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ABSTRACT

In *Exact dynamical decay rate for the almost Mathieu operator* by Jitomirskaya *et al.* [Math. Res. Lett. **27**(3), 789–808 (2020)], the authors analysed the dynamical decay in expectation for the supercritical almost-Mathieu operator in function of the coupling parameter, showing that it is equal to the Lyapunov exponent of its transfer matrix cocycle, and asked whether the same is true for the 1d Anderson model. We show that this is never true for bounded potentials when the disorder parameter is sufficiently large.

I. INTRODUCTION

Consider the one-dimensional Anderson model, i.e., the operator H acting on a dense subset of $\ell^2(\mathbb{Z})$ via

$$(H\psi)(x) = \psi(x+1) + V_x\psi(x) + \psi(x-1), \quad x \in \mathbb{Z}, \tag{1}$$

where V_x are i.i.d. random variables. We assume that the distribution of V_0 is bounded and not concentrated at one point (in most of the discussion below, the first assumption can be relaxed to the existence of a finite fractional moment $\mathbb{E}|V_0|^\eta < \infty$). Carmona–Klein–Martinelli[5] showed that under these assumptions H exhibits Anderson localisation, i.e., almost surely H has pure point spectrum, and moreover

$$\mathbb{P}\left\{\forall(\lambda,\psi) \in \mathcal{E} \ \limsup_{x\to\pm\infty} \frac{1}{|x|}\log|\psi(x)| = -\gamma(\lambda)\right\} = 1, \tag{2}$$

where $\mathcal{E} = \mathcal{E}[H]$ is the collection of eigenpairs of H (the spectrum is almost surely simple, so $|\psi|$ is well-defined), and $\gamma(\lambda)$ is the Lyapunov exponent of H at energy λ. Under more restrictive assumptions on H, the pure point nature of the spectrum was first proved by Goldsheid–Molchanov–Pastur[8] and by Kunz–Souillard;[12] the exponential decay of the eigenfunctions was first established by Molchanov.[15]

While the proof of Ref. 5 employs multi-scale analysis, single-scale proofs have recently been found by Bucaj *et al.*,[4] Gorodetski–Kleptsyn,[9] and Jitomirskaya–Zhu.[11] Generalisations to models with off-diagonal disorder and to matrix-valued potentials are studied in Refs. 13 and 16.

A stronger notion of Anderson localisation involves the notion of eigenfunction correlator, introduced by Aizenman.[1] Denote

$$Q_H(x,y) = \sup\{|f(H)(x,y)| \ : \ f:\mathbb{R}\to\mathbb{C}, \ \sup|f| \leqslant 1\},$$

where the supremum is taken over Borel functions. If H has pure point spectrum, the correlator takes the form

$$Q_H(x,y) = \sum_{(\lambda,\psi)\in\mathcal{E}} |\psi_\lambda(x)||\psi_\lambda(y)|. \tag{3}$$

Then there exists $\gamma > 0$ such that for any x,

$$\mathbb{P}\left\{\limsup_{y\to\pm\infty} \frac{1}{|y-x|} \log Q(x,y) \leqslant -\gamma\right\} = 1. \tag{4}$$

In fact, in the current setting (4) holds with $\gamma = \gamma_{\inf}$, where

$$\gamma_{\inf} = \inf_{\lambda\in\sigma(H)} \gamma_\lambda(H) \tag{5}$$

and $\sigma(H)$ is the spectrum of H (a deterministic set)—see Ref. 11. This strong form of (4) implies (2), as well as dynamical localisation, decay of the Fermi projection as well as other properties of relevance in quantum dynamics.

Ge and Zhao built on the work[11] and proved the following:

Theorem 1.1 (Ge–Zhao). *For the operator H of (1) with V_0 bounded and not concentrated at one point, one has, for any $x \in \mathbb{Z}$,*

$$\gamma^{\mathbb{E}} = -\limsup_{y\to\pm\infty} \frac{1}{|y-x|} \log \mathbb{E}Q(x,y) > 0. \tag{6}$$

In Sec. II we give another, arguably, simpler, proof of this result, adopting an argument from Ref. 6.

Jitomirskaya *et al.* [10] studied the validity of (6) in the almost-periodic setting, namely, for the supercritical almost-Mathieu operator with Diophantine frequency, and showed that in that setting $\gamma^{\mathbb{E}}$ can be taken to be equal to $\gamma_{\inf}$. They asked whether the same is true for the Anderson model. We show that this is not the case. A first counterexample comes from the Anderson–Bernoulli model:

Theorem 1.2. *For $a > 0$, consider the operator $H^a = H_0 + aV$ with V_x being a bounded random variable having an atom at 0. Then $\gamma^{\mathbb{E}}$ is bounded from above uniformly in a.*

In particular, if V_x is a Bernoulli random variable with parameter p, by a result of Martinelli and Micheli,[14] $\gamma_{\inf} \geqslant c\log a$ for sufficiently large a. Therefore, by the above theorem, $\gamma^{\mathbb{E}}(H^a) \neq \gamma_I(H^a)$ for a large enough.

Furthermore, the above theorem remains true for any bounded random potential satisfying mild conditions, at sufficiently high disorder:

Theorem 1.3. *Let $V = \{V_i\}_{i\in\mathbb{Z}}$ be a nondeterministic, bounded, i.i.d. random potential, and let $H^a := H_0 + aV$. Then, for any a large enough*

$$\inf_{E\in\sigma(H^a)} \gamma_E(H^a) > \gamma^{\mathbb{E}}(H^a).$$

II. PROOF OF THEOREM 1.1

For $a, b \in \mathbb{Z}$, denote by $H_{[a,b]}$ the restriction of H to $[a,b]$ (with Dirichlet boundary conditions), and let $G_E[H_{[a,b]}] = (H_{[a,b]} - E)^{-1}$. Let $\tau > 0$, $E \in \mathbb{R}$ and $N \geqslant 1$. A site $x \in \mathbb{Z}$ is called (τ, E, N)-nonresonant $[x \notin \mathrm{Res}(\tau,E,N)]$ if

$$|G_E[H_{[x-N,x+N]}](x, x\pm N)| \leqslant e^{-(\gamma(E)-\tau)N}.$$

Otherwise, x is called (τ,E,N)-resonant $[x \in \mathrm{Res}(\tau,E,N)]$. The proof of the theorem uses the following.

Claim 1. Assume that V_0 is bounded and not concentrated at one point. Then for any $\tau > 0$ there exist $C, c > 0$ such that

$$\mathbb{P}\{\forall E \in \mathbb{R}\ \mathrm{diam}(\mathrm{Res}(\tau.E,N) \cap [-N^2,N^2]) > 2N\} \leqslant Ce^{-cN}.$$

See Ref. 13, Proposition 2.1 for this formulation (in the more general case of matrix potentials) and Ref. 11, Theorem 4.1, for a similar statement in the pure one-dimensional case.

Next, we need a representation for the eigenfunction correlator as a singular integral [see (7.4) at p. 102 of Ref. 2]:

$$Q(x,y) = \lim_{L\to\infty} Q^L(x,y), \qquad Q^L(x,y) = \lim_{\epsilon\to 0^+} \frac{\epsilon}{2}\int |G_E[H_{[-L,L]}](x,y)|^{1-\epsilon} dE \leqslant 1. \tag{7}$$

Having these two ingredients, we argue as follows. Without loss of generality we can assume that $x = 0$. Set $\tau = \frac{1}{2}\min_{E\in\sigma(H)} \gamma(E)$ and $N = \lfloor \frac{y}{10} \rfloor$, and consider the event

$$\mathcal{R} = \{\exists E \in \mathbb{R}\ :\ 0, y \in \mathrm{Res}(\tau,E,N)\}.$$

According to Claim 1, $\mathbb{P}(\mathcal{R}) \leqslant Ce^{-cN}$. On the complement $\Omega \backslash \mathcal{R}$, we have $\mathbb{R} = A \cup B$, where

$$A = \{E \in \mathbb{R} \ : \ 0 \notin \mathrm{Res}(\tau, E, N)\}, \qquad B = \{E \in \mathbb{R} \ : \ y \notin \mathrm{Res}(\tau, E, N)\}.$$

For $E \in A$,

$$\begin{aligned} G_E[H_{[-L,L]}](0,y) &= G_E[H_{[-N,N]}](0,N) G_E[H_{[-L,L]}](N+1,y) \\ &\quad + G_E[H_{[-N,N]}](0,-N) G_E[H_{[-L,L]}](-N-1,y), \end{aligned}$$

whence

$$|G_E[H_{[-L,L]}](0,y)| \leqslant \ e^{-\tau N}(|G_E[H_{[-L,L]}](N+1,y)| + |G_E[H_{[-L,L]}](-N-1,y)|.$$

and an analogous bound can be deduced for B.

Thus

$$\lim_{\epsilon \to +0} \frac{\epsilon}{2} \left[\int_A + \int_B \right] |G_E[H_{[-L,L]}](0,y)|^{1-\epsilon} \mathrm{d}E \leqslant 4e^{-\tau N}.$$

Finally,

$$\begin{aligned} \mathbb{E}Q(0,y) &= \mathbb{E}(Q(0,y)|\mathcal{R}) \ \mathbb{P}(\mathcal{R}) + \mathbb{E}(Q(0,y)|\,\Omega \backslash \mathcal{R})(1 - \mathbb{P}(\mathcal{R})) \\ &\leqslant \mathbb{P}(\mathcal{R}) + \mathbb{E}(Q(0,y)|\,\Omega \backslash \mathcal{R}) \leqslant Ce^{-cN} + 2e^{-\tau N}. \end{aligned}$$

Thus, $\gamma^{\mathbb{E}} \geqslant \min(c, \tau) > 0$.

Remark. This proof can be extended to quasi-one-dimensional operator, such as the Anderson model on the strip of width W or the more general model studied in Ref. 13. A slightly weaker version of (7) is still true in this case (see Ref. 6):

$$Q^L(x,y) \leqslant \lim_{\epsilon \to 0^+} \frac{\epsilon}{2} \int \|G_E[H_{[-L,L]}](x,y)\|^{1-\epsilon} \mathrm{d}E \leqslant W,$$

and the argument above follows with minor modifications.

III. PROOF OF THEOREM 1.2

Let $K > 0$ be a large numerical constant (independent of any parameters), to be specified later. For $x > 0$, consider the event

$$\Omega_{K,x} = \{\forall y \in [-Kx, (K+1)x], \ V_y = 0\}.$$

We shall prove the following: for any $\epsilon > 0$, one has on $\Omega_{K,x}$ for sufficiently large x:

$$Q(0,x) \geqslant e^{-c|x|}. \tag{8}$$

Since $\mathbb{P}(\Omega_{K,x}) = (1-p)^{2Kx+1}$, this this would imply that

$$\gamma^{\mathbb{E}} \leqslant -\lim_{x \to \infty} \frac{1}{x} \log(1-p)^{2Kx+1} e^{-cx} \leqslant c - 2K \log(1-p) \tag{9}$$

as claimed.

We now turn to the proof of (8). Since the argument is uniform in a, we will use H for H^a. Observe that for any $\delta > 0$,

$$Q(x,y) \geqslant \delta |G_\delta[H](x,y)|. \tag{10}$$

In fact,

$$Q(x,y) = \sum_{(\lambda,\,\psi) \in \mathcal{E}} |\psi_\lambda(0)||\psi_\lambda(x)| \geqslant \delta \sum_{(\lambda,\,\psi) \in \mathcal{E}} \frac{|\psi_\lambda(0)| \ |\psi_\lambda(x)|}{|i\delta - \lambda|} \geqslant \delta |G_{i\delta}[H](0,x)|.$$

Let $T = H_0$ be the free Laplacian [obtained by setting $V_x \equiv 0$ in (1)], and let $T_{K,x}$ be the restriction of T to the finite volume $[-Kx, (K+1)x]$. Then, by applying the resolvent identity and the reverse triangle inequality twice, we get

$$\begin{aligned}
&|G_{i\delta}[H](0,x)| \\
&\quad \geqslant |G_{i\delta}[T_{K,x}](0,x)| \\
&\qquad - |G_{i\delta}[T_{K,x}](0,-Kx)||G_{i\delta}[H](-Kx-1,x)| \\
&\qquad - |G_{i\delta}[T_{K,x}](0,(K+1)x)||G_{i\delta}[H]((K+1)x+1,x)| \\
&\quad \geqslant |G_{i\delta}[T](0,x)| \\
&\qquad - |G_{i\delta}[T_{K,x}](0,-Kx)||G_{i\delta}[H](-Kx-1,x)| \\
&\qquad - |G_{i\delta}[T_{K,x}](0,(K+1)x)||G_{i\delta}[H]((K+1)x+1,x)| \\
&\qquad - |G_{i\delta}[T_{K,x}](0,-Kx)||G_{i\delta}[T](-Kx-1,x)| \\
&\qquad - |G_{i\delta}[T_{K,x}](0,(K+1)x)||G_{i\delta}[T]((K+1)x+1,x)|.
\end{aligned}$$

By the Combes–Thomas estimate (Ref. 2, Theorem 10.5), we deduce that for $\delta \in (0,1)$,

$$|G_{i\delta}[H](0,x)| \geqslant |G_{i\delta}[T](0,x)| - \frac{C_1}{\delta} \mathrm{e}^{-c_1(2K+1)\ \delta|x|}.$$

Now,

$$|G_{i\delta}[T](0,x)| \geqslant C_2 \mathrm{e}^{-c_2\delta|x|}. \tag{11}$$

Indeed, $g(x) = G_{i\delta}[T](0,x)$ is by definition the square-summable solution to the equation

$$g(x+1) + g(x-1) + i\delta g(x) = \delta_{x,0}.$$

Plugging in the ansatz $g(x) = \alpha \mathrm{e}^{-\xi|x|}$, we find that this is indeed a solution provided that

$$\xi = \operatorname{arccosh}\left(\frac{i\delta}{2}\right); \qquad \alpha = (2 + 2\mathrm{e}^{-\xi} + \delta)^{-1},$$

hence $|g(x)| \geqslant |\alpha| \mathrm{e}^{\,-(\Re\xi)|x|}$, where $\Re\xi \leqslant c_2\delta$ for $\delta > 0$ small enough, as

$$\Re\xi = \log\left(\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + 1}\right) = \log\left(1 + \frac{\delta}{2} + o(\delta^2)\right) = \frac{\delta}{2} + o(\delta^2)$$

in a neighbourhood of 0.

Having set $c = c_2\delta$ and $2K + 1 = \lceil 100c_2/c_1 \rceil$, we obtain (8).

IV. PROOF OF THEOREM 1.3: LOGARITHMIC DIVERGENCE FOR GENERAL POTENTIALS

A version of the Martinelli–Micheli bound the 1d Anderson model with absolutely continuous, bounded potential has been proven in 1983 by Avron *et al.* in Ref. 3.

Theorem 4.1 (Avron *et al.*[3]). *Let $H^a = H_0 + aV$ be a random Schrödinger operator where V is a bounded random potential with absolutely continuous density. Then the Lyapunov exponent γ_a of H^a is such that*

$$\gamma_a \geqslant \log(a) - K,$$

where K is a finite constant.

We will adapt the proof in Ref. 3 to any potential having finite first moment (not necessarily absolutely continuous).[7] Avron *et al.*'s proof relies on the Thouless' formula for the Lyapunov exponent of a Schrödinger operator H, stating that

$$\gamma_H(E) = \int \log|E - E'|\, \mathrm{d}\kappa_H(E'), \tag{12}$$

where $\kappa_H(E')$ denotes the *integrated density of states* of the operator H. They proceed then to bound the negative part of the logarithm in 12 using the *Wegner estimate*: If $H = H_0 + V$ is a random Schrödinger operator with i.i.d. potential, then

$$\frac{\mathrm{d}\kappa_H(E)}{\mathrm{d}E} \leqslant \|\rho_V\|_\infty,$$

which is unfortunately proven true only when the distribution ρ_V of the potential is absolutely continuous.

Fortunately, Shubin *et al.* proved in Ref. 17 a slightly weaker bound for the IDS of a random Schrödinger operator whose potential satisfies the conditions of Theorem 1.3: if $\mathbb{E}[|V_0|] < \infty$ (condition that we have automatically since the distribution of V_0 is bounded), then

$$|\kappa_H(E) - \kappa_H(E')| \leqslant C|E - E'|^\alpha \tag{13}$$

for some $\alpha \in (0,1)$ and some constant $C > 0$. This bound is sufficient to let the Avron–Craig–Simon argument work in the present generality.

By applying the Thouless formula to H^a and splitting the logarithm into its positive and negative parts, we get that

$$\begin{aligned}
&\int \log|E - E'|\,\mathrm{d}\kappa_{H^a}(E') \\
&= \int \log_+|E - E'|\,\mathrm{d}\kappa_{H^a}(E') - \int \log_-|E - E'|\,\mathrm{d}\kappa_{H^a}(E') \\
&\geqslant \int_{a/2}^{\infty} \log_+|E - E'|\,\mathrm{d}\kappa_{H^a}(E') - \int_0^{\infty} \kappa_{H^a}\{E' : |E - E'| \leqslant \mathrm{e}^{-t}\}\,\mathrm{d}t.
\end{aligned}$$

We claim that

$$\int_0^{\infty} \kappa_{H^a}\{E' : |E - E'| \leqslant \mathrm{e}^{-t}\}\,\mathrm{d}t \leqslant C\,; \tag{14}$$

for some $C > 0$ uniform in a, and that

$$\int_{a/2}^{\infty} \log_+|E - E'|\,\mathrm{d}\kappa_{H^a}(E') \geqslant c\,\log a \tag{15}$$

for some positive constant c.

The first bound (14) is proven by using inequality (13):

$$\begin{aligned}
\int_0^{\infty} \kappa_{H^a}\{E' : |E - E'| \leqslant \mathrm{e}^{-t}\}\,\mathrm{d}t &\leqslant \|V\|_\infty \int_0^{\infty} \mathrm{e}^{-t}\mathrm{d}t^\alpha \\
&\leqslant C\int_0^{\infty} \alpha t^{\alpha-1}\mathrm{e}^{-t}\mathrm{d}t \\
&\leqslant C\,\Gamma(\alpha) \leqslant C'.
\end{aligned}$$

Remark. We strongly believe that the main result of this section (and thus Theorem 1.3 as a whole) can be extended to the most general setting for which 1-d localisation has been proven (nondeterministic potential with any finite fractional moment). However, generalising (13) to the case where one only has a generic fractional moment appears to be nasty.

An alternative to this approach could be extending the proof of the logarithmic divergence for the Anderson–Bernoulli model in Ref. 14 to the general case; however, even if this seems to be doable and should not present major technical difficulties, additional estimates would be needed to make Martinelli's and Micheli's already six page long proof work for generic potentials, and many formulas would get much longer and nastier.

In conclusion, the length of the present paper would likely get doubled by such an attempt, therefore we avoid it to keep the paper short and more readable while keeping the result reasonably general.

V. PROOF OF THEOREM 1.3: GREEN FUNCTION ESTIMATES

Since the above argument uses crucially the fact that a Bernoulli random variable is 0 with positive probability, one might suspect that the presence of an atom at zero is required for the annealed dynamical decay to be non-Lyapunov. However, in this section we will use a simple trick to eliminate the atom at 0. The trick relies on the observation that it is possible to decompose any dilated random variable aX having an atom as a sum of two (not necessarily independent) random variables, one of which is bounded in a, and the other has basically the same distribution as aX with the difference that the atom has been subtracted some mass. If we subtract in this way mass from the atom at a sufficient rate, and control the error given by the bounded addend, then we can show that the growth of the annealed decay rate in a is much slower than the logarithmic lower bound prescribed by the results of Avron–Craig–Simon and Shubin–Vakilian–Wolff.

Proof. In order to exploit the case of a potential with an atom at 0, we will make use of the following observation.

Observation 5.1. *Let X be a bounded random variable, and let $\bar{x} \in \operatorname{supp}(X) \subseteq [-R, R]$. Suppose that X is absolutely continuous in a neighborhood of $\bar{x}$ and denote by $\widetilde{X}^\epsilon$ the random variable having the same density as X, except for the fact that it has an atom at $\bar{x}$ of mass ϵ, suitably renormalised. Then there exists a bounded random variable $\tilde{\eta}$ (not necessarily independent on X) such that*

$$aX \stackrel{d}{=} a'\widetilde{X}^{\epsilon} + \tilde{\eta}$$

for some $a' > 0$.

This observation basically asserts that we can remove (or, by extension, subtract mass to) an atom from the distribution of a random variable at the cost of adding another (dependent) random variable uniformly bounded in the coupling.

It follows by simply observing that if $\tilde{\eta}$ has the same distribution of X and $\widetilde{X}^{\epsilon}$ is chosen to take the same values as $\tilde{\eta}$ (so that $\widetilde{X}^{\epsilon}$ would retain its usual law and its atom at $\bar{x}$, but becoming totally dependent on $\tilde{\eta}$), then $a\widetilde{X}^{\epsilon} + \tilde{\eta} \stackrel{d}{=} (a+1)X$.

Observation 5.2. *Without loss of generality, we can take* $R = \delta/10$.

In fact, δ is by construction always positive and we can always multiply the potential by any finite constant and incorporate such constant into the disorder parameter a.

We now use these two observations to prove the general result. Take V such that $\mathrm{supp}(V) \subseteq [-R, R]$ and set $R = \delta/10$. Apply Observation 5.1 to the potential V with $\epsilon = \epsilon(a) \gg a^{-\beta}$ for all $\beta > 0$, and decompose $aV = a\widetilde{V}^{\epsilon(a)} + \tilde{\eta}$, where $\tilde{\eta} \stackrel{d}{=} V$, and $\widetilde{V}^{\epsilon(a)}$ is a random variable having the same distribution as V except for having an atom at 0 of mass $\epsilon(a)$ (with the necessary renormalisation). Then $H^a = T^R + a\widetilde{V}^{\epsilon(a)}$, where $T^R = T_0 + \tilde{\eta}$.

Thus again, if $Q(0,x) \geqslant e^{-c|x|}$ we get, as in (9),

$$\gamma^{\mathbb{E}} \leqslant c + 2K \log(\epsilon(a)) \ll \log(a). \tag{16}$$

Furthermore, we call $T^R_{K,x}$ the restriction of T^R to the box $[-Kx, (K+1)x]$, analogously as before. We will shift T^R by $-2 \cdot \mathbf{1}$ so that the spectrum of the resulting operator lies below $\delta/2$. A double resolvent expansion analogue to the one performed in the Proof of Theorem 1.2 and the Combes-Thomas bound yield

$$\begin{aligned} Q(0,x) &\geqslant \delta\ G_\delta[H^a - 2\cdot\mathbf{1}](0,x) \\ &\geqslant \delta\ G_\delta[T^R - 2\cdot\mathbf{1}](0,x) - C_3 e^{-c_3(2K+1)\sqrt{\delta/2}|x|} \end{aligned}$$

on the event

$$\Omega_{x,K;a} := \{V_y^{\epsilon(a)} \equiv 0 \quad \forall\, y \in [-Kx, (K+1)x]\}, \qquad \mathbb{P}\{\Omega_{x,K;a}\} = [\epsilon(a)]^{(2K+1)x}.$$

Remark. Notice that this time we chose δ instead of $i\delta$ as a spectral parameter. The reason for this choice is that we need to use the negativity of the shifted Laplacian to compare its Green's function to that of the (negative) shifted operator.

Eventually, the only thing left to us to show is that

$$G_\delta[T^R - 2\cdot\mathbf{1}](0,x) \geqslant \widetilde{C} e^{-\tilde{c}\sqrt{\delta}|x|}.$$

By writing down the Neumann series for $G_\delta[T - 2 \cdot \mathbf{1}]$, we get the following inequalities:

$$\begin{aligned} |G_\delta[T^R - 2\cdot\mathbf{1}](x,y)| &= \left[-T_0(2+\delta-\tilde{\eta})^{-1}\right]^{-1}(x,y)\cdot\frac{1}{2+\delta-\tilde{\eta}} \\ &= \left|\left(\sum_{n=0}^{\infty}(2+\delta-\tilde{\eta})^{-n}(T_0)^n(x,y)\right)\right|\cdot\frac{1}{2+\delta-\tilde{\eta}} \\ &\geqslant \frac{1}{2+\frac{\delta}{2}}\left|\sum_{n=0}^{\infty}T_0^n(x,y)\left(2+\frac{\delta}{2}\right)^{-n}\right| \\ &= |G_{\delta/2}[T_0 - 2\cdot\mathbf{1}](x,y)|. \end{aligned}$$

We can compute $G_{\delta/2}[T_0 - 2 \cdot \mathbf{1}](0,x)$ explicitly via the same method used in the proof of 1.2. In this case, we get that

$$|G_{\delta/2}[T_0 - 2\cdot\mathbf{1}](x,y)| \geqslant |\alpha| e^{-\xi|x|}, \ \text{with}\ \alpha = \left(2e^{-\xi} + \frac{\delta}{2} - 2\right)^{-1}, \qquad \xi = \operatorname{arccosh}\left(1 + \frac{\delta}{4}\right).$$

In particular, when δ is small, $G_{\delta/2}[T_0 - 2 \cdot \mathbf{1}](x,y)$ decays exponentially with rate of order $\sqrt{\delta}$ and

$$G_\delta[T^R - 2\cdot\mathbf{1}](0,x) \geqslant G_{\delta/2}\,(T_0 - 2\cdot\mathbf{1})](0,x) \geqslant \widetilde{C} e^{-\tilde{c}\sqrt{\delta/2}\ |x|}.$$

Setting, again, K large enough so that $2K+1 = \lceil 100\,\tilde{c}/c_3 \rceil$, and setting $c = \tilde{c}\sqrt{\frac{\delta}{2}}$, we finally conclude that

$$Q(0,x) \geqslant \frac{\delta}{2} G_\delta[T^R - 2\cdot\mathbf{1}](0,x) \geqslant e^{-c|x|}.$$

This, combined with (16) and the Avron–Craig–Simon bound for general bounded potentials proven in Paragraph 4, implies the thesis. □

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Davide Macera: Conceptualization (equal); Investigation (equal); Validation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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