faithful gauge group, i.e., the global structure of the group, is to be uncovered. The matter content and charges under the gauge and global symmetries are displayed in table 1, and all fermions are taken to be left-handed Weyls. The anomalies associated with the $\mathrm{U}(1)_B$ and $\mathrm{U}(1)_L$ symmetries are given by: $\mathrm{U}(1)_B \times [\mathrm{SU}(2)]^2 = \mathrm{U}(1)_L \times [\mathrm{SU}(2)]^2 = 1$, $\mathrm{U}(1)_B \times [\mathrm{SU}(3)]^2 = \mathrm{U}(1)_L \times [\mathrm{SU}(3)]^2 = 0$, $\mathrm{U}(1)_B \times [\mathrm{U}(1)]^2 = \mathrm{U}(1)_L \times [\mathrm{U}(1)]^2 = -18$. Thus, we see that $\mathrm{U}(1)_{B-L}$ symmetry is anomaly-free symmetry (we neglect gravity in this context). Under a $\mathrm{U}(1)_{B+L}$ rotation, the path integral picks up an ABJ phase

$$\exp\left(i\alpha \cdot N_f(2c_2(F) - 36c_2(f))\right) , \tag{4.13}$$

where $N_f$ is the number of families, $c_2(F)$ is the second Chern class for SU(2) and $c_2(f)$ is the second Chern class for U(1). The ABJ anomaly breaks the $\mathrm{U}(1)_{B+L}$ down to a $\mathbb{Z}^{B+L}_{\gcd(2,36)N_f} = \mathbb{Z}^{B+L}_{2N_f}$ symmetry. Notice that SU(3) does not play a role in the ABJ anomaly.

The matter content is consistent with the existence of an electric $\mathbb{Z}_6^{(1)}$ 1-form global symmetry [54, 55]. The cocycle conditions satisfied by SM on $\mathbb{T}^4$ with a gauged $\mathbb{Z}_6^{(1)}$ are given by [55]:

$$\begin{aligned}
\Omega_{(3)\mu}(x^\nu = L^\nu)\Omega_{(3)\nu}(x^\mu = 0) &= e^{2\pi i \frac{n^{(3)}_{\mu\nu}}{3}} \Omega_{(3)\nu}(x^\mu = L^\mu)\Omega_{(3)\mu}(x^\nu = 0) \,, \\
\Omega_{(2)\mu}(x^\nu = L^\nu)\Omega_{(2)\nu}(x^\mu = 0) &= e^{2\pi i \frac{n^{(2)}_{\mu\nu}}{2}} \Omega_{(2)\nu}(x^\mu = L^\mu)\Omega_{(2)\mu}(x^\nu = 0) \,, \\
\omega_{(1)\mu}(x^\nu = L^\nu)\omega_{(1)\nu}(x^\mu = 0) &= e^{-2\pi i \left(\frac{n^{(3)}_{\mu\nu}}{3} + \frac{n^{(2)}_{\mu\nu}}{2}\right)} \omega_{(1)\nu}(x^\mu = L^\mu)\omega_{(1)\mu}(x^\nu = 0) \,.
\end{aligned} \tag{4.14}$$

$\Omega_{(i)}$, $i = 2, 3$, and $\omega_{(1)}$ are the transition functions of the gauge bundles, $n^{(i)}_{\mu\nu}$ are the 't Hooft twists, and the superscript/subscript $(i) = (3), (2), (1)$ denote the condition for the $\mathrm{SU}(3), \mathrm{SU}(2), \mathrm{U}(1)$ gauge groups respectively. The electric $\mathbb{Z}_6^{(1)}$ symmetry is generated by a



JHEP03(2024)169

combinations of the SU(3) center, $T_j^{(3)}$, the SU(2) center, $T_j^{(2)}$, and the U(1) center $t_j$, such that the full $\mathbb{Z}_6^{(1)}$ symmetry generator is given by $T_j^{(3)}T_j^{(2)}t_j$, $j=x,y,z$.

The anomalous $\mathrm{U}(1)_{B+L}$ current conservation law is given by

$$\partial_\mu j^\mu_{B+L} - 2N_f\partial_\mu K^\mu_{\mathrm{SU}(2)}(A) + \frac{36N_f}{8\pi^2}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu a^\nu\partial^\lambda a^\sigma = 0\,, \tag{4.15}$$

where $K^\mu_{\mathrm{SU}(2)}$ is the SU(2) topological current. The corresponding unbroken $\mathbb{Z}_{2N_f}^{B+L}$ symmetry operator on $\mathbb{T}^3$ is given by:

$$U_{\mathbb{Z}_{2N_f},\ell} = \exp\left[i\frac{2\pi\ell}{2N_f}Q_5\right]\,, \tag{4.16}$$

where the conserved charge $Q_5$ is given by (here we turn on a $\mathbb{Z}_6$ magnetic twist)

$$\begin{aligned}Q_5 = &\int_{\mathbb{T}^3} d^3x\left[j^0_{B+L} - 2N_fK^0_{\mathrm{SU}(2)}(A) + \frac{36N_f}{8\pi^2}\epsilon^{ijk}a_i\partial_j a_k\right]\\ &-\frac{18N_f}{4\pi}\left(N_z+\frac{1}{6}n_z\right)\left[\int_0^{L_y}\frac{dy}{L_y}\int_0^{L_z}dz a_z(x=0,y,z) + \int_0^{L_x}\frac{dx}{L_x}\int_0^{L_z}dz a_z(x,y=0,z)\right]\\ &+\sum_{\text{cyclic}}(x\to y\to z\to x)\,.\end{aligned} \tag{4.17}$$

Under a U(1) gauge transformation, $U_{\mathbb{Z}_{2N_f},\ell}$ transforms as

$$U_{\mathbb{Z}_{2N_f},\ell} \longrightarrow U_{\mathbb{Z}_{2N_f},\ell}\, e^{-i2\pi\left(\frac{18\ell N_f}{N_f}\left(N_x+\frac{n_x}{6}\right)\right)+(x\to y)+(x\to z)} = U_{\mathbb{Z}_{2N_f},\ell}\,. \tag{4.18}$$

Therefore, $U_{\mathbb{Z}_{2N_f},\ell}$ is U(1) gauge invariant, as required. Further, we examine $U_{\mathbb{Z}_{2N_f},\ell}$ after gauging the electric $\mathbb{Z}_6^{(1)}$ 1-form center by sandwiching $U_{\mathbb{Z}_{2N_f},\ell}$ between its generators (this is a generalization of eq. (4.5)):

$$\begin{aligned}T_x^{(3)}T_x^{(2)}t_x U_{\mathbb{Z}_{2N_f},\ell}\left(T_x^{(3)}T_x^{(2)}t_x\right)^{-1} &= \underbrace{e^{-i\frac{2\pi\ell(2N_f)}{2N_f}\frac{n_x^{(2)}}{2}}}_{\text{from } K^0_{\mathrm{SU}(2)}(A)}\underbrace{e^{i2\pi\ell\frac{36N_f}{2N_f}\left(\frac{1}{6}\right)\left(N_x+\frac{n_x^{(2)}}{2}+\frac{n_x^{(3)}}{3}\right)}}_{\text{from } \epsilon^{ijk}a_i\partial_j a_k} U_{\mathbb{Z}_{2N_f},\ell}\\ &= U_{\mathbb{Z}_{2N_f},\ell}\,.\end{aligned} \tag{4.19}$$

We used eq. (2.33), setting $k_x = m_x = 1$, to find the first exponent. The second exponent is found by applying eq. (2.44) and using $n=1$, $N=6$. Here, $n_x^{(2)}$, $n_x^{(3)}$, and $N_x$ are the SU(2) and SU(3) fractional twists and U(1) integral magnetic flux, respectively. This analysis shows that SM does not possess noninvertible symmetries in its nongravitational sectors. Our findings are consistent with [31].

## 5 Coupling gauge theories to axions and noninvertible symmetries

In this section, we introduce axions into the game, taking $\mathbb{T}^4$ to be larger than any scale in the theory. To be specific, we take $\mathrm{SU}(N)/\mathbb{Z}_p$ or $\mathrm{SU}(N)\times\mathrm{U}(1)/\mathbb{Z}_p$ gauge theories of the previous

JHEP03(2024)169

sections and follow the setup of [56] by adding a complex scalar $\Phi$ that is neutral under the gauge groups but couples to the fermions. Thus, we add the following terms to the Lagrangian:

$$\mathcal{L} \supset |\partial_\mu \Phi|^2 + V(\Phi) + y\Phi\tilde{\psi}\psi + \text{h.c.}\,, \tag{5.1}$$

where $\psi, \tilde{\psi}$ are two left-handed Weyl fermions in representations $\mathcal{R}$ and its complex conjugate $\bar{\mathcal{R}}$, respectively, and $y$ is a Yukawa coupling. The potential of the complex field is $V(\Phi) = \lambda(|\Phi|^2 - v^2)^2$, where $\lambda$ is $\mathcal{O}(1)$ dimensionless parameter. We take the scalar field v.e.v. $v \gg \Lambda$, where $\Lambda$ the strong scale of the gauge sector. We shall pretend that we did not know about the noninvertible symmetries or how to construct them, and let us see if we can reproduce them in the IR.

Let us first consider the SU($N$) gauge theory before gauging U(1) and the electric $\mathbb{Z}_p^{(1)}$ symmetry. Under $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ and U(1) baryon number, the different fields transform as

$$\begin{aligned}
\mathbb{Z}^\chi_{2T_\mathcal{R}} : \quad & \Phi \longrightarrow e^{i\frac{-2\pi}{T_\mathcal{R}}}\Phi\,, \psi \longrightarrow e^{i\frac{2\pi}{2T_\mathcal{R}}}\psi\,, \tilde{\psi} \longrightarrow e^{i\frac{2\pi}{2T_\mathcal{R}}}\tilde{\psi}\,,\\
\text{U}(1) : \quad & \Phi \longrightarrow \Phi\,, \psi \longrightarrow e^{i\alpha}\psi\,, \tilde{\psi} \longrightarrow e^{-i\alpha}\tilde{\psi}\,, \alpha \in [0, 2\pi)\,.
\end{aligned} \tag{5.2}$$

If we write $\Phi$ as $\Phi = \rho e^{i\varphi}$, where $\varphi$ is the axion, then $\varphi$ transforms under $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ as

$$\varphi \longrightarrow \varphi - \frac{2\pi}{T_\mathcal{R}} \tag{5.3}$$

and notice that the axion is inert under the $\mathbb{Z}_2^F$ fermion number subgroup of $\mathbb{Z}^\chi_{2T_\mathcal{R}}$.

Next, we consider SU($N$)/$\mathbb{Z}_p$ or SU($N$) $\times$ U(1)/$\mathbb{Z}_p$ gauge theories with axions. Flowing to an energy scale below $v$, the radial degree of freedom $\rho$ freezes in, i.e., we set $\rho = v$, and the fermions acquire a mass $\sim yv$ and decouple. What remains is the light degree of freedom, the axion $\varphi$. However, the axion should reproduce all the UV anomalies. Thus, we can write the following IR effective Lagrangian of $\varphi$:

$$\mathcal{L}_\varphi = v^2\,(\partial_\mu \varphi)^2 + T_\mathcal{R}\varphi\frac{\text{tr}(F\wedge F)}{8\pi^2} + d_\mathcal{R}\varphi\frac{f\wedge f}{8\pi^2}\,. \tag{5.4}$$

Variation of $\mathcal{L}_\varphi$ w.r.t $\varphi$ produces the anomalous current conservation law:

$$\partial_\mu j^\mu_{(\varphi)} - T_\mathcal{R}\partial_\mu K^\mu(A) - \frac{d_\mathcal{R}}{8\pi^2}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu a^\nu \partial^\lambda a^\sigma = 0\,, \tag{5.5}$$

where $j^\mu_{(\varphi)} = v^2\partial^\mu\varphi$. This is exactly the anomalous current conservation law we had previously, now written down for the axion current. Therefore, everything we said in the previous sections applies here. In particular, we can define an operator of the $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ symmetry as:

$$\mathcal{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} = \exp\left[i\frac{2\pi\ell}{T_\mathcal{R}}\int_{\mathbb{T}^3}(j^0_{(\varphi)} - T_\mathcal{R}K^0(A)) + \ldots\right]\,, \tag{5.6}$$

where the dots denote the contribution from the U(1) gauge field (see eq. (2.43)). We used a calligraphic letter for the operator to emphasize that it is constructed in the IR. Yet, all the anomalies and failure of invariance under gauge symmetries that lead to the noninvertibility of the UV operators apply here as well. Thus, similar to what we did before, we can construct the noninvertible operator $\tilde{\mathcal{U}}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$, which implements the noninvertible symmetry $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ in



JHEP03(2024)169

the IR. Such operators shall project onto magnetic sectors and also exhibit mixed anomalies with the global 1-form electric center symmetry, exactly as we discussed previously.

It was pointed out in [57] that $SU(N)/\mathbb{Z}_p$ theories with axions have noninvertible symmetries. However, our construction shows that such a conclusion is not general and depends on the UV completion. Consider two theories $SU(4k)/\mathbb{Z}_2$ and $SU(4k+2)/\mathbb{Z}_2$ with a Dirac fermion in the 2-index antisymmetric representation and coupled to a complex scalar field $\Phi$ as above. As we flow to the IR, we can construct the operators corresponding to the chiral symmetries. We discussed in section 3.3.1 that $SU(4k)/\mathbb{Z}_2$ theory does not exhibit an anomaly between its chiral symmetry and the 1-form symmetry of the corresponding $SU(4k)$ theory, and hence, the chiral symmetry operator is invertible. Therefore, an axion domain wall (DW), implemented by the action of $\tilde{\mathcal{U}}_{\mathbb{Z}_{8k-4},\ell}$, will not be dressed with TQFT degrees of freedom. On the contrary, $SU(4k+2)/\mathbb{Z}_2$ exhibits an anomaly between its chiral symmetry and the 1-form center of the corresponding $SU(4k+2)$ theory, and thus, the minimal chiral symmetry operator $\tilde{\mathcal{U}}_{\mathbb{Z}_{8k},\ell=1}$ is noninvertible. The axion DW implemented by $\tilde{\mathcal{U}}_{\mathbb{Z}_{8k},\ell=1}$ must be dressed with a fractional quantum Hall TQFT.

We may also consider axions in $SU(N)\times U(1)/\mathbb{Z}_p$ theory of section 4. Everything we said there is transcendent to the IR axion domain walls. In particular, for $p=1$, the operator $\tilde{\mathcal{U}}_{\mathbb{Z}_{2T_\mathcal{R}},\ell=1}$ annihilates the Hilbert space sectors characterized by vanishing fractional $\boldsymbol{n}=0$ and integral magnetic fluxes $\boldsymbol{N}\notin T_\mathcal{R}\mathbb{Z}^3/\gcd(T_\mathcal{R},d_\mathcal{R})$. It will be interesting to examine what happens to the axion domain walls of this theory as we place them in such an external magnetic field.

## Acknowledgments

We would like to thank Erich Poppitz and Tin Sulejmanpasic for various illuminating discussions. We also thank Erich Poppitz for comments on the manuscript. This work is supported by STFC through grant ST/T000708/1.



## References

[1] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators 1. Partition functions*, *Nucl. Phys. B* **646** (2002) 353 [hep-th/0204148] [INSPIRE].

[2] Z. Komargodski, K. Ohmori, K. Roumpedakis and S. Seifnashri, *Symmetries and strings of adjoint QCD$_2$*, *JHEP* **03** (2021) 103 [arXiv:2008.07567] [INSPIRE].

[3] M. Nguyen, Y. Tanizaki and M. Ünsal, *Semi-Abelian gauge theories, non-invertible symmetries, and string tensions beyond N-ality*, *JHEP* **03** (2021) 238 [arXiv:2101.02227] [INSPIRE].

[4] M. Nguyen, Y. Tanizaki and M. Ünsal, *Noninvertible 1-form symmetry and Casimir scaling in 2D Yang-Mills theory*, *Phys. Rev. D* **104** (2021) 065003 [arXiv:2104.01824] [INSPIRE].

[5] Y. Choi et al., *Noninvertible duality defects in $3+1$ dimensions*, *Phys. Rev. D* **105** (2022) 125016 [arXiv:2111.01139] [INSPIRE].

[6] J. Wang and Y.-Z. You, *Gauge Enhanced Quantum Criticality Between Grand Unifications: Categorical Higher Symmetry Retraction*, arXiv:2111.10369 [INSPIRE].

[7] L. Bhardwaj, L.E. Bottini, S. Schafer-Nameki and A. Tiwari, *Non-invertible higher-categorical symmetries*, *SciPost Phys.* **14** (2023) 007 [arXiv:2204.06564] [INSPIRE].

[8] Y. Choi et al., *Non-invertible Condensation, Duality, and Triality Defects in* $3+1$ *Dimensions*, *Commun. Math. Phys.* **402** (2023) 489 [arXiv:2204.09025] [INSPIRE].

[9] J. Kaidi, G. Zafrir and Y. Zheng, *Non-invertible symmetries of* $\mathcal{N}=4$ *SYM and twisted compactification*, *JHEP* **08** (2022) 053 [arXiv:2205.01104] [INSPIRE].

[10] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Global Symmetries in the Standard Model*, *Phys. Rev. Lett.* **129** (2022) 161601 [arXiv:2205.05086] [INSPIRE].

[11] C. Cordova and K. Ohmori, *Noninvertible Chiral Symmetry and Exponential Hierarchies*, *Phys. Rev. X* **13** (2023) 011034 [arXiv:2205.06243] [INSPIRE].

[12] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Time-Reversal Symmetry*, *Phys. Rev. Lett.* **130** (2023) 131602 [arXiv:2208.04331] [INSPIRE].

[13] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible Symmetries and Higher Representation Theory I*, arXiv:2208.05993 [INSPIRE].

[14] J.J. Heckman, M. Hübner, E. Torres and H.Y. Zhang, *The Branes Behind Generalized Symmetry Operators*, *Fortsch. Phys.* **71** (2023) 2200180 [arXiv:2209.03343] [INSPIRE].

[15] C. Cordova, S. Hong, S. Koren and K. Ohmori, *Neutrino Masses from Generalized Symmetry Breaking*, arXiv:2211.07639 [INSPIRE].

[16] A. Karasik, *On anomalies and gauging of* U(1) *non-invertible symmetries in 4d QED*, *SciPost Phys.* **15** (2023) 002 [arXiv:2211.05802] [INSPIRE].

[17] I. García Etxebarria and N. Iqbal, *A Goldstone theorem for continuous non-invertible symmetries*, *JHEP* **09** (2023) 145 [arXiv:2211.09570] [INSPIRE].

[18] Y. Choi, H.T. Lam and S.-H. Shao, *Non-invertible Gauss law and axions*, *JHEP* **09** (2023) 067 [arXiv:2212.04499] [INSPIRE].

[19] R. Yokokura, *Non-invertible symmetries in axion electrodynamics*, arXiv:2212.05001 [INSPIRE].

[20] L. Bhardwaj, S. Schafer-Nameki and A. Tiwari, *Unifying constructions of non-invertible symmetries*, *SciPost Phys.* **15** (2023) 122 [arXiv:2212.06159] [INSPIRE].

[21] L. Bhardwaj, S. Schafer-Nameki and J. Wu, *Universal Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200143 [arXiv:2208.05973] [INSPIRE].

[22] L. Bhardwaj, L.E. Bottini, S. Schafer-Nameki and A. Tiwari, *Non-invertible symmetry webs*, *SciPost Phys.* **15** (2023) 160 [arXiv:2212.06842] [INSPIRE].

[23] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible Symmetries and Higher Representation Theory II*, arXiv:2212.07393 [INSPIRE].

[24] J.J. Heckman et al., *Top down approach to topological duality defects*, *Phys. Rev. D* **108** (2023) 046015 [arXiv:2212.09743] [INSPIRE].

[25] A. Apte, C. Cordova and H.T. Lam, *Obstructions to gapped phases from noninvertible symmetries*, *Phys. Rev. B* **108** (2023) 045134 [arXiv:2212.14605] [INSPIRE].

[26] F. Apruzzi, I. Bah, F. Bonetti and S. Schafer-Nameki, *Noninvertible Symmetries from Holography and Branes*, *Phys. Rev. Lett.* **130** (2023) 121601 [arXiv:2208.07373] [INSPIRE].




JHEP03(2024)169

[27] I. García Etxebarria, *Branes and Non-Invertible Symmetries*, Fortsch. Phys. **70** (2022) 2200154 [arXiv:2208.07508] [INSPIRE].

[28] F. Apruzzi, F. Bonetti, D.S.W. Gould and S. Schafer-Nameki, *Aspects of Categorical Symmetries from Branes: SymTFTs and Generalized Charges*, arXiv:2306.16405 [INSPIRE].

[29] C. Delcamp and A. Tiwari, *Higher categorical symmetries and gauging in two-dimensional spin systems*, arXiv:2301.01259 [INSPIRE].

[30] J. Kaidi, E. Nardoni, G. Zafrir and Y. Zheng, *Symmetry TFTs and anomalies of non-invertible symmetries*, JHEP **10** (2023) 053 [arXiv:2301.07112] [INSPIRE].

[31] P. Putrov and J. Wang, *Categorical Symmetry of the Standard Model from Gravitational Anomaly*, arXiv:2302.14862 [INSPIRE].

[32] M. Dierigl, J.J. Heckman, M. Montero and E. Torres, *R7-branes as charge conjugation operators*, Phys. Rev. D **109** (2024) 046004 [arXiv:2305.05689] [INSPIRE].

[33] Y. Choi, M. Forslund, H.T. Lam and S.-H. Shao, *Quantization of Axion-Gauge Couplings and Non-Invertible Higher Symmetries*, arXiv:2309.03937 [INSPIRE].

[34] S. Schafer-Nameki, *ICTP lectures on (non-)invertible generalized symmetries*, Phys. Rept. **1063** (2024) 1 [arXiv:2305.18296] [INSPIRE].

[35] S.-H. Shao, *What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetry*, arXiv:2308.00747 [INSPIRE].

[36] M.M. Anber and E. Poppitz, *Noninvertible anomalies in* $\mathrm{SU}(N)\times\mathrm{U}(1)$ *gauge theories*, JHEP **08** (2023) 149 [arXiv:2305.14425] [INSPIRE].

[37] R. Argurio and R. Vandepopeliere, *When* $\mathbb{Z}_2$ *one-form symmetry leads to non-invertible axial symmetries*, JHEP **08** (2023) 205 [arXiv:2306.01414] [INSPIRE].

[38] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, JHEP **08** (2013) 115 [arXiv:1305.0318] [INSPIRE].

[39] G. 't Hooft, *Aspects of Quark Confinement*, Phys. Scripta **24** (1981) 841 [INSPIRE].

[40] P. van Baal, *Twisted Boundary Conditions: A Nonperturbative Probe for Pure Nonabelian Gauge Theories*, Other thesis, Utrecht University, 3508 TA Utrecht, Netherlands (1984) [INSPIRE].

[41] A.A. Cox, E. Poppitz and F.D. Wandler, *The mixed 0-form/1-form anomaly in Hilbert space: pouring the new wine into old bottles*, JHEP **10** (2021) 069 [arXiv:2106.11442] [INSPIRE].

[42] D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, *Theta, Time Reversal, and Temperature*, JHEP **05** (2017) 091 [arXiv:1703.00501] [INSPIRE].

[43] M.M. Anber and E. Poppitz, *The gaugino condensate from asymmetric four-torus with twists*, JHEP **01** (2023) 118 [arXiv:2210.13568] [INSPIRE].

[44] M.M. Anber and E. Poppitz, *Multi-fractional instantons in* $\mathrm{SU}(N)$ *Yang-Mills theory on the twisted* $\mathbb{T}^4$, JHEP **09** (2023) 095 [arXiv:2307.04795] [INSPIRE].

[45] E. Poppitz and F.D. Wandler, *Gauge theory geography: charting a path between semiclassical islands*, JHEP **02** (2023) 014 [arXiv:2211.10347] [INSPIRE].

[46] Y. Tanizaki and M. Ünsal, *Center vortex and confinement in Yang-Mills theory and QCD with anomaly-preserving compactifications*, PTEP **2022** (2022) 04A108 [arXiv:2201.06166] [INSPIRE].

[47] M.M. Anber and E. Poppitz, *On the baryon-color-flavor (BCF) anomaly in vector-like theories*, JHEP **11** (2019) 063 [arXiv:1909.09027] [INSPIRE].

JHEP03(2024)169

[48] M.M. Anber, *Condensates and anomaly cascade in vector-like theories*, *JHEP* **03** (2021) 191 [arXiv:2101.04132] [INSPIRE].

[49] H. Reinhardt, *On 't Hooft's loop operator*, *Phys. Lett. B* **557** (2003) 317 [hep-th/0212264] [INSPIRE].

[50] S. Yamaguchi, *'t Hooft anomaly matching condition and chiral symmetry breaking without bilinear condensate*, *JHEP* **01** (2019) 014 [arXiv:1811.09390] [INSPIRE].

[51] M.M. Anber, *Self-conjugate QCD*, *JHEP* **10** (2019) 042 [arXiv:1906.10315] [INSPIRE].

[52] M.M. Anber and S.Y.L. Chan, *2-index chiral gauge theories*, *JHEP* **10** (2023) 025 [arXiv:2308.08052] [INSPIRE].

[53] M.M. Anber, S. Hong and M. Son, *New anomalies, TQFTs, and confinement in bosonic chiral gauge theories*, *JHEP* **02** (2022) 062 [arXiv:2109.03245] [INSPIRE].

[54] D. Tong, *Line Operators in the Standard Model*, *JHEP* **07** (2017) 104 [arXiv:1705.01853] [INSPIRE].

[55] M.M. Anber and E. Poppitz, *Nonperturbative effects in the Standard Model with gauged 1-form symmetry*, *JHEP* **12** (2021) 055 [arXiv:2110.02981] [INSPIRE].

[56] M.M. Anber and E. Poppitz, *Deconfinement on axion domain walls*, *JHEP* **03** (2020) 124 [arXiv:2001.03631] [INSPIRE].

[57] C. Cordova, S. Hong and L.-T. Wang, *Axion Domain Walls, Small Instantons, and Non-Invertible Symmetry Breaking*, arXiv:2309.05636 [INSPIRE].




JHEP03(2024)169